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LECTURE NOTES 6

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Nonlinear Elasticity with Application to Material Modelling

Centre of Excellence for
Advanced Materials and Structures

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Preface

This volume contains the text of 12 lectures given between March and May 2002 at the Centre of Excellence for Advanced Materials and Structures in the Institute of Fundamental Technological Research, Polish Academy of Sciences, Warsaw. It begins by summarizing the main ingredients of the theory of elasticity, including a description of the kinematics of deformation and the associated deformation and strain tensors, a summary of the equations governing the motion of a continuum and a discussion of the various stress tensors used in the analysis. This discussion is followed by a description of constitutive laws for elastic materials, with reference to restrictions placed on the form of the constitutive law by the requirements of objectivity and material symmetry. Internal constraints are also discussed. Particular attention is focussed on the special case of isotropy since this is important in applications to many materials, including rubberlike materials, and on the incompressibility constraint. Some illustrative boundary-value problems involving homogeneous or non-homogeneous deformations are examined in detail. Some aspects of the application of the theory to the characterization of the elastic properties rubberlike solids are then reviewed.

The constitutive theory next concentrates on applications to the modelling of fibre-reinforced materials. The cases of one and two families of fibres are considered separately. Fibre reinforcement is particularly important in soft biological tissues (for example, arteries), where the fibres typically are in the form of collagen, and some discussion of this area of application is included. Again, some simple boundary-value problems are examined, but, additionally, the effect of residual stresses is also included since these have a critical influence on the response of tissues under both typical and abnormal physiological conditions.

Finally, a detailed analysis is given of a representative boundary-value problem for compressible isotropic elastic materials in order to illustrate how forms of strain-energy function may be generated by making assumptions on the kinematics. In this case the problem of azimuthal shear of a thick-walled circular cylindrical tube is examined under the assumption that the deformation is isochoric.

There are many general texts that can be referred to for more details of the material in Chapters 1–3 and parts of the other chapters. We mention here, for example, the books by Chadwick [1], Ciarlet [2], Holzapfel [8] and Ogden [18]. Many other references can be found in these works and for the most part we do not therefore give detailed lists of references. We mention here, however, the recent volume by Fu and Ogden [4], which contains several review articles on different aspects of nonlinear elasticity and many references to the original literature. A number of specific references are given in the later chapters in order to provide pointers to the literature that might not otherwise be readily available.

I would like to express my appreciation to Professors Włodzimierz Domański, Witold Kosiński, Zenon Mróz, Zbigniew Olesiak, Kazimierz Sobczyk, J. Joachim Telega, and Henryk Zorski for their kindness and hospitality during my visit to Warsaw. To my host, Professor Henryk Petryk, I am especially grateful, for he contributed so much to making my two-month visit very enjoyable and stimulating.

Glasgow, January 2003 *Ray Ogden*

Chapter 1

Kinematics

1.1. Bodies, configurations and motions

Definition. A *body* $\mathcal{B}$ is a set whose elements can be put into one-to-one correspondence with points of a region B in three-dimensional Euclidean point space. The elements of $\mathcal{B}$ are called *particles* (or *material points*) and B is called a *configuration* of $\mathcal{B}$.

As the body moves the configuration changes with time. Let $t \in I \subset \mathbb{R}$ denote time, where I is an interval in $\mathbb{R}$. If, with each $t \in I$, we associate a unique configuration B_t of $\mathcal{B}$ then the family of configurations $\{\mathrm{B}_t : t \in I\}$ is called a *motion* of $\mathcal{B}$. We assume that as $\mathcal{B}$ moves continuously then B_t changes continuously.

It is convenient to identify a *reference configuration*, B_r say, which is an arbitrarily chosen fixed configuration. Then, any particle P of $\mathcal{B}$ may be labelled by its position vector $\mathbf{X}$ in B_r relative to some origin O. Let $\mathbf{x}$ be the position vector of P in the configuration B_t at time t relative to an origin o (which need not coincide with O), as depicted in Fig. 1.1.

We say that $\mathcal{B}$ *occupies* the configuration B_t at time t – B_t is also referred to as the *current configuration*. Note that B_r need not be a configuration actually occupied by $\mathcal{B}$ during the motion, but is often chosen to be the configuration occupied by $\mathcal{B}$ at some prescribed time.

Since B_r and B_t are configurations of $\mathcal{B}$ there exists a bijection mapping $\chi : \mathrm{B}_r \to \mathrm{B}_t$ such that

$$\mathbf{x} = \chi(\mathbf{X}) \quad \text{for all } \mathbf{X} \in \mathrm{B}_r, \qquad \mathbf{X} = \chi^{-1}(\mathbf{x}) \quad \text{for all } \mathbf{x} \in \mathrm{B}_t. \tag{1.1}$$

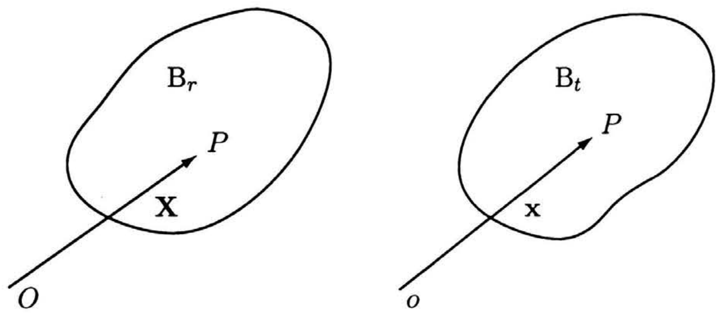

FIGURE 1.1. Reference configuration B_r and current configuration B_t with position vectors $\mathbf{X}$ and $\mathbf{x}$ of a material point P.

The mapping χ is called the *deformation* of the body *from* B_r *to* B_t. Since B_t depends on t we write

$$\mathbf{x} = \chi_t(\mathbf{X}), \quad \mathbf{X} = \chi_t^{-1}(\mathbf{x}) \tag{1.2}$$

instead of (1.1), or

$$\mathbf{x} = \chi(\mathbf{X}, t) \quad \text{for all } \mathbf{X} \in B_r,\ t \in I. \tag{1.3}$$

For each particle P (with label $\mathbf{X}$) this describes the motion of P with t as parameter, and hence the motion of $\mathcal{B}$. It is usual to assume that $\chi(\mathbf{X}, t)$ is twice-continuously differentiable with respect to position and time, although there are situations where this requirement needs to be relaxed. For example, across a phase boundary where one or more of the first or second derivatives of χ is discontinuous.

Example: *Rigid motion*

A motion is said to be *rigid* if the distance between any two particles of $\mathcal{B}$ does not change during the motion.

The motion defined by

$$\mathbf{x} \equiv \chi(\mathbf{X}, t) = \mathbf{c}(t) + \mathbf{Q}(t)\mathbf{X}, \tag{1.4}$$

where $\mathbf{c}(t)$ is a vector and $\mathbf{Q}(t)$ is a proper orthogonal second-order tensor, is a rigid motion. To show this we consider $\mathbf{Y} \in B_r$ so that $\mathbf{y} = \mathbf{c}(t) + \mathbf{Q}(t)\mathbf{Y}$.

