Developments in Fuzzy Sets,
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Volume I: Foundations

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Dependencies between fuzzy negation, disjunction and implication

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Abstract

This paper deals with some dependencies between weak fuzzy connectives of different types. In particular, a fuzzy implication generated from a fuzzy negation and disjunction is considered. In the case of a fuzzy disjunction only border conditions and monotonicity are assumed. The results are illustrated by examples of weak fuzzy connectives.

Keywords: fuzzy negation, fuzzy disjunction, (D,N)-implication.

1 Introduction

Multivalued logic with truth values in [0,1] was developed after the paper of J. Łukasiewicz [7]. Fuzzy set theory introduced by L.A. Zadeh [10] brought new applications of multivalued logic and new directions in examination of logical connectives. After the contribution of B. Schweizer and A. Sklar [9] the notions of the triangular norm and conorm have played the role of a fuzzy conjunction and disjunction. J. Fodor and M. Roubens [6], M. Baczyński and B. Jayaram [1] examined families of multivalued connectives based on triangular norms and conorms. However, some authors (e.g. I. Batyrshin and O. Kaynak [3], F. Durante et al. [5]) underline that the assumptions made on these multivalued connectives are sometimes too strong and difficult to obtain. Thus, some of the conditions are omitted.

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In this contribution the way of generating of fuzzy implication from a fuzzy negation and a fuzzy disjunction is considered. Implications created by the use of such method are considered in the literature in the case when the disjunction is a triangular conorm by J. Fodor and M. Roubens [6] (S-implications), M. Baczyński and B. Jayaram [1] ((S,N)-implications). Also, the case when a disjunction is replaced by a disjunctive uninorm was examined by M. Baczyński and B. Jayaram [2] ((U,N)-implications). In these consideration only border conditions and monotonicity are required for a fuzzy disjunction. This is why such implications will be called (D,N)-implications. This name should not be confused with Dishkant implications (D-implications), see e.g. [8].

In the following section the definitions and examples of fuzzy connectives used in the sequel are presented. Next, in Section 3, fuzzy negations generated from fuzzy implications are recalled. Finally, Section 4, presents the results concerning (D,N)-implications.

2 Basic definitions

First, the notion and examples of a fuzzy negation are recalled.

Definition 1 ([1], p. 14)**.** *A decreasing function $N\colon [0,1] \to [0,1]$ is called a fuzzy negation if*

$$N(0)=1, \qquad N(1)=0.$$

A fuzzy negation is called:
• a strict negation if it is continuous and strictly decreasing,
• a strong negation if it an involution, i.e. it fulfills the condition $N(N(x))=x$, $x\in[0,1]$,
• a non-vanishing negation, if $N(x)=0 \Leftrightarrow x=1$, $x\in[0,1]$,
• a non-filling negation, if $N(x)=1 \Leftrightarrow x=0$, $x\in[0,1]$.

Remark 1. *Strict and strong negations are continuous functions. A strong negation is a strict one. A strict or a strong negation is both non-vanishing and non-filling.*

Example 1. *The classical fuzzy negation $N_C(x)=1-x$ for $x\in[0,1]$ fulfills all four additional assumptions of the above definition. The negation $N(x)=1-x^2$ for $x\in[0,1]$ is strict.*
The least and the greatest fuzzy negations are of the form:

$$N_0(x)=\begin{cases} 1, & gdy\ \ x=0 \\ 0, & gdy\ \ x>0 \end{cases}, \qquad N_1(x)=\begin{cases} 1, & gdy\ \ x<1 \\ 0, & gdy\ \ x=1 \end{cases}.$$

The negation N_0 is non-filling, N_1 is non-vanishing.

Now, the definition of a fuzzy disjunction is presented.

Definition 2 ([4])**.** *An operation $D : [0,1]^2 \to [0,1]$ is called a fuzzy disjunction if it is increasing with respect to each variable and*

$$D(0,0) = 0, \quad D(0,1) = D(1,0) = D(1,1) = 1.$$

Corollary 1. *A fuzzy disjunction has a zero element* 1.

