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Umberto Viaro

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Prof. Umberto Viaro
Dipartimento di Ingegneria Elettrica, Gestionale e Meccanica
Università degli Studi di Udine
via delle Scienze 208, 33100 Udine, Italy
email: viaro@uniud.it

Preface by Prof. Wiesław Krajewski
Systems Research Institute
Polish Academy of Sciences
01-447 Warsaw, Newelska 6
Wieslaw.Krajewski@ibspan.waw.pl

Papers Reviewers:

Prof. Jerzy Klamka
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Chapter 5

Delay approximation by feedback

In modelling dynamic systems for control purposes, it is often necessary to account for pure time delays related, e.g., to transport phenomena or distributed–parameter components. Since many control system synthesis techniques refer to rational transfer functions, the transcendental transfer function e^{-Ts} of an element introducing a delay equal to T must frequently be approximated by means of a rational function. This approximation problem has a century–old history [1] but still receives considerable attention (see, e.g., [2]–[8] and bibliographies therein).

In many practical applications the requirements of physical realizability and stability of the approximant limit the choice of the approximant to proper rational functions with real coefficients and a Hurwitz denominator. These requirements are satisfied by the Blaschke products:

$$B(s) = \frac{\prod_{i=1}^{n}(s - a_i)}{\prod_{i=1}^{n}(s + a_i)}, \quad \mathbb{R}\mathrm{e}[a_i] > 0, \tag{5.1}$$

whose poles are either real or appear in complex conjugate pairs. Funtion (5.1) exhibits two desirable properties: (i) it is allpass, i.e., $|B(\jmath\omega)| = |e^{-\jmath T\omega}| = 1$, $\forall\omega$, and (ii) $\arg[B(\jmath\omega)]$ is monotonically decreasing with ω like $\arg[e^{-\jmath T\omega}] = -T\omega$. Obviously, a criterion for selecting the poles or zeros of (5.1) must be provided.

The method illustrated in the following sections naturally leads to an approximant of this form. Note that the Padé technique, which is still the most widely adopted method for approximating the transfer function of a delay element, does not even ensure the stability of strictly–proper approximants [9]; on the other hand, exactly–proper Padé approximants lead to the best aproximation around $s = 0$ only.

5.1 Approximation criterion

A reasonable approach to model reduction for control system synthesis consists in referring directly to the desired *closed–loop* characteristics [10], [11]. In fact, the actual feedback control system may turn out to be unstable or fragile if the controller is designed by referring to a reduced model of the plant that is obtained without consideration of the closed–loop specifications [12].

Approximating a system with transfer function $G(s)$ from an approximation of the transfer function

$$W(s) = \frac{G(s)}{1 \pm G(s)} \tag{5.2}$$

of the negative/positive unity–feedback system whose forward–path transfer function is $G(s)$, may be convenient for other reasons, too. For instance, if the poles of $G(s)$ cannot be separated into a set of dominant poles and a set of remote poles, a reduction procedure based on the retention of the dominant modes is not applicable to $G(s)$. Instead, in most cases a pole–retention technique can be applied to $W(s)$ because, often, its poles get far apart as the loop gain increases and some of them become definitely dominant over the others.

Denoting by $\hat{W}(s)$ the approximation of $W(s)$, the required approximation of $G(s)$ will then be obtained according to

$$\hat{G}(s) = \frac{\hat{W}(s)}{1 \mp \hat{W}(s)}. \tag{5.3}$$

It is shown next that another case in which this approach is profitable is the rational approximation of $G(s) = e^{-Ts}$.

