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**Thermodynamics
of a Rate Sensitive
Plastic Material**

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THERMODYNAMICS OF A RATE SENSITIVE PLASTIC MATERIAL

1. Introduction

The basic object of the present paper is the formulation of the thermodynamic theory of a rate sensitive plastic material within the framework of thermodynamics of material with internal state variables[⌘].

In thermodynamic theory of plastic solids at finite strains there are two basic difficulties. The first of them is connected with kinematic description of plastic deformation. In this description it is usually assumed the additivity of the elastic and inelastic parts of the deformation tensor^{⌘⌘}. The second difficulty in thermodynamic description of plastic deformation is connected with the problem of choice of thermodynamic variables of state. This choice in thermodynamics of plastic deformation is not unique^{⌘⌘⌘}. The important question rises here whether the plastic deformation tensor may be treated as the thermodynamic state variable.

We intend to establish the thermodynamic theory of an inelastic material in which both these difficulties may be

⌘ Thermodynamics with internal state variables has been presented independently by Coleman and Gurtin [1967, 1] and Valanis [1967, 6]. The application of the concept of internal state parameters to the description of the properties of viscoelastic materials may be found in several papers /Vid. Schapery [1964, 2], Meixner [1966, 2], and Valanis [1966, 5], [1967, 5], for example/.

⌘⌘ Green and Naghdi [1965, 2] assumed the postulate of additivity. Backman [1964, 1] and Lee and Liu [1967, 2] presented a different approach to this problem. Perzyna [1966, 4] has formulated the theory of elastic-viscoplastic materials in which no division of the deformation tensor into two parts is needed. In the papers [1966, 3], [1967, 4] and [1967, 7] the additivity postulate have been also assumed.

⌘⌘⌘ Cf. Green and Naghdi [1965, 2] and Kestin [1966, 1], for example.

taken into account. Assuming the deformation tensor and temperature as thermodynamic state variables and the components of the inelastic deformation tensor as internal state parameters /hidden parameters/ the thermodynamic theory of a rate sensitive plastic material is formulated. Thus, the basic conception is to treat a rate sensitive plastic material as a material with internal state variables. No connection between the deformation tensor and the inelastic deformation tensor is postulated. The deformation tensor is implied by kinematics of the given motion for a body \mathcal{B} and the inelastic deformation tensor is determined by the solution of the initial-value problem for ordinary first order differential equation.

The full system of reduced constitutive equations describing the behaviour of an elastic/viscoplastic material^{*} in thermodynamic process has been given. The restrictions imposed by thermodynamic postulate of a nonnegative entropy production have been investigated. The conditions under which the equilibrium state may be reached have been discussed. It has been shown that the asymptotically stable equilibrium state may be reached for an elastic/viscoplastic material in the isothermal relaxation process. The mathematical description of the isothermal relaxation process has been presented. For isotropic material the polynomial representations for response functions have been given. The limit case which leads to an elastic-plastic material has been discussed. The full discussion of thermodynamic theory of an elastic-plastic material is shown.

2. Constitutive assumptions for a material with internal state variables

Let us consider a body \mathcal{B} with particles \mathcal{X} and assume

* For the definition of an elastic/viscoplastic material vide infra.

that this body may deform and conduct heat. We shall assume that couple stresses and body couples are absent.

The thermodynamic process of a body \mathcal{B} is described by nine functions $\{\underline{\chi}, \underline{\mathbb{I}}, \underline{b}, \psi, \underline{q}, \tau, \eta, \vartheta, \underline{A}\}$ of the particle X and time t . These functions have following interpretations. The function of motion $\underline{\chi}(X,t)$ determines the spatial position occupied by the material point X at time t , which in reference configuration \mathcal{R} occupied the position \underline{X} , i.e.,

$$\underline{x} = \underline{\chi}(\underline{X}, t). \quad /2.1/$$

The components of the function $\underline{\chi}$ are assumed to be continuously differentiable. The function $\underline{\mathbb{I}}(X,t)$ is symmetric Cauchy stress tensor, $\underline{b}(X,t)$ is the body force per unit mass, $\psi(\underline{X},t)$ denotes specific free energy per unit mass, $\underline{q}(X,t)$ the heat flux vector, $\tau(X,t)$ the heat supply per unit mass and unit time, $\eta(X,t)$ is the specific entropy, $\vartheta(X,t)$ the local absolute temperature and $\underline{A}(X,t)$ is the internal state tensor. We assume that \underline{A} is symmetric second order tensor.

Since we identify the material point X with its position \underline{X} in the reference configuration \mathcal{R} , hence the deformation gradient \underline{F} is determined by

$$\underline{F} = \text{Grad } \underline{\chi}(\underline{X}, t), \quad /2.2/$$

where Grad is computed with respect to material coordinates \underline{X} . We shall introduce the second Piola-Kirchhoff stress tensor

$$\underline{\bar{\mathbb{I}}} \equiv \int \underline{F}^{-1} \underline{\mathbb{I}} (\underline{F}^{-1})^T, \quad /2.3/$$

where $\int = \det \underline{F} > 0$. Similarly, the heat flux vector per unit surface in the reference configuration \mathcal{R} will be define as follows

$$q_R \equiv \int F^{-1} q. \quad /2.4/$$

The system of nine functions $\{X, \tilde{T}, b, \psi, q_R, \tau, \eta, \mathcal{F}, A\}$ defined for every particle X in \mathcal{B} and for any time t is called the thermodynamic process in \mathcal{B} if, and only if, it is compatible with the condition for the balance of linear momentum /Cauchy's first law of motion/

