Developments in Fuzzy Sets, Intuitionistic Fuzzy Sets, Generalized Nets and Related Topics. Volume I: Foundations

Recent Advances in Fuzzy Sets, Intuitionistic Fuzzy Sets, Generalized Nets and Related Topics Volume I: Foundations

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IBS PAN SRI PAS

Systems Research Institute
Polish Academy of Sciences
Newelska 6, 01-447 Warsaw, Poland
www.ibspan.waw.pl

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The Inclusion-Exclusion Principle in semigroups

Jana Kelemenová
Faculty of Natural Sciences, Matej Bel University
Banská Bystrica, Slovakia
kelemen@fpv.umb.sk

Abstract

This paper contains a generalization of a Grzegorzewski theorem [1]. He has proved the inclusion- exclusion principle for a special case of IF-events. We prove it for mappings with values in semigroups.

Keywords: inclusion-exclusion principle, semigroup, IF-event.

1 Introduction

Grzegorzewski [1] has worked the probability version of the inclusion-exclusion principle and made a generalization for IF-events. He had applied two versions of the generalized formula, corresponding to different t-conorms and so defined the union of IF-events. The probability $\mathcal{P}(\bigcup_{k=1}^{n} A_k)$ of the union of n ($n \geq 2$) IF-events $A_1, \ldots, A_n$ is obtained as a solution of the equation

$$\mathcal{P}(\bigcup_{k=1}^{n} A_k) + \sum_{k=1}^{n/2} S_{2k}^{(n)} = \sum_{k=1}^{n/2} S_{2k-1}^{(n)}$$

if n is even and a solution of the equation

$$\mathcal{P}(\bigcup_{k=1}^{n} A_k) + \sum_{k=1}^{(n+1)/2-1} S_{2k}^{(n)} = \sum_{k=1}^{(n+1)/2} S_{2k-1}^{(n)}$$

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if n is odd.

$$S_k^{(n)} = \sum_{1 \le i_1 < i_2 < \ldots < i_k \le n} \mathcal{P}(A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_k}).$$

2 Assumptions

Let $(\mathcal{G}, +, \cdot)$ be an algebraic system, where $(\mathcal{G}, +)$ be a commutative semigroup, and $\cdot$ be a commutative binary operation satisfying the following conditions

$$(a+b)\cdot c = a\cdot c + b\cdot c \quad \forall a, b, c \in G c \cdot c = c \qquad \forall c \in G$$

Let (H,+) be a commutative semigroup, and $m : \mathcal{G} \to H$ be a mapping satisfying the valuation property

$$m(a+b) + m(a\cdot b) = m(a) + m(b) \tag{1}$$

We want to generalize the Grzegorzewski theorem for the function $\cdot$.

3 Examples

Example 1 *Let $a, b, c \in G$. Then,*

$$m\left((a+b)\cdot c\right) = m\left(a\cdot c + b\cdot c\right),$$

hence for $n = 3$

$$m((a+b)+c) + m((a+b)\cdot c) = m(a+b) + m(c)$$
$$m(a+b+c) + m(a\cdot c + b\cdot c) + m(a\cdot b) = m(a+b) + m(c) + m(a\cdot b)$$
$$m(a+b+c)+m(a\cdot c+b\cdot c)+m(a\cdot b\cdot c)+m(a\cdot b) =$$
$$= m(a)+m(b)+m(c)+m(a\cdot b\cdot c)$$
$$m(a+b+c)+m(a\cdot c)+m(b\cdot c)+m(a\cdot b) = m(a)+m(b)+m(c)+m(a\cdot b\cdot c)$$