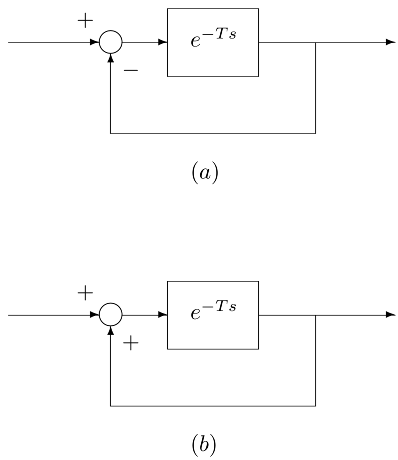

Figure 5.1: Delay element in the forward path of: (a) a negative feedback system, and (b) a positive feedback system .

5.2 Approximation procedure

The approximation of e^{-Ts} will be illustrated separately for the case of negative feedback (Fig. 5.1a) and the case of positive feedback (Fig. 5.1b). The first leads to approximants of even order, while the second leads to approximants of odd order.

Even–order approximants

The step response $w_{-1}(t)$ of the negative feedback system of Fig. 5.1a is represented in Fig. 5.2. For $t > 0$, this response is given by the difference between the step function $\frac{1}{2}\,\delta_{-1}(t)$ of amplitude $\frac{1}{2}$ and a square wave of period $2T$ and amplitude $\frac{1}{2}$ that is equal to $-\frac{1}{2}$ in the first half–period. Expanding this square wave into Fourier series, the step response of the feedback system can be written as

$$w_{-1}(t) = \frac{1}{2}\,\delta_{-1}(t) - \frac{2}{\pi}\sum_{i=1}^{\infty}\frac{1}{2i-1}\sin\left[(2i-1)\frac{\pi t}{T}\right]. \qquad (5.4)$$

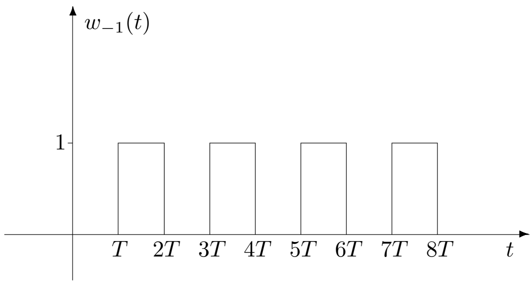

Figure 5.2: Step response $w_{-1}(t)$ of the negative feedback system of Fig. 5.1a.

By transforming (5.4) term by term, the Laplace transform $W_{-1}(s)$ of $w_{-1}(t)$ turns out to be

$$W_{-1}(s) = \frac{1}{2s} - \frac{2}{T}\sum_{i=1}^{\infty}\frac{1}{s^2 + [(2i-1)\pi/T]^2}. \tag{5.5}$$

Therefore, the transfer function of the negative feedback system can be expressed as

$$W(s) = \frac{1}{2} - \frac{2}{T}\sum_{i=1}^{\infty}\frac{s}{s^2 + [(2i-1)\pi/T]^2}. \tag{5.6}$$

By truncating the series in (5.6), the following approximant of even order $2k$ is obtained:

$$\hat{W}_{2k}(s) = \frac{1}{2} - \frac{2}{T}\sum_{i=1}^{k}\frac{s}{s^2 + [(2i-1)\pi/T]^2} \tag{5.7}$$

or, more compactly,

$$\hat{W}_{2k}(s) = \frac{1}{2} - \frac{\hat{N}_{2k-1}(s)}{\hat{D}_{2k}(s)}, \tag{5.8}$$

where

$$\hat{N}_{2k-1}(s) = \frac{2}{T}\sum_{i=1}^{k} s \prod_{j=1, j\neq i}^{k}\left\{s^2 + \left[\frac{(2j-1)\pi}{T}\right]^2\right\} \tag{5.9}$$

and

$$\hat{D}_{2k}(s) = \prod_{i=1}^{k} \left\{ s^2 + \left[\frac{(2j-1)\pi}{T} \right]^2 \right\}. \tag{5.10}$$

