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Structural optimization – a one level approach

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Reliability-oriented optimization is an essential tool for maximizing the benefit of technical facilities or making efficient use of natural resources. The optimization problem is generally divided in two parts. The first level is the determination of certain design parameters, e.g. initial cost or weight of a structure under a reliability constraint. The calculation of failure probabilities in order to find the expected failure cost and the reliability of a structure requires solution of another optimization problem if modern FORM/SORM is used. Instead of solving two optimization tasks, both are combined in a one-level approach by adding the first order Kuhn-Tucker optimality conditions for the design point of the reliability problem as constraints to the cost optimization problem using first order reliability methods (FORM) in standard space. Solution techniques have been developed for component and series system problems in time-invariant and time-variant case using locally stationary load models.

Key words: *structural reliability, one level optimization, outcrossing approach*

1. Introduction

The calculation of failure probabilities or reliability indices for given sets of basic variables, limit state functions and deterministic parameters are well-known within the context of FORM/SORM, essentially involving a task of optimization. The determination of a certain parameter set in order to maximize benefits or to make efficient use of resources, is much more difficult and involves another optimization task. Both tasks can, however, be combined in finding optimal designs with or without reliability restrictions. In this one level approach the first-order Kuhn-Tucker optimality conditions are added as constraints to the cost optimization problem, based on an idea by Madsen and Friis Hansen [6]. Methods have been developed for time-invariant [9, 10] and time-variant [11] component problems including SORM improvements [13] and separable series system problems [12].

Time-variant optimization concepts have been proposed as early as 1971 by Rosenblueth/Mendoza [19] with special reference to earthquake resistant design. Two reconstruction policies are distinguished: no reconstruction after failure or systematic reconstruction. Moreover, capitalization and appropriate interest rates must be included. These ideas are used in this paper to optimize structural systems under reliability constraints in the time-variant case using first order reliability methods (FORM) in standard space. Two different locally stationary load models, rectangular wave renewal processes and differentiable Gaussian processes, are used. Examples demonstrate the proposed methodology.

2. Structural Reliability

2.1. Time-invariant and time-variant failure probabilities

The methods to compute time-invariant probabilities are well known. Let the quantities connected with uncertainty be modelled as stochastic variables in an n-dimensional basic variable vector $\mathbf{X} = (X_1, \ldots, X_n)^T$ with continuously differentiable distribution function $F_{\mathbf{X}}(\mathbf{x})$. The (differentiable) state function is denoted as $G(\mathbf{X}, \mathbf{p})$, which depends on the random variables $\mathbf{X}$ and a d-dimensional vector $\mathbf{p}$ of design parameters. $G(\mathbf{X}, \mathbf{p}) < \mathbf{0}$ corresponds to failure states, $G(\mathbf{X}, \mathbf{p}) = \mathbf{0}$ to the limit state and $G(\mathbf{X}, \mathbf{p}) > \mathbf{0}$ to safe states. Time-invariant probability of failure is then given by

$$P_f(\mathbf{p}) = \int\limits_{G(\mathbf{x},\mathbf{p})\leq 0} f_{\mathbf{X}}(\mathbf{x})\, d\mathbf{x} \tag{2.1}$$

where $f_{\mathbf{X}}(\mathbf{x})$ is the probability density of $\mathbf{X}$. Approximations for this integral can be obtained by modern first order reliability methods (FORM). A FORM - analysis usually introduces a transformation $\mathbf{X} = \mathbf{T}(\mathbf{U})$, which maps the random variables $\mathbf{X} = (X_1, \ldots, X_n)^T$ from original space into a $\mathbf{U}$-space of independent, standardized and normally distributed variables $\mathbf{U} = (U_1, \ldots, U_n)^T$, [8]. With $G(\mathbf{x}, \mathbf{p}) = G(\mathbf{T}(\mathbf{u}), \mathbf{p}) = g(\mathbf{u}, \mathbf{p})$ the probability of failure can be written as follows:

$$P_f(\mathbf{p}) = \int\limits_{g(\mathbf{u},\mathbf{p})\leq 0} \varphi_{\mathbf{U}}(\mathbf{u})\; d\mathbf{u} \tag{2.2}$$

where $\varphi_{\mathbf{U}}(\mathbf{u})$ is the standard normal density. $\mathbf{u}^*$ (most likely failure, design or β-point) is the solution of the following optimization problem:

$$\begin{aligned} \textbf{minimize}: &\quad \|\mathbf{u}\|, \\ \textbf{subject to}: &\quad g(\mathbf{u}, \mathbf{p}) \leq 0. \end{aligned}$$

Then, the FORM approximation of the failure probability is

$$P_f(\mathbf{p}) \approx \Phi(-\beta_{\mathbf{p}}) \tag{2.3}$$

with $\beta_{\mathbf{p}} = \|\mathbf{u}^*\|$. Once the FORM-solution is found it is possible to improve the result either by the second-order reliability method (SORM) or by importance sampling which, however requires additional numerical effort.

Time-variant reliability is more difficult to compute than time-invariant component reliability. T is the random time of exit into the failure domain, e.g. the probability of first passage into the failure domain. If failure occurs at a random time ($t > 0$) the distribution function of T has to be known. The probability of entering the failure domain for the first time given that the component was in the safe state at $t = 0$ in the time interval $[0, t]$ is then given by

$$P_f(\mathbf{p},t) = P(T \leq t|\ \mathbf{p}) = 1 - P(\{\forall\tau \in [0,t] : G(\mathbf{x}, \mathbf{p},\tau) > 0\})$$

where T is the so called first passage time. Unfortunately, exact solutions are only available for some few special cases of little interest in practice. Also simulation methods are by far too time-consuming. Therefore, in practical applications the outcrossing approach must be used.

The limit state function $G(\mathbf{x}, \mathbf{p},t)$ in time-variant reliability analysis contains a vector of deterministic parameter $\mathbf{p}$ and time-dependent and time-independent random variables $\mathbf{X}(t)$.

The rate of outcrossings into the failure domain $F = \{G(\mathbf{X}(t, \mathbf{p}, t) \leq 0\}$ is defined by

$$\nu^+(F,\tau) = \lim_{\triangle\to\mathbf{0}} \frac{P(\{G(\mathbf{X}(\tau), \mathbf{p},\tau) > 0\} \cap \{G(\mathbf{X}(\tau+\triangle), \mathbf{p},\tau+\triangle) \leq 0\})}{\triangle}.$$

The point process of crossings has to be a regular process. The mean number of outcrossings from time t_1 to t_2 is then determined by

$$E\big(N^+(t_1,t_2)\big) = \int_{t_1}^{t_2} \nu^+(F,\tau)\, d\tau \tag{2.4}$$

where $N^+(t_1,t_2)$ is the number of outcrossings in $[t_1,t_2]$ ($t_2 > t_1$).