Example 2 *For $n = 4$*

$$m((a+b+c)+d) + m((a+b+c)\cdot d) = m(a+b+c) + m(d)$$
$$m(a+b+c+d) + m(a\cdot c) + m(b\cdot c) + m(a\cdot b) + m(a\cdot d + b\cdot d + c\cdot d) =$$
$$= m(a+b+c) + m(d) + m(a\cdot c) + m(b\cdot c) + m(a\cdot b)$$
$$m(a+b+c+d) + m(a\cdot c) + m(b\cdot c) + m(a\cdot b) + m(a\cdot d + b\cdot d + c\cdot d) =$$
$$= m(a) + m(b) + m(c) + m(d) + m(a\cdot b\cdot c)$$

$$m(a+b+c+d)+m(a\cdot c)+m(b\cdot c)+m(a\cdot b)+m(a\cdot d+b\cdot d+c\cdot d)+$$
$$+m(a\cdot d\cdot b\cdot d)+m(a\cdot d\cdot c\cdot d)+m(b\cdot d\cdot c\cdot d)=$$
$$=m(a)+m(b)+m(c)+m(d)+m(a\cdot b\cdot c)+m(a\cdot b\cdot d)+m(a\cdot c\cdot d)+$$
$$+m(b\cdot c\cdot d)m(a+b+c+d)+m(a\cdot c)+m(b\cdot c)+m(a\cdot b)+m(a\cdot d)+$$
$$+m(b\cdot d)++m(c\cdot d)++m(a\cdot b\cdot c\cdot d)=$$
$$=m(a)+m(b)+m(c)+m(d)+m(a\cdot b\cdot c)+m(a\cdot b\cdot d)+m(a\cdot c\cdot d)+$$
$$+m(b\cdot c\cdot d)$$

4 Main result

Theorem 1 *Let $m:\mathcal{G}\to H$ satisfying the condition (1) for any $a,b\in\mathcal{G}$.*
Then for n even we have

$$m(a_1+a_2+\ldots+a_n)+\sum_{k=1}^{n/2}S_{2k}^{(n)}=\sum_{k=1}^{n/2}S_{2k-1}^{(n)},$$

where

$$S_k^{(n)}=\sum_{1\le i_1<i_2<\ldots<i_k\le n}m(a_{i_1}\cdot a_{i_2}\cdot\ldots\cdot a_{i_k})$$

For n odd we have

$$m(a_1+a_2+\ldots+a_n)+\sum_{k=1}^{(n+1)/2-1}S_{2k}^{(n)}=\sum_{k=1}^{(n+1)/2}S_{2k-1}^{(n)},$$

where

$$S_k^{(n)}=\sum_{1\le i_1<i_2<\ldots<i_k\le n}m(a_{i_1}\cdot a_{i_2}\cdot\ldots\cdot a_{i_k})$$

Proof.

We shall prove it by induction. We have seen that Theorem holds for $n=2$ (assumption (1)) and $n=3$ (Example 1)

Let n be even and the principle holds for n, we shall prove it for $n+1$.

Since

$$m\left(\sum_{k=1}^{n+1}a_k)=m((a_1+a_2+\ldots+a_n)+a_{n+1}\right)$$

then, by (1) we have:

$$m\left((\sum_{k=1}^{n} a_k) + a_{n+1}\right) + m\left(\sum_{k=1}^{n} a_k \cdot a_{n+1}\right) = m\left(\sum_{k=1}^{n} a_k\right) + m\left(a_{n+1}\right) \quad (2)$$

Since n is even, then

$$m(a_1 + a_2 + \ldots + a_n) + \sum_{k=1}^{n/2} S_{2k}^{(n)} = \sum_{k=1}^{n/2} S_{2k-1}^{(n)}$$

Induction assumption:

$$m\left(\sum_{k=1}^{n} a_k\right) + \sum_{k=1}^{n/2} \sum_{1\le i_1<i_2<\ldots<i_{2k}\le n} m\left(a_{i_1} \cdot a_{i_2} \cdot \ldots \cdot a_{i_{2k}}\right) =$$

$$= \sum_{k=1}^{n/2} \sum_{1\le i_1<i_2<\ldots<i_{2k-1}\le n} m\left(a_{i_1} \cdot a_{i_2} \cdot \ldots \cdot a_{i_{2k-1}}\right) \quad (3)$$