Therefore, according to (5.3) with the minus sign, the approximant turns out to be

$$\hat{G}_{2k}(s) = \frac{\dfrac{1}{2} - \dfrac{\hat{N}_{2k-1}(s)}{\hat{D}_{2k}(s)}}{1 - \left[\dfrac{1}{2} - \dfrac{\hat{N}_{2k-1}(s)}{\hat{D}_{2k}(s)} \right]} = \frac{\dfrac{1}{2} - \dfrac{\hat{N}_{2k-1}(s)}{\hat{D}_{2k}(s)}}{\dfrac{1}{2} + \dfrac{\hat{N}_{2k-1}(s)}{\hat{D}_{2k}(s)}} \tag{5.11}$$

so that

$$\hat{G}_{2k}(s) = \frac{\hat{D}_{2k}(s) - 2\hat{N}_{2k-1}(s)}{\hat{D}_{2k}(s) + 2\hat{N}_{2k-1}(s)} \tag{5.12}$$

whose numerator and denominator have the same even part $\hat{D}_{2k}(s)$ and opposite odd parts $\pm\hat{N}_{2k-1}(s)$.
On the basis of the previous derivation, the following result can be stated.

Proposition 5.2.1 *$\hat{G}_{2k}(s)$ is a stable Blaschke product.*

Proof The fraction:

$$\frac{\hat{N}_{2k-1}(s)}{\hat{D}_{2k}(s)} \tag{5.13}$$

ia an odd positive–real function, being the sum of functions of this kind [2]. Therefore, it can assume the value $-1/2$, thus annihilating the denominator of $\hat{G}_{2k}(s)$, only for $\mathbb{R}\mathrm{e}[s] < 0$. □

Odd–order approximants

The step response $w_{-1}(t)$ of the positive feedback system of Fig. 5.1b is represented in Fig. 5.3. For $t > 0$, this response is the sum of the ramp $\frac{1}{T}\delta_{-2}(t)$,the step $\frac{1}{2}\delta_{-1}(t)$ and a saw–tooth wave, as shown in Fig. 5.4. Expanding the periodic component into Fourier series, the step response can be written as

$$w_{-1}(t) = \frac{1}{T}\,\delta_{-2}(t) - \frac{1}{2}\,\delta_{-1}(t) + \sum_{i=1}^{\infty} \frac{1}{\pi i} \sin\left[\frac{2\pi i t}{T}\right] \tag{5.14}$$

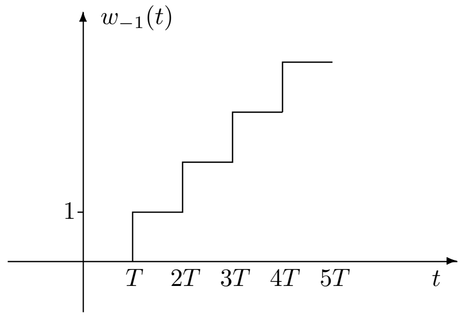

Figure 5.3: Step response $w_{-1}(t)$ of the positive feedback system of Fig. 5.1b.

whose Laplace transform is

$$W_{-1}(s) = \frac{1}{Ts^2} - \frac{1}{2s} + \frac{2}{T}\sum_{i=1}^{\infty} \frac{1}{s^2 + (2\pi i/T)^2}. \tag{5.15}$$

Therefore, the transfer function of the positive feedback system is

$$W(s) = \frac{1}{Ts} - \frac{1}{2} + \frac{2}{T}\sum_{i=1}^{\infty} \frac{s}{s^2 + (2\pi i/T)^2}. \tag{5.16}$$

By retaining the first k terms of the series in (5.16), the following rational approximant of odd order $2k+1$ is obtained:

$$\hat{W}_{2k+1}(s) = \frac{1}{Ts} - \frac{1}{2} + \frac{2}{T}\sum_{i=1}^{k} \frac{s}{s^2 + (2\pi i/T)^2}. \tag{5.17}$$

or, more compactly,

$$\hat{W}_{2k+1}(s) = \frac{1}{Ts} - \frac{1}{2} + \frac{\hat{N}_{2k-1}(s)}{\hat{D}_{2k}(s)}, \tag{5.18}$$

where

$$\hat{N}_{2k-1}(s) = \frac{2}{T}\sum_{i=1}^{k} s \prod_{j=1, j\neq i}^{k} \left\{ s^2 + \left[\frac{2\pi j}{T}\right]^2 \right\} \tag{5.19}$$