In practical applications upper and lower bounds as solution for the failure probability $P_f(\mathbf{p},t_1,t_2)$ are in most cases sufficient. An upper bound solution is proposed in [2]. Thus failure occurs at t_1 or if there is at least one outcrossing into the failure domain in $[t_1,t_2]$. The upper bound derived can be written as follows:

$$P_f(\mathbf{p},t_1,t_2) \leq P_f(\mathbf{p},t_1) + E\big(N^+(t_1,t_2)\big) \tag{2.5}$$

where $P_f(\mathbf{p}, t_1)$ is calculated as in time-invariant case with a fixed time parameter t_1, usually a very small number. It is further assumed that for all calculations a probability distribution transformation into standard space is performed and the random processes starts with random initial conditions.

In the following the outcrossing rates of rectangular wave renewal and Gaussian processes will be developed in order to compute the time-variant failure probability. Then this is used in the reliability oriented optimization of structural components and separable series systems in a one-level approach.

2.2. Outcrossing rates for rectangular wave renewal processes

It is assumed that the rectangular wave renewal vector processes are regular, i.e. the probability of more than one renewals in a small time interval is negligible. Furthermore, renewals of each component of the vector are independent and the amplitudes are independent. In this paper only the stationary case is dealt with. Therefore the process is characterized by its amplitude distributions and time-independent jump rates $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_n)$, where n is the number of components of the vector process. Breitung and Rackwitz [3] have shown that the outcrossing rate can be evaluated as a summation of the products of the jump rate $\boldsymbol{\lambda}$ and the probability of having jumps from the safe domain into the failure domain in each component of a vector process. According to first order reliability methodology the outcrossing rate can be determined explicitly for linear failure surfaces $\delta F = \alpha^T \mathbf{u} + \beta_{\mathbf{p}} = 0$ with $\alpha = \nabla g(\mathbf{u}, \mathbf{p}) / \|\nabla g(\mathbf{u}, \mathbf{p})\|$:

$$\nu^+(F, \tau) = \sum_{i=1}^{n} \lambda_i [\Phi(-\beta_{\mathbf{p}}) - \Phi_2(-\beta_{\mathbf{p}}, -\beta_{\mathbf{p}}; 1 - \alpha_i^2)] \leq \sum_{i=1}^{n} \lambda_i \Phi(-\beta_{\mathbf{p}}) \quad (2.6)$$

where $\Phi_2(\,.\,, \,.\,; \,.\,)$ is the two-dimensional normal integral.

The asymptotic result is derived neglecting the probability of having jumps from the failure domain into the failure domain, which is often negligibly small. Improvements of this result is possible by using second order reliability methods. Further results including solutions for the non-stationary case can be found in [16].

2.3. Outcrossing rates for differentiable Gaussian processes

A differentiable Gaussian vector process is completely defined by its mean values $m(t)$ and the positive definite, symmetric matrix of covariance functions

$$C(t_1, t_2) = \left\{\sigma_{ij}(t_1, t_2); \quad i, j = 1, \ldots, n\right\}$$

where n is the number of components of the vector process. Alternatively it is characterized by its variance function $\sigma_i^2(t)$, $i = 1, \ldots, n_s$, and the autocorrelation coefficient function:

$$\rho_{ij}(t_1, t_2) = \frac{\sigma_{ij}(t_1, t_2)}{\sigma_i(t_1)\sigma_j(t_2)}.$$

The process is stationary, if the corresponding density is time independent, i.e. $m = m(t)$, $\sigma_i^2 = \sigma_i^2(t)$, $\rho_{ij}(\tau) = \rho_{ij}(t_1, t_2)$ with $\tau = |t_1 - t_2|$.

The outcrossing rate of a stationary, standardized Gaussian process can be written according to [23] by using first order methodology:

$$\nu^+(F) = \varphi(\beta_{\mathbf{p}})\frac{\omega_0}{\sqrt{2\pi}} \quad \text{with} \quad (\omega_0)^2 = -\alpha^T \ddot{R}\,\alpha \tag{2.7}$$

where $\beta_{\mathbf{p}}$ is the reliability index defined in Sec. 2, ω_0 the cycle rate and $\ddot{R}$ the matrix of second derivatives of the matrix of correlation functions:

$$\ddot{R} = \left\{ \frac{\partial^2}{\partial\tau_1\partial\tau_2}\rho_{ij}(\tau_1, \tau_2)_{|\tau=\tau_1=\tau_2} \quad i, j = 1, \ldots, n \right\}.$$

Further results including second order improvements can be found in [4] or for the more complicated non-stationary case in [17]. The outcrossing rate due to rectangular wave processes and differentiable Gaussian processes can simply be added due to regularity of the outcrossing process. For a series system it is frequently sufficient to take just the sum of all componental outcrosssing rates.

3. One-Level Approach

In practical applications structural optimization with respect to various design parameters, such as cost, weight or volume under additional reliability restrictions is frequently of interest. The problem is to combine these conflicting aims. Here only cost optimization, possibly including initial cost and expected cost of failure, subject to a given minimum reliability and other structural performance requirements will be discussed. In order to combine both levels the first-order Kuhn–Tucker optimality conditions for design points of the reliability optimization problem will be added as constraints to the cost optimization problem. This means, that at the β-point the following conditions have to be fulfilled:

$$g(\mathbf{u}^*, \mathbf{p}) = 0,$$

$$\frac{\mathbf{u}^*}{\|\mathbf{u}^*\|} + \frac{\nabla_{\mathbf{u}} g(\mathbf{u}^*, \mathbf{p})}{\|\nabla_{\mathbf{u}} g(\mathbf{u}^*, \mathbf{p})\|} = 0.$$

The second condition consists of $n-1$ separate equations, where n is the dimension of the vector $\mathbf{U} = (U_1, \ldots, U_n)^T$ of stochastic variables. The Kuhn–Tucker optimality conditions mean geometrically that the vector $\mathbf{u}^*$ is perpendicular to the limit state surface of the constraint $g(\mathbf{u}^*, \mathbf{p}) = 0$.