Example 2. *Operations D_0 and D_1 are the least and the greatest fuzzy disjunction, respectively, where*

$$D_0(x,y) = \begin{cases} 1, & \textit{if } x = 1 \textit{ or } y = 1 \\ 0, & \textit{else} \end{cases},$$

$$D_1(x,y) = \begin{cases} 0, & \textit{if } x = y = 0 \\ 1, & \textit{else} \end{cases}.$$

The following are other examples of fuzzy disjunctions. The well-known triangular conorms are denoted in the traditional way.

$$D_2(x,y) = \begin{cases} y, & \text{if } x = 0 \\ 1, & \text{if } x > 0 \end{cases}, \qquad S_M(x,y) = \max(x,y),$$

$$D_3(x,y) = \begin{cases} x, & \text{if } y = 0 \\ 1, & \text{if } y > 0 \end{cases}, \qquad S_P(x,y) = x + y - xy,$$

$$D_4(x,y) = \begin{cases} 1, & \text{if } x + y \geqslant 1 \\ y, & \text{if } x + y < 1 \end{cases}, \qquad S_L(x,y) = \min(x+y, 1),$$

$$D_5(x,y) = \begin{cases} 1, & \text{if } x + y \geqslant 1 \\ x, & \text{if } x + y < 1 \end{cases}, \qquad S_D(x,y) = \begin{cases} x, & \text{for } y = 0 \\ y, & \text{for } x = 0 \\ 1 & \text{else} \end{cases}.$$

Finally, the notion of a fuzzy implication is given.

Definition 3 ([1], pp. 2,9)**.** *A function $I \colon [0,1]^2 \to [0,1]$ is called a fuzzy implication if it is decreasing with respect to the first variable and increasing with respect to the second variable and*

$$I(0,0) = I(0,1) = I(1,1) = 1, \quad I(1,0) = 0.$$

We say that a fuzzy implication I fulfills:
• *neutral property (NP) if $I(1,y) = y,\ y \in [0,1]$,*

• *exchange principle (EP) if* $I((x, I(y,z)) = I(y, I(x,z)),\ x,y,z \in [0,1]$,
• *identity principle (IP) if* $I(x,x) = 1,\ x \in [0,1]$,
• *ordering property (OP) if* $I(x,y) = 1 \Leftrightarrow x \leq y,\ x,y \in [0,1]$.

Example 3 ([1], pp. 4,5)**.** *Operations* I_0 *and* I_1 *are the least and the greatest fuzzy implication, respectively, where*

$$I_0(x,y) = \begin{cases} 1, & \textit{if } x = 0 \textit{ or } y = 1 \\ 0, & \textit{else} \end{cases},$$

$$I_1(x,y) = \begin{cases} 0, & \textit{if } x = 1, y = 0 \\ 1, & \textit{else} \end{cases}.$$

The following are other examples of fuzzy implications.

$$I_{\text{ŁK}}(x,y) = \min(1-x+y, 1), \quad I_{\text{GG}}(x,y) = \begin{cases} 1, & \text{if } x \leq y \\ \frac{y}{x}, & \text{if} x > y \end{cases},$$

$$I_{\text{GD}}(x,y) = \begin{cases} 1, & \text{if } x \leq y \\ y, & \text{if } x > y \end{cases}, \quad I_{\text{RS}}(x,y) = \begin{cases} 1, & \text{if } x \leq y \\ 0, & \text{if } x > y \end{cases},$$

$$I_{\text{RC}}(x,y) = 1 - x + xy, \quad I_{\text{YG}}(x,y) = \begin{cases} 1, & \text{if } x,y = 0 \\ y^x, & \text{if else} \end{cases},$$

$$I_{\text{DN}}(x,y) = \max(1-x, y), \quad I_{\text{FD}}(x,y) = \begin{cases} 1, & \text{if } x \leq y \\ \max(1-x,y), & \text{if } x > y \end{cases}.$$

3 Negation generated from implication

A fuzzy negation can be generated from a fuzzy implication by means of a simple dependence as in the following theorem.

