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ON THE STABILITY OF SPRING PENDULUM VIBRATION WITH THE MOVABLE SUSPENSION POINT

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Preface.

A study of the dynamic behavior of pendulum vibration with the movable point may begin logically with measuring of many physical magnitudes when the measuring instruments are connected with the test body. It is well known, the support motion has a great quantitative and qualitative influence on the engine vibration. The work of the mechanical vibrator, which is connected with medium, is the other exemple of this problem.

The purpose of this paper is to develop the subject of pendulum vibration with a movable suspension point conside-red by G. Gorelik and A. Witt [1], J.P. Den Hartog [2], N. Minorsky [3], L.A. Pipes [4], R.A. Arnold and L. Maunder [5].

1. The general equation of pendulum vibrations.

We consider the motion of the system shown diagramatically in Fig. 1. The system consists of a heavy uniform rod AB of mass m_1 and of length 2b whose upper end A is constrained to move horizontally with displacement $x = x(t)$ and vertically with displacement $y = y(t)$. On this rod moves the other mass m_2 which has a center of gravity denoted by m_2. The spring is attached to the rod at A, is of stiffness k_1 and has an unstressed length approximatelly ξ_0. We denote by $\tilde{\xi}(t)$ the distance between point A and center mass m_2: thus the elongation of the spring in each time is $\xi_0 - \tilde{\xi}(t)$.

Viscous friction exists between the mass m_2 and rod m_1 of the value c_1 per unit of relative velocity.

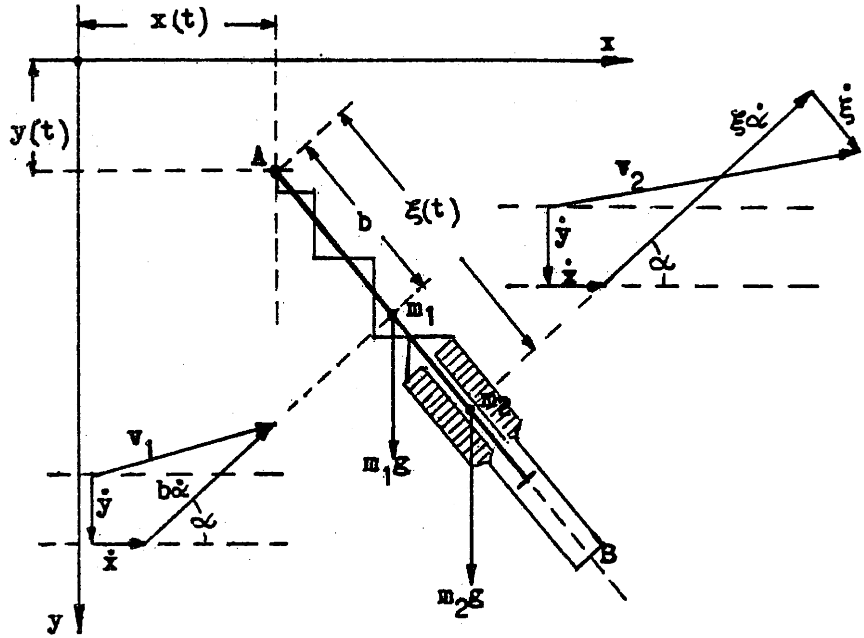

Fig. 1

Viscous friction by pendulum motion around the suspension point A is assumed in the value of c_2 per unit of angular velocity of rod. The distance $\tilde{\xi}(t)$ of the mass m_2 from A and the angle $\alpha(t)$ between AB and the vertical may be chosen as our generalized coordinates.

The absolute speeds of point m_1 and m_2 may be obtained from the expressions

$$(1)\qquad \begin{aligned} v_1^2 &= \dot{x}^2+\dot{y}^2+(b\dot{\alpha})^2+2b\dot{\alpha}(\dot{x}\cos\alpha-\dot{y}\sin\alpha), \\ v_2^2 &= \dot{x}^2+\dot{y}^2+(\tilde{\xi}\dot{\alpha})^2+2\tilde{\xi}\dot{\alpha}(\dot{x}\cos\alpha-\dot{y}\sin\alpha)+\dot{\tilde{\xi}}^2+2\dot{\tilde{\xi}}(\dot{x}\sin\alpha+\dot{y}\cos\alpha), \end{aligned}$$

which are calculated from geometry of vectors as shown in Fig. 1.

The kinetic energy of the system may be obtained from the equation

$$(2) \qquad T = \frac{1}{2}\left(m_1 v_1^2 + m_2 v_2^2 + I\dot{\alpha}^2\right),$$

where $I = I_1^0 + I_2^0$ and I_1^0 - the moment of m_1 inertia in regard to m_1, I_2^0 - the moment of m_2 inertia in regard to m_2.