3.1. Objective functions

The general objective function $Z(\mathbf{p})$ will be maximized in the structural optimization:

$$Z(\mathbf{p}) = B(\mathbf{p}) - C(\mathbf{p}) - D(\mathbf{p}) \tag{3.1}$$

where $B(\mathbf{p})$ is the benefit derived from the structure, $C(\mathbf{p})$ the construction cost and $D(\mathbf{p})$ the damage cost. All monetary units of the structure, which will eventually fail after some time, are expected values and need to be capitalized down to the decision point $t = 0$. A continuous discount function $\delta(t) = \exp(-\gamma t)$ with interest rate γ is chosen which is sufficient in practical applications. It will be assumed that the benefit is independent of the parameter vector $\mathbf{p}$ and constant in time. With $t_s \to \infty$ as the expected time of use the benefit is determined as $B(\mathbf{p}) = \frac{b}{\gamma}$.

In addition, several replacement strategies have been defined: (1) the facility is given up after completion of mission or after failure or (2) the facility is systematically rebuilt after failure. Further, it is possible to distinguish between structures that fail upon completion or never and structures that fail at a random point in time. The appropriate objective functions are given in Table 1.

TABLE 1. Objective functions for replacement strategies and failure models.

Replacement strategy	**Objective function** $Z(p)$
Failure due to:	
- time invariant loads	$\frac{b}{\gamma} - C(p) - (C(p)+H)\frac{P_f(\mathbf{p})}{1-P_f(\mathbf{p})}$
- Structure is given up after failure [18]	$\frac{b}{\gamma+\lambda(\mathbf{p})} - C(p) - H\frac{\lambda(\mathbf{p})}{\gamma+\lambda(\mathbf{p})}$
Systematic reconstruction:	
- General asymptotic result [18]	$\frac{b}{\gamma} - C(p) - (C(p)+H)\frac{1}{\gamma E[T]}$
- Poissonian failures [20, 18]	$\frac{b}{\gamma} - C(p) - (C(p)+H)\frac{\lambda(\mathbf{p})}{\gamma}$
- Poissonian disturbances [7]	$\frac{b}{\gamma} - C(p) - (C(p)+H)\frac{\lambda P_f(\mathbf{p})}{\gamma+\lambda P_f(\mathbf{p})}$

It is assumed that Poissonian failures occcur with exponential failure times with parameter $\lambda(\mathbf{p})$ which can be controlled by $\mathbf{p}$. Furthermore renewal theory requires that the random times between failures are independent and identically distributed (except possibly the first failure) and the same design rules are used to reconstruct the structure, whereas the construction time is negligibly short compared to the interarrival times of failure.

3.2. Numerical optimization

The complete reliability-oriented cost optimization problem of structural systems in time-invariant case with systematic rebuilding and FORM-approximation of the probability of failure can be written as follows:

$$\begin{aligned}
&\textbf{minimize}: \\
&\qquad -Z(p) = -\frac{b}{\gamma} + C(p) + (C(p)+H)\frac{P_f(\mathbf{p})}{1-P_f(\mathbf{p})}, \\
&\textbf{subject to}: \\
&\qquad g(\mathbf{u},\mathbf{p}) = 0, \\
&\qquad u_i \, \|\nabla_{\mathbf{u}} g(\mathbf{u},\mathbf{p})\| + \nabla_{\mathbf{u}} g(\mathbf{u},\mathbf{p})_i \, \|\mathbf{u}\| = 0, \quad i = 1,\ldots,n-1, \\
&\qquad P_f(\mathbf{p}) - P_f^{\max} \le 0,
\end{aligned} \tag{3.2}$$

where the Kuhn-Tucker conditions are added as constraints to the optimization problem. Moreover, a maximum admissible failure probability is added, which can be absent. The advantage of the one level approach is that only one optimization algorithm is needed and convergence can be proven easily. It is important to reduce the set of the gradient conditions in the Kuhn-Tucker conditions by one. Otherwise the system of Kuhn-Tucker conditions is overdetermined. It is also important that the remaining Kuhn-Tucker conditions are retained under all circumstances, for example, if one or more become co-linear with one or more of the other restrictions.

Additionally the mathematical and physical admissibility of the design parameter vector and simple lower and upper bounds for the the transformed basic variable vector and the design vector must be observed:

$$\begin{aligned}
&\textbf{subject to}: \\
&\qquad h_j(\mathbf{p}) = 0\,, && j = 1,\ldots,m'\,, \\
&\qquad h_\ell(\mathbf{p}) \le 0\,, && \ell = m'+1,\ldots,m\,, \\
&\qquad u_{\min,i} \le u_i \le u_{\max,i}\,, && i = 1,\ldots,n\,, \\
&\qquad p_{\min,k} \le p_k \le p_{\max,k}\,, && k = 1,\ldots,n_p\,,
\end{aligned}$$

where $\mathbf{h}_i(\mathbf{p})$, $i = 1, \ldots, m$ contains m' equality and $m - m'$ inequality constraints on design parameters. $u_{\min}$, $u_{\max}$ and $p_{\min}$, $p_{\max}$ represent simple lower and upper bounds on stochastic variables resp. design parameters.

It is now assumed that failure occurs at a random time. Structural failures should be rare and independent events. Therefore, if the events follow a stationary Poisson process, the intensity $\lambda(\mathbf{p})$ can be replaced by the outcrossing rate $\nu^+(F, \mathbf{p})$, which has been evaluated in Secs. 2.2 and 2.3 in case of stationary rectangular wave renewal and differentiable Gaussian processes. Then it is possible to impose a maximum admissible failure rate and replace the reliability constraint in the reliability oriented structural optimization:

$$\nu^+(F, \mathbf{p}) \leq \nu_{\text{admissible}} \, .$$

The time-variant optimization problem in a one-level approach can be formulated similarly. The β-point may be identified as the point of a maximum outcrossing rate implying that the distance of the failure surface to the origin also dominates the local outcrossing rate. The complete optimization problem for systematic reconstruction can be written as follows [11]:

$$\begin{aligned} &\textbf{minimize}: \\ &\quad - Z(p) = -\frac{b}{\gamma} + C(p) + (C(p) + H)\frac{\nu^+(F, \mathbf{p})}{\gamma}, \\ &\textbf{subject to}: \\ &\quad g(\mathbf{u}, \mathbf{p}) = 0, \\ &\quad u_i \, \|\nabla_{\mathbf{u}} g(\mathbf{u}, \mathbf{p})\| + \nabla_{\mathbf{u}} g(\mathbf{u}, \mathbf{p})_i \, \|\mathbf{u}\| = 0, \quad i = 1, \ldots, n-1, \\ &\quad \nu^+(F, \mathbf{p}) - \nu_{\text{admissible}} \leq 0. \end{aligned} \tag{3.3}$$

Other cases are given in Table 1.

