Developments in Fuzzy Sets, Intuitionistic Fuzzy Sets, Generalized Nets and Related Topics. Volume I: Foundations

Recent Advances in Fuzzy Sets, Intuitionistic Fuzzy Sets, Generalized Nets and Related Topics
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Properties of intuitionistic fuzzy preference relations

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Abstract

We consider some properties of intuitionistic fuzzy preference relations. We pay attention to preservation of preference relation by lattice operations, composition and some Atanassov's operators like $F_{\alpha,\beta}$, $P_{\alpha,\beta}$, $Q_{\alpha,\beta}$, where $\alpha, \beta \in [0, 1]$. We also define some semi-properties of intuitionistic fuzzy relations. These properties are: reflexivity, irreflexivity, connectedness, asymmetry, transitivity. Moreover, we study under which assumptions intuitionistic fuzzy preference relations fulfil these properties.

Keywords: intuitionistic fuzzy preference relations, properties of intuitionistic fuzzy relations.

1 Introduction

We deal with Atanassov's intuitionistic fuzzy relations which were introduced by Atanassov [1] as a generalization of the idea of fuzzy relations defined by Zadeh [18]. We will also call them intuitionistic fuzzy relations. Fuzzy sets and relations have many applications in diverse types of areas, for example in data bases, pattern recognition, neural networks, fuzzy modelling, economy, medicine, multicriteria decision making. Moreover, multiattribute decision making using intuitionistic fuzzy sets is possible [10]. If it comes to the composition of intuitionistic fuzzy

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relations the effective approach to deal with decision making in medical diagnosis was proposed [5]. We take into account intuitionistic fuzzy preference relations which are applied in group decision making problems where a solution from the individual preferences over some set of options should be derived. The concept of a preference relation was considered by many authors, in the crisp case for example by [13] and in the fuzzy environment by [4]. The first authors who generalized the concept of preference from the fuzzy case to the intuitionistic fuzzy one, were Szmidt and Kacprzyk [14]. Next, other papers were devoted to this topic, for example [16], [15], [17].
In this work we recall some concepts and results useful in our further considerations (section 2). Next, we put results connected with the preservation of preference relation by lattice operations, composition and Atanassov's operators (section 3). Finally, we define some new properties of intuitionistic fuzzy relations and we check when such properties are fulfilled by intuitionistic preference relations (section 4).

2 Basic definitions

Now we recall some definitions which will be helpful in our investigations.

Definition 1 ([1]). *Let $X, Y \neq \emptyset$, $R,\ R^d : X \times Y \to [0,1]$ be fuzzy relations fulfilling the condition*

$$R(x,y) + R^d(x,y) \leq 1, \ \ (x,y) \in (X \times Y). \qquad (1)$$

A pair $\rho = (R, R^d)$ is called an Atanassov's intuitionistic fuzzy relation. The family of all Atanassov's intuitionistic fuzzy relations described in the given sets X, Y is denoted by $AIFR(X \times Y)$. In the case $X = Y$ we will use the notation $AIFR(X)$.

The boundary elements in *AIFR*$(X \times Y)$ are $\mathbf{1} = (1,0)$ and $\mathbf{0} = (0,1)$, where $0, 1$ are the constant fuzzy relations. Basic operations for $\rho = (R, R^d)$, $\sigma = (S, S^d) \in AIFR(X \times Y)$ are the union and the intersection

$$\rho \vee \sigma = (R \vee S, R^d \wedge S^d), \ \rho \wedge \sigma = (R \wedge S, R^d \vee S^d). \qquad (2)$$

Similarly, for arbitrary set $T \neq \emptyset$

$$(\bigvee_{t \in T} \rho_t)(x,y) = (\bigvee_{t \in T} R_t(x,y), \bigwedge_{t \in T} R_t^d(x,y)),$$

$$(\bigwedge_{t\in T}\rho_t)(x,y)=(\bigwedge_{t\in T}R_t(x,y),\bigvee_{t\in T}R_t^d(x,y)).$$

Moreover, the order is defined by

$$\rho\le\sigma\ \Leftrightarrow\ (R\le S,\ S^d\le R^d). \tag{3}$$

The pair $(AIFR(X\times Y),\le)$ is a partially ordered set. Operations $\vee,\wedge$ are the binary supremum and infimum in the family $AIFR(X\times Y)$, respectively. The family $(AIFR(X\times Y),\vee,\wedge)$ is a complete, distributive lattice. Now, let us recall the notion of the composition in its standard form