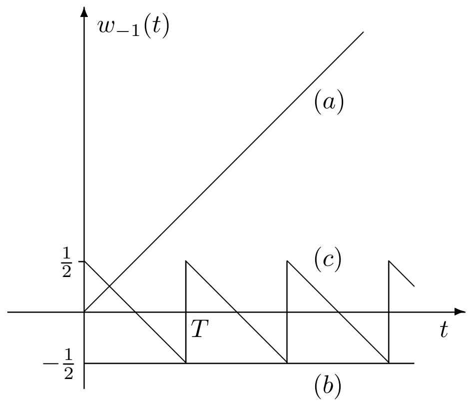

Figure 5.4: Components of the step response in Fig. 5.3: (a) ramp $\frac{1}{T}\delta_{-2}(t)$, (b) step $-\frac{1}{2}\delta_{-1}(t)$, (c) saw–tooth wave $\frac{1}{\pi}\sum_{i=1}^{\infty}\frac{1}{i}\sin[(2\pi it)/T]$.

and

$$\hat{D}_{2k}(s) = \prod_{i=1}^{k}\left\{s^2 + \left[\frac{2\pi j}{T}\right]^2\right\}. \tag{5.20}$$

According to (5.3) with the plus sign, the approximant is

$$\hat{G}_{2k+1}(s) = \frac{\dfrac{1}{Ts} - \dfrac{1}{2} + \dfrac{\hat{N}_{2k-1}(s)}{\hat{D}_{2k}(s)}}{1 + \dfrac{1}{Ts} - \dfrac{1}{2} + \dfrac{\hat{N}_{2k-1}(s)}{\hat{D}_{2k}(s)}} \tag{5.21}$$

so that

$$\hat{G}_{2k+1}(s) = \frac{2[\hat{D}_{2k}(s) + Ts\hat{N}_{2k-1}(s)] - Ts\hat{D}_{2k}(s)}{2[\hat{D}_{2k}(s) + Ts\hat{N}_{2k-1}(s)] + Ts\hat{D}_{2k}(s)} \tag{5.22}$$

which, by arguments similar to those adopted in Proposition 5.2.1, can be proved to be a stable Blaschke product.

5.3 Comparison

The magnitude of the frequency response $\hat{G}_h(\jmath\omega)$ of the approximants of both even and odd order h derived in Section 5.2 is equal to 1 for

order h	B_h (rad/sec)
2	2.35
3	5
4	7.8
5	10.6

Table 5.1: Frequencies B_h above which $\delta\phi_h(\omega) < \delta\phi_{P,h}(\omega)$ for $T = 1$.

any ω like the magnitude of $e^{-\jmath T\omega}$, and their phase tends to $-h\pi$. The same features are exhibited by the standard allpass Padé approximants $\hat{G}_{P,h}(s)$ of the same order. However, the phase deviation of the Padé approximants:

$$\delta\phi_{P,h}(\omega) := \arg[\hat{G}_{P,h}(\jmath\omega)] - \arg[e^{-\jmath T\omega}] \tag{5.23}$$

is a monotonically increasing positive function, whereas

$$\delta\phi_h(\omega) := \arg[\hat{G}_h(\jmath\omega)] - \arg[e^{-\jmath T\omega}] \tag{5.24}$$

is a nonmonotonic positive function that is equal to zero at $\omega = (2i+1)\pi$, $i = 0, \ldots, \frac{h}{2}-1$, for i even and at $\omega = 2i\pi$, $i = 0, \ldots, \frac{h-1}{2}-1$, for i odd [5]. As is expected, at the low frequencies $\delta\phi_{P,h}(\omega) < \delta\phi_h(\omega)$, while at the high frequencies $\delta\phi_h(\omega) < \delta\phi_{P,h}(\omega)$. Table 5.1 shows the frequencies B_h above which $\delta\phi_h(\omega) < \delta\phi_{P,h}(\omega)$, $\forall\omega > B_h$, for $h = 2, 3, 4, 5$ and $T = 1$.