Another application of the proposed approach is the extension to time-invariant and time-variant separable series systems. Separability of the system implies independence of the components. Let $\mathbf{U}_k$ be an independent vector of stochastic variables for each mode k. Separability of the series system makes it possible to fulfill the componental Kuhn–Tucker optimality conditions in each failure mode simultaneously. Furthermore, it is assumed that the total expected failure cost of the system is bounded from above by the summation of the expected cost in every individual failure mode:

$$D(\mathbf{p}) = \sum_{k=1}^{s} \left[(C(\mathbf{p}) + H_k) \frac{P_f(\mathbf{p})_k}{1 - P_f(\mathbf{p})_k} \right],$$

where s is the number of separate failure modes which simplifies if the same failure cost $H = H_k$, $k = 1, \ldots, s$, are chosen for each mode. Thus, the complete reliability-oriented cost optimization problem in time-invariant case for separable series systems with systematic rebuilding and a FORM-approximation of the probability of failure can be written as follows [12]:

$$\begin{aligned}
&\textbf{minimize}:\\
&\quad -Z(p) \le -\frac{b}{\gamma} + C(p) + (C(p)+H)\frac{\sum_{k=1}^{s} P_f(\mathbf{p})_k}{1-\sum_{k=1}^{s} P_f(\mathbf{p})_k},\\
&\textbf{subject to}:\\
&\quad g_k(\mathbf{u}_k,\mathbf{p}) = 0,\\
&\quad (u_k)_i\,\|\nabla_{\mathbf{u}_k} g_k(\mathbf{u}_k,\mathbf{p})\| + \nabla_{\mathbf{u}_k} g_k(\mathbf{u}_k,\mathbf{p})_i\,\|\mathbf{u}_k\| = 0,\\
&\qquad\qquad\qquad i = 1,\ldots,n-1, \quad k = 1,\ldots,s,\\
&\quad P_f(\mathbf{p})_k - (P_f^{\max})_k \le 0.
\end{aligned} \tag{3.4}$$

Optimization of time-variant separable series systems can be done as follows for stationary Poissonian failures with systematic reconstruction if the same failure cost H_k, $k = 1,\ldots,s$, are chosen:

$$\begin{aligned}
&\textbf{minimize}:\\
&\quad -Z(p) = -\frac{b}{\gamma} + C(p) + (C(p)+H)\frac{\sum_{k=1}^{s} \nu^{+}(F,\mathbf{p})_k}{\gamma},\\
&\textbf{subject to}:\\
&\quad g_k(\mathbf{u}_k,\mathbf{p}) = 0,\\
&\quad (u_k)_i\,\|\nabla_{\mathbf{u}_k} g_k(\mathbf{u}_k,\mathbf{p})\| + \nabla_{\mathbf{u}_k} g_k(\mathbf{u}_k,\mathbf{p})_i\,\|\mathbf{u}_k\| = 0,\\
&\qquad\qquad\qquad i = 1,\ldots,n-1, \quad k = 1,\ldots,s,\\
&\quad \nu_k^{+}(F,\mathbf{p}) - (\nu_{\text{admissible}})_k \le 0.
\end{aligned} \tag{3.5}$$

Alternatively, one can specify a maximum admissible failure rate for the whole system. It is seen that the optimization problem for stationary time-variant problems differs from the time-invariant equivalent only by the way in which reliability measures are calculated. It should be clear, however, that the number of uncertain variables grows linearly with the size of the system and, therefore, increased numerical effort must be expected.

The proposed optimization tasks can conveniently be solved by a constrained sequential quadratic programming procedure. Therefore, in the next section some details of the new optimization algorithm JOINT 5 are presented [22] based on an earlier algorithm proposed by Enevoldsen/Sorensen [5].

4. The special algorithm JOINT 5 for reliability-oriented optimization

We want to solve the general optimization problem:

$$\begin{aligned} &\textbf{minimize}: && f(\mathbf{p}), \\ &\textbf{subject to}: && h_i(\mathbf{u},\mathbf{p}) = 0, \quad i = 1,\dots,m', \\ & && h_j(\mathbf{u},\mathbf{p}) \le 0, \quad j = m'+1,\dots,m. \end{aligned} \tag{4.1}$$

Forming the Lagrangian, using the Kuhn-Tucker conditions and performing one Newton-Raphson step leads to the following quadratic subproblem [14, 1]:

$$\begin{pmatrix} \mathbf{d}_{u,p}^k \\ \boldsymbol{\lambda}_{u,p}^{k+1} \end{pmatrix} = \begin{bmatrix} \nabla_{u,p}^2 L(\mathbf{u}^k,\mathbf{p}^k,\boldsymbol{\lambda}^k) & \nabla_{u.p}\mathbf{h}(\mathbf{u}^k,\mathbf{p}^k) \\ \nabla_{u.p}\mathbf{h}(\mathbf{u}^k,\mathbf{p}^k)^T & \mathbf{0} \end{bmatrix}^{-1} \begin{pmatrix} -\nabla_{u,p} f(\mathbf{p}^k) \\ -\mathbf{h}(\mathbf{u}^k,\mathbf{p}^k) \end{pmatrix} \tag{4.2}$$

where $\mathbf{d}^k$ is the search direction in the k-th iteration, $\lambda_{u,p}^{k+1}$ the Lagrangian multipliers, $\nabla_{u,p}^2 L(\mathbf{u}^k,\mathbf{p}^k,\boldsymbol{\lambda}^k)$ the Hessian of the Lagragian function, $\nabla_{u.p}\mathbf{h}(\mathbf{u}^k,\mathbf{p}^k)$ the gradient of the active constraints and $\nabla_{u,p} f(\mathbf{p}^k)$ the gradient of the twice differentiable objective function.