Definition 2 (cf. [9],[3])**.** *Let $\sigma=(S,S^d)\in AIFR(X\times Y)$, $\rho=(R,R^d)\in AIFR(Y\times Z)$. By the composition of relations σ and ρ we call the relation $\sigma\circ\rho\in$ AIFR$(X\times Z)$,*

$$(\sigma\circ\rho)(x,z)=((S\circ R)(x,z),(S^d\circ' R^d)(x,z)),$$

where

$$(S\circ R)(x,z)=\bigvee_{y\in Y}(S(x,y)\wedge R(y,z)), \tag{4}$$

$$(S^d\circ' R^d)(x,z)=\bigwedge_{y\in Y}(S^d(x,y)\vee R^d(y,z)). \tag{5}$$

The fuzzy relation $\pi_\rho\colon X\times Y\to[0,1]$ is associated with each Atanassov's intuitionistic fuzzy relation $\rho=(R,R^d)$, where

$$\pi_\rho(x,y)=1-R(x,y)-R^d(x,y),\qquad x\in X,y\in Y. \tag{6}$$

The number $\pi_\rho(x,y)$ is called an index of an element (x,y) in an Atanassov's intuitionistic fuzzy relation ρ. It is also described as an index (a degree) of hesitation whether x and y are in the relation ρ or not. This value is also regarded as a measure of non-determinacy or uncertainty (see [11]) and is useful in applications. Intuitionistic indices allow to calculate the best final result and the worst one that may be expected in a process leading to a final optimal decision (see [11]).

If we consider decision making problems in the intuitionistic fuzzy environment we deal with the finite set of alternatives $X=\{x_1,\dots,x_n\}$ and an expert who needs to provide his/her preference information over alternatives. In the sequel, we will consider a preference relation on a finite set $X=\{x_1,\dots,x_n\}$. In this situation intuitionistic fuzzy relations may be represented by matrices.

Definition 3 ([16], cf. [14]). *Let* $\overline{\overline{X}} = n$. *An intuitionistic fuzzy preference relation* ρ *on the set* X *is represented by a matrix* $\rho = (\rho_{ij})_{n\times n}$ *with* $\rho_{ij} = (R(i,j), R^d(i,j))$, *for all* $i,j = 1,...,n$, *where* ρ_{ij} *is an intuitionistic fuzzy value, composed by the degree* $R(i,j)$ *to which* x_i *is preferred to* x_j, *the degree* $R^d(i,j)$ *to which* x_i *is non-preferred to* x_j, *and the uncertainty degree* $\pi(i,j)$ *to which* x_i *is preferred to* x_j. *Furthermore,* $R(i,j), R^d(i,j)$ *satisfy the following characteristics:*

$$0 \le R(i,j) + R^d(i,j) \le 1, \quad R(i,j) = R^d(j,i), \quad R(j,i) = R^d(i,j),$$
$$R(i,i) = R^d(i,i) = 0.5 \quad for \quad all\ i,j = 1,...,n.$$

Directly from this definition it follows $\pi(i,j) = \pi(j,i)$ for all $i,j = 1,...,n$.

3 Operations on preference relations

Lattice operations and the composition in the family $AIFR(X)$ do not preserve a preference relation, i.e. if ρ and σ are intuitionistic preference relations, then their sum, intersection and composition need not have this property.

Example 1. *Let* $card\ X = 2$ *and* $\rho = (R, R^d), \sigma = (S, S^d) \in AIFR(X)$ *be preference relations represented by the matrices:*

$$\rho = \begin{bmatrix} (0.5, 0.5) & (0.3, 0.6) \\ (0.6, 0.3) & (0.5, 0.5) \end{bmatrix}, \qquad \sigma = \begin{bmatrix} (0.5, 0.5) & (1, 0) \\ (0, 1) & (0.5, 0.5) \end{bmatrix}.$$

Then according to (2), (3), (4), (5), *we obtain*

$$\rho \vee \sigma = \begin{bmatrix} (0.5, 0.5) & (1, 0) \\ (0.6, 0.3) & (0.5, 0.5) \end{bmatrix}, \qquad \rho \wedge \sigma = \begin{bmatrix} (0.5, 0.5) & (0.3, 0.6) \\ (0, 1) & (0.5, 0.5) \end{bmatrix},$$

$$\rho \circ \sigma = \begin{bmatrix} (0.5, 0.5) & (0.5, 0.5) \\ (0.5, 0.5) & (0.6, 0.3) \end{bmatrix}, \qquad \rho \circ \rho = \begin{bmatrix} (0.5, 0.5) & (0.3, 0.6) \\ (0.5, 0.5) & (0.5, 0.5) \end{bmatrix}.$$

We see that none of the relations $\rho \vee \sigma$, $\rho \wedge \sigma$, $\rho \circ \sigma$, $\rho \circ \rho$ *is a preference relation.*

Now we put definitions of some Atanassov's operators

Definition 4 ([2]). *Let* $\rho \in AIFR(X \times Y)$, $\rho = (R, R^d)$, $\alpha,\ \beta \in [0,1]$, $\alpha + \beta \le 1$. *The operators* $F_{\alpha,\beta}, P_{\alpha,\beta}, Q_{\alpha,\beta} : AIFR(X \times Y) \to AIFR(X \times Y)$ *are defined as follows*

$$F_{\alpha,\beta}(\rho(x,y)) = (R(x,y) + \alpha\pi_\rho(x,y), R^d(x,y) + \beta\pi_\rho(x,y)),$$
$$P_{\alpha,\beta}(\rho(x,y)) = (\max(\alpha, R(x,y)), \min(\beta, R^d(x,y))),$$
$$Q_{\alpha,\beta}(\rho(x,y)) = (\min(\alpha, R(x,y)), \max(\beta, R^d(x,y))).$$

We examine whether Atanassov's operators preserve intuitionistic fuzzy preference relations.