Then

$$\begin{aligned}|\mathbf{x}-\mathbf{y}|^2 &= (\mathbf{x}-\mathbf{y})\cdot(\mathbf{x}-\mathbf{y}) = [\mathbf{Q}(\mathbf{X}-\mathbf{Y})]\cdot[\mathbf{Q}(\mathbf{X}-\mathbf{Y})]\\ &= [\mathbf{Q}^T\mathbf{Q}(\mathbf{X}-\mathbf{Y})]\cdot(\mathbf{X}-\mathbf{Y}) = (\mathbf{X}-\mathbf{Y})\cdot(\mathbf{X}-\mathbf{Y})\\ &= |\mathbf{X}-\mathbf{Y}|^2,\end{aligned} \tag{1.5}$$

where we have used $\mathbf{Q}^T\mathbf{Q} = \mathbf{I}$. In fact, although we have not proved it here, *every* rigid motion can be expressed in the form (1.4). Note that $\mathbf{c}(t)$ represents a *translation* and $\mathbf{Q}(t)$ a *rotation.*

In the development of the basic principles of continuum mechanics a body $\mathcal{B}$ is endowed with various physical properties which are represented by scalar, vector and tensor fields defined on *either* B_r *or* B_t (for example, density, temperature, shape of surface, velocity, strain). In the case of B_r the position vector $\mathbf{X}$ and time t serve as independent variables, and the fields are then said to be defined in terms of the *referential* or *material* description. Alternatively, in the case of B_t, $\mathbf{x}$ and t are used and the description is said to be *spatial.* The terminologies *Lagrangian* and *Eulerian descriptions* are also used in respect of B_r and B_t respectively.

Rectangular Cartesian coordinate systems with basis vectors $\{\mathbf{E}_i\}$ and $\{\mathbf{e}_i\}$ are chosen for B_r and B_t respectively, with *material coordinates* X_i and *spatial coordinates* x_i $(i = 1, 2, 3)$. Thus, relative to the origins O and o respectively, we have

$$\mathbf{X} = X_i\mathbf{E}_i, \quad \mathbf{x} = x_i\mathbf{e}_i. \tag{1.6}$$

In (1.6) the summation convention over repeated indices applies. It will also apply henceforth except where stated otherwise. In general, $\mathbf{E}_i$ and $\mathbf{e}_i$ may be chosen to have different orientations, but it is often convenient to let them coincide.

1.2. The material time derivative

The *velocity* $\mathbf{v}$ of a particle P is defined as

$$\mathbf{v} \equiv \dot{\mathbf{x}} = \frac{\partial}{\partial t}\chi(\mathbf{X}, t), \tag{1.7}$$

i.e. the rate of change of position of P (or $\partial/\partial t$ at fixed $\mathbf{X}$). The *acceleration* $\mathbf{a}$ of P is

$$\mathbf{a} \equiv \dot{\mathbf{v}} \equiv \ddot{\mathbf{x}} = \frac{\partial^2}{\partial t^2}\chi(\mathbf{X}, t). \tag{1.8}$$

In each case a superposed dot indicates differentiation with respect to t at fixed $\mathbf{X}$.

Let ϕ be a scalar field defined on B_t, i.e. $\phi(\mathbf{x},t)$. Since $\mathbf{x}=\chi(\mathbf{X},t)$, we may write

$$\phi(\mathbf{x},t)=\phi[\chi(\mathbf{X},t),t]\equiv\Phi(\mathbf{X},t), \tag{1.9}$$

which defines the notation Φ. Thus, any field defined on B_t (respectively B_r) can, through (1.2) or its inverse, equally be defined on B_r (respectively B_t).

The *material derivative* of ϕ is the rate of change of ϕ at fixed *material point* P, i.e. at fixed $\mathbf{X}$. We write the material derivative as $\dot{\phi}$ or $D\phi/Dt$.

By definition, we have

$$\dot{\phi}=\frac{\partial}{\partial t}\Phi(\mathbf{X},t),$$

and by the chain rule for partial derivatives we then obtain

$$\frac{\partial}{\partial t}\Phi(\mathbf{X},t)=\frac{\partial}{\partial t}\phi(\mathbf{x},t)+\frac{\partial x_i}{\partial t}\frac{\partial}{\partial x_i}\phi(\mathbf{x},t)=\frac{\partial}{\partial t}\phi(\mathbf{x},t)+\frac{\partial \mathbf{x}}{\partial t}\cdot\nabla\phi(\mathbf{x},t),$$

where ∇ denotes the gradient operator with respect to $\mathbf{x}$. Using (1.7) we thus have

$$\underbrace{\frac{\partial}{\partial t}\Phi(\mathbf{X},t)}_{\text{material description}}\equiv\dot{\phi}=\underbrace{\frac{\partial}{\partial t}\phi+\mathbf{v}\cdot\nabla\phi}_{\text{spatial description}}. \tag{1.10}$$

Similarly, for a vector field

$$\mathbf{u}(\mathbf{x},t)=\mathbf{u}[\chi(\mathbf{X},t),t]=\mathbf{U}(\mathbf{X},t), \tag{1.11}$$

wherein $\mathbf{U}$ is defined, we obtain

$$\frac{\partial}{\partial t}\mathbf{U}(\mathbf{X},t)\equiv\dot{\mathbf{u}}=\frac{\partial \mathbf{u}}{\partial t}+(\mathbf{v}\cdot\nabla)\mathbf{u}. \tag{1.12}$$

In particular, the acceleration $\mathbf{a}=\dot{\mathbf{v}}$ is given by

$$\mathbf{a}=\dot{\mathbf{v}}=\frac{\partial \mathbf{v}}{\partial t}+(\mathbf{v}\cdot\nabla)\mathbf{v}. \tag{1.13}$$

1.3. Differentiation of Cartesian tensor fields

Let $\phi,\mathbf{u},\mathbf{T}$ be scalar, vector and tensor functions of position $\mathbf{x}$. The operation of the gradient operator, grad or ∇, on these functions with respect to the basis $\{\mathbf{e}_i\}$ is defined as follows:

$$\operatorname{grad}\phi\equiv\nabla\phi=\frac{\partial\phi}{\partial x_i}\mathbf{e}_i, \tag{1.14}$$

$$\operatorname{grad}\mathbf{u} \equiv \nabla \otimes \mathbf{u} = \frac{\partial \mathbf{u}}{\partial x_q} \otimes \mathbf{e}_q = \frac{\partial}{\partial x_q}(u_p \mathbf{e}_p) \otimes \mathbf{e}_q = \frac{\partial u_p}{\partial x_q}\mathbf{e}_p \otimes \mathbf{e}_q, \tag{1.15}$$

$$\operatorname{grad}\mathbf{T} \equiv \nabla \otimes \mathbf{T} = \frac{\partial}{\partial x_i}\mathbf{T} \otimes \mathbf{e}_i = \frac{\partial}{\partial x_i}(T_{pq}\mathbf{e}_p \otimes \mathbf{e}_q) \otimes \mathbf{e}_i = \frac{\partial T_{pq}}{\partial x_i}\mathbf{e}_p \otimes \mathbf{e}_q \otimes \mathbf{e}_i, \tag{1.16}$$

and similarly for higher-order tensors. Note that the operation of grad increases the order of the tensor by one. On the other hand, the operation of *contraction* reduces the order of a tensor by two. For example, $\operatorname{grad}\mathbf{u}$ contracts to give $\nabla \cdot \mathbf{u}$, so that a second-order tensor reduces to a scalar.