$$\text{Div}(F \tilde{T}) + \zeta_R b = \zeta_R \ddot{X} \quad /2.5/$$

and with the balance of energy /the first law of thermodynamics/

$$\frac{1}{2} \text{tr}(\tilde{T} \dot{C}) - \text{Div} q_R - \zeta_R (\dot{\psi} + \mathcal{F} \dot{\eta} + \dot{\mathcal{F}} \eta) + \zeta_R \tau = 0, \quad /2.6/$$

where operator Div is computed with respect to material coordinates X , ζ_R is the mass density in the reference configuration \mathcal{R} , the dot denotes the material differentiation with respect to time t , and C is the right Cauchy-Green deformation tensor

$$C = F^T F. \quad /2.7/$$

In order to define a thermodynamic process in \mathcal{B} , it suffices to prescribe the seven functions $\{X, \tilde{T}, \psi, q_R, \eta, \mathcal{F}, A\}$. The two remaining functions b and τ are then uniquely determined by equations /2.5/ and /2.6/.

We shall require that for any time t for a thermodynamic process in \mathcal{B} the thermodynamic postulate will be satisfied. This is equivalent to the following inequality^{*}:

* This inequality is implied by the Clausius-Duhem inequality /cf. Truesdell and Noll [1965, 1]/.

$$-\dot{\psi} - \dot{\mathcal{J}}\eta + \frac{1}{2\zeta_R} \text{tr}(\bar{\mathbb{T}} \dot{\underline{C}}) - \frac{1}{\zeta_R \mathcal{J}} q_R \cdot \text{Grad } \mathcal{J} \geq 0, \quad /2.8 /$$

which must be satisfied for every particle X in \mathcal{B} .

Similarly, we shall assume that all constitutive equations describing the physical properties of a material satisfy the principle of material frame-indifference formulated by Noll[‡].

A material with internal state variables is characterized in particle X by five constitutive equations^{‡‡‡}

$$\begin{aligned} \psi &= \hat{\psi}(\underline{C}, \mathcal{J}, \text{Grad } \mathcal{J}, \underline{A}), \\ \eta &= \hat{\eta}(\underline{C}, \mathcal{J}, \text{Grad } \mathcal{J}, \underline{A}), \\ \bar{\mathbb{T}} &= \hat{\bar{\mathbb{T}}}(\underline{C}, \mathcal{J}, \text{Grad } \mathcal{J}, \underline{A}), \\ q_R &= \hat{q}_R(\underline{C}, \mathcal{J}, \text{Grad } \mathcal{J}, \underline{A}), \\ \dot{\underline{A}} &= \hat{\dot{\underline{A}}}(\underline{C}, \mathcal{J}, \text{Grad } \mathcal{J}, \underline{A}). \end{aligned} \quad /2.9/$$

It is obvious that in constitutive assumptions /2.9/ we take use of the principle of equipresence formulated by Truesdell^{‡‡‡}.

For definiteness we assume here that the internal state tensor \underline{A} remains invariant upon a change of frame, i.e.,

$$\underline{A} \rightarrow \underline{A}$$

We shall say that the thermodynamic process in \mathcal{B} described by the system of functions $\{\underline{X}, \bar{\mathbb{T}}, \psi, q_R, \eta, \mathcal{J}, \underline{A}\}$

‡ Cf. Noll [1958, 1] and Truesdell and Noll [1965, 1].

‡‡ Vid. Coleman and Gurtin [1967, 1] and Valanis [1967, 6]. In the basic system of constitutive equations describing material with internal state variables Coleman and Gurtin [1967, 1] and Valanis [1967, 6] assumed the vector parameter $\underline{\alpha} = (\alpha_1, \dots, \alpha_n)$ defining the internal state. For our aim it is convenient to introduce the tensor internal parameter \underline{A} .

‡‡‡ Cf. Truesdell and Noll [1965, 1].

is called an admissible process in \mathcal{B} if it is compatible with constitutive equations /2.9/ at each point X of \mathcal{B} and for all time t .

To investigate the restrictions which are imposed on the constitutive equations /2.9/ by the thermodynamic postulate /2.8/ we compute

$$\dot{\psi} = \text{tr}(\partial_{\underline{c}} \hat{\psi} \dot{\underline{c}}) + \partial_{\mathcal{J}} \hat{\psi} \dot{\mathcal{J}} + \partial_{\text{Grad } \mathcal{J}} \hat{\psi} \cdot \overline{\text{Grad } \mathcal{J}} + \text{tr}(\partial_{\underline{A}} \hat{\psi} \dot{\underline{A}}). \quad /2.10/$$

Substitution the result /2.10/ into the inequality /2.8/ gives

$$\begin{aligned} \frac{1}{2\zeta_R} \text{tr}[(\bar{\Gamma} - 2\zeta_R \partial_{\underline{c}} \hat{\psi}) \dot{\underline{c}}] - (\partial_{\mathcal{J}} \hat{\psi} + \eta) \dot{\mathcal{J}} - \partial_{\text{Grad } \mathcal{J}} \hat{\psi} \cdot \overline{\text{Grad } \mathcal{J}} \\ - \text{tr}(\partial_{\underline{A}} \hat{\psi} \dot{\underline{A}}) - \frac{1}{\zeta_R \mathcal{J}} \hat{q}_R \cdot \text{Grad } \mathcal{J} \geq 0. \end{aligned} \quad /2.11/$$

Choosing arbitrary values $\dot{\underline{c}}$, $\dot{\mathcal{J}}$ and $\overline{\text{Grad } \mathcal{J}}$ it is possible to determine an admissible thermodynamic process in \mathcal{B} ^{*}. Hence, to satisfy the inequality /2.11/ one has to assume

$$\partial_{\text{Grad } \mathcal{J}} \hat{\psi} = 0, \quad /2.12/$$

$$\bar{\Gamma} = 2\zeta_R \partial_{\underline{c}} \hat{\psi}(\underline{c}, \mathcal{J}, \underline{A}), \quad /2.13/$$

$$\eta = -\partial_{\mathcal{J}} \hat{\psi}(\underline{c}, \mathcal{J}, \underline{A}), \quad /2.14/$$

$$\text{tr}[\partial_{\underline{A}} \hat{\psi}(\underline{c}, \mathcal{J}, \underline{A}) \dot{\underline{A}}] + \frac{1}{\zeta_R \mathcal{J}} \hat{q}_R(\underline{c}, \mathcal{J}, \text{Grad } \mathcal{J}, \underline{A}) \cdot \text{Grad } \mathcal{J} \leq 0. \quad /2.15/$$

* Vid. Coleman and Gurtin [1967, 1] and Valanis [1967, 6].