Proposition 1. *Let* $\rho \in AIFR(X)$, $\overline{\overline{X}} = n$, $\alpha,\ \beta \in [0,1]$, $\alpha + \beta \leq 1$ *and* $\rho = (R, R^d)$ *be an intuitionistic fuzzy preference relation.*
• $F_{\alpha,\alpha}(\rho)$ *is an intuitionistic fuzzy preference relation.*
• *If* $\alpha \leq R(i,j) \leq \beta$ *for all* $i, j = 1, ..., n$*, then* $P_{\alpha,\beta}(\rho)$ *is also an intuitionistic fuzzy preference relation.*
• *If* $\beta \leq R(i,j) \leq \alpha$ *for all* $i, j = 1, ..., n$*, then* $Q_{\alpha,\beta}(\rho)$ *is also an intuitionistic fuzzy preference relation.*

Proof. First we consider operation $F_{\alpha,\alpha}(\rho)$ and we observe for
$1 \leq i, j \leq n$ that

$$F_{\alpha,\alpha}(\rho_{ii}) = (R(i,i) + \alpha\pi_\rho(i,i), R^d(i,i) + \alpha\pi_\rho(i,i)) =$$

$$(R(i,i), R^d(i,i)) = (0.5, 0.5).$$

Moreover

$$R(i,j) + \alpha\pi_\rho(i,j) = R^d(j,i) + \alpha\pi_\rho(j,i).$$

Thus $F_{\alpha,\alpha}(\rho)$ preserves preference property.
For $\alpha \leq R(i,j) \leq \beta$ we have

$$\max(\alpha, R(i,j)) = R(i,j) = R^d(j,i) = \min(\beta, R^d(j,i)).$$

This proves that $P_{\alpha,\beta}(\rho)$ preserves preference property and the case of $Q_{\alpha,\beta}(\rho)$ we can prove in a similar way. □

4 Properties of intuitionistic preference relations

In this section we consider some properties of intuitionistic fuzzy relations and intuitionistic fuzzy preference relations. First, we recall the concept of a partially included relation.

Definition 5 (cf. [3])**.** *An intuitionistic fuzzy relation* $\rho = (R, R^d) \in AIFR(X)$ *is partially included, if*

$$\underset{1\leq i,j,k\leq n}{\forall} sgn(R(i,j) - R(j,k)) = sgn(R^d(j,k) - R^d(i,j)).$$

Definition 6 (cf. [8])**.** *An intuitionistic fuzzy relation* $\rho = (R, R^d) \in AIFR(X)$ *is transitive, if* $\rho \circ \rho \leq \rho$*.*

Thus we have

Lemma 1 (cf. [12]). *Let $\rho \in AIFR(X)$, $\alpha, \beta \in [0,1]$, $\alpha + \beta \leq 1$. If ρ is partially included and transitive, then $F_{\alpha,\beta}(\rho)$ is transitive.*

By above lemma and by adequate condition: $\rho_{ij} + \rho_{ji} = (1,1)$, which means that $R(i,j) + R(j,i) = 1$ and $R^d(i,j) + R^d(j,i) = 1$, we obtain

Proposition 2. *Let $\rho \in AIFR(X)$, $\overline{\overline{X}} = n$ and $\alpha, \beta \in [0,1]$. If $\rho = (R, R^d)$ is an intuitionistic fuzzy preference relation fulfilling the property $\rho_{ij} + \rho_{ji} = (1,1)$ for all $i,j = 1,...,n$ and the transitivity property, then $F_{\alpha,\beta}(\rho)$ $(F_{\alpha,\alpha}(\rho))$ is also an intuitionistic fuzzy transitive relation (intuitionistic fuzzy transitive preference relation).*

Proof. If $\rho_{ij} + \rho_{ji} = (1,1)$, then for an intuitionistic fuzzy preference relation $(R(i,j) + R(j,i) = 1) \Leftrightarrow (R^d(i,j) + R^d(j,i) = 1)$ and ρ is partially included, i.e.

$$sgn(R(i,j) - R(j,k)) = sgn(1 - R(j,i) - (1 - R(k,j))) =$$
$$sgn(R(k,j) - R(j,i)) = sgn(R^d(j,k) - R^d(i,j)).$$

By Lemma 1 we see that $F_{\alpha,\beta}(\rho)$ is transitive, moreover by the Proposition 1, $F_{\alpha,\beta}(\rho)$ for $\alpha = \beta$ is an intuitionistic fuzzy transitive preference relation. □

Now we define the notion of equivalent fuzzy relations.