Theorem 1 ([1], p. 18)**.** *Let* I *be a fuzzy implication. The function* $N_I\colon [0,1] \to [0,1]$, *where*

$$N_I(x) = I(x,0), \quad x \in [0,1] \tag{1}$$

is a fuzzy negation.

Proof. Let N_I be defined by the formula (1). We have

$$N_I(0) = I(0,0) = 1, \qquad N_I(1) = I(1,0) = 0.$$

The monotonicity of the function N_I follows from the monotonicity of the fuzzy implication with respect to the first variable. □

The next theorem shows which of additional assumptions made on the fuzzy implication cause the membership of the generated negation to each of the four classes of fuzzy negations from Definition 1.

Theorem 2. *Let I be a fuzzy implication. Then*
(i) if I is strictly decreasing and continuous with respect to the first variable, then N_I is strict,
(ii) if I is a fuzzy implication continuous with respect to the first variable fulfilling (EP) and (OP), then N_I is a strong negation,
(iii) N_I is non-vanishing negation if and only if the following condition holds

$$I(x,0) > 0 \quad x \in [0,1),$$

(iv) N_I is non-filling negation if and only if the following condition holds

$$I(x,0) < 1 \quad x \in (0,1].$$

Proof. Parts (i), (iii) i (iv) follow directly by the definition of the fuzzy negation N_I. To prove (ii) it is sufficient to show that N_I is an involution. Let $x \in [0,1]$. By (EP) and (OP) one obtains

$$I(x, N_I(N_I(x))) = I(x, I(I(x,0),0)) = I(I(x,0), I(x,0)) = 1.$$

that is, again by (OP),

$$x \leq N_I(N_I(x)). \tag{2}$$

By the monotonicity of N_I and by (2) one has

$$N_I(N_I(N_I(x))) \leq N_I(x).$$

On the other hand

$$\begin{aligned} I(N_I(x), N_I(N_I(N_I(x)))) &= I(I(x,0), I(I(I(x,0),0),0)) \\ &= I(I(I(x,0),0), I(I(x,0),0)) = 1, \end{aligned}$$

so $N_I(N_I(N_I(x))) \geqslant N_I(x)$. Thus,

$$N_I(N_I(N_I(x))) = N_I(x), \quad x \in [0,1]. \tag{3}$$

By continuity with respect to the first variable of the fuzzy implication I the continuity of N_I follows. That means that for an arbitrary $y \in [0,1]$ there exists $x \in [0,1]$ such that $y = N_I(x)$. Hence and from (3) we obtain

$$N_I(N_I(y)) = N_I(N_I(N_I(x))) = N_I(x) = y, \quad y \in [0,1],$$

so N_I is an involution. □

Example 4 ([1], p. 18)**.** *The following table shows fuzzy negations with their generators (cf. Examples 1, 3).*

Implication I	*Negation* N_I
I_{LK}	N_C
I_{GD}	N_0
I_{RC}	N_C
I_{DN}	N_C
I_{GG}	N_0
I_{RS}	N_0
I_{YG}	N_0
I_{FD}	N_C

For example one can check that the implication I_{LK} generates the negation N_C. For an arbitrary $x \in [0, 1]$ we have

$$N_{I_{LK}}(x) = I_{LK}(x, 0) = \min(1, 1 - x) = 1 - x.$$

4 Implication generated from negation and disjunction

One can use the connection between the implication, disjunction and negation in the classical propositional calculus

$$p \to q \Leftrightarrow \sim p \vee q$$

in order to define a class of fuzzy implication.