The relation /2.12/ implies that the response function of free energy ψ is independent on $\text{Grad } \mathcal{J}$. The equations /2.13/ and /2.14/ determine the response functions for stress tensor $\hat{\mathbb{T}}$ and for entropy $\hat{\eta}$ by means of the response function $\hat{\psi}(\underline{C}, \mathcal{J}, \underline{A})$. The inequality /2.15/ is called the general dissipation inequality.

We introduce the definition of internal dissipation function as follows

$$\begin{aligned} \sigma &= \hat{\sigma}(\underline{C}, \mathcal{J}, \text{Grad } \mathcal{J}, \underline{A}) \\ &= -\frac{1}{\mathcal{J}} \text{tr} \left[\partial_{\underline{A}} \hat{\psi}(\underline{C}, \mathcal{J}, \underline{A}) \dot{\underline{A}}(\underline{C}, \mathcal{J}, \text{Grad } \mathcal{J}, \underline{A}) \right]. \end{aligned} \quad /2.16/$$

The general dissipation inequality /2.15/ implies that for $\text{Grad } \mathcal{J} = 0$ the internal dissipation inequality is satisfied

$$\hat{\sigma}(\underline{C}, \mathcal{J}, \underline{0}, \underline{A}) \geq 0. \quad /2.17/$$

3. Description of an elastic/viscoplastic material

The experimental investigations of dynamical properties of materials have shown that many materials behave in different way under dynamic loading and under static loading. These results have also proved that the basic reason of these differences is the strain rate sensitivity of the material investigated^{*}. The influence of strain rate effect may be taken into account within the framework of viscoplasticity. We assume that before yielding material has only elastic properties, and after yielding has viscoelastic and plastic properties. This is why we shall call the rate sensitive plastic material an elastic/viscoplastic.

* A discussion of the results of experimental investigations in the domain of dynamic loads acting on metals can be found in the paper [1966, 5].

The aim of the present paper is the description of an elastic/viscoplastic material within the framework of thermodynamics with internal state variables. From the relation for internal dissipation /2.16/ one can see that the internal state tensor \underline{A} is introduced to describe the dissipation of the material. For an elastic/viscoplastic material such a parameter will be the inelastic deformation tensor \underline{c} . From previous researches in viscoplasticity^{**} we know that the rate of the inelastic deformation tensor $\dot{\underline{c}}$ for an elastic/viscoplastic material is proportional to the function $\phi(\mathcal{F})$, where \mathcal{F} is the statical yield function. Since an elastic/viscoplastic material before yielding has only elastic properties, hence the initial yield condition can be assumed to be similar as in inviscid theory of plasticity. Thus, the statical yield condition can be defined as follows

$$\mathcal{F} = \frac{\psi(\bar{I}, \mathcal{J}, \underline{c})}{\alpha} - 1, \quad /3.1/$$

where the isotropic work-hardening parameter α is determined by^{**}

$$\alpha = \text{tr}\{N(\bar{I}, \mathcal{J}, \underline{c})\dot{\underline{c}}\}. \quad /3.2/$$

The functions ψ and N are tensor functions. The function $\phi(\mathcal{F})$ may be chosen to represent the results of tests on the dynamical behaviour of materials. At the same time the proper choice of $\phi(\mathcal{F})$ enables a description of the influence of the rate of deformation and the temperature on the yield limit of the material. It is postulated that the following differential equations determines the tensor internal state parameter \underline{c} for an elastic/viscoplastic material

* Cf. Perzyna [1963, 1,2], [1967, 4], [1968, 1], Perzyna and Wojno [1966, 3] and Wojno [1967, 7].

** Cf. Green and Naghdi [1965, 2].

$$\dot{\underline{c}} = \nu(\vartheta) \langle \Phi(\mathcal{F}) \rangle \underline{M}(\bar{\underline{T}}, \vartheta, \underline{c}), \quad /3.3/$$

where $\nu(\vartheta)$ is a temperature dependent viscosity coefficient of a material, the symbol $\langle \Phi(\mathcal{F}) \rangle$ is defined as follows

$$\langle \Phi(\mathcal{F}) \rangle = \begin{cases} 0 & \text{for } \mathcal{F} \leq 0, \\ \Phi(\mathcal{F}) & \text{for } \mathcal{F} > 0, \end{cases} \quad /3.4/$$

and \underline{M} denotes the symmetric tensor function, i.e., $\underline{M} = \underline{M}^T$. To ensure that the internal state tensor \underline{c} is invariant upon a change of frame it is sufficient to assume that the tensor function \underline{M} is objective.

The equation /3.3/ postulates that the rate of the inelastic deformation tensor $\dot{\underline{c}}$ is the function of the excess of stress over the static yield condition. We notice here that the equation /3.3/ is the generalization of the known relation for shear strain rate in the physical theory of dislocation for face-centred cubic crystals based on thermal activation process^{*}.