Substituting the value v_1^2 and v_2^2 from (1) in (2) gives

$$(3) \qquad \begin{aligned} T = {} & \frac{1}{2}(m_1+m_2)(\dot{x}^2+\dot{y}^2)+\frac{1}{2}(m_1 b^2+m_2\xi^2+I)\dot{\alpha}^2+\dot{x}\dot{\alpha}\cos\alpha(m_1 b+m_2\xi)+ \\ & + \frac{1}{2}m_2\dot{\xi}^2+m_2\dot{x}\dot{\xi}\sin\alpha-\dot{x}\dot{y}\sin\alpha(m_1 b+m_2\xi)+\dot{\xi}\dot{y}m_2\cos\alpha . \end{aligned}$$

The potential energy of the system is

$$(4) \qquad V = -m_1 g(b\cos\alpha+y)-m_2 g(\xi\cos\alpha+y)+\frac{1}{2}k_1(\xi-\xi_0)^2 .$$

Applaying the Lagrangian equations

$$\frac{d}{dt}\frac{\partial T}{\partial\dot{\xi}}-\frac{\partial T}{\partial\xi}+\frac{\partial V}{\partial\xi}=-c_1\dot{\xi}, \quad \frac{d}{dt}\frac{\partial T}{\partial\dot{\alpha}}-\frac{\partial T}{\partial\alpha}+\frac{\partial V}{\partial\alpha}=-c_2\dot{\alpha}$$

after calculating we obtained the following nonlinear differential equations of motion

$$(5) \qquad \begin{aligned} & m_2\ddot{\xi}+c_1\dot{\xi}+k_1(\xi-\xi_0)+m_2\left[\ddot{x}\sin\alpha-(g-\ddot{y})\cos\alpha\right]-m_2\dot{\alpha}^2\xi=0, \\ & (I+m_1 b^2+m_2\xi^2)\ddot{\alpha}+c_2\dot{\alpha}+(m_1 b+m_2\xi)\left[\ddot{x}\cos\alpha+(g-\ddot{y})\sin\alpha\right]+2m_2\xi\dot{\xi}\dot{\alpha}=0 . \end{aligned}$$

If $y = 0$ this system has the identical form as in the book of R.N. Arnold and L. Maunder mentioned in references [5].

Let $x = m_1 = I = 0$ and $y = a\sin\omega t$, where ω is frequency and a the amplitude of the pendulum suspension point A in the vertical direction. After introducing these values into the system (5) the equation of the simplified pendulum are thus

$$(6)\quad \begin{aligned} &\ddot{\xi} + \frac{c_1}{m}\dot{\xi} + \frac{k_1}{m}\xi = a\omega^2 \sin\omega t\cos\alpha + (1+\xi)\dot{\alpha}^2 - g(1-\cos\alpha)\,, \\ &\ddot{\alpha} + \frac{c_2}{ml^2}\dot{\alpha} + \frac{g}{l}\sin\alpha = -\frac{1+\xi}{l^2}\left(2\dot{\alpha}\dot{\xi} + a\omega^2\sin\omega t\sin\alpha\right) - \frac{2l+\xi}{l^2}\xi\ddot{\alpha} - \frac{g}{l^2}\xi\sin\alpha\,, \end{aligned}$$

where $m = m_2$, $\xi = \tilde{\xi} - 1$ and $k_1(1-\xi_0) = mg$, l - length of the spring in the static equilibrium in vertical position.

It is not proposed to attempt a solution to the full system (5) of motion equation. A stady in this work will be made, however, of the simplified cases described by (6).

2. Solution of a simplified deflected pendulum.

Introduce to the system (6) the dimensionless time $\tau = \omega t$, dimensionless coordinate $\zeta = \frac{\xi}{a}$ and denote by

$\omega_1^2 = \frac{k_1}{m}$ - the natural frequency of vibration of mass m through rod,

$\omega_0^2 = \frac{g}{l}$ - the natural frequency of angular vibration,

$h_1 = \frac{c_1}{c_{cr}}$ - damping ratio,

$c_{cr} = 2m\omega_1$ - "critical" damping,

$\Omega = \frac{\omega}{\omega_0}$ - forced frequency ratio,

$\rho = \frac{\omega_1}{\omega_0}$ - natural frequency ratio,

$\mu = \frac{a}{l}$ - parameter, which we call "small",

$K = \frac{1}{a}\frac{\omega_0}{\omega}$, $\frac{c_2}{ml^2\omega} = 2\mu K h_2$, $h_2 = \frac{c_2}{2ml^2\omega}$ - coefficients.