Definition 7 (cf. [7]). *Fuzzy relations $R, S : X \times X \to [0,1]$ are equivalent ($R \sim S$), if*

$$\forall_{x,y,u,v \in X} \; R(x,y) \leqslant R(u,v) \;\Leftrightarrow\; S(x,y) \leqslant S(u,v). \tag{7}$$

Analogously, for intuitionistic fuzzy relations we have

Definition 8 ([8]). *Let $\rho = (R, R^d), \sigma = (S, S^d) \in AIFR(X)$. We say that ρ and σ are equivalent ($\rho \sim \sigma$), if for all $x, y, u, v \in X$*

$$R(x,y) \leqslant R(u,v) \;\Leftrightarrow\; S(x,y) \leqslant S(u,v)$$

and

$$R^d(x,y) \leqslant R^d(u,v) \;\Leftrightarrow\; S^d(x,y) \leqslant S^d(u,v).$$

Directly from definition it follows that the relation "$\sim$" is an equivalence relation in the family $AIFR(X)$. This fact enables to classify intuitionistic fuzzy information and find some subordinations between this information.

Corollary 1 ([8])**.** *Let* $\rho = (R, R^d), \sigma = (S, S^d) \in AIFR(X)$. *Then*

$$\rho \sim \sigma \ \Leftrightarrow \ (R \sim S \text{ and } R^d \sim S^d).$$

Now, let us turn to considerations involving the operations supremum and infimum. These results may be applied in verifying the equivalence between given intuitionistic fuzzy relations.

Theorem 1 ([8])**.** *Let* $\rho = (R, R^d), \sigma = (S, S^d) \in AIFR(X)$. *If* $\rho \sim \sigma$, *then for every non-empty subset* P *of* $X \times X$ *and each* $x, y, z, t \in P$ *the following conditions are fulfilled*

$$\begin{cases} R(x,y) = \bigvee\limits_{(u,v)\in P} R(u,v) \ \Leftrightarrow \ S(x,y) = \bigvee\limits_{(u,v)\in P} S(u,v) & \text{and} \\ R^d(z,t) = \bigvee\limits_{(u,v)\in P} R^d(u,v) \ \Leftrightarrow \ S^d(z,t) = \bigvee\limits_{(u,v)\in P} S^d(u,v) & \end{cases}, \quad (8)$$

$$\begin{cases} R(x,y) = \bigwedge\limits_{(u,v)\in P} R(u,v) \ \Leftrightarrow \ S(x,y) = \bigwedge\limits_{(u,v)\in P} S(u,v) & \text{and} \\ R^d(z,t) = \bigwedge\limits_{(u,v)\in P} R^d(u,v) \ \Leftrightarrow \ S^d(z,t) = \bigwedge\limits_{(u,v)\in P} S^d(u,v) & \end{cases}, \quad (9)$$

$$\begin{cases} R(x,y) = \bigvee\limits_{(u,v)\in P} R(u,v) \ \Leftrightarrow \ S(x,y) = \bigvee\limits_{(u,v)\in P} S(u,v) & \text{and} \\ R^d(z,t) = \bigwedge\limits_{(u,v)\in P} R^d(u,v) \ \Leftrightarrow \ S^d(z,t) = \bigwedge\limits_{(u,v)\in P} S^d(u,v) & \end{cases}, \quad (10)$$

$$\begin{cases} R(x,y) = \bigwedge\limits_{(u,v)\in P} R(u,v) \ \Leftrightarrow \ S(x,y) = \bigwedge\limits_{(u,v)\in P} S(u,v) & \text{and} \\ R^d(z,t) = \bigvee\limits_{(u,v)\in P} R^d(u,v) \ \Leftrightarrow \ S^d(z,t) = \bigvee\limits_{(u,v)\in P} S^d(u,v) & \end{cases}. \quad (11)$$

Let us notice that the converse statement to Theorem 1 is true and it is enough to assume that only one of the conditions (8) - (11) is fulfilled for finite subsets P of $X \times X$.

Theorem 2 ([8])**.** *Let* $\rho = (R, R^d), \sigma = (S, S^d) \in AIFR(X)$. *If for every finite, non-empty subset* P *of* $X \times X$ *and each* $x, y, z, t \in P$ *one of the conditions* (8) - (11) *holds, then* $\rho \sim \sigma$.

Equivalent relations have connection with transitivity property.

Theorem 3 ([8])**.** *Let* $\rho = (R, R^d), \sigma = (S, S^d) \in AIFR(X)$. *If* $\rho \sim \sigma$, *then* ρ *is transitive if and only if* σ *is transitive.*

For intuitionistic fuzzy preference relations we can weaken assumptions from the above theorem.