The constitutive assumption /3.3/ yields to the following dynamical yield criterion

$$f(\bar{\underline{T}}, \vartheta, \underline{c}) = \alpha \left\{ 1 + \Phi^{-1} \left[\frac{(\text{tr } \underline{c}^2)^{1/2}}{\nu(\vartheta)} (\text{tr } \underline{M}^2)^{-1/2} \right] \right\}. \quad /3.5/$$

This relation may be interpreted as a description of actual change of the yield surface during the thermodynamic process. This change is caused by isotropic and anisotropic work-hardening effects and by influence of the rate of inelastic

* Vid. Seeger [1965, 1]. The physical foundations of an elastic/viscoplastic material have been fully discussed in the paper [1968, 1]. /Of. also Lindholm [1967, 3]/.

deformation tensor and temperature on the yield limit of a material[‡].

Notice that the relation /3.5/ is a base for experimental investigations which aim to examine the theoretical assumptions.

Due to previous results and the constitutive assumptions introduced the full system of the constitutive equations for an elastic/viscoplastic material may be written as follows

$$\psi = \hat{\psi}(\underline{c}, \mathcal{J}, {}^i\underline{c}), \quad /3.6/$$

$$\underline{\tilde{T}} = 2 \varrho_R \partial_{\underline{c}} \hat{\psi}(\underline{c}, \mathcal{J}, {}^i\underline{c}), \quad /3.7/$$

$$\eta = - \partial_{\mathcal{J}} \hat{\psi}(\underline{c}, \mathcal{J}, {}^i\underline{c}), \quad /3.8/$$

$$\underline{q}_R = \hat{q}_R(\underline{c}, \mathcal{J}, \text{Grad } \mathcal{J}, {}^i\underline{c}), \quad /3.9/$$

$${}^i\dot{\underline{c}} = \nu(\mathcal{J}) \langle \phi(\mathcal{F}) \rangle \underline{M}(\underline{\tilde{T}}, \mathcal{J}, {}^i\underline{c}). \quad /3.10/$$

Thus, an elastic/viscoplastic material is described in a thermodynamic process by the response functions $\hat{\psi}$, \hat{q}_R , \underline{M} , $\phi(\mathcal{F})$ and by viscosity coefficient $\nu(\mathcal{J})$.

Due to relation /3.7/ the differential equation /3.10/ which determined the tensor parameter of internal state ${}^i\underline{c}$ can be written in the following form

$${}^i\dot{\underline{c}} = \underline{G}(\underline{c}, \mathcal{J}, {}^i\underline{c}), \quad /3.11/$$

where tensor function \underline{G} is a symmetric tensor and is defined by function \underline{M} and $\phi(\mathcal{F})$.

The equation /3.11/ may be treated as the definition of the inelastic deformation tensor ${}^i\underline{c}$. This equation

[‡] In physical theory of Seeger [1955, 1] exists similar relation between actual stress, temperature and strain rate.

shows that the present theory of an elastic/viscoplastic material takes account of the history of deformation and temperature. This is implied by the fact that to integrate the differential equation /3.11/ and to determine the actual value of the inelastic deformation tensor ${}^i\mathbb{C}(t)$ at X in \mathcal{B} we have to know the initial value ${}^i\mathbb{C}^0$ and the full histories of the inelastic deformation tensor ${}^i\mathbb{C}$, the deformation tensor \mathbb{C} and temperature \mathcal{T} at X .

In the present theory the general dissipation inequality has form

$$\text{tr}[\partial_{i_c} \hat{\Psi}(\mathbb{C}, \mathcal{T}, {}^i\mathbb{C}) \mathbb{G}(\mathbb{C}, \mathcal{T}, {}^i\mathbb{C})] + \frac{1}{\mathbb{C}_R} \hat{q}_R(\mathbb{C}, \mathcal{T}, \text{Grad } \mathcal{T}, {}^i\mathbb{C}) \cdot \text{Grad } \mathcal{T} \leq 0 \quad /3.12/$$

and ensures the fulfilment of the thermodynamic postulate /2.8/.

The internal dissipation function $\hat{\mathbb{G}}$ is determined by the relation

$$\begin{aligned} \hat{\mathbb{G}} &= \hat{\mathbb{G}}(\mathbb{C}, \mathcal{T}, {}^i\mathbb{C}) & /3.13/ \\ &= -\frac{1}{\mathcal{T}} \mathcal{V}(\mathcal{T}) \langle \Phi(\mathcal{F}) \rangle + \text{tr}[\partial_{i_c} \hat{\Psi}(\mathbb{C}, \mathcal{T}, {}^i\mathbb{C}) \mathbb{M}(\mathbb{T}, \mathcal{T}, {}^i\mathbb{C})] \\ &= -\frac{1}{\mathcal{T}} \text{tr}[\partial_{i_c} \hat{\Psi}(\mathbb{C}, \mathcal{T}, {}^i\mathbb{C}) \mathbb{G}(\mathbb{C}, \mathcal{T}, {}^i\mathbb{C})]. \end{aligned}$$

4. Equilibrium state and the relaxation process

The triple $(\mathbb{C}^*, \mathcal{T}^*, {}^i\mathbb{C}^*)$ which satisfies the condition

$$\mathbb{G}(\mathbb{C}^*, \mathcal{T}^*, {}^i\mathbb{C}^*) = \underline{0} \quad \text{and} \quad \text{Grad } \mathcal{T} = \underline{0} \quad /4.1/$$

is called the equilibrium state of a material at X . Since for an elastic/viscoplastic material, after using the relation /3.7/, we have

$$\mathbb{G}(\mathbb{C}, \mathcal{T}, {}^i\mathbb{C}) = \mathcal{V}(\mathcal{T}) \langle \Phi(\mathcal{F}) \rangle \mathbb{M}(\mathbb{T}, \mathcal{T}, {}^i\mathbb{C}), \quad /4.2/$$

then the condition /4.1/ can be satisfied if

$$\phi(\mathcal{F}) = 0 \quad \text{i.e.,} \quad \mathcal{F} = 0. \quad /4.3/$$

The tensor function \underline{M} in whole domain of variability of \underline{C} , \mathcal{F} and ${}^i\underline{C}$ can not be equal to zero. This is implied by physical feature of an elastic/viscoplastic material.