The computation of the Hessian of the Lagrangian function may cause considerable effort, is generally too expensive and may not remain positive definite during iteration. Therefore, an approximation is used. The JOINT 5 algorithm is based on a linearization method developed by Pshenichnyj, which substitutes the second derivative of the Lagrangian function $\nabla_{u,p}^2 L(\mathbf{u}^k,\mathbf{p}^k,\lambda^k)$ by the identity matrix all the time, cf. [15]). Equation (4.2) can then be written as follows:

$$\begin{pmatrix} \mathbf{d}_{u,p}^k \\ \boldsymbol{\lambda}_{u,p}^{k+1} \end{pmatrix} = \begin{bmatrix} \mathbf{I} & \nabla_{u.p}\mathbf{h}(\mathbf{u}^k,\mathbf{p}^k) \\ \nabla_{u.p}\mathbf{h}(\mathbf{u}^k,\mathbf{p}^k)^T & \mathbf{0} \end{bmatrix}^{-1} \begin{pmatrix} -\nabla_{u,p} f(\mathbf{p}^k) \\ -\mathbf{h}(\mathbf{u}^k,\mathbf{p}^k) \end{pmatrix} \tag{4.3}$$

Nevertheless it is possible that the quadratic subalgorithm fails or produces solutions in an infeasible domain. In particular, the linearized constraints can become linearly dependent although a solution of the optimization problem exists. In that case an 'extended' equation system (cf. [21]) will be solved. The optimization problem is therefore one dimension higher and can be written as follows:

$$\begin{aligned} &\textbf{minimize}: && \frac{1}{2}(\mathbf{d}_{u,p}^k)^T \mathbf{I} \mathbf{d}_{u,p}^k + \nabla_{u,p} f(\mathbf{p}^k)^T \mathbf{d}_{u,p}^k + \frac{1}{2} r^k (q^k)^2 \\ &\textbf{subject to}: && \nabla_{u.p}\mathbf{h}(\mathbf{u}^k,\mathbf{p}^k)^T \mathbf{d}_{u,p}^k + (1-q)\nabla_{u.p}\mathbf{h}(\mathbf{u}^k,\mathbf{p}^k) = \varepsilon, \end{aligned} \tag{4.4}$$

where only active constraints are considered and $\varepsilon > 0$ ($\varepsilon < 10^{-3}$ recommended). r^k is estimated by

$$r^k = \max\left\{r^0, \frac{r^*((\mathbf{d}_{u,p}^{k-1})^T \nabla_{u.p}\mathbf{h}(\mathbf{u}^{k-1},\ \mathbf{p}^{k-1})\boldsymbol{\lambda}_{u,p}^{k})^2}{(1-q^{k-1})^2(\mathbf{d}_{u,p}^{k-1})^T \mathbf{I}(\mathbf{d}_{u,p}^{k-1})}\right\} \tag{4.5}$$

with $r^* \geq 1$ if $k > 0$. The solution is $(\mathbf{d}_{u,p}^k, \boldsymbol{\lambda}_{u,p}^{k+1}, q^k)$. The point $(\mathbf{d}_{u,p}, q) = (\mathbf{0}, 1)$ obviously satisfies the constraints of (4.4) and can be used as a feasible starting point. The quadratic subproblem is then solvable almost always with solution:

$$\begin{aligned}\boldsymbol{\lambda}_{u,p}^{k+1} = \Big(-\nabla_{u.p}\mathbf{h}(\mathbf{u}^k,\mathbf{p}^k)^T\nabla_{u.p}\mathbf{h}(\mathbf{u}^k,\mathbf{p}^k) - \frac{1}{r^k}\mathbf{h}(\mathbf{u}^k,\mathbf{p}^k)\mathbf{h}(\mathbf{u}^k,\mathbf{p}^k)^T\Big)^{-1}&\\ \times\Big(\nabla_{u.p}\mathbf{h}(\mathbf{u}^k,\mathbf{p}^k)^T\nabla_{u,p}f(\mathbf{p}^k) - \mathbf{h}(\mathbf{u}^k,\mathbf{p}^k) + \varepsilon\Big),& \qquad (4.6)\end{aligned}$$

$$\mathbf{d}_{u,p}^k = -\nabla_{u.p}\mathbf{h}(\mathbf{u}^k,\mathbf{p}^k)\boldsymbol{\lambda}_{u,p}^{k+1} - \nabla_{u,p}f(\mathbf{p}^k), \tag{4.7}$$

$$q^k = \frac{1}{r^k}\mathbf{h}(\mathbf{u}^k,\mathbf{p}^k)^T\boldsymbol{\lambda}_{u,p}^{k+1}. \tag{4.8}$$

In a second step, when the search direction $\mathbf{d}^k$ is known, the step length has to be found. The new iteration point is determined by $\mathbf{x}^{k+1} = \mathbf{x}^k + \alpha^k\mathbf{d}_{u,p}^k$ with the step length parameter α^k ($0 \leq \alpha^k \leq 1$). α^k can be determined by a suitable line search method. Pure sequential quadratic optimization algorithms use either an exact or an approximate (quadratic interpolation) one-dimensional line search, cf. [21]. In the JOINT 5 algorithm a more robust strategy is used. With a descent function by Pshenichnyj ([15], [1]) and bisection of α^k the algorithm slows down. Starting from $\alpha^k = 1$ the following inequality is calculated:

$$\begin{aligned}&f(\mathbf{x}^k + \alpha^k\mathbf{d}_{u,p}^k) + t^k \max_{j\in M_A}\left\{\left\|g(\mathbf{x}^k + \alpha^k\mathbf{d}_{u,p}^k)\right\|\right\}\\ &\qquad\leq f(\mathbf{x}^k) + t^k \max_{j\in M_A}\left\{\left\|g(\mathbf{x}^k)\right\|\right\} - \delta\alpha^k\left\|\mathbf{d}_{u,p}^k\right\|^2 \qquad (4.9)\end{aligned}$$

where M_A denotes the number of active constraints, δ a parameter with $0 < \delta < 1$ ($\delta \approx 0.25$ recommended) and t^k a penalty parameter. If the inequality is not fulfilled, α^k will be half of the former value and the evaluation is repeated. This continues until a given finite number of bisections is done. The penalty parameter t^k will be set greater than or equal to the sum of all the Lagragian multipliers: $t^k \geq \gamma\sum_{j\in M_A}\left|\lambda_j^{k+1}\right|$ with $1 < \gamma \leq 2$. Additionally, the algorithm requires a careful strategy to avoid locally co-linear constraint where the Kuhn–Tucker conditions of the reliability problem must always be retained. Also, an efficient active set strategy must be implemented. It is important to note that all those devices are necessary for the problem at hand. Off-the-shelf algorithms will most likely fail.

As it can be seen, the algorithm needs first derivatives of the objective and all active constraints. In case of cost optimization under reliability constraints first order Kuhn–Tucker optimality conditions for a design point are restrictions to the optimization problem. In these equations first derivatives of the limit state function are already required. Thus, the solution of the quadratic subproblem needs second derivatives, i.e. the complete Hessian of $g(\mathbf{u}, \mathbf{p})$. The determination of the Hessian in each iteration step is laborious and can be numerically inexact. In order to avoid this, an approximation by iteration is proposed. One of the possibilities of replacing the Hessian is to preset it with zeros all the time. Note that linear limit state functions always have a zero Hessian. This implies some loss of efficiency, but the overall numerical effort needs not to rise, because calculation of the Hessian is no more necessary. In order to improve the results in case of nonlinear limit state functions, it is possible to evaluate the Hessian after the first optimization run and restart the algorithm. The solution is the new starting point and the Hessian matrix is fixed for the run and keep it fixed for the second run, and so on. This iterative improvement with subsequent restarts continues until the results differ only with respect to a given precision which is usually after very few steps.