Proposition 3. *Let $\rho, \sigma \in AIFR(X)$, $\overline{\overline{X}} = n$. If $\rho = (R, R^d), \sigma = (S, S^d)$ are intuitionistic fuzzy preference relations and for arbitrary non-empty set $P \subset X \times X$ and $(i, j) \in P$ holds:*

$$R(i,j) = \bigvee_{(v,w)\in P} R(v,w) \Leftrightarrow S(i,j) = \bigvee_{(v,w)\in P} S(v,w) \qquad (12)$$

or

$$R(i,j) = \bigwedge_{(v,w)\in P} R(v,w) \Leftrightarrow S(i,j) = \bigwedge_{(v,w)\in P} S(v,w), \qquad (13)$$

then ρ is transitive if and only if σ is transitive.

Proof. For intuitionistic fuzzy preference relations and conditions (12) and (13) we obtain dual conditions for relations R^d, S^d. Moreover, from definition of an intuitionistic fuzzy preference relation and equivalence relation we observe, that if $\rho = (R, R^d), \sigma = (S, S^d)$ are intuitionistic fuzzy preference relations and $R \sim S$, then $R^d \sim S^d$. As a result, if $\rho = (R, R^d), \sigma = (S, S^d)$ are intuitionistic fuzzy preference relations and $R \sim S$, then $\rho \sim \sigma$. Now by assumptions (12), (13) and Theorems 1- 3 we have transitivity property both for ρ and σ. □

In the sequel, we will use the following properties of intuitionistic fuzzy relations in a finite set X.

Definition 9 ([16]). *Let $\overline{\overline{X}} = n$. An intuitionistic fuzzy relation $\rho = (R, R^d) \in AIFR(X)$ is weakly transitive, if*

$$\forall_{1\le i,j,k\le n} \rho(i,k) \ge (0.5, 0.5), \rho(k,j) \ge (0.5, 0.5) \Rightarrow \rho(i,j) \ge (0.5, 0.5). \quad (14)$$

Definition 10. *Let $\overline{\overline{X}} = n$. An intuitionistic fuzzy relation $\rho = (R, R^d) \in AIFR(X)$ is said to be a relation with strictly dominating upper (lower) triangle, if*

$$\forall_{1\le i,j\le n, i<j (i>j)} \rho(i,j) > 0.5. \qquad (15)$$

Proposition 4. *Let $\overline{\overline{X}} = n$. If $\rho = (R, R^d) \in AIFR(X)$ is an intuitionistic fuzzy preference relation with strictly dominating lower (upper) triangle, then it is weakly transitive.*

Proof. Let $\rho = (R, R^d)$ be an intuitionistic fuzzy preference relation with strictly dominating upper triangle.
If $i = j$, then $\rho(i, j) = (0.5, 0.5)$. Thus implication (14) is true.

If $i \neq j$, then we consider the following cases:
1. For $i > j$ we have by (15) $\rho(i,j) < (0.5, 0.5)$ and we examine:
• If $i \geq k > j$, then $\rho(k,j) < (0.5, 0.5)$;
• If $k > i > j$, then $\rho(k,j) < (0.5, 0.5)$;
• If $i > j \geq k$, then $\rho(i,k) < (0.5, 0.5)$.
In all these cases we obtained false antecedent and consequence, so implication (14) is true.
2. For $i < j$ we have $\rho(i,j) > (0.5, 0.5)$ so implication (14) is true. The proof for a relation with strictly dominating lower triangle is similar and the intuitionistic fuzzy preference relation $\rho = (R, R^d)$ is weakly transitive. □

Now, we define parameterized versions of intuitionistic fuzzy relation properties. We follow the concept of such properties given by Drewniak [6] for fuzzy relations but we restrict ourselves only to parameter $\alpha = 0.5$. This is why we will call these properties *semi-properties*.

Definition 11. *An intuitionistic fuzzy relation $\rho = (R, R^d) \in AIFR(X)$ is called:*
• *semi-reflexive if*

$$\underset{x \in X}{\forall} \rho(x,x) \geqslant (0.5, 0.5), \tag{16}$$

• *semi-irreflexive if*

$$\underset{x \in X}{\forall} \rho(x,x) \leqslant (0.5, 0.5), \tag{17}$$

• *semi-symmetric if*

$$\underset{x,y \in X}{\forall} \rho(x,y) \geqslant (0.5, 0.5) \Rightarrow \rho(y,x) = \rho(x,y), \tag{18}$$

• *semi-asymmetric if*

$$\underset{x,y \in X}{\forall} \rho(x,y) \wedge \rho(y,x) \leqslant (0.5, 0.5), \tag{19}$$

• *semi-antisymmetric if*

$$\underset{x,y \in X, x \neq y}{\forall} \rho(x,y) \wedge \rho(y,x) \leqslant (0.5, 0.5), \tag{20}$$