For this material ${}^i\underline{C} = \underline{0}$ if, and only if, $\mathcal{F} \leq 0$ and ${}^i\underline{C} \neq \underline{0}$ for $\mathcal{F} > 0$.

After Coleman and Gurtin [1967, 1] we introduce the definition of the domain of attraction of the equilibrium state $(\underline{C}^*, \mathcal{F}^*, {}^i\underline{C}^*)$ at constant strain and temperature as a set $\mathcal{D}(\underline{C}^*, \mathcal{F}^*, {}^i\underline{C}^*)$ of all initial values ${}^i\underline{C}^0$ such that the solution ${}^i\underline{C} = {}^i\underline{C}(t)$ of the initial-value problem

$${}^i\underline{C} = \underline{G}(\underline{C}^*, \mathcal{F}^*, {}^i\underline{C}), \quad {}^i\underline{C}(0) = {}^i\underline{C}^0, \quad /4.4/$$

exists for all $t \geq 0$ and tends to ${}^i\underline{C}^*$, i.e.,

$${}^i\underline{C}(t) \rightarrow {}^i\underline{C}^* \quad \text{if} \quad t \rightarrow \infty. \quad /4.5/$$

The equilibrium state $(\underline{C}^*, \mathcal{F}^*, {}^i\underline{C}^*)$ is called asymptotically stable at constant strain and temperature* if the domain of attraction of equilibrium state at constant strain and temperature $\mathcal{D}(\underline{C}^*, \mathcal{F}^*, {}^i\underline{C}^*)$ contains a neighbourhood of ${}^i\underline{C}^*$, i.e., if there exists a $\varphi > 0$ such that every tensor of internal state ${}^i\underline{C}^0$ satisfying the condition

$$|{}^i\underline{C}^0 - {}^i\underline{C}^*| < \varphi \quad /4.6/$$

is in $\mathcal{D}(\underline{C}^*, \mathcal{F}^*, {}^i\underline{C}^*)$.

Let us assume that there exists a triple $(\underline{C}^*, \mathcal{F}^*, {}^i\underline{C}^*)$ defining the equilibrium state and investigate the solution of the initial-value problem /4.4/ with condition /4.6/. From physical features of an elastic/viscoplastic material

* Vid. the definition given by Coleman and Gurtin [1967, 1].

we know that $\dot{i}\underline{c} \geq 0$. The thermodynamic process considered is characterized by constancy of the deformation tensor $\underline{c} = \underline{c}^*$ and temperature $\mathcal{J} = \mathcal{J}^*$ in interval of time $[t_0, \infty)$. The inelastic deformation tensor in such thermodynamic process is described by the equation /4.4/ and the stress tensor by^{*}

$$\underline{T} = 2 \rho_R \partial_{\underline{c}} \hat{\Psi}(\underline{c}^*, \mathcal{J}^*, \underline{c}). \quad /4.7/$$

The process described by /4.4/ and /4.7/ is the isothermal relaxation process for an elastic/viscoplastic material.

For such process

$$\dot{i}\underline{c} \rightarrow 0 \quad \text{if} \quad t \rightarrow \infty \quad /4.8/$$

and

$$\dot{i}\underline{c} \rightarrow \dot{i}\underline{c}^* = \hat{\underline{c}}^*(\underline{c}^*, \mathcal{J}^*), \quad /4.9/$$

where the function $\hat{\underline{c}}^*$ is determined by the condition

$$\underline{F} = 0. \quad /4.10/$$

Thus, we have proved the following remark:

The asymptotically stable equilibrium state $(\underline{c}^*, \mathcal{J}^*, \dot{i}\underline{c}^*)$ for an elastic/viscoplastic material is only reached in the isothermal relaxation process.

For the isothermal relaxation process and for $\text{Grad } \mathcal{J} = 0$ from the thermodynamic postulate /2.8/ yields

$$\dot{\Psi} \leq 0. \quad /4.11/$$

* From the equation /4.7/ we can easily compute the Cauchy stress tensor

$$\underline{T} = 2 \gamma^{*-1} \rho_R \underline{F}^* \partial_{\underline{c}} \hat{\Psi}(\underline{c}^*, \mathcal{J}^*, \underline{c}) \underline{F}^{*\top}.$$

Thus, we can write

$$\hat{\psi}(\underline{c}^*, \vartheta^*, \underline{c}) \geq \hat{\psi}(\underline{c}^*, \vartheta^*, \underline{c}^*) \quad /4.12/$$

for all internal state tensor parameters \underline{c} in a neighbourhood of \underline{c}^* . Hence

$$\partial_{\underline{c}} \hat{\psi} \Big|_{\underline{c} = \underline{c}^*} = 0 \quad /4.13/$$

The equation /4.13/ is called the equation of the internal equilibrium of an elastic/viscoplastic material.

This implies the following remark:

In the isothermal relaxation process for an elastic/viscoplastic material the free energy has minimum value at the asymptotically stable equilibrium state.