There are several possible contractions of $\nabla \otimes \mathbf{T}$. We *define* $\operatorname{div}\mathbf{T}$ by

$$\operatorname{div}\mathbf{T} \equiv \frac{\partial T_{pq}}{\partial x_i}\mathbf{e}_q(\mathbf{e}_p \cdot \mathbf{e}_i),$$

which is the p–i contraction. Since $\mathbf{e}_p \cdot \mathbf{e}_i = \delta_{ip}$ we obtain

$$\operatorname{div}\mathbf{T} = \frac{\partial T_{pq}}{\partial x_p}\mathbf{e}_q. \tag{1.17}$$

In the notation defined here we note that (1.12) can be written as

$$\dot{\mathbf{u}} = \frac{\partial \mathbf{u}}{\partial t} + (\nabla \otimes \mathbf{u})\mathbf{v}. \tag{1.18}$$

1.4. Deformation and velocity gradients

Let Grad, Div, Curl (respectively grad, div, curl) denote the gradient, divergence and curl operators in the reference (respectively current) configuration, i.e. with respect to $\mathbf{X}$ (respectively $\mathbf{x}$). Then, we define the *deformation gradient tensor* $\mathbf{F}$ as

$$\mathbf{F}(\mathbf{X}, t) = \operatorname{Grad}\mathbf{x} \equiv \operatorname{Grad}\boldsymbol{\chi}(\mathbf{X}, t). \tag{1.19}$$

With respect to the chosen basis vectors and with use of (1.15) we have

$$\mathbf{F} = \frac{\partial}{\partial X_j}(x_i \mathbf{e}_i) \otimes \mathbf{E}_j = \frac{\partial x_i}{\partial X_j}\mathbf{e}_i \otimes \mathbf{E}_j$$

or, in component form,

$$F_{ij} = \frac{\partial x_i}{\partial X_j} \tag{1.20}$$

with $x_i = \chi_i(\mathbf{X}, t)$.

We assume that $\det \mathbf{F} \neq 0$ (to be justified shortly) so that $\mathbf{F}$ has an inverse $\mathbf{F}^{-1}$, given by

$$\mathbf{F}^{-1} = \operatorname{grad} \mathbf{X}, \tag{1.21}$$

with components

$$(\mathbf{F}^{-1})_{ij} = \frac{\partial X_i}{\partial x_j}. \tag{1.22}$$

This may be checked by means of the calculation

$$(\mathbf{F}\mathbf{F}^{-1})_{ij} = F_{ik}(\mathbf{F}^{-1})_{kj} = \frac{\partial x_i}{\partial X_k}\frac{\partial X_k}{\partial x_j} = \frac{\partial x_i}{\partial x_j} = \delta_{ij}.$$

It follows from (1.20) that

$$F_{ij}dX_j = \frac{\partial x_i}{\partial X_j}dX_j = dx_i,$$

i.e.

$$d\mathbf{x} = \mathbf{F}d\mathbf{X}, \tag{1.23}$$

which has inverse

$$d\mathbf{X} = \mathbf{F}^{-1}d\mathbf{x}. \tag{1.24}$$

Equation (1.23) describes how small *line elements* $d\mathbf{X}$ of material at $\mathbf{X}$ transform under the deformation into $d\mathbf{x}$ (which consists of the same material as $d\mathbf{X}$) at $\mathbf{x}$. It shows that *line elements* transform *linearly* since $\mathbf{F}$ depends on $\mathbf{X}$ (and not on $d\mathbf{X}$). Thus, at each $\mathbf{X}$, $\mathbf{F}$ is a *linear mapping* (i.e. a second-order tensor).

We justify taking $\mathbf{F}$ to be *non-singular* ($\det \mathbf{F} \neq 0$) by noting that $\mathbf{F}d\mathbf{X} \neq \mathbf{0}$ if $d\mathbf{X} \neq \mathbf{0}$, i.e. a line element cannot be annihilated by the deformation process.

Example

Let $\phi, \mathbf{u}, \mathbf{T}$ respectively be scalar, vector, and second-order tensor fields associated with a moving body. We now establish the following very useful formulas:

$$\operatorname{Grad} \phi = \mathbf{F}^T \operatorname{grad} \phi, \quad \operatorname{Grad} \mathbf{u} = (\operatorname{grad} \mathbf{u})\mathbf{F}, \tag{1.25}$$

$$\operatorname{Div} \mathbf{u} = J\operatorname{div}(J^{-1}\mathbf{F}\mathbf{u}), \quad \operatorname{Div} \mathbf{T} = J\operatorname{div}(J^{-1}\mathbf{F}\mathbf{T}), \tag{1.26}$$

where J is defined as

$$J = \det \mathbf{F}. \tag{1.27}$$

First, we calculate

$$\begin{aligned}\mathbf{F}^T \operatorname{grad}\phi &= \Big(\frac{\partial x_i}{\partial X_j}\mathbf{e}_i\otimes\mathbf{E}_j\Big)^T\frac{\partial\phi}{\partial x_p}\mathbf{e}_p = \frac{\partial x_i}{\partial X_j}(\mathbf{E}_j\otimes\mathbf{e}_i)\mathbf{e}_p\frac{\partial\phi}{\partial x_p}\\ &= \frac{\partial x_i}{\partial X_j}\frac{\partial\phi}{\partial x_p}\mathbf{E}_j\delta_{ip} = \frac{\partial x_p}{\partial X_j}\frac{\partial\phi}{\partial x_p}\mathbf{E}_j = \frac{\partial\phi}{\partial X_j}\mathbf{E}_j = \operatorname{Grad}\phi.\end{aligned}$$

Next,

$$\begin{aligned}(\operatorname{grad}\mathbf{u})\mathbf{F} &= \Big(\frac{\partial u_i}{\partial x_j}\mathbf{e}_i\otimes\mathbf{e}_j\Big)\Big(\frac{\partial x_p}{\partial X_q}\mathbf{e}_p\otimes\mathbf{E}_q\Big) = \frac{\partial u_i}{\partial x_j}\frac{\partial x_p}{\partial X_q}(\mathbf{e}_i\otimes\mathbf{e}_j)(\mathbf{e}_p\otimes\mathbf{E}_q)\\ &= \frac{\partial u_i}{\partial x_j}\frac{\partial x_p}{\partial X_q}\mathbf{e}_i\otimes\mathbf{E}_q\delta_{jp} = \frac{\partial u_i}{\partial x_p}\frac{\partial x_p}{\partial X_q}\mathbf{e}_i\otimes\mathbf{E}_q = \frac{\partial u_i}{\partial X_q}\mathbf{e}_i\otimes\mathbf{E}_q\\ &= \operatorname{Grad}\mathbf{u}.\end{aligned}$$

For the right-hand side of the first equation in (1.26), we calculate

$$J\operatorname{div}(J^{-1}\mathbf{F}\mathbf{u}) = J\frac{\partial}{\partial x_p}(J^{-1}F_{pq}u_q) = F_{pq}\frac{\partial u_q}{\partial x_p} + Ju_q\frac{\partial}{\partial x_p}(J^{-1}F_{pq}). \tag{1.28}$$

But,

$$\begin{aligned}\frac{\partial}{\partial x_p}(J^{-1}F_{pq}) &= \frac{\partial X_r}{\partial x_p}\frac{\partial}{\partial X_r}(J^{-1}F_{pq}) = -J^{-2}\frac{\partial J}{\partial X_r}\frac{\partial X_r}{\partial x_p}F_{pq} + J^{-1}\frac{\partial X_r}{\partial x_p}\frac{\partial F_{pq}}{\partial X_r}\\ &= -J^{-2}\Big(J(\mathbf{F}^{-1})_{ts}\frac{\partial F_{st}}{\partial X_r}\Big)\underbrace{\frac{\partial X_r}{\partial x_p}\frac{\partial x_p}{\partial X_q}}_{\delta_{rq}} + J^{-1}\frac{\partial X_r}{\partial x_p}\frac{\partial^2 x_p}{\partial X_q\partial X_r},\end{aligned}$$

which requires the formula

$$\frac{\partial}{\partial X_r}(\det\mathbf{F}) = (\det\mathbf{F})\operatorname{tr}\left(\mathbf{F}^{-1}\frac{\partial\mathbf{F}}{\partial X_r}\right). \tag{1.29}$$