• *totally semi-connected if*

$$\underset{x,y \in X}{\forall} \rho(x,y) \vee \rho(y,x) \geqslant (0.5, 0.5), \tag{21}$$

• *semi-connected if*

$$\underset{x,y \in X, x \neq y}{\forall} \rho(x,y) \vee \rho(y,x) \geqslant (0.5, 0.5), \tag{22}$$

• *semi-transitive if*

$$\underset{x,y,z\in X}{\forall} \rho(x,y) \wedge \rho(y,z) \geqslant (0.5, 0.5) \Rightarrow \rho(x,z) \geqslant \rho(x,y) \wedge \rho(y,z). \qquad (23)$$

¿From definition of semi-transitivity and definition of the composition of intuitionistic fuzzy relations it follows

Lemma 2. *Let $\rho = (R, R^d) \in AIFR(X)$ be an intuitionistic fuzzy relation. Relation ρ is semi-transitive if and only if*

$$\underset{x,z\in X}{\forall} \rho^2(x,z) \geqslant (0.5, 0.5) \Rightarrow \rho(x,z) \geqslant \rho^2(x,z). \qquad (24)$$

Proof. If $\rho = (R, R^d)$ is semi-transitive, then by (23), definition of the order (3) and by applying the tautologies for quantifiers we obtain

$$\underset{x,y,z\in X}{\forall} R(x,y) \wedge R(y,z) \geqslant 0.5 \Rightarrow R(x,z) \geqslant R(x,y) \wedge R(y,z)$$

and

$$\underset{x,y,z\in X}{\forall} R^d(x,y) \vee R^d(y,z) \leqslant 0.5 \Rightarrow R^d(x,z) \leqslant R^d(x,y) \vee R^d(y,z).$$

As a result

$$\underset{x,z\in X}{\forall} (\underset{y\in X}{\forall} R(x,y) \wedge R(y,z) \geqslant 0.5 \Rightarrow \underset{y\in X}{\forall} R(x,z) \geqslant R(x,y) \wedge R(y,z))$$

and

$$\underset{x,z\in X}{\forall} (\underset{y\in X}{\forall} R^d(x,y) \vee R^d(y,z) \leqslant 0.5 \Rightarrow \underset{y\in X}{\forall} R^d(x,z) \leqslant R^d(x,y) \vee R^d(y,z)).$$

This implies

$$\underset{x,z\in X}{\forall} \sup_{y\in X}(R(x,y) \wedge R(y,z)) \geqslant 0.5 \Rightarrow R(x,z) \geqslant \sup_{y\in X}(R(x,y) \wedge R(y,z)) \qquad (25)$$

and

$$\underset{x,z\in X}{\forall} \inf_{y\in X}(R^d(x,y) \vee R^d(y,z)) \leqslant 0.5 \Rightarrow R^d(x,z) \leqslant \inf_{y\in X}(R^d(x,y) \vee R^d(y,z)), \qquad (26)$$

so by the definition of composition we get (24).
Let us assume that condition (24) is fulfilled which is equivalent to conditions (25) and (26). We will show that ρ is semi-transitive. Let $x, y, z \in X$ and the

antecedent in (23) be fulfilled. As a result $R(x,y) \wedge R(y,z) \geqslant 0.5$ and $R^d(x,y) \vee R^d(y,z) \leqslant 0.5$. By definition of supremum and infimum we obtain

$$\sup_{y \in X}(R(x,y) \wedge R(y,z)) \geqslant R(x,y) \wedge R(y,z) \geqslant 0.5$$

and

$$\inf_{y \in X}(R^d(x,y) \vee R^d(y,z)) \leqslant R^d(x,y) \vee R^d(y,z) \leqslant 0.5.$$

¿From (25), (26) and definition of supremum and infimum we have

$$R(x,z) \geqslant \sup_{y \in X}(R(x,y) \wedge R(y,z)) \geqslant R(x,y) \wedge R(y,z)$$

and

$$R^d(x,z) \leqslant \inf_{y \in X}(R^d(x,y) \vee R^d(y,z)) \leqslant R^d(x,y) \vee R^d(y,z)$$

This by definition of an intuitionistic fuzzy relation and the order (3) finishes the proof. □

Now, we will check under which assumptions an intuitionistic fuzzy preference relation has each of the semi-property. Directly by the definition of an intuitionistic preference relation we obtain

Corollary 2. *Each intuitionistic fuzzy preference relation is semi-reflexive and semi-irreflexive.*

Theorem 4. *Let* $\overline{\overline{X}} = n$, $\rho = (R, R^d) \in AIFR(X)$ *be an intuitionistic fuzzy preference relation. If*

$$\forall_{i,j \in \{1,...,n\}, i \neq j} \max(R(i,j), R^d(i,j)) \geqslant 0.5, \tag{27}$$

then ρ *is totally semi-connected, semi-connected, semi-asymmetric, semi-antisymmetric.*