Moreover,

$$m\left((\sum_{k=1}^{n} a_k) \cdot a_{n+1}\right) = m\left(\sum_{k=1}^{n} (a_k \cdot a_{n+1})\right)$$

so we get

$$m\left(\sum_{k=1}^{n} (a_k \cdot a_{n+1})\right) + \sum_{k=1}^{n/2} \sum_{1\le i_1<i_2<\ldots<i_{2k}\le n} m\left(a_{i_1} \cdot a_{i_2} \cdot \ldots \cdot a_{i_{2k}} \cdot a_{n+1}\right) =$$

$$= \sum_{k=1}^{n/2} \sum_{1\le i_1<i_2<\ldots<i_{2k-1}\le n} m\left(a_{i_1} \cdot a_{i_2} \cdot \ldots \cdot a_{i_{2k-1}} \cdot a_{n+1}\right) \quad (4)$$

By (2) and (4) we obtain:

$$m\left(\sum_{k=1}^{n+1} a_k\right) + m\left(\sum_{k=1}^{n} (a_k \cdot a_{n+1})\right) + \sum_{k=1}^{n/2} S_{2k}^{(n)} +$$

$$+ \sum_{k=1}^{n/2} \sum_{1\le i_1<i_2<\ldots<i_{2k}\le n} m\left(a_{i_1} \cdot a_{i_2} \cdot \ldots \cdot a_{i_{2k}} \cdot a_{n+1}\right) = m\left(\sum_{k=1}^{n} a_k\right) +$$

$$+ m\left(a_{n+1}\right) + \sum_{k=1}^{n/2} S_{2k}^{(n)} + \sum_{k=1}^{n/2} \sum_{1\le i_1<i_2<\ldots<i_{2k}\le n} m\left(a_{i_1} \cdot a_{i_2} \cdot \ldots \cdot a_{i_{2k}} \cdot a_{n+1}\right) \quad (5)$$

By (5), (3):

$$m\left(\sum_{k=1}^{n+1} a_k\right)+\sum_{k=1}^{n/2} S_{2k}^{(n)}+m\left(\sum_{k=1}^{n} a_k\cdot a_{n+1}\right)+$$

$$+\sum_{k=1}^{n/2}\sum_{1\le i_1<i_2<\ldots<i_{2k}\le n} m\left(a_{i_1}\cdot a_{i_2}\cdot\ldots\cdot a_{i_{2k}}\cdot a_{n+1}\right)=m(a_{n+1})+$$

$$+\sum_{k=1}^{n/2} S_{2k-1}^{(n)}+\sum_{k=1}^{n/2}\sum_{1\le i_1<i_2<\ldots<i_{2k}\le n} m\left(a_{i_1}\cdot a_{i_2}\cdot\ldots\cdot a_{i_{2k}}\cdot a_{n+1}\right) \qquad (6)$$

By (6) and (4) we get

$$m\left(\sum_{k=1}^{n+1} a_k\right)+\sum_{k=1}^{n/2} S_{2k}^{(n)}$$

$$+\sum_{k=1}^{n/2}\sum_{1\le i_1<i_2<\ldots<i_{2k-1}\le n} m\left(a_{i_1}\cdot a_{i_2}\cdot\ldots\cdot a_{i_{2k-1}}\cdot a_{n+1}\right)=\sum_{k=1}^{n/2} S_{2k-1}^{(n)}+$$

$$+m(a_{n+1})+\sum_{k=1}^{n/2}\sum_{1\le i_1<i_2<..<i_{2k}\le n} m\left(a_{i_1}\cdot a_{i_2}\cdot\ldots\cdot a_{i_{2k}}\cdot a_{n+1}\right) \qquad (7)$$

hence

$$m\left(\sum_{k=1}^{n+1} a_k\right)+\sum_{k=1}^{(n+2)/2-1}\sum_{1\le i_1<i_2<\ldots<i_{2k}\le n+1} m\left(a_{i_1}\cdot a_{i_2}\cdot\ldots\cdot a_{i_{2k}}\right)=$$

$$=\sum_{k=1}^{(n+2)/2}\sum_{1\le i_1<i_2<\ldots<i_{2k-1}\le n+1} m\left(a_{i_1}\cdot a_{i_2}\cdot\ldots\cdot a_{i_{2k-1}}\right)$$