The results can be simultaneously improved by including second-order corrections during reiteration, see [13]. Any other more exact improvement can be taken into account. The results for separable systems are slightly conservative. However, the same reiteration can adjust for this conservatism, if necessary.

5. Examples

5.1. Short column with rectangular cross section

In the first example a short column with rectangular cross section is considered. The dimensions, width b and depth h, determine as design parameters the total cost function $C_{\text{tot}}(\mathbf{p})$. It shall not contain any expected cost of failure and can then be written as $C_{\text{tot}}(\mathbf{p}) = 1.0\left[\frac{\text{CU}}{\text{m}^2}\right] bh$. No further constraints on the design parameters are imposed. They only had to satisfy the following upper and lower bounds: $b \in [5\text{m}; 15\text{m}]$, $h \in [15\text{m}; 25\text{m}]$. The limit state function of the proposed problem in terms of the parameter vector $\mathbf{p} = (b, h)$ and the vector of stochastic variables $\mathbf{x} = (P, M, Y)$ is given by

$G(\mathbf{x},\mathbf{p}) = 1 - \frac{4M}{bh^2Y} - \frac{P^2}{(bhY)^2}$ where the random variables $\mathbf{x} = (P, M, Y)$ are determined by stochastic variables presented in Table 2. The allowable failure probability is $1.0 \cdot 10^{-3}$ and the corresponding reliability index β equal to 3.090.

TABLE 2.

Stochastic variable	Distribution	Mean/St.deriv.	Unit
Axial force P	Normal	500/100	$\frac{\mathrm{N}}{\mathrm{mm}^2}$
Bending moment M	Normal	2000/400	MNm
Yield stress Y	Lognormal	5/0.5	$\frac{\mathrm{N}}{\mathrm{mm}^2}$

Optimization of total cost $C_{\mathrm{tot}}(\mathbf{p})$ will be performed with two different strategies. In the first strategy the complete Hessian of the second derivatives of the constraints is calculated numerically. In the second strategy the Hessian is approximated by a zero matrix. The results of the optimization problem

TABLE 3.

Evaluation of the Hessian matrix:	Numerical	Zero matrix
Results at the optimal point:		
Total cost [CU]	238.5	238.5
Final failure probability	$1.0 \cdot 10^{-3}$	$1.0 \cdot 10^{-3}$
Final reliability index	3.090	3.090
Vector of stochastic variables	$(731; 2705; 4.1)$	$(731; 2705; 4.1)$
Vector of cost variables	$(9.54; 25.0)$	$(9.54; 25.0)$

TABLE 4.

Numerical effort:	Numerical	Zero matrix
Number of iterations	5	5
Number of state function calls	152	50
Number of state function gradient calls:		
in basic variables x	8	8
in design variables p	8	8
Radii of curvature in U-space:	26.426; 138.934	

and the numerical effort for the two calculation alternatives are shown in Tables 3 and 4.

It can be seen that the results at the optimal point are the same in each case. But the numerical effort is substantially different. The number of state function calls using the zero matrix is distinctively smaller than the one where the Hessian is evaluated numerically in every iteration step. This is readily explained by the relatively large radii of curvature of the limit state surface in the solution point. The numerical computation of the second derivatives requires state function calls.

The algorithm works efficiently in this case. It should be mentioned that the problem is rather sensitive to the given design parameters and a good starting solution and suitable bounds have to be found in order to reduce the numerical effort.

5.2. Ten Bar Truss

In a second example structural optimization of a steel truss consisting of ten pin-jointed bars will be performed. The statical system is shown in Fig. 1.

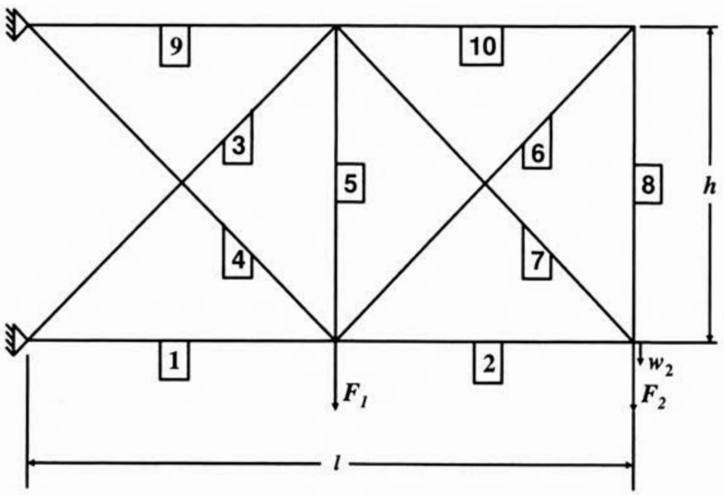

FIGURE 1. Statically system of the ten bar truss.

The cost function is connected with the weight of the structure and therefore with the cross section area of the bars. The design parameters p_i and a constant factor $A_0 = 19.6\,\mathrm{mm}^2$ determine the mean values of the ten cross section areas $A_i = p_i A_0$, $i = 1,..,10$. The interval of the upper and lower bounds of the design parameter is defined as $p_i \in [0.1; 100]$, $i = 1, \dots, 10$. The objective function is the total weight of the structure and will be minimized:

$$C(\mathbf{p}) = \rho A_0 \left(\sqrt{h^2 + \frac{l^2}{4}} (p_3 + p_4 + p_6 + p_7) + \frac{l}{2}(p_1 + p_2 + p_9 + p_{10}) + h(p_5 + p_8) \right).$$

The stochastic characteristics of the cross section areas, the stochastic nodal point loads F_1, F_2 and the limit of the displacement at node 2 w_{limit} are given in Table 5. The constant factors are $A_0 = 19.6\,\text{mm}^2$, $h = 1000.0\,\text{mm}$, $l = 2000.0\,\text{mm}$, $E = 210000.0\frac{\text{N}}{\text{mm}^2}$, $\rho = 8.0 \cdot 10^{-3} \frac{\text{N}}{\text{mm}^3}$ with Youngs modulus E and density ρ. The limit state function consists of the displacement $w_2(\mathbf{x}, \mathbf{p})$ at node 2 and the uncertain limit w_{limit} which may not be exceeded:

$$G(\mathbf{x}, \mathbf{p}) = w_{\text{limit}} - w_2(\mathbf{x}, \mathbf{p})$$

where $w_2(\mathbf{x}, \mathbf{p})$ is evaluated analytically in a statically indeterminate computation.