After performing some algebra equation (6) is transformed into

$$\zeta'' + 2h_1\frac{\varrho}{\Omega}\zeta' + \left(\frac{\varrho}{\Omega}\right)^2\zeta = \sin\tau\cos\alpha + \frac{1}{\mu}\alpha'^2 + \alpha''\zeta - \mu K^2\left(1-\cos\alpha\right),$$

$$(7)\quad \alpha'' + 2\mu K h_2\alpha' + \mu K^2\sin\alpha = -\mu\sin\tau\sin\alpha - 2\mu\alpha'\zeta' - \mu^2\zeta\sin\tau\sin\alpha -$$

$$-2\mu^2\zeta\zeta'\alpha' - 2\mu\zeta\alpha'' - \mu^3K^2\sin\alpha ,$$

where $' = \frac{d}{d\tau}$ is the symbol of differentiation in respect to dimensionless undependable variable τ .

The first equation of system (7) for $\alpha = 0$ takes form

$$\zeta'' + 2h_1\frac{\varrho}{\Omega}\zeta' + \left(\frac{\varrho}{\Omega}\right)^2\zeta = \sin\tau .$$

The amplitude of forced vibration will be less of a unit when

$$\frac{1}{\sqrt{\left[\left(\frac{\varrho}{\Omega}\right)^2 - 1\right]^2 + 4h_1^2\left(\frac{\varrho}{\Omega}\right)^2}} \leqslant 1,$$

thus

$$\left[\left(\frac{\varrho}{\Omega}\right)^2 - 1\right]^2 + 4h_1^2\left(\frac{\varrho}{\Omega}\right)^2 \geqslant 1$$

or

$$\left(\frac{\varrho}{\Omega}\right)\left[\frac{\varrho}{\Omega} - 2\left(1 - 2h_1^2\right)\right] \geqslant 0$$

from which $1 - 2h_1^2 \leqslant 0$ and $h_1 \geqslant \frac{\sqrt{2}}{2}$. Therefore, the amplitude of forced vibration is less of a unit for any Ω when $h_1 \geqslant \frac{\sqrt{2}}{2}$.

In our case we are interested only in such case when $h_1 \geqslant \frac{\sqrt{2}}{2}$ or when the damping coefficient in relative motion through rod AB $c_1 \geqslant \frac{\sqrt{2}}{2} c_{cr}$. Which means that $|\xi| \leqslant a$.

The solution of the system (7) due to the first approximation we want to find in the form

$$\zeta = X,$$

(8) $$\alpha = Y + \mu \sin\tau \sin Y,$$

$$\frac{d\alpha}{d\tau} = \mu Y_1 + \mu \cos\tau \sin Y .$$

Applying this to the equation (7), we write

(9) $$X'' + 2h_1 \frac{\varrho}{\Omega} X' + \left(\frac{\varrho}{\Omega}\right)^2 X = \sin\tau \cos Y + \mu[\quad] + \ldots,$$

$$Y' = \mu Y_1 + \mu^2[\quad] + \ldots,$$

(10) $$Y_1' = -2\mu K h_2 Y_1 - K^2 \sin Y - \sin^2\tau \sin Y \cos Y - 2\mu Y_1 X' -$$

$$-2\mu X' \cos\tau \sin Y + \mu X \sin\tau \sin Y + \mu^2[\quad] + \ldots .$$

Assuming $\mu \ll 1$ from the equation (9) we obtain

(11) $$X'' + 2h_1 \frac{\varrho}{\Omega} X' + \left(\frac{\varrho}{\Omega}\right)^2 X = \sin\tau \cos Y,$$

where Y is constant by the steady state motion.

The solution of the equation (11) by these assumptions we may write

(12) $$X = U \cos Y \sin(\tau - \varphi),$$

where

$$U = \frac{1}{\sqrt{\left[\left(\frac{\varrho}{\Omega}\right)^2 - 1\right]^2 + 4h_1^2 \left(\frac{\varrho}{\Omega}\right)^2}}, \quad \tan\varphi = \frac{2h_1 \frac{\varrho}{\Omega}}{\left(\frac{\varrho}{\Omega}\right)^2 - 1} .$$

Substituting the solution (12) into (10) and averaging through dimensionless time τ in this period gives

$$(13) \qquad Y'' + 2\mu K h_2 Y' + \mu^2\left[K^2 + \frac{1}{2}\left(1 + U\cos\varphi\right)\cos Y\right]\sin Y = 0.$$

Now, we are investigating the stability of the down equilibrium position of the pendulum, in other words, stability of the zero solution of the equation (13) . In such a case from the equation (13) we obtain the following variational equation

$$\frac{d^2\delta Y}{d\tau^2} + 2\mu K h_2 \frac{d\delta Y}{d\tau} + \mu^2\left[K^2 + \frac{1}{2}\left(1 + U\cos\varphi\right)\right]\delta Y = 0,$$

and the stability condition

$$(14) \qquad K^2 + \frac{1}{2}\left(1 + U\cos\varphi\right) > 0.$$

The unequalities (14) are satisfied for the all finite magnitudes of kinematic restorting force frequency ω and φ according to the previous assumptions, becomes $0 \leqslant U \leqslant 1$.