5.4 Input–dependent approximants

The approximation procedure described in the previous sections determines first a particular form of the Laplace transform $W_{-1}(s)$ of the feedback–system step response. Then, the approximant of the transfer function $\hat{W}(s)$ of the feedback system is obtained by: (i) dividing $W_{-1}(s)$ by the transform $1/s$ of the step function, thus arriving at an expression of the original transfer function $W(s)$ and (ii) truncating the series appearing in this particular expression.

Using the terminology in [13] (see also Chapter 8), the rationale of such a procedure consists in retaining the *input component* (that is, the aperiodic component "resembling" the input) and truncating the system component (that is, the component with the same poles as $W(s)$) of the step response. Finally, the approximant $\hat{G}(s)$ of the transfer function of the delay element is determined according to (5.3). Therefore, $\hat{G}(s)$ in turn depends on the chosen input, i.e., the step function.

However, other inputs could be used. In [6], the family of canonical inputs $\{u(t) = t^m, t > 0, m \in \mathbb{N}\}$ is considered. To improve the approximation within a suitable frequency band, in [14] an input with a Laplace transform equal to

$$U(s) = \frac{1}{1 + 2\xi\dfrac{s}{\omega_n} + \dfrac{s^2}{\omega_n^2}}, \tag{5.25}$$

with ξ suitably small, is chosen instead. In this way, a frequency band centred at ω_n is privileged.

5.5 Concluding remarks

The step response of a unity–feedback system with a delay element in the forward path is formed by an aperiodic component plus a zero–mean periodic component, whereas a periodic term is not present in the step response of the delay element. To obtain a rational approximation of the feedback system response, it is natural to retain the aperiodic component as well as some harmonics of its periodic component. From the approximate response of the feedback system, it is then immediate to derive a rational approximant of the delay element itself according to (5.3).

As shown in Section 5.2, the approximants turn out to be stable Blaschke products whose frequency response is significantly better than that of the standard Padé approximants at medium and high frequencies (Section 5.3).

The procedure has been illustrated with reference to step inputs, but other inputs can also be adopted to improve the approximation within specific frequency bands (Section 5.4).

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Often, short papers tend to be sharper than longer works because they focus on a single theme without lingering on unessential aspects, thus showing clearly the signicance of a contribution or an idea. The author of this book had the privilege of collaborating for over a quarter of a century with Antonio Lepschy (1931-2005), a recognized leader of the Italian control community.

Lepschy had a liking for the brief paper format, so that many results obtained by his research team were published in this way. The present compilation tells a few of these short stories, duly updated, trying to preserve their original flavour.

Umberto Viaro (http://umbertoviaro.blogspot.com/) has been professor of System and Control Theory at the University of Udine, Italy, since 1994. His 25-year-long collaboration with Antonio Lepschy resulted in more than 100 joint papers and two books. An essential role in this research activity was played by Wiesław Krajewski of the Systems Research Institute, Polish Academy of Sciences. The current research interests of Umberto Viaro concern optimal model reduction, robust control, switching and LPV control. He is the author or coauthor of 4 books and about 180 research papers.

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SYSTEMS RESEARCH INSTITUTE
POLISH ACADEMY OF SCIENCES
Phone: (+48) 22 3810246 / 22 3810241 / 22 3810273
tel. (22) 3810 277; e-mail: biblioteka@ibspan.waw.pl