Let us assume that for any arbitrary pair of deformation and temperature $(\underline{c}^*, \vartheta^*)$ exists only one internal state tensor \underline{c}^* such that /4.1/ is satisfied, i.e., the triple $(\underline{c}^*, \vartheta^*, \underline{c}^*)$ is the equilibrium state of a material at point X . This is a case for an elastic/viscoplastic material if the condition /4.3/ is satisfied, hence exists the relation

$$\underline{c}^* = \hat{\underline{c}}^*(\underline{c}^*, \vartheta^*), \quad /4.14/$$

which may be called the equilibrium response function for the internal state tensor \underline{c}^* . Using the equations /3.6/ - /3.9/ we can determine the equilibrium response functions $\hat{\psi}^*$, $\hat{\eta}^*$, $\hat{\underline{T}}^*$ and $\hat{\underline{q}}_{\underline{r}}^*$ as functions of \underline{c}^* and ϑ^* only

$$\psi^* = \hat{\psi}^*(\underline{c}^*, \vartheta^*) = \hat{\psi}^*(\underline{c}^*, \vartheta^*, \hat{\underline{c}}^*(\underline{c}^*, \vartheta^*)), \quad /4.15/$$

$$\eta^* = \hat{\eta}^*(\underline{c}^*, \vartheta^*), \quad /4.16/$$

$$\underline{T}^* = \hat{\underline{T}}^*(\underline{c}^*, \vartheta^*), \quad /4.17/$$

$$q_r^* = \hat{q}_r^*(c^*, j^*). \quad /4.18/$$

It can be proved that $q_r^* = 0$ [‡]. Computation of $\partial_j \hat{\psi}^*$ gives

$$\partial_j \hat{\psi}^*(c, j) = \partial_j \hat{\psi}(c, j, c) + tr[\partial_{c_i} \hat{\psi}(c, j, c) \partial_j c^i(c, j)]. \quad /4.19/$$

Similar relation is satisfied for $\partial_c \hat{\psi}^*$

Thus, due to the relation /4.13/ we have following remark:

If a triple (c^*, j^*, c^*) is the asymptotically stable equilibrium state, for which the condition of internal equilibrium /4.13/ is satisfied, then the following relations for stress tensor and for entropy are valid

$$\bar{T}^* = 2 \zeta_r \partial_c \hat{\psi}^*(c^*, j^*), \quad /4.20/$$

$$\eta^* = -\partial_j \hat{\psi}^*(c^*, j^*). \quad /4.21/$$

5. Isotropic material

The constitutive equations /3.6/ - /3.10/ describing the properties of an elastic/viscoplastic material are valid for arbitrary initial anisotropy. We shall now assume that material considered is initially isotropic. In this case there may exist for the response tensor functions the polynomial representations^{***}.

The fundamental response function $\hat{\psi}(c, j, c)$ may have

[‡] The proof is similar to that given by Coleman and Gurtin [1967, 1].

^{***} Cf. Truesdell and Noll [1965, 1].

the polynomial representation of ten invariants

$$\begin{aligned} & \text{tr } \underline{c}, \quad \text{tr } \underline{c}^2, \quad \text{tr } \underline{c}^3, \quad \text{tr } \dot{\underline{c}}, \quad \text{tr } \dot{\underline{c}}^2, \quad \text{tr } \dot{\underline{c}}^3, \\ & \text{tr } \underline{c} \dot{\underline{c}}, \quad \text{tr } \underline{c}^2 \dot{\underline{c}}, \quad \text{tr } \underline{c} \dot{\underline{c}}^2, \quad \text{tr } \underline{c}^2 \dot{\underline{c}}^2. \end{aligned} \quad /5.1/$$

Thus we can write

$$\begin{aligned} \hat{\psi}(\underline{c}, \mathcal{T}, \dot{\underline{c}}) = & \hat{\psi}_0(\mathcal{T}) + \hat{\psi}_1(\mathcal{T}) \text{tr } \underline{c} + \hat{\psi}_2(\mathcal{T}) \text{tr } \underline{c}^2 + \hat{\psi}_3(\mathcal{T}) \text{tr } \underline{c}^3 \\ & + \hat{\psi}_4(\mathcal{T}) \text{tr } \dot{\underline{c}} + \hat{\psi}_5(\mathcal{T}) \text{tr } \dot{\underline{c}}^2 + \hat{\psi}_6(\mathcal{T}) \text{tr } \dot{\underline{c}}^3 \\ & + \hat{\psi}_7(\mathcal{T}) \text{tr } \underline{c} \dot{\underline{c}} + \hat{\psi}_8(\mathcal{T}) \text{tr } \underline{c} \dot{\underline{c}}^2 + \hat{\psi}_9(\mathcal{T}) \text{tr } \underline{c}^2 \dot{\underline{c}} + \hat{\psi}_{10}(\mathcal{T}) \text{tr } \underline{c}^2 \dot{\underline{c}}^2. \end{aligned} \quad /5.2/$$

For stress tensor and entropy, due to equations /3.7/ - /3.8/ and /5.2/, we deduce following relations

$$\begin{aligned} \bar{\mathbb{T}} = & 2 \varrho_R [\hat{\psi}_1(\mathcal{T}) \underline{1} + 2 \hat{\psi}_2(\mathcal{T}) \underline{c} + 3 \hat{\psi}_3(\mathcal{T}) \underline{c}^2 \\ & + \hat{\psi}_4(\mathcal{T}) \dot{\underline{c}} + \hat{\psi}_8(\mathcal{T}) \dot{\underline{c}}^2 + 2 \hat{\psi}_9(\mathcal{T}) \underline{c} \dot{\underline{c}} + 2 \hat{\psi}_{10}(\mathcal{T}) \underline{c} \dot{\underline{c}}^2] \end{aligned} \quad /5.3/$$