By the upper equilibrium position of the pendulum ($Y = \pi$) after a short calculation, the stability condition becomes

$$(15) \qquad K^2 - \frac{1}{2}\left(1 + U\cos\varphi\right) < 0.$$

By using the notes for K and U we perform these unequalities into

$$(16) \qquad y_1 < y_2 ,$$

where

$$y_1 = \frac{2\varepsilon^2}{\left(\frac{\Omega}{\varrho}\right)^2} , \qquad y_2 = 1 + \frac{\left(\frac{\Omega}{\varrho}\right)^2\left[1 - \left(\frac{\Omega}{\varrho}\right)^2\right]}{\left[1 - \left(\frac{\Omega}{\varrho}\right)^2\right]^2 + 4h_1^2\left(\frac{\Omega}{\varrho}\right)^2}$$

This both functions y_1 and y_2 are shown in Fig. 2 as a plot of the function $\frac{\Omega}{\varrho} = \omega\sqrt{\frac{m}{k_1}}$ for the definite systems

$$\delta^2 = \left(\frac{1}{a}\right)^2\left(\frac{\omega_o}{\omega_1}\right)^2 = \frac{1}{4}, \frac{1}{2}, 1, 2 \text{ and } h_1 = \frac{c_1}{c_{cr}} = \frac{\sqrt{2}}{2}, 1, \infty.$$

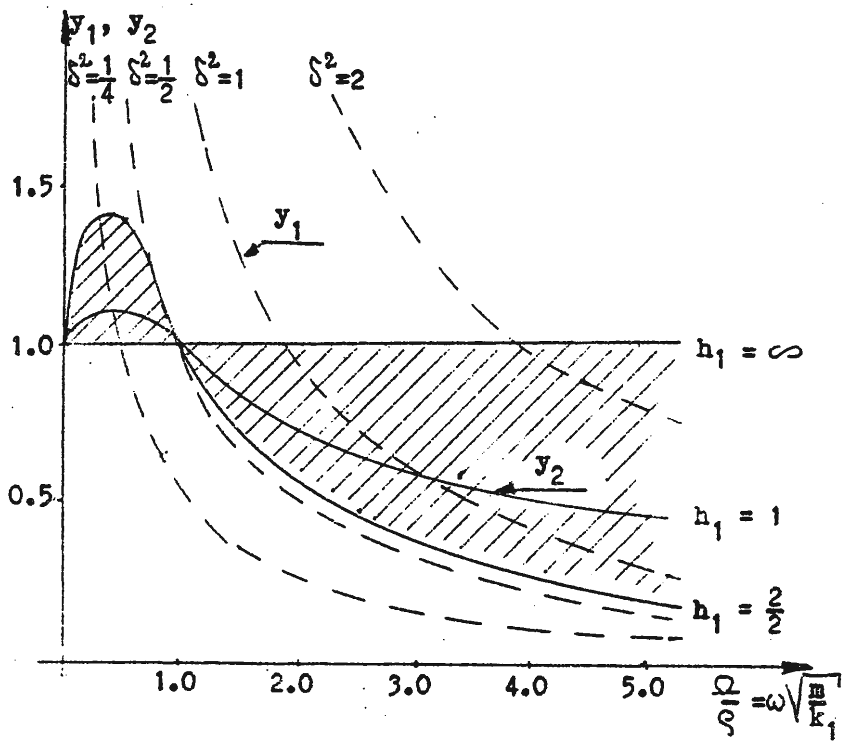

Fig. 2

The region of value functions y_2 is contained between two curves y_2 for $h_1 = \frac{\sqrt{2}}{2}$ and $h_1 = \infty$.

For example, we are trying the stability of small motion of the pendulum for which a = 2 cm, mg = 1 kG, k_1 = 1 kG/cm, $c_1 = 0.71\ c_{cr}$. Thus $\delta^2 = \frac{1}{4}$ and from the plot we find $\omega\sqrt{\frac{m}{k_1}} > 0.4$. In other words, by $\omega > 12.6$ the upper equilibrium position is stable.

In this way we can try stability conditions of spring pendulum vibration in this position.

When h_1 or k_1 tends to infinity (i.e., the spring of pendulum occur in the inverted mathematical pendulum with rigid rod) the unequalities (15) than becomes

$$K^2 - \frac{1}{2} < 0 .$$

These results agree with works [1], [3], [4].

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