Thus,

$$\frac{\partial}{\partial x_p}(J^{-1}F_{pq}) = -J^{-1}\frac{\partial X_t}{\partial x_s}\frac{\partial^2 x_s}{\partial X_q\partial X_t} + J^{-1}\frac{\partial X_r}{\partial x_p}\frac{\partial^2 x_p}{\partial X_q\partial X_r} = 0.$$

Hence, (1.28) gives

$$J\operatorname{div}(J^{-1}\mathbf{F}\mathbf{u}) = F_{pq}\frac{\partial u_q}{\partial x_p} = \frac{\partial x_p}{\partial X_q}\frac{\partial u_q}{\partial x_p} = \frac{\partial u_q}{\partial X_q} = \operatorname{Div}\mathbf{u}.$$

Similarly,

$$
\begin{aligned}
J\operatorname{div}(J^{-1}\mathbf{F}\mathbf{T}) &= J\frac{\partial}{\partial x_p}(J^{-1}F_{pq}T_{qr}\mathbf{E}_r) \\
&= \underbrace{J\frac{\partial}{\partial x_p}(J^{-1}F_{pq})T_{qr}\mathbf{E}_r}_{=0} + F_{pq}\frac{\partial T_{qr}}{\partial x_p}\mathbf{E}_r \\
&= \frac{\partial x_p}{\partial X_q}\frac{\partial T_{qr}}{\partial x_p}\mathbf{E}_r = \frac{\partial T_{qr}}{\partial X_q}\mathbf{E}_r = \operatorname{Div}\mathbf{T}.
\end{aligned}
$$

1.5. Deformation of area and volume elements

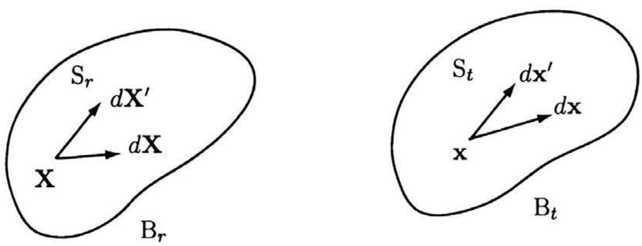

FIGURE 1.2. Infinitesimal line elements at $\mathbf{X}$ on the surface S_r in the reference configuration B_r and their images at $\mathbf{x}$ on the deformed surface S_t in the current configuration B_t.

Consider a surface S_r in B_r which deforms into the surface S_t in B_t, as depicted in Fig. 1.2. Let $\mathbf{X}$ be a point on S_r and $\mathbf{x}$ the corresponding point on S_t. Let $d\mathbf{X}$ and $d\mathbf{X}'$ be line elements of material on S_r based at $\mathbf{X}$ with images $d\mathbf{x}$ and $d\mathbf{x}'$ on S_t under the deformation. Strictly, the line elements are tangential to the surface and only approximately lie *in* the surface. If $\mathbf{F}$ denotes the deformation gradient, then

$$d\mathbf{x} = \mathbf{F}d\mathbf{X}, \quad d\mathbf{x}' = \mathbf{F}d\mathbf{X}'. \tag{1.30}$$

Let dA and da be surface area elements on S_r and S_t respectively, and let $\mathbf{N}$ and $\mathbf{n}$ be unit normals at $\mathbf{X}$ and $\mathbf{x}$ respectively. For the parallelogram with sides $d\mathbf{X}, d\mathbf{X}'$ we have

$$\mathbf{N}dA = d\mathbf{X} \times d\mathbf{X}'.$$

Under the deformation this becomes a parallelogram with sides $d\mathbf{x}, d\mathbf{x}'$ and area

$$\mathbf{n}da = d\mathbf{x} \times d\mathbf{x}'.$$

► From (1.30) we obtain

$$\mathbf{F}^T\mathbf{n}da = \mathbf{F}^T[(\mathbf{F}d\mathbf{X}) \times (\mathbf{F}d\mathbf{X}')] = (\det \mathbf{F})\, d\mathbf{X} \times d\mathbf{X}'.$$

Hence

$$\mathbf{n}da = J(\mathbf{F}^T)^{-1}\mathbf{N}dA,$$

where $J = \det \mathbf{F}$. With the notation

$$\mathbf{F}^{-T} = (\mathbf{F}^T)^{-1} = (\mathbf{F}^{-1})^T,$$

this becomes

$$\mathbf{n}da = J\mathbf{F}^{-T}\mathbf{N}dA. \tag{1.31}$$

This is an important result, known as *Nanson's formula*, and it describes how elements of surface area deform. It applies to area elements of arbitrary shape, not just the parallelogram considered here

Next, consider the parallelepiped in B_r formed by line elements $d\mathbf{X}$, $d\mathbf{X}'$, $d\mathbf{X}''$ at $\mathbf{X}$. Its volume dV is given by

$$dV = d\mathbf{X} \cdot (d\mathbf{X}' \times d\mathbf{X}'') = \det \begin{pmatrix} d\mathbf{X} & d\mathbf{X}' & d\mathbf{X}'' \end{pmatrix}.$$

The corresponding volume dv in B_t is

$$\begin{aligned} dv = d\mathbf{x} \cdot (d\mathbf{x}' \times d\mathbf{x}'') &= \det \begin{pmatrix} d\mathbf{x} & d\mathbf{x}' & d\mathbf{x}'' \end{pmatrix} \\ &= \det \begin{pmatrix} \mathbf{F}d\mathbf{X} & \mathbf{F}d\mathbf{X}' & \mathbf{F}d\mathbf{X}'' \end{pmatrix} = \det(\mathbf{F}) \det \begin{pmatrix} d\mathbf{X} & d\mathbf{X}' & d\mathbf{X}'' \end{pmatrix}, \end{aligned}$$

i.e.

$$dv = JdV. \tag{1.32}$$

Recalling that $\mathbf{F}$ is nonsingular, it is appropriate, by convention, to define volume elements to be positive, so that

$$J \equiv \det \mathbf{F} > 0. \tag{1.33}$$

► From (1.32) we see that J is a measure of the change in volume under the deformation. If the deformation is such that there is no change in volume then the deformation is said to be *isochoric*, and then

$$J \equiv \det \mathbf{F} = 1. \tag{1.34}$$

For some materials many deformations are such that (1.34) holds to a good approximation, and (1.34) is adopted as an *idealization*. An (ideal) material for which (1.34) holds for *all* deformations is called an *incompressible material.*

In order to analyze further the local nature of the deformation, i.e. of $\mathbf{F}$, we require some properties of second-order tensors.

1.6. Some results from tensor algebra

1.6.1. The square root theorem

If $\mathbf{S}$ is a positive definite, symmetric second-order tensor then there exists a unique, positive definite, symmetric second-order tensor, $\mathbf{U}$ say, such that $\mathbf{U}^2 = \mathbf{S}$.