$$\begin{aligned} \eta = & - [\hat{\psi}'_0(\mathcal{T}) + \hat{\psi}'_1(\mathcal{T}) \text{tr } \underline{c} + \hat{\psi}'_2(\mathcal{T}) \text{tr } \underline{c}^2 + \hat{\psi}'_3(\mathcal{T}) \text{tr } \underline{c}^3 \\ & + \hat{\psi}'_4(\mathcal{T}) \text{tr } \dot{\underline{c}} + \hat{\psi}'_5(\mathcal{T}) \text{tr } \dot{\underline{c}}^2 + \hat{\psi}'_6(\mathcal{T}) \text{tr } \dot{\underline{c}}^3 \\ & + \hat{\psi}'_7(\mathcal{T}) \text{tr } \underline{c} \dot{\underline{c}} + \hat{\psi}'_8(\mathcal{T}) \text{tr } \underline{c} \dot{\underline{c}}^2 + \hat{\psi}'_9(\mathcal{T}) \text{tr } \underline{c}^2 \dot{\underline{c}} + \hat{\psi}'_{10}(\mathcal{T}) \text{tr } \underline{c}^2 \dot{\underline{c}}^2], \end{aligned} \quad /5.4/$$

where $\hat{\psi}'_0(\mathcal{T}), \dots, \hat{\psi}'_{10}(\mathcal{T})$ are the derivatives with respect to temperature \mathcal{T} of the temperature dependent coefficients $\hat{\psi}_0(\mathcal{T}), \dots, \hat{\psi}_{10}(\mathcal{T})$.

The differential equation determining the internal state tensor $\dot{\underline{c}}$ /3.10/ may be written in the form

$$\begin{aligned} \dot{\underline{c}} = & \mathcal{F}(\mathcal{T}) \langle \Phi(\mathbb{F}) \rangle [\varphi_0 \underline{1} + \varphi_1 \bar{\mathbb{T}} + \varphi_2 \underline{c} + \varphi_3 \bar{\mathbb{T}}^2 + \varphi_4 \dot{\underline{c}}^2 \\ & + \varphi_5 (\bar{\mathbb{T}} \dot{\underline{c}} + \dot{\underline{c}} \bar{\mathbb{T}}) + \varphi_6 (\bar{\mathbb{T}}^2 \underline{c} + \underline{c} \bar{\mathbb{T}}^2) \\ & + \varphi_7 (\bar{\mathbb{T}} \dot{\underline{c}}^2 + \dot{\underline{c}}^2 \bar{\mathbb{T}}) + \varphi_8 (\bar{\mathbb{T}}^2 \underline{c}^2 + \underline{c}^2 \bar{\mathbb{T}}^2)] \end{aligned} \quad /5.5/$$

where the functions \mathbb{F} and $\varphi_0, \dots, \varphi_8$ depend on temperature \mathcal{T} and ten invariants

$$\begin{aligned} & \text{tr } \bar{\mathbb{T}}, \text{tr } \bar{\mathbb{T}}^2, \text{tr } \bar{\mathbb{T}}^3, \text{tr } \dot{\mathbb{C}}, \text{tr } \dot{\mathbb{C}}^2, \text{tr } \dot{\mathbb{C}}^3, \\ & \text{tr } \bar{\mathbb{T}} \dot{\mathbb{C}}, \text{tr } \bar{\mathbb{T}}^2 \dot{\mathbb{C}}, \text{tr } \bar{\mathbb{T}} \dot{\mathbb{C}}^2, \text{tr } \bar{\mathbb{T}}^2 \dot{\mathbb{C}}^2. \end{aligned} \quad /5.6/$$

6. Elastic-plastic material

The dynamical yield condition /3.5/ implies that an elastic/viscoplastic material loses its strain rate sensitivity if, and only if, the viscosity coefficient $\Psi(\dot{\mathcal{J}}) \rightarrow \infty$. In this case the statical yield condition

$$\mathbb{F} = 0 \quad /6.1/$$

is satisfied, the material loses its viscosity properties and behaves as an elastic-plastic material. From the definition of the symbol $\langle \Phi(\mathbb{F}) \rangle$ /cf. the relation /3.4// we see that the differential equation determining the plastic deformation tensor $\dot{\mathbb{C}}$ takes the following form

$$\dot{\mathbb{C}} = \Lambda \mathbb{M}(\bar{\mathbb{T}}, \dot{\mathcal{J}}, \dot{\mathbb{C}}), \quad /6.2/$$

where the parameter $\Lambda = \Psi(\dot{\mathcal{J}}) \langle \Phi(\mathbb{F}) \rangle$ may be determined from the condition that the point representing in temperature-stress space the actual state of temperature and stress lies on the yield surface

$$f(\bar{\mathbb{T}}, \dot{\mathcal{J}}, \dot{\mathbb{C}}) = \alpha, \quad /6.3/$$

where the work-hardening parameter is determined by the relation /cf. the definition /3.2//

$$\alpha = \text{tr} \{ \mathbb{N}(\bar{\mathbb{T}}, \dot{\mathcal{J}}, \dot{\mathbb{C}}) \dot{\mathbb{C}} \}. \quad /6.4/$$

From the yield condition /6.3/ and the definition /6.4/ one can deduce the following criterion of loading

$$f = \alpha \quad \text{and} \quad \text{tr}(\partial_{\bar{T}} f \dot{\bar{T}}) + \partial_{\dot{\gamma}} f \dot{\gamma} > 0. \quad /6.5/$$

Similarly, the criteria

$$f = \alpha \quad \text{and} \quad \text{tr}(\partial_{\bar{T}} f \dot{\bar{T}}) + \partial_{\dot{\gamma}} f \dot{\gamma} \leq 0 \quad /6.6/$$

define unloading and neutral state, respectively.