The allowable failure probability is $0.5 \cdot 10^{-2}$ and the corresponding reliability index β equal to 2.576. The failure cost H have been set at the relatively large value of 50000 [CU]. For smaller H the reliability constraint would become active.

Structural optimization under reliability constraints will be performed with the complete numerical evaluation of the Hessian and the approximation by the zero-matrix. The results at the optimal point and the numerical effort are shown in Tables 6 and 7.

As it can be seen the optimization program works well for a zero Hessian matrix. The results at the optimal point coincide. The number of iterations for a zero Hessian matrix is slightly larger than for the numerical Hessian but the number of state function calls is significantly reduced. The difference in the number of state function gradient calls results from a higher number of steps in the bisection line search strategy (see 4.9).

A certain difficulty in computing this example originates from the very low value of the bounds for the design parameters. The structure is statically indeterminate. This means that not all bars are necessary to transfer the load to the support. Design parameters which are set at the lower bound in the optimization process show that the corresponding bars are not necessary and could be omitted. Practically, different load combinations have to be taken into account which then may justify their existence. The optimized structure is shown in Fig. 2.

TABLE 5.

Stochastic variable	Symbol [Unit]	Distribution	Mean/St.deviat.
Cross section area 1-10	X_i [mm^2]	Lognormal	$p_i A_0/0.1$
Nodal Load	F_1 [N]	Normal	10000/1000
Nodal Load	F_2 [N]	Normal	10000/1000
Limit on displacement	w_{limit} [mm]	Normal	3.5/0.35

TABLE 6.

Evaluation of the Hessian matrix:	Numerical	Zero matrix
Results at the optimal point:		
Total cost [CU]	1428	1428
Final failure probability	$1.2 \cdot 10^{-3}$	$1.2 \cdot 10^{-3}$
Final reliability index	3.024	3.024
Vector of stochastic variables	(25.8;17.8;26.3;8.4; 1.8;1.3;25.1;1.4;37; 1.4;26;11380;10326)	(25.8;17.8;26.3;8.4; 1.8;1.3;25.1;1.4;37; 1.4;26;11380;10326)
Vector of cost variables	(1.3;0.9;1.4;0.5;0.1; 0.1;1.3;0.1;1.9;0.1)	(1.3;0.9;1.4;0.5;0.1; 0.1;1.3;0.1;1.9;0.1)

TABLE 7.

Numerical effort:	Numerical	Zero Matrix
Number of iterations	18	22
Number of state function calls	5156	812
Number of state function gradient calls:		
in basic variables x	22	40
in design variables p	21	25

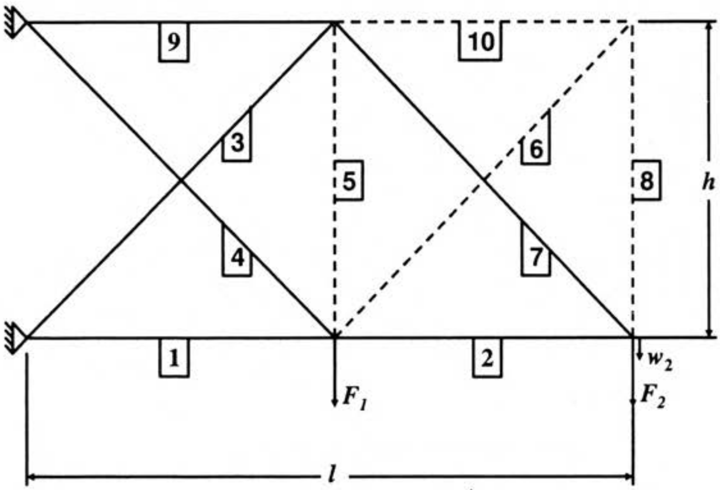

FIGURE 2. Optimized ten bar bar truss (dashed bars are zero bars).

5.3. Two-bay frame

In the last example a two-bay frame as shown in Fig. 3 using rigid-plastic theory with random horizontal and vertical loading and random plastic moments at nodes $1, \ldots, 10$ will be optimized under reliability constraints.

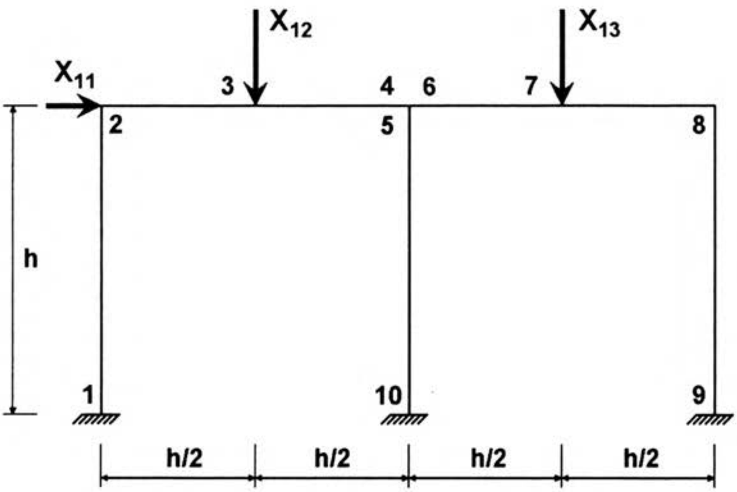

FIGURE 3. Loads and statical system of the frame.

Plastic hinges may form at nodes 1 to 10. Therefore, the structure can fail in eight different failure modes as shown in Fig. 4. The first three failure events are elementary mechanisms, the others combined mechanisms.

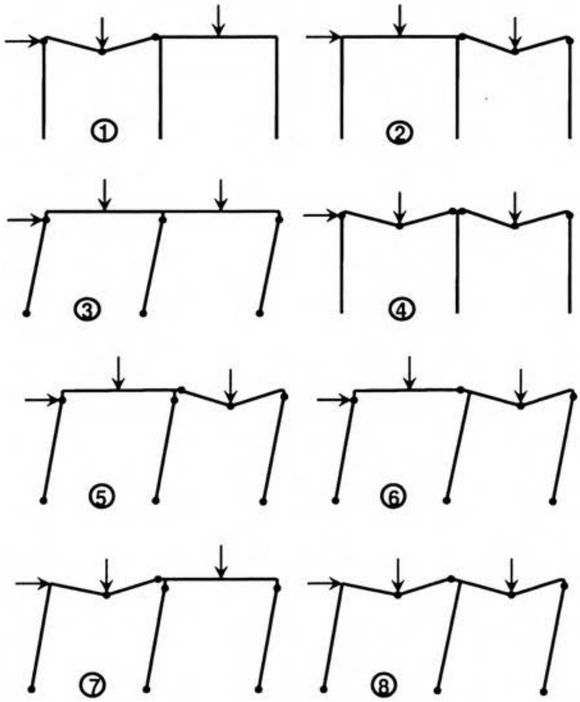

FIGURE 4. Failure modes of the frame.