Proof. Let $i, j \in \{1, ..., n\}$. Firstly, we will prove total semi-connectedness of ρ (then semi-connectedness will be obvious). If $i = j$, then condition (21) is fulfilled by definition of a preference relation. Let $i \neq j$. Since ρ is a preference relation $R^d(i,j) = R(j,i)$, so we have

$$\max(R(i,j), R(j,i)) \geqslant 0.5. \tag{28}$$

Relation ρ is the intuitionistic one, so by (27) it follows that $\min(R(i,j), R^d(i,j)) \leqslant 0.5$. Moreover, ρ is a preference relation, so we have $R(i,j) = R^d(j,i)$. As a result

$$\min(R^d(j,i), R^d(i,j)) \leqslant 0.5. \tag{29}$$

Finally, by (28), (29) and the definition of order for intuitionistic relations we get $\rho(i,j) \vee \rho(j,i) \geqslant (0.5, 0.5)$. It proves that ρ is totally semi-connected (semi-connected). We will show that ρ is semi-asymmetric (then semi-antisymmetry will be obvious). By assumptions and because of (1) we also have

$$\min(R(i,j), R(j,i)) \leqslant 0.5. \tag{30}$$

and similarly

$$\max(R^d(j,i), R^d(i,j)) \geqslant 0.5. \tag{31}$$

Finally, by (30), (31) and the definition of order for intuitionistic relations we obtain $\rho(i,j) \wedge \rho(j,i) \leqslant (0.5, 0.5)$, so relation ρ is semi-asymmetric (semi-antisymmetric).

□

Theorem 5. *Let* $\overline{\overline{X}} = n$, $\rho = (R, R^d) \in AIFR(X)$ *be an intuitionistic fuzzy preference relation. If*

$$\underset{i,j\in\{1,\dots,n\}, i\neq j}{\forall} (\rho(i,j) = (0.5, 0.5) \textit{ or } \max(R(i,j), R^d(i,j)) < 0.5), \tag{32}$$

then ρ *is semi-symmetric.*

Proof. Let $i, j \in \{1, ..., n\}$. If $i = j$, then condition (18) is fulfilled by definition of a preference relation. Let $i \neq j$. If $\rho(i,j) = (0.5, 0.5)$, then since ρ is a preference $R(j,i) = R^d(i,j)$ and $R^d(j,i) = R(i,j)$. As a result $\rho(j,i) = (0.5, 0.5)$ and $\rho(i,j) = \rho(j,i)$. If $\max(R(i,j), R^d(i,j)) < 0.5$, then we have two cases: 1^0) $\max(R(i,j), R^d(i,j)) = R(i,j) < 0.5$. In this case the antecedent of the implication in (18) is false, so the implication is true. 2^0) $\max(R(i,j), R^d(i,j)) = R^d(i,j) < 0.5$. By assumption $R^d(i,j) = R(j,i)$, so $R(j,i) < 0.5$. In this case the antecedent of the implication for the pair (j,i) in (18) is false, so the implication is true.

□

Now, we turn to considerations connected with semi-transitivity which is a stronger property than weak transitivity discussed before. By Lemma 2 determination of the relation ρ^2 is helpful in checking whether ρ is semi-transitive.

Theorem 6. *Let* $\overline{\overline{X}} = n$, $\rho = (R, R^d) \in AIFR(X)$ *be an intuitionistic fuzzy preference relation. If*

$$\forall_{i,j\in\{1,...,n\}} (\rho^2(i,j) < (0.5, 0.5) \textit{ or } \rho(i,j) \geqslant \rho^2(i,j)), \tag{33}$$

then ρ *is semi-transitive.*

Proof. Let $i, j \in \{1, ..., n\}$. If $\rho^2(i,j) < (0.5, 0.5)$, then the antecedent of the implication is false in (24), so the implication is true. If $\rho(i,j) \geqslant \rho^2(i,j)$, then then the consequence of the implication is true in (24) and this implication is true. By Lemma 2 this finishes the proof. □

5 Conclusion

In this paper we considered properties of intuitionistic fuzzy preference relations in the context of preservation of this property by lattice operations, the composition and by Atanassov's operators. We also introduced semi-properties of intuitionistic fuzzy relations and we investigated fulfilment of these properties by preference relations. In our further considerations we want to study other transitivity properties of intuitionistic fuzzy preference relations introduced in [16].

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References

[1] K.T. Atanassov, Intuitionistic fuzzy sets, Fuzzy Sets Syst. 20 (1986), 87–96.

[2] K.T. Atanassov, Intuitionistic Fuzzy Sets: Theory and Applications, Springer-Verlag, 1999.

[3] P. Burillo, H. Bustince, Intuitionistic Fuzzy Relations. Effect of Atanassov's Operators on the Properties of the Intuitionistic Fuzzy Relations, Math. Soft Comp. 2 (1995), 117–148.