So,

$$m\left(\sum_{k=1}^{n+1} a_k\right)+\sum_{k=1}^{(n+2)/2-1} S_{2k}^{(n+1)}=\sum_{k=1}^{(n+2)/2-1} S_{2k-1}^{(n+1)}$$

Let n be odd, hence the induction assumption gives

$$m\left(\sum_{k=1}^{n} a_k\right)+\sum_{k=1}^{(n+1)/2-1} S_{2k}^{(n)}=\sum_{k=1}^{(n+1)/2} S_{2k-1}^{(n)} \qquad (8)$$

Induction assumption implies

$$m\left(\sum_{k=1}^{n}(a_k\cdot a_{n+1})\right)+$$

$$+\sum_{k=1}^{(n+1)/2-1}\sum_{1\le i_1<i_2<\ldots<i_{2k}\le n} m\left(a_{i_1}\cdot a_{i_2}\cdot\ldots\cdot a_{i_{2k}}\cdot a_{n+1}\right)=$$

$$=\sum_{k=1}^{(n+1)/2}\sum_{1\le i_1<i_2<\ldots<i_{2k-1}\le n} m\left(a_{i_1}\cdot a_{i_2}\cdot\ldots\cdot a_{i_{2k-1}}\cdot a_{n+1}\right). \tag{9}$$

From (2) we have:

$$m\left(\sum_{k=1}^{(n+1)} a_k\right)+m\left(\sum_{k=1}^{n}(a_k\cdot a_{n+1})\right)+\sum_{k=1}^{n/2} S_{2k}^{(n)}+$$

$$+\sum_{k=1}^{(n+1)/2-1}\sum_{1\le i_1<i_2<\ldots<i_{2k}\le n} m\left(a_{i_1}\cdot a_{i_2}\cdot\ldots\cdot a_{i_{2k}}\cdot a_{n+1}\right)=$$

$$=m\left(\sum_{k=1}^{n} a_k\right)+\sum_{k=1}^{n/2} S_{2k}^{(n)}+$$

$$\sum_{k=1}^{(n+1)/2-1}\sum_{1\le i_1<i_2<..<i_{2k}\le n} m\left(a_{i_1}\cdot a_{i_2}\cdot\ldots\cdot a_{i_{2k}}\cdot a_{(n+1)}\right)+m(a_{n+1}) \tag{10}$$

By (10), (9) and (8)

$$m\left(\sum_{k=1}^{n+1} a_k\right)+$$

$$\sum_{k=1}^{(n+1)/2}\sum_{1\le i_1<i_2<\ldots<i_{2k-1}\le n} m\left(a_{i_1}\cdot a_{i_2}\cdot\ldots\cdot a_{i_{2k-1}}\cdot a_{n+1}\right)+\sum_{k=1}^{n/2} S_{2k}^{(n)}=$$

$$=\sum_{k=1}^{n+1/2} S_{2k-1}^{(n)}+m(a_{n+1})+$$

$$\sum_{k=1}^{(n+1)/2-1}\sum_{1\le i_1<i_2<\ldots<i_{2k}\le n} m\left(a_{i_1}\cdot a_{i_2}\cdot\ldots\cdot a_{i_{2k}}\cdot a_{n+1}\right)$$

Hence,

$$m\left(\sum_{k=1}^{n+1} a_k\right) + \sum_{k=1}^{(n+1)/2} \sum_{1\le i_1<i_2<\ldots<i_{2k}\le n+1} m\left(a_{i_1}\cdot a_{i_2}\cdot\ldots\cdot a_{i_{2k}}\right) =$$
$$= \sum_{k=1}^{(n+1)/2} \sum_{1\le i_1<i_2<\ldots<i_{2k-1}\le n+1} m\left(a_{i_1}\cdot a_{i_2}\cdot\ldots\cdot a_{i_{2k-1}}\right).$$

$$m\left(\sum_{k=1}^{(n+1)} a_k\right) + \sum_{k=1}^{(n+1)/2} S_{2k}^{(n)} = \sum_{k=1}^{(n+1)/2} S_{2k-1}^{(n)}.$$

5 Conclusions

The inclusion- exclusion principle works for semigroups, but we can see many applications also on IF-events as in the paper [1],with Gödel connectives.