Theorem 3. *Let N and D be a fuzzy negation and disjunction, respectively. Then the function $I_{D,N}$ of the form*

$$I_{D,N}(x, y) = D(N(x), y), \quad x, y \in [0, 1]$$

*is a fuzzy implication. Such implication will be called (D, N)-*implication*.*

Proof. The function $I_{D,N}$ is decreasing with respect to the first variable and increasing with respect to the second variable as the composition of the monotonic functions N and D. Moreover, directly from Definitions 1,2 one gets

$$I_{D,N}(0, 0) = D(N(0), 0) = D(1, 0) = 1,$$

$$I_{D,N}(0, 1) = D(N(0), 1) = D(1, 1) = 1,$$

$$I_{D,N}(1, 1) = D(N(1), 1) = D(0, 1) = 1,$$

$$I_{D,N}(1, 0) = D(N(1), 0) = D(0, 0) = 0,$$

what proves that $I_{D,N}$ is a fuzzy implication. $\square$

Example 5. *The following table presents (D, N)-implications together with their generators (cf. Examples 1 - 3).*

Disjunction D	*Negation* N	*Implication* $I_{D,N}$
D_0	N *- a non-fulfilling negation*	I_0
D_1	N *- a non-vanishing negation*	I_1
S_{LK}	N_C	I_{LK}
S_P	N_C	I_{RC}
S_M	N_C	I_{DN}

For example one can generate the least fuzzy implication I_0. Let N be an arbitrary non-filling negation, D_0 the least fuzzy disjunction. Then

$$I_{D_0,N}(0, y) = D_0(N(0), y) = D_0(1, y) = 1,$$

$$I_{D_0,N}(x, 1) = D_0(N(x), 1) = 1.$$

In the case of $y \neq 1$ and $x \neq 0$ one has $N(x) \neq 1$ and hence

$$I_{D_0,N}(x, y) = D_0(N(x), y) = 0.$$

Altogether one obtain $I_{D_0,N} = I_0$.

Theorem 4. *Let N be a fuzzy negation. The implication $I_{D,N}$ fulfills (NP) if and only if 0 is a left neutral element of D.*

Proof. Let us see, that for all $y \in [0, 1]$ one has

$$I_{D,N}(1, y) = D(N(1), y) = D(0, y).$$

This is why the condition (NP) is fulfilled if and only if

$$D(0, y) = y, \quad y \in [0, 1],$$

that is 0 is a left neutral element of the operation D. □

Theorem 5. *Let N be a fuzzy negation and D be an associative and commutative fuzzy disjunction. Then $I_{D,N}$ fulfills (EP).*

Proof. For arbitrary $x, y, z \in [0, 1]$, from associativity and commutativity of the operation D one has

$$I_{D,N}(x, I_{D,N}(y, z)) = D(N(x), D(N(y), z)) = D(D(N(x), N(y)), z)) =$$

$$= D(D(N(y), N(x)), z)) = D(N(y), D(N(x), z)) = I_{D,N}(y, I_{D,N}(x, z)).$$

□

Example 6. *Let us observe that (D, N)-implications do not need to fulfill (OP) and (IP). (D, N)-implication I_{RC} fulfills none of these conditions. We have $0.5 \leqslant 0.5$, but*

$$I_{RC}(0.5, 0.5) = 1 - 0.5 + (0.5)^2 = 0,75 \neq 1.$$

So the conditions (OP) and (IP) are not fulfilled.

Theorem 6. *(D,N)-implication $I_{D_1,N}$ fulfills (IP).*

Proof. Let us consider the (D,N)-implication $I_{D_1,N}$ generated from the greatest fuzzy disjunction and an arbitrary fuzzy negation. One has

$$I_{D_1,N}(0,0) = D_1(1,0) = 1.$$

Additionally, for arbitrary $x \in (0, 1]$ one can obtain

$$I_{D_1,N}(x, x) = D_1(N(x), x) = 1$$

for the sake of non-zero second variable. □

Theorem 7. *Let $I_{D,N}$ be an implication generated from a fuzzy negation N and fuzzy disjunction D. The equality $N_{I_{D,N}} = N$ holds if and only if 0 is the right neutral element of the operation $D|_{Range(N)}$.*