[4] F. Chiclana, E. Herrera-Viedma, S. Alonso, R. A. M. Pereira, Preferences and consistency issues in group decision making, in: H. Bustince at al. (Eds.), Fuzzy Sets and Their Extensions: Representation, Aggregation and Models, Springer-Verlag, Berlin, 2008, 219–237.

[5] G. Deschrijver, E.E. Kerre, On the composition of intuitionistic fuzzy relations, Fuzzy Sets Syst. 136(3) (2003), 333–361.

[6] J. Drewniak, Fuzzy Relation Calculus, Silesian University, Katowice, 1989.

[7] J. Drewniak, U. Dudziak, Local properties of fuzzy relations, in: O. Hryniewicz et al. (Eds.), Issues in Soft Computing. Decisions and Operations Research, EXIT, Warszawa, 2005, 161–170.

[8] U. Dudziak, B. Pękala, Equivalent bipolar fuzzy relations, Fuzzy Sets Syst. 161 (2010), 234–253.

[9] A. Goguen, L-fuzzy sets, J. Math. Anal. Appl. 18 (1967), 145–174.

[10] D.-F. Li, Multiattribute decision making models and method using intuitionistic fuzzy sets, J. Comp. Syst. Sci. 70 (2005), 73–85.

[11] L. Lin, X-H. Yuan, Z-Q. Xia, Multicriteria fuzzy decision-making methods based on intuitionistic fuzzy sets, J. Comp. Syst. Sci. 73 (2007), 84–88.

[12] B. Pękala, Properties of Interval-Valued Fuzzy Relations, Atanassov's Operators and Decomposable Operations, in: E. Hüllermeier, R. Kruse, F. Hoffmann (Eds.), Information Processing and Management of Uncertainty in Knowledge-Based Systems (13th International Conference, IPMU 2010, Dortmund, Germany, June 28 - July 2, 2010, Proceedings, Part I), Communications in Computer and Information Science 80, Springer, Berlin Heidelberg, 2010, 647-655.

[13] M. Roubens, P. Vincke, Preference Modelling, Springer-Verlag, Berlin, 1985.

[14] E. Szmidt, J. Kacprzyk, Using intuitionistic fuzzy sets in group decision making, Control Cybernet. 31 (2002), 1037–1053.

[15] E. Szmidt, J. Kacprzyk, Atanassov's Intuitionistic Fuzzy Sets as a Promising Tool for Extended Fuzzy Decision Making Models, in: H. Bustince at al. (Eds.), Fuzzy Sets and Their Extensions: Representation, Aggregation and Models, Springer-Verlag, Berlin, 2008, 335–355.

[16] Z. Xu, Intuitionistic preference relations and their application in group decision making, Inform. Sci. 177 (2007), 2363–2379.

[17] R. R. Yager, Z. Xu, Intuitionistic and interval-valued intuitionistic fuzzy preference relations and their measures of similarity for the evaluation of agreement within a group, Fuzzy Optim. Decis. Mak. 8 (2009), 123–139.

[18] L.A. Zadeh, Fuzzy sets, Inform. Control 8 (1965), 338–353.

The papers presented in this Volume 1 constitute a collection of contributions, both of a foundational and applied type, by both well-known experts and young researchers in various fields of broadly perceived intelligent systems.

It may be viewed as a result of fruitful discussions held during the Ninth International Workshop on Intuitionistic Fuzzy Sets and Generalized Nets (IWIFSGN-2010) organized in Warsaw on October 8, 2010 by the Systems Research Institute, Polish Academy of Sciences, in Warsaw, Poland, Institute of Biophysics and Biomedical Engineering, Bulgarian Academy of Sciences in Sofia, Bulgaria, and WIT - Warsaw School of Information Technology in Warsaw, Poland, and co-organized by: the Matej Bel University, Banska Bystrica, Slovakia, Universidad Publica de Navarra, Pamplona, Spain, Universidade de Tras-Os-Montes e Alto Douro, Vila Real, Portugal, and the University of Westminster, Harrow, UK:

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The consecutive International Workshops on Intuitionistic Fuzzy Sets and Generalized Nets (IWIFSGNs) have been meant to provide a forum for the presentation of new results and for scientific discussion on new developments in foundations and applications of intuitionistic fuzzy sets and generalized nets pioneered by Professor Krassimir T. Atanassov. Other topics related to broadly perceived representation and processing of uncertain and imprecise information and intelligent systems have also been included. The Ninth International Workshop on Intuitionistic Fuzzy Sets and Generalized Nets (IWIFSGN-2010) is a continuation of this undertaking, and provides many new ideas and results in the areas concerned.

We hope that a collection of main contributions presented at the Workshop, completed with many papers by leading experts who have not been able to participate, will provide a source of much needed information on recent trends in the topics considered.

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