Proof

Since $\mathbf{S}$ is symmetric we may write it in the spectral form

$$\mathbf{S} = \sum_{i=1}^{3} s_i \mathbf{e}'_i \otimes \mathbf{e}'_i,$$

where s_i are the (real) eigenvalues of $\mathbf{S}$ and $\{\mathbf{e}'_i\}$ are the (unit) eigenvectors. Since $\mathbf{S}$ is positive definite, we have $s_i > 0$. Now define $\mathbf{U}$ by

$$\mathbf{U} = \sum_{i=1}^{3} \sqrt{s_i} \mathbf{e}'_i \otimes \mathbf{e}'_i.$$

Then, $\mathbf{U}$ is positive definite and symmetric and $\mathbf{U}^2 = \mathbf{S}$, as required. Uniqueness is obvious.

1.6.2. The polar decomposition theorem

Let $\mathbf{F}$ be a second-order Cartesian tensor such that $\det \mathbf{F} > 0$. Then there exist unique, positive definite, symmetric tensors, $\mathbf{U}$ and $\mathbf{V}$, and a unique proper orthogonal tensor $\mathbf{R}$ such that

$$\mathbf{F} = \mathbf{RU} = \mathbf{VR}. \tag{1.35}$$

Proof

The tensors $\mathbf{F}\mathbf{F}^T$ and $\mathbf{F}^T\mathbf{F}$ are symmetric and positive definite. Hence, by the square root theorem, there exist unique positive definite symmetric tensors $\mathbf{U}$, $\mathbf{V}$ such that

$$\mathbf{V}^2 = \mathbf{F}\mathbf{F}^T, \quad \mathbf{U}^2 = \mathbf{F}^T\mathbf{F}.$$

Now define $\mathbf{R} = \mathbf{F}\mathbf{U}^{-1}$. We need to prove that $\mathbf{R}$ is proper orthogonal. First, we calculate

$$\mathbf{R}^T\mathbf{R} = (\mathbf{F}\mathbf{U}^{-1})^T(\mathbf{F}\mathbf{U}^{-1}) = \mathbf{U}^{-1}\mathbf{F}^T\mathbf{F}\mathbf{U}^{-1} = \mathbf{U}^{-1}\mathbf{U}^2\mathbf{U}^{-1} = \mathbf{I},$$

and hence we deduce that $\mathbf{R}$ is orthogonal. Second, we calculate

$$\det\mathbf{R} = \det(\mathbf{F}\mathbf{U}^{-1}) = (\det\mathbf{F})(\det\mathbf{U})^{-1} > 0,$$

and it follows that $\mathbf{R}$ is *proper* orthogonal.

Since $\mathbf{U}$ is unique, $\mathbf{R}$ is unique and hence $\mathbf{F} = \mathbf{R}\mathbf{U}$. Similarly, $\mathbf{F} = \mathbf{V}\mathbf{S}$, where $\mathbf{S}$ is proper orthogonal. Thus,

$$\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{S} = \mathbf{R}\mathbf{U}\mathbf{R}^T\mathbf{R}.$$

By uniqueness it follows that $\mathbf{S} = \mathbf{R}$ and hence (1.35) holds. Note that $\mathbf{V} = \mathbf{R}\mathbf{U}\mathbf{R}^T$.

Corollary. If $\mathbf{U}$ has eigenvalues λ_i and eigenvectors $\mathbf{u}^{(i)}$, $i \in \{1, 2, 3\}$, then $\lambda_i > 0$ and λ_i are also the eigenvalues of $\mathbf{V}$ with eigenvectors $\mathbf{R}\mathbf{u}^{(i)}$.

Proof

It follows from symmetry and from positive definiteness of $\mathbf{U}$ that $\lambda_i > 0$. Also, we have

$$\mathbf{V}(\mathbf{R}\mathbf{u}^{(i)}) = \mathbf{V}\mathbf{R}\mathbf{u}^{(i)} = \mathbf{R}\mathbf{U}\mathbf{u}^{(i)} = \mathbf{R}(\lambda_i\mathbf{u}^{(i)}) = \lambda_i(\mathbf{R}\mathbf{u}^{(i)}),$$

which shows that $\mathbf{R}\mathbf{u}^{(i)}$ are the eigenvectors of $\mathbf{V}$.

1.7. Analysis of deformation

1.7.1. Stretch, extension, shear and strain

Let $\mathbf{M}$ and $\mathbf{m}$ be unit vectors along $d\mathbf{X}$ and $d\mathbf{x}$ respectively, so that $d\mathbf{X} = \mathbf{M}|d\mathbf{X}|, d\mathbf{x} = \mathbf{m}|d\mathbf{x}|$ and (1.23) gives $\mathbf{m}|d\mathbf{x}| = \mathbf{F}\mathbf{M}|d\mathbf{X}|$. Thus

$$|d\mathbf{x}|^2 = (\mathbf{F}\mathbf{M}) \cdot (\mathbf{F}\mathbf{M})|d\mathbf{X}|^2 = (\mathbf{F}^T\mathbf{F}\mathbf{M}) \cdot \mathbf{M}|d\mathbf{X}|^2 \tag{1.36}$$

and hence

$$\frac{|d\mathbf{x}|}{|d\mathbf{X}|} = |\mathbf{FM}| = [\mathbf{M}\cdot(\mathbf{F}^T\mathbf{FM})]^{1/2} \equiv \lambda(\mathbf{M}), \tag{1.37}$$

which defines $\lambda(\mathbf{M})$, called the *stretch in the direction* $\mathbf{M}$ at $\mathbf{X}$. Note that $0 < \lambda(\mathbf{M}) < \infty$ for all unit vectors $\mathbf{M}$.

Now consider a pair of line elements $d\mathbf{X}_1, d\mathbf{X}_2$ based at $\mathbf{X}$, so that

$$d\mathbf{x}_1 = \mathbf{F}d\mathbf{X}_1, \quad d\mathbf{x}_2 = \mathbf{F}d\mathbf{X}_2.$$

Let Θ be the angle between them before deformation and θ the corresponding angle after deformation. Then,

$$\cos\Theta = \mathbf{M}_1\cdot\mathbf{M}_2, \quad \cos\theta = \frac{\mathbf{M}_1\cdot(\mathbf{F}^T\mathbf{F}\mathbf{M}_2)}{\lambda(\mathbf{M_1})\lambda(\mathbf{M_2})}.$$

The decrease in angle $\Theta - \theta$ (which may be positive or negative) is called the *shear* of the direction $\mathbf{M}_1, \mathbf{M}_2$ in the plane of $\mathbf{M}_1, \mathbf{M}_2$.

Next, from (1.36), we have

$$|d\mathbf{x}|^2 - |d\mathbf{X}|^2 = d\mathbf{X}\cdot(\mathbf{F}^T\mathbf{F} - \mathbf{I})d\mathbf{X}. \tag{1.38}$$

The material is said to be *unstrained* at $\mathbf{X}$ if no line element changes length, i.e.

$$d\mathbf{X}\cdot(\mathbf{F}^T\mathbf{F} - \mathbf{I})d\mathbf{X} = 0 \quad \text{for all } d\mathbf{X},$$

or, equivalently,

$$\lambda(\mathbf{M}) = 1 \quad \text{for all } \textit{unit} \text{ vectors } \mathbf{M}.$$

It follows that $\mathbf{F}^T\mathbf{F} - \mathbf{I} = \mathbf{O}$, the zero tensor. This allows the possibility that $\mathbf{F}$ is just a rotation $\mathbf{R}$, since, for orthogonal $\mathbf{R}$, we have $\mathbf{R}^T\mathbf{R} = \mathbf{I}$.