Proof. For an arbitrary $x \in [0, 1]$ we have

$$N_{I_{D,N}}(x) = I_{D,N}(x, 0) = D(N(x), 0).$$

Now, it is easy to observe that $D(N(x), 0) = 0$ if and only if

$$D(y, 0) = y \quad \text{for} \quad y \in Range(N).$$

□

5 Conclusion

In this contribution a generalization of Boolean formula of the implication which connects the implication with the negation and the disjunction is examined. Many particular dependencies between fuzzy logical connectives can be found in the monograph [1] which can be a source of further generalizations.

References

[1] M. Baczyński, B. Jayaram, Fuzzy implications, Springer, Berlin 2008.

[2] M. Baczyński, B. Jayaram, (U,N)-implications and their characterizations, Fuzzy Sets Syst. **160** (2009), 2049–2062.

[3] I. Batyrshin, O. Kaynak, Parametric classes of generalized conjunction and disjunction operations for fuzzy modelling, IEEE Trans. Fuzzy Syst. **7** (1999), 586–595.

[4] J. Drewniak, A. Król, A survey of weak connectives and the preservation of their properties by aggregations, Fuzzy Sets Syst. **161** (2010), 202–215.

[5] F. Durante, E.P. Klement, R. Mesiar, C. Sempi, Conjunctors and their residual implicators: characterizations and construction methods, Mediterr. J. Math. **4** (2007), 343–356.

[6] J.C. Fodor, M. Roubens, Fuzzy preference modelling and multicriteria decision support, Kluwer, Dordrecht 1994.

[7] J. Łukasiewicz, Interpretacja liczbowa teorii zda, Ruch Filozoficzny **7** (1923), 92–93. (translated in: L. Borkowski, (Ed.), Jan Łukasiewicz Selected Works, North Holland, Amsterdam, Polish Sci. Publ., Warszawa, 1970 129–130.

[8] M. Mas, M. Monserrat, J. Torrens, QL-implications versus D-implications. Kybernetika, **42** (2006), 351–366.

[9] B. Schweizer, A. Sklar, Probabilistic metric spaces, North Holland, New York 1983.

[10] L.A. Zadeh, Fuzzy sets, Inform. Control **8** (1965), 338–353.

The papers presented in this Volume 1 constitute a collection of contributions, both of a foundational and applied type, by both well-known experts and young researchers in various fields of broadly perceived intelligent systems.
It may be viewed as a result of fruitful discussions held during the Ninth International Workshop on Intuitionistic Fuzzy Sets and Generalized Nets (IWIFSGN-2010) organized in Warsaw on October 8, 2010 by the Systems Research Institute, Polish Academy of Sciences, in Warsaw, Poland, Institute of Biophysics and Biomedical Engineering, Bulgarian Academy of Sciences in Sofia, Bulgaria, and WIT - Warsaw School of Information Technology in Warsaw, Poland, and co-organized by: the Matej Bel University, Banska Bystrica, Slovakia, Universidad Publica de Navarra, Pamplona, Spain, Universidade de Tras-Os-Montes e Alto Douro, Vila Real, Portugal, and the University of Westminster, Harrow, UK:

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The consecutive International Workshops on Intuitionistic Fuzzy Sets and Generalized Nets (IWIFSGNs) have been meant to provide a forum for the presentation of new results and for scientific discussion on new developments in foundations and applications of intuitionistic fuzzy sets and generalized nets pioneered by Professor Krassimir T. Atanassov. Other topics related to broadly perceived representation and processing of uncertain and imprecise information and intelligent systems have also been included. The Ninth International Workshop on Intuitionistic Fuzzy Sets and Generalized Nets (IWIFSGN-2010) is a continuation of this undertaking, and provides many new ideas and results in the areas concerned.

We hope that a collection of main contributions presented at the Workshop, completed with many papers by leading experts who have not been able to participate, will provide a source of much needed information on recent trends in the topics considered.

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