A limit state function for each failure mode is available using the energy theorem:

$$G_1(x,p) = X_2+2X_3+X_4-X_{12}\cdot\tfrac{h}{2},$$
$$G_2(x,p) = X_6+2X_7+X_8-X_{13}\cdot\tfrac{h}{2},$$
$$G_3(x,p) = X_1+X_2+X_5+X_8+X_9+X_{10}-X_{11}\cdot h,$$
$$G_4(x,p) = X_2+2X_3+X_4+X_6+2X_7+X_8-X_{12}\cdot\tfrac{h}{2}-X_{13}\cdot\tfrac{h}{2},$$
$$G_5(x,p) = X_1+X_2+X_5+X_6+2X_7+2X_8+X_9+X_{10}-X_{11}\cdot h - X_{13}\cdot\tfrac{h}{2},$$
$$G_6(x,p) = X_1+X_2+X_4+2X_7+2X_8+X_9+X_{10}-X_{11}\cdot h - X_{13}\cdot\tfrac{h}{2},$$
$$G_7(x,p) = X_1+2X_3+X_4+X_5+X_8+X_9+X_{10}-X_{11}\cdot h - X_{12}\cdot\tfrac{h}{2},$$
$$G_8(x,p) = X_1+2X_3+2X_4+2X_7+2X_8+X_9+X_{10}-X_{11}\cdot h - X_{12}\cdot\tfrac{h}{2}-X_{13}\cdot\tfrac{h}{2},$$

where X_i, $i = 1,\ldots,10$, are the plastic moments of the frame at node i. X_{11},

X_{12} and X_{13} are stochastic loads at node 2, 3 and 7. The stochastic properties of the random variables X_i are given in the following table. The random plastic moments at node X_i, $i = 1, \ldots, 10$, are assumed to be lognormally distributed, the stochastic loads X_{11}, X_{12} and X_{13} are normally distributed (see Table 8).

TABLE 8.

Stochastic variable	[Unit]	**Mean/St.deviat.**
Plastic moment at node $1, 2, 5, 8, 9, 10$	X_i [kNm]	$p1/0.1 \cdot p1$
Plastic moment at node $3, 4, 6, 7$	X_i [kNm]	$p2/0.1 \cdot p2$
Load at node 2	X_{11} [kN]	2/0.6
Load at node 3	X_{12} [kN]	4/1.2
Load at node 7	X_{13} [kN]	6/1.8

The loads at node 3 and 7 are modeled as stationary rectangular wave renewal processes with jump rates $\lambda_{12} = \lambda_{13} = 0.5$ [1/year]. The load at node 2 is modelled as stationary differentiable Gaussian process with autocorrelation function $\rho_{ij}(\tau) = \exp(-\tau^2)$. The design parameters $p1$ and $p2$ are the mean values of the appropriate stochastic variables. The bounds for $p1$ and $p2$ are as follows: $p1 \in [5.0; 80.0]$ kNm, $p2 \in [5.0; 80.0]$ kNm. The objective function, which will be minimized in the optimization program, is defined as construction cost depending on the mean values of the plastic moments at nodes $1, \ldots, 10$ as $C(\mathbf{p}) = p1 + 2.0 \cdot p2$. The failure cost are $H = 1000$ and the interest rate is $\gamma = 0.02$. The optimization problem contains of 106 optimization variables. The optimal cost parameter for time- variant cost optimization under reliability constraints of this separable series system with $h = 20\,\mathrm{m}$ and a time interval of one year are

$$p1^* = 16.95, \qquad p2^* = 38.45,$$

and the optimal cost are $C_{\mathrm{tot}}(\mathbf{p}) = 93.85$ [CU]. The time-variant upper bound failure probability in each mode is computed as: $\big(P_{f,1}(\mathbf{p}^*), P_{f,2}(\mathbf{p}^*), P_{f,3}(\mathbf{p}^*), P_{f,4}(\mathbf{p}^*), P_{f,5}(\mathbf{p}^*), P_{f,6}(\mathbf{p}^*), P_{f,7}(\mathbf{p}^*), P_{f,8}(\mathbf{p}^*)\big) = (9.76 \cdot 10^{-11}, 2.21 \cdot 10^{-4}, 2.53 \cdot 10^{-6}, 2.37 \cdot 10^{-11}, 4.13 \cdot 10^{-8}, 1.82 \cdot 10^{-6}, 9.66 \cdot 10^{-10}, 1.38 \cdot 10^{-9})$. The equivalent reliability indices can be derived as: $(\beta_1^*, \beta_2^*, \beta_3^*, \beta_4^*, \beta_5^*, \beta_6^*, \beta_7^*, \beta_8^*) = (6.37, 3.51, 4.56, 6.58, 5.36, 4.63, 6.00, 5.95)$. The system failure probability is $2.25 \cdot 10^{-4}$ with the corresponding equivalent reliability index 3.51.

6. Conclusion

A method for reliability-oriented time-invariant and time-variant structural optimization of components and separable (independent) series systems in a special one level approach using first order reliability methods (FORM) in standard space has been derived. Approximations for time-variant failure probabilities are computed via the outcrossing method for locally stationary rectangular wave renewal and differentiable Gaussian processes.

The optimization problem is solved by the newly developed gradient based algorithm JOINT 5. It includes a reliable and robust slow down strategy to improve stability of the algorithm instead of an exact line search. Furthermore, it is possible to solve an 'extended' equation system in case of failure in the quadratic subalgorithm, e.g. linear dependency of the linearized constraints. It requires second derivatives of the limit state functions. This can be avoided by iteration. In the first iteration the Hessian is approximated by a zero matrix corresponding to linear limit state functions. In the second iteration the Hessian is determined once and kept fixed. The results can thus be improved by reiteration of the complete optimization task. Generalization to intermittent random vector processes in the one-level approach is theoretically possible as well as including maintenance considerations and a non-constant benefit into the objective function. If random loads with non-stationary or non-Poissonian failure models are considered only bi-level methods are applicable at the moment. Further research is required.

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