Strain is measured locally by changes in the lengths of line elements, i.e. by the value of (1.38). Thus, the tensor $\mathbf{F}^T\mathbf{F} - \mathbf{I}$ is a measure of strain. The so-called *Green strain tensor* $\mathbf{E}$ is defined by

$$\mathbf{E} = \frac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{I}). \tag{1.39}$$

Using the polar decomposition (1.35) for the deformation gradient $\mathbf{F}$, we may also form the following tensor measures of deformation:

$$\mathbf{C} = \mathbf{F}^T\mathbf{F} = \mathbf{U}^2, \qquad \mathbf{B} = \mathbf{F}\mathbf{F}^T = \mathbf{V}^2. \tag{1.40}$$

We refer to $\mathbf{C}$ and $\mathbf{B}$ as the *right* and *left Cauchy-Green deformation tensors* respectively. Note that $\mathbf{E}$ may be written as

$$\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I}) = \frac{1}{2}(\mathbf{U}^2 - \mathbf{I}). \tag{1.41}$$

Since $\mathbf{U}$ is positive definite and symmetric there exist (unit) eigenvectors $\mathbf{u}^{(i)}$ such that

$$\mathbf{U} = \sum_{i=1}^{3} \lambda_i \mathbf{u}^{(i)} \otimes \mathbf{u}^{(i)}, \tag{1.42}$$

where $\lambda_i > 0$ are the *principal stretches* of the deformation and $\mathbf{u}^{(i)}$ are the *principal directions*. Note that, in accordance with the definition (1.37), $\lambda_i = \lambda(\mathbf{u}^{(i)})$ – hence the terminology *principal stretch*.

The tensors $\mathbf{U}$ and $\mathbf{V}$ are called the *right* and *left stretch tensors* respectively. The deformation $\mathbf{F}$ rotates the principal axes of $\mathbf{U}$ into those of $\mathbf{V}$ as well as stretching along those directions. The principal axes of $\mathbf{U}$ and $\mathbf{V}$ are sometimes referred to as the *Lagrangian* and *Eulerian principal axes* respectively.

Other strain tensors based on $\mathbf{U}$ may be defined. For example, we define $\mathbf{E}^{(m)}$ as follows:

$$\mathbf{E}^{(m)} = \frac{1}{2}(\mathbf{U}^m - \mathbf{I}) \quad m \neq 0, \tag{1.43}$$

$$\mathbf{E}^{(0)} = \ln \mathbf{U}, \tag{1.44}$$

where m is a real number, not necessarily an integer. These are Lagrangian tensors, all coaxial with $\mathbf{U}$, and have eigenvalues $(\lambda_i^m - 1)/m$ for $m \neq 0$ and $\ln \lambda_i$ for $m = 0$. Corresponding Eulerian tensors, here denoted $\mathbf{e}^{(m)}$ and based on $\mathbf{V}$, are defined by

$$\mathbf{e}^{(m)} = \frac{1}{2}(\mathbf{V}^m - \mathbf{I}) \quad m \neq 0, \tag{1.45}$$

$$\mathbf{e}^{(0)} = \ln \mathbf{V}, \tag{1.46}$$

and we note that, on recalling the connection $\mathbf{V} = \mathbf{R}\mathbf{U}\mathbf{R}^T$, $\mathbf{e}^{(m)} = \mathbf{R}\mathbf{E}^{(m)}\mathbf{R}^T$ for each m. Thus, $\mathbf{E}^{(m)}$ and $\mathbf{e}^{(m)}$ have the same eigenvalues.

Finally in this section it is useful to note that the *displacement* $\mathbf{u}$ of a particle is defined as

$$\mathbf{u} = \mathbf{x} - \mathbf{X},$$

so that

$$\mathbf{x} = \mathbf{X} + \mathbf{u}$$

and

$$\mathbf{F} = \operatorname{Grad}\mathbf{x} = \mathbf{I} + \operatorname{Grad}\mathbf{u}, \tag{1.47}$$

where Grad $\mathbf{u}$ is the *displacement gradient* (recall that Grad $\mathbf{X} = \mathbf{I}$, the identity tensor.)

1.7.2. Homogeneous deformations

If $\mathbf{F}$ is independent of $\mathbf{X}$ then the deformation is said to be *homogeneous* (the same at each point of the body). The most general form of homogeneous deformation is given by $\mathbf{x} = \mathbf{F}\mathbf{X} + \mathbf{c}$, with $\mathbf{F}$ independent of $\mathbf{X}$ and $\mathbf{c}$ a constant vector. The following examples are all special cases of this.

Simple elongation Consider the uniform axial extension of a solid right circular cylinder (with lateral contraction). For this deformation $\mathbf{F} = \mathbf{U} = \mathbf{V}$ and there is no change in the orientation of the principal axes of $\mathbf{U}$ during the deformation. Let the principal axis $\mathbf{u}^{(1)}$ lie along the cylinder axis and correspond to principal stretch λ_1. Then, since there is symmetry perpendicular to the axis, $\lambda_2 = \lambda_3$ and hence the deformation gradient may be written

$$\mathbf{F} = \mathbf{U} = \lambda_1 \mathbf{u}^{(1)} \otimes \mathbf{u}^{(1)} + \lambda_2(\mathbf{u}^{(2)} \otimes \mathbf{u}^{(2)} + \mathbf{u}^{(3)} \otimes \mathbf{u}^{(3)}).$$

Pure dilatation This is defined by $\lambda_1 = \lambda_2 = \lambda_3$, $\mathbf{F} = \lambda_1 \mathbf{I}$ and might be associated with, for example, the deformation of a cube into a cube of a different size or a sphere into another sphere.

Pure shear This is an isochoric deformation defined by

$$\mathbf{F} = \lambda \mathbf{u}^{(1)} \otimes \mathbf{u}^{(1)} + \lambda^{-1} \mathbf{u}^{(2)} \otimes \mathbf{u}^{(2)} + \mathbf{u}^{(3)} \otimes \mathbf{u}^{(3)},$$

with the principal axes independent of λ. It is an example of a *plane strain* deformation and is such that $\lambda_1 = \lambda, \lambda_2 = \lambda^{-1}, \lambda_3 = 1$.

Simple shear Simple shear is defined by the equations

$$x_1 = X_1 + \gamma X_2, \quad x_2 = X_2, \quad x_3 = X_3, \tag{1.48}$$

where γ (constant) is called the *amount of shear*, $\tan^{-1}\gamma$ is the shear of the directions $\mathbf{e}_1, \mathbf{e}_2$, and the same basis vectors are used for both reference and current coordinates, see Fig. 1.3.

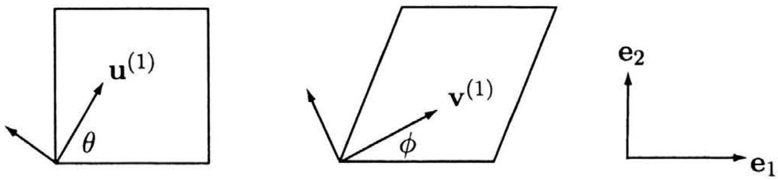

FIGURE 1.3. Simple shear in the (X_1, X_2) plane showing the orientation angles θ and ϕ of the Lagrangian and Eulerian principal axes.

