FRACTIONAL CALCULUS AND PATH INTEGRAL METHOD FOR NONLINEAR SYSTEMS UNDER WHITE NOISE PROCESSES

M. Di Paola
Dipartimento di Ingegneria Strutturale Aerospaziale e Geotecnica (DISAG)
Università degli Studi di Palermo, Viale delle Scienze 90128 Palermo, Italy

1. Introduction

One of the most challenging problem in the field of nonlinear systems under stochastic agencies is to find statistics of the response process. In this framework the Path Integral (PI) method is an effective tool to provide a step-by-step integration technique in terms of Probability density function [see e.g. [1]]. The method starts from the Chapman Kolmogorov equation combined with the so called short-time Gaussian approximation (STGA). Keeping this in mind, the probability density function (PDF) at time $(t+\tau)$ is a convolution integral involving the PDF at time t and a Gaussian kernel that represents the conditional density of the response PDF. The latter is the solution of the Fokker Planck equation with assigned deterministic initial condition at time t, that may be easily evaluated by considering that for (τ) small such a response is Gaussian distributed (STGA). Even though the formulation is quite simple numerical problems arise to evaluate the convolution integrals especially for numerous degree of freedom systems. Moreover working in terms of response moments is impossible since an hierarchy of moments immediately appears.
Recently it has been shown that functional moments that is moments of the type $E[iX^{-\gamma}]$, $\gamma \in C$, $0<\mathrm{Re}(\gamma)<1$ always produce a representation of both PDF and Characteristic Function (CF) in the whole domain of existence of the two domains. It has also been shown that $E[-iX^{-\gamma}]$ coincides with the Riemann Liouville fractional derivative of the CF in zero. By using Melling transform it has also been shown that

(1)
$$\phi_X(\pm\vartheta) = \frac{1}{2\pi i}\int_{\rho-i\infty}^{\rho+i\infty}\Gamma(\gamma)E\left[(\mp iX)^{-\gamma}\right]\vartheta^{-\gamma}d\gamma$$

(2)
$$p_X(x) = \frac{1}{2\pi^2 \mathrm{i}}\mathrm{Re}\left\{\int_{\rho-\mathrm{i}\infty}^{\rho+\mathrm{i}\infty}\Gamma(\gamma)\Gamma(1-\gamma)E\left[(-\mathrm{i}X)^{-\gamma}\right](\mathrm{i}x)^{\gamma-1}\,\mathrm{d}\gamma\right\}$$

Where $\phi_X(\vartheta)$ is the CF, $p_X(x)$ is the PDF, $\rho = \mathrm{Re}(\gamma)$ and $\Gamma(.)$ is the Euler Gamma function. It has to be remarked that in eq. (1) and (2) integrals are performed along the imaginary axis and then the remain finite also for α - stable processes, for which the integer moments of order greater than two remain divergent ones. Discretization of integrals (1) and (2) produces quite good results as demonstrated in [1]. Keeping these results in mind in the paper it is shown that the PIS remains an amenable problem also from computational point of view [2, 3]. In the latter approaches the PIS was implemented in terms of fractional moments. In the proposed paper a different strategy is proposed to evaluate probability density function and CF at time $(t+\tau)$ by knowing the PDF or the CF at the previous time instant t.

In order to aim at this as a first step the PIS is converted in terms of CF by means of the Fourier transform of the Chapman Kolmogorov equation, then the CF at time t is converted in terms of fractional moments as shown in eq.(1) and then the fractional moments at time $(t+\tau)$ may be easily evaluated by eq.(2).

2. References

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[2] G. Cottone and M. Di Paola (2009). On the Use of Fractional Calculus for the Probabilistic Characterization of Random Variables, *Probabilistic Engineering Mechanics* , **24**, 321-330.

[3] G. Cottone, M. Di Paola and A. Pirrotta (2008). Path integral solution by fractional calculus, *Journal of Physics*: Conference Series **96** pp. 012007-0112007.