To satisfy the condition that the point representing the actual state of loading and temperature lies on the yield surface it is sufficient to fulfil $f = \alpha$, i.e.,

$$\text{tr}(\partial_{\bar{T}} f \dot{\bar{T}}) + \partial_{\dot{\gamma}} f \dot{\gamma} + \text{tr}(\partial_{p_c} f \dot{p}_c) = \text{tr}\{N(\bar{T}, \dot{\gamma}, p_c) \dot{p}_c\}. \quad /6.7/$$

Using /6.2/ and /6.4/ from /6.7/ we have

$$\Lambda = \lambda [\text{tr}(\partial_{\bar{T}} f \dot{\bar{T}}) + \partial_{\dot{\gamma}} f \dot{\gamma}], \quad /6.8/$$

where

$$\lambda = \{ \text{tr}[(N(\bar{T}, \dot{\gamma}, p_c) - \partial_{p_c} f) M(\bar{T}, \dot{\gamma}, p_c)] \}^{-1}. \quad /6.9/$$

We shall assume the condition

$$\lambda > 0. \quad /6.10/$$

The relation for Λ /6.8/ and the criteria of loading, unloading and neutral state have shown that the differential equation determining the internal state tensor p_c for an elastic-plastic material can be written as follows

$$\dot{p}_c = \lambda \langle \text{tr}(\partial_{\bar{T}} f \dot{\bar{T}}) + \partial_{\dot{\gamma}} f \dot{\gamma} \rangle M(\bar{T}, \dot{\gamma}, p_c), \quad /6.11/$$

where symbol $\langle \text{tr}(\partial_{\dot{\mathbb{I}}} \dot{\mathbb{I}}) + \partial_{\dot{\mathcal{J}}} \dot{\mathcal{J}} \rangle$ is defined by

$$\langle [\text{tr}(\partial_{\dot{\mathbb{I}}} \dot{\mathbb{I}}) + \partial_{\dot{\mathcal{J}}} \dot{\mathcal{J}}] \rangle = \begin{cases} [] & \text{if } \dot{\mathcal{J}} = \alpha \text{ and } [] > 0, \\ 0 & \text{if } \dot{\mathcal{J}} = \alpha \text{ and } [] \leq 0 \text{ or if } \dot{\mathcal{J}} < \alpha. \end{cases} \quad /6.12/$$

The full system of constitutive equations describing the behaviour of an elastic-plastic material in material point X in \mathcal{B} during thermodynamic process has the form

$$\psi = \hat{\psi}(\underline{c}, \mathcal{J}, {}^p\underline{c}), \quad /6.13/$$

$$\dot{\mathbb{I}} = 2\epsilon_R \partial_{\underline{c}} \hat{\psi}(\underline{c}, \mathcal{J}, {}^p\underline{c}), \quad /6.14/$$

$$\eta = -\partial_{\mathcal{J}} \hat{\psi}(\underline{c}, \mathcal{J}, {}^p\underline{c}), \quad /6.15/$$

$$\underline{q}_R = \hat{q}_R(\underline{c}, \mathcal{J}, \text{Grad } \mathcal{J}, {}^p\underline{c}), \quad /6.16/$$

$$\dot{{}^p\underline{c}} = \underline{H}_1(\underline{c}, \mathcal{J}, {}^p\underline{c})[\dot{\underline{c}}] + \underline{H}_2(\underline{c}, \mathcal{J}, {}^p\underline{c})\dot{\mathcal{J}} \quad /6.17/$$

where the tensor functions \underline{H}_1 and \underline{H}_2 are symmetric, and can be determined by substitution of the relation /6.14/ into the right-hand side of the equation /6.11/.

The equation /6.17/ is the definition of the plastic deformation tensor ${}^p\underline{c}$.

It should be pointed out that the equation /6.17/ is invariant under a change of time-scale.

To satisfy the thermodynamic postulate by an elastic-plastic material the constitutive equations /6.13/-/6.17/ should fulfil the general dissipation inequality

$$\begin{aligned} & \text{tr}[\partial_{\underline{c}} \hat{\psi}(\underline{c}, \mathcal{J}, {}^p\underline{c}) (\underline{H}_1(\underline{c}, \mathcal{J}, {}^p\underline{c})[\dot{\underline{c}}] + \underline{H}_2(\underline{c}, \mathcal{J}, {}^p\underline{c})\dot{\mathcal{J}})] \\ & + \frac{1}{\epsilon_R} \hat{q}_R(\underline{c}, \mathcal{J}, \text{Grad } \mathcal{J}, {}^p\underline{c}) \cdot \text{Grad } \mathcal{J} \leq 0. \end{aligned} \quad /6.18/$$

The internal dissipation function \hat{G} for an elastic-plastic material is determined by the relation

$$\hat{G} = -\frac{1}{\nu} \lambda \langle \text{tr}(\partial_{\dot{\mathbf{T}}} \dot{\mathbf{T}}) + \partial_{\dot{\nu}} \dot{\nu} \rangle \text{tr}[\partial_{\mathbf{c}} \hat{\Psi}(\mathbf{c}, \nu, \dot{\mathbf{c}}) \mathbf{M}]$$

$$= -\frac{1}{\nu} \text{tr}[\partial_{\mathbf{c}} \hat{\Psi}(\mathbf{c}, \nu, \dot{\mathbf{c}}) (\mathbf{H}_1(\mathbf{c}, \nu, \dot{\mathbf{c}}) [\dot{\mathbf{c}}] + \mathbf{H}_2(\mathbf{c}, \nu, \dot{\mathbf{c}}) \dot{\nu})]. \quad /6.19/$$

Similarly as for an elastic/viscoplastic material one can write the polynomial representations for the elastic-plastic response functions.

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