The deformation gradient $\mathbf{F}$ has matrix of components, denoted F, given by

$$\mathsf{F} = \left(\frac{\partial x_i}{\partial X_j}\right) = \begin{bmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

To find the Lagrangian principal axes we consider, in matrix form,

$$\mathsf{U}^2 = \mathsf{F}^T\mathsf{F} = \begin{bmatrix} 1 & \gamma & 0 \\ \gamma & 1+\gamma^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{1.49}$$

The characteristic equation for $\mathbf{U}^2$, from which the eigenvalues λ^2 are determined, is

$$\det(\mathbf{U}^2 - \lambda^2\mathbf{I}) = 0,$$

i.e.

$$\begin{vmatrix} 1-\lambda^2 & \gamma & 0 \\ \gamma & 1-\lambda^2+\gamma^2 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 0,$$

or, when expanded out,

$$(\lambda^2 - 1)[\lambda^4 - (2+\gamma^2)\lambda^2 + 1] = 0.$$

Let the values of λ be denoted by λ_1, λ_2 and $\lambda_3 = 1$. Then

$$\lambda_1^2 + \lambda_2^2 = 2 + \gamma^2, \quad \lambda_1^2\lambda_2^2 = 1.$$

Now set $\lambda_1 = \lambda \geq 1$, $\lambda_2 = \lambda^{-1}$ so that

$$\lambda^2 + \lambda^{-2} = 2 + \gamma^2$$

and hence

$$\gamma = \lambda - \lambda^{-1}, \quad \lambda = \frac{1}{2}\gamma + \sqrt{1 + \frac{1}{4}\gamma^2},$$

in which we have taken $\gamma \geq 0$ to correspond to $\lambda \geq 1$.

Let

$$\mathbf{u}^{(1)} = \cos\theta\,\mathbf{e}_1 + \sin\theta\,\mathbf{e}_2, \quad \mathbf{u}^{(2)} = -\sin\theta\,\mathbf{e}_1 + \cos\theta\,\mathbf{e}_2.$$

Then, when referred to the Cartesian axes, the representation

$$\mathbf{U}^2 = \lambda_1^2\mathbf{u}^{(1)} \otimes \mathbf{u}^{(1)} + \lambda_2^2\mathbf{u}^{(2)} \otimes \mathbf{u}^{(2)} + \lambda_3^2\mathbf{u}^{(3)} \otimes \mathbf{u}^{(3)}$$

yields the matrix U^2, given by

$$\lambda^2 \begin{bmatrix} \cos^2\theta & \sin\theta\cos\theta & 0 \\ \sin\theta\cos\theta & \sin^2\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} + \lambda^{-2} \begin{bmatrix} \sin^2\theta & -\sin\theta\cos\theta & 0 \\ -\sin\theta\cos\theta & \cos^2\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Comparison with (1.49) shows that

$$\begin{aligned} &\lambda^2\cos^2\theta + \lambda^{-2}\sin^2\theta = 1, \\ &\lambda^2\sin^2\theta + \lambda^{-2}\cos^2\theta = 1 + \gamma^2, \\ &(\lambda^2 - \lambda^{-2})\sin\theta\cos\theta = \gamma, \end{aligned}$$

from which we may deduce that

$$\tan 2\theta = -\frac{2}{\gamma} \quad \left(\frac{\pi}{4} \leq \theta < \frac{\pi}{2}\right). \tag{1.50}$$

The corresponding angle for the principal axes of $\mathbf{F}\mathbf{F}^T = \mathbf{V}^2$ is calculated in a similar way. Let $\mathbf{v}^{(1)} = \cos\phi\,\mathbf{e}_1 + \sin\phi\,\mathbf{e}_2$, $\mathbf{v}^{(2)} = -\sin\phi\,\mathbf{e}_1 + \cos\phi\,\mathbf{e}_2$. The result is

$$\tan\phi = \frac{2}{\gamma} \quad \left(0 < \phi \leq \frac{\pi}{4}\right). \tag{1.51}$$

1.8. Analysis of motion

Recalling that the velocity is denoted $\mathbf{v}$, we define the *velocity gradient tensor*, denoted $\mathbf{L}$, as

$$\mathbf{L} = \operatorname{grad}\mathbf{v}, \tag{1.52}$$

which has components

$$L_{ij} = \frac{\partial v_i}{\partial x_j} \tag{1.53}$$

with respect to the basis $\{\mathbf{e}_i\}$.

Using the second identity in (1.25), we obtain

$$\text{Grad}\,\mathbf{v} = (\text{grad}\,\mathbf{v})\mathbf{F} = \mathbf{LF}.$$

Since $\mathbf{v} = \dot{\mathbf{x}}$ we also have

$$\text{Grad}\,\dot{\mathbf{x}} = \frac{\partial}{\partial t}\text{Grad}\,\mathbf{x} = \dot{\mathbf{F}},$$

recalling that the superposed dot represents the material time derivative. Hence, we have the important connection

$$\dot{\mathbf{F}} = \mathbf{LF}. \tag{1.54}$$

Using the result

$$\frac{\partial}{\partial t}(\det\mathbf{F}) = (\det\mathbf{F})\,\text{tr}\,(\mathbf{F}^{-1}\dot{\mathbf{F}})$$

together with (1.54) we deduce that

$$\frac{\partial}{\partial t}(\det\mathbf{F}) = (\det\mathbf{F})\,\text{tr}\,(\mathbf{L})$$

or, equivalently,

$$\dot{J} = J\,\text{tr}\,(\mathbf{L}) = J\,\text{div}\,\mathbf{v}, \tag{1.55}$$

remembering that $J = \det\mathbf{F}$, $\text{tr}\,(\mathbf{L}) = L_{ii} = \partial v_i/\partial x_i = \text{div}\,\mathbf{v}$.

Thus, div $\mathbf{v}$ measures the rate at which volume changes during the motion. For an *isochoric* motion $J \equiv 1$, $\dot{J} = 0$ and hence

$$\text{div}\,\mathbf{v} = 0. \tag{1.56}$$

It should also be noted that from (1.54) and the fact that $\mathbf{FF}^{-1} = \mathbf{I}$ it follows that

$$\frac{\partial}{\partial t}(\mathbf{F}^{-1}) = -\mathbf{F}^{-1}\mathbf{L}. \tag{1.57}$$

1.8.1. Stretching and spin

The deformation gradient $\mathbf{F}$ describes how material line elements change their length and orientation during deformation; the velocity gradient $\mathbf{L}$ describes the rate of these changes. Note that while $\mathbf{F}$ relates B_t to B_r, $\mathbf{L}$ is independent of B_r.

Let us write

$$\mathbf{L} = \mathbf{D} + \mathbf{W}, \tag{1.58}$$

where

$$\mathbf{D} = \underbrace{\frac{1}{2}(\mathbf{L} + \mathbf{L}^T)}_{\text{symmetric}}, \quad \mathbf{W} = \underbrace{\frac{1}{2}(\mathbf{L} - \mathbf{L}^T)}_{\text{skewsymmetric}}. \tag{1.59}$$

In order to interpret $\mathbf{D}$ and $\mathbf{W}$ we consider the line element $d\mathbf{X} \to d\mathbf{x} = \mathbf{F}d\mathbf{X}$. Then, we calculate

$$\begin{aligned} d\mathbf{x} \cdot d\mathbf{x} - d\mathbf{X} \cdot d\mathbf{X} &= (\mathbf{F}d\mathbf{X}) \cdot (\mathbf{F}d\mathbf{X}) - d\mathbf{X} \cdot d\mathbf{X} \\ &= d\mathbf{X} \cdot (\mathbf{F}^T\mathbf{F}d\mathbf{X}) - d\mathbf{X} \cdot d\mathbf{X} \\ &= d\mathbf{X} \cdot (\mathbf{F}^T\mathbf{F} - \mathbf{I})d\mathbf{X}. \end{aligned}$$

▶ From (1.54) it follows that

$$\begin{aligned} \frac{\partial}{\partial t}(d\mathbf{x} \cdot d\mathbf{x} - d\mathbf{X} \cdot d\mathbf{X}) &= d\mathbf{X} \cdot \frac{\partial}{\partial t}(\mathbf{F}^T\mathbf{F} - \mathbf{I})d\mathbf{X} = d\mathbf{X} \cdot (\mathbf{F}^T\dot{\mathbf{F}} + \dot{\mathbf{F}}^T\mathbf{F})d\mathbf{X} \\ &= d\mathbf{X} \cdot (\mathbf{F}^T\mathbf{L}\mathbf{F} + \mathbf{F}^T\mathbf{L}^T\mathbf{F})d\mathbf{X} \\ &= (\mathbf{F}d\mathbf{X}) \cdot (\mathbf{L} + \mathbf{L}^T)\mathbf{F}d\mathbf{X} = 2d\mathbf{x} \cdot (\mathbf{D}d\mathbf{x}). \end{aligned}$$

This shows that $\mathbf{D}$ measures the rate at which line elements are changing their lengths. It is called the (*Eulerian*) *strain-rate tensor* or *rate of stretching tensor*. The motion is *rigid* if and only if $\mathbf{D} = \mathbf{O}$.