5.1. Gödel connectives
Let use the operations

$$A \vee B = (\max(\mu_A,\mu_B), \min(\nu_A,\nu_B))$$
$$A \wedge B = (\min(\mu_A,\mu_B), \max(\nu_A,\nu_B)).$$

For any $A_1,\ldots,A_n \in F, n \in N$ there holds

$$\mathcal{P}(\bigvee_{i=1}^{n} A_i) = \sum_{k=1}^{n}(-1)^{k+1} \sum_{1\le j_1<\ldots<j_k\le n} \mathcal{P}(\bigwedge_{i=1}^{k} A_{j_i}).$$

Another application is on the algebraic system with two binary operations: $(M,+,\cdot)$, and a mapping $m : M \to< 0,1 >$, where M is any commutative and associative algebraic system.

5.2. Algebraic system $(M,+,\cdot)$, where

- The operations +, . are commutative and associative
- The distributivity law holds and $c\cdot c = c \quad \forall c\in M$

For $a_1, \dots, a_n \in M$ we have

$$m(a_1 + \ldots + a_n) = \sum_{i=1}^{n} m(a_i) - \sum_{i<j}^{n} m(a_i \cdot a_j) +$$
$$+ \sum_{i<j<k}^{n} m(a_i \cdot a_j \cdot a_k) - \ldots + (-1)^{n+1} m\left(\prod_{i=1}^{n} a_i\right).$$

For the proof see [3].

References

[1] Grzegorzewski, P. (2011) The Inclusion-Exclusion Principle for IF - Events, Information Sciences, Volume 181, Issue 3, 536-546.

[2] Atanassov, K. (1999) Intuitionistic Fuzzy Sets: Theory and Applications, Physica- Verlag.

[3] Kuková, M. (2011) The Inclusion-Exclusion Principle on some algebraic structures, To appear.

The papers presented in this Volume 1 constitute a collection of contributions, both of a foundational and applied type, by both well-known experts and young researchers in various fields of broadly perceived intelligent systems.

It may be viewed as a result of fruitful discussions held during the Ninth International Workshop on Intuitionistic Fuzzy Sets and Generalized Nets (IWIFSGN-2010) organized in Warsaw on October 8, 2010 by the Systems Research Institute, Polish Academy of Sciences, in Warsaw, Poland, Institute of Biophysics and Biomedical Engineering, Bulgarian Academy of Sciences in Sofia, Bulgaria, and WIT - Warsaw School of Information Technology in Warsaw, Poland, and co-organized by: the Matej Bel University, Banska Bystrica, Slovakia, Universidad Publica de Navarra, Pamplona, Spain, Universidade de Tras-Os-Montes e Alto Douro, Vila Real, Portugal, and the University of Westminster, Harrow, UK:

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The consecutive International Workshops on Intuitionistic Fuzzy Sets and Generalized Nets (IWIFSGNs) have been meant to provide a forum for the presentation of new results and for scientific discussion on new developments in foundations and applications of intuitionistic fuzzy sets and generalized nets pioneered by Professor Krassimir T. Atanassov. Other topics related to broadly perceived representation and processing of uncertain and imprecise information and intelligent systems have also been included. The Ninth International Workshop on Intuitionistic Fuzzy Sets and Generalized Nets (IWIFSGN-2010) is a continuation of this undertaking, and provides many new ideas and results in the areas concerned.

We hope that a collection of main contributions presented at the Workshop, completed with many papers by leading experts who have not been able to participate, will provide a source of much needed information on recent trends in the topics considered.

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