Since

$$\frac{\partial}{\partial t}d\mathbf{x} = \dot{\mathbf{F}}d\mathbf{X} = \mathbf{L}\mathbf{F}d\mathbf{X} = \mathbf{L}d\mathbf{x} = (\mathbf{D} + \mathbf{W})d\mathbf{x}$$

and we have an interpretation of $\mathbf{D}$, as discussed above, it remains to interpret $\mathbf{W}$. We do this by setting $\mathbf{D} = \mathbf{O}$, so that

$$\frac{\partial}{\partial t}d\mathbf{x} = \mathbf{W}d\mathbf{x} = \mathbf{w} \times d\mathbf{x},$$

where $\mathbf{w}$ is the axial vector of $\mathbf{W}$. This shows that the motion is locally a rigid rotation and $\mathbf{W}$ is a measure of the *rate of rotation* (or *spin*) of line elements and it is called the *body spin*. The combination of $\mathbf{D}$ and $\mathbf{W}$ shows that the motion consists of stretching and rotation (analogous to the interpretation of $\mathbf{U}$ and $\mathbf{R}$). Note, however, that if $\mathbf{D} \neq 0$ then it contributes a rotation to line elements and the interpretation of $\mathbf{W}$ requires modification.

1.9. Integration of tensors

We first summarize some results from vector calculus that will be needed subsequently. The *divergence theorem* is written

$$\int_R \operatorname{div} \mathbf{v} dV = \int_{\partial R} \mathbf{v} \cdot \mathbf{n} dA, \tag{1.60}$$

where R is a domain in $\mathbb{R}^3$ and ∂R is its boundary (a closed surface), and $\mathbf{v}$ is a vector field. An alternative (and equivalent) form of the theorem is

$$\int_R \nabla \phi dV = \int_{\partial R} \phi \mathbf{n} dA, \tag{1.61}$$

where ϕ is a scalar field, or, in index notation,

$$\int_R \frac{\partial \phi}{\partial x_i} dV = \int_{\partial R} \phi n_i dA. \tag{1.62}$$

In particular, (1.62) applies to the *components* (which are scalar fields) $T_{pqr\ldots}$ of any Cartesian tensor. Thus,

$$\int_R \frac{\partial T_{pqr\ldots}}{\partial x_i} dV = \int_{\partial R} T_{pqr\ldots} n_i dA. \tag{1.63}$$

In tensor notation (1.59) is equivalent to

$$\int_R \nabla \otimes \mathbf{T} dV = \int_{\partial R} \mathbf{T} \otimes \mathbf{n} dA. \tag{1.64}$$

If, in particular, $\mathbf{T}$ is a second-order Cartesian tensor then contraction of (1.63), with $i = p$, gives

$$\int_R \frac{\partial T_{pq}}{\partial x_p} dV = \int_{\partial R} T_{pq} n_p dA \tag{1.65}$$

or, in tensor notation,

$$\int_R \operatorname{div} \mathbf{T} dV = \int_{\partial R} \mathbf{T}^T \mathbf{n} dA. \tag{1.66}$$

This is an *important formula* and will occur frequently in the remaining chapters of this volume.

1.10. Transport formulas

Let C_t, S_t and R_t denote curves, surfaces and regions in B_t, the current configuration of the body. Then, the following identities hold:

$$\frac{d}{dt}\int_{C_t}\phi d\mathbf{x} = \int_{C_t}(\dot{\phi}d\mathbf{x} + \phi\mathbf{L}d\mathbf{x}), \tag{1.67}$$

$$\frac{d}{dt}\int_{S_t}\phi\mathbf{n}da = \int_{S_t}\{[\dot{\phi} + \phi\mathrm{tr}\,(\mathbf{L})]\mathbf{n} - \phi\mathbf{L}^T\mathbf{n}\}da, \tag{1.68}$$

$$\frac{d}{dt}\int_{R_t}\phi dv = \int_{R_t}[\dot{\phi} + \phi\mathrm{tr}\,(\mathbf{L})]dv, \tag{1.69}$$

$$\frac{d}{dt}\int_{C_t}\mathbf{u}\cdot d\mathbf{x} = \int_{C_t}(\dot{\mathbf{u}} + \mathbf{L}^T\mathbf{u})\cdot d\mathbf{x}, \tag{1.70}$$

$$\frac{d}{dt}\int_{S_t}\mathbf{u}\cdot\mathbf{n}da = \int_{S_t}[\dot{\mathbf{u}} + \mathbf{u}\mathrm{tr}\,(\mathbf{L}) - \mathbf{L}\mathbf{u}]\cdot\mathbf{n}da, \tag{1.71}$$

$$\frac{d}{dt}\int_{R_t}\mathbf{u}dv = \int_{R_t}[\dot{\mathbf{u}} + \mathrm{tr}\,(\mathbf{L})\mathbf{u}]dv. \tag{1.72}$$

Proof

Use the formulas $d\mathbf{x} = \mathbf{F}d\mathbf{X}, \mathbf{n}da = J\mathbf{F}^{-T}\mathbf{N}dA, dv = JdV$ to convert the integrals over C_t, S_t, R_t in B_t to integrals over C_r, S_r, R_r in B_r, together with expressions for $\dot{\mathbf{F}}$ and $\dot{J}$. We illustrate the process by proving (1.71).

$$\begin{aligned}
\frac{d}{dt}\int_{S_t}\mathbf{u}\cdot\mathbf{n}da &= \frac{d}{dt}\int_{S_r}\mathbf{u}\cdot(J\mathbf{F}^{-T}\mathbf{N})dA \\
&\qquad \text{(note the integral is now over } S_r) \\
&= \frac{d}{dt}\int_{S_r}(J\mathbf{F}^{-1}\mathbf{u})\cdot\mathbf{N}dA \\
&\qquad \text{(using the definition of transpose)} \\
&= \int_{S_r}\underbrace{\frac{\partial}{\partial t}(J\mathbf{F}^{-1}\mathbf{u})}_{\text{at fixed } \mathbf{X}}\cdot\mathbf{N}dA \\
&= \int_{S_r}[J\mathbf{F}^{-1}\dot{\mathbf{u}} + \dot{J}\mathbf{F}^{-1}\mathbf{u} + J\partial(\mathbf{F}^{-1})/\partial t\,\mathbf{u}]\cdot\mathbf{N}dA.
\end{aligned}$$

► From (1.55) we have $\dot{J} = J\mathrm{tr}\,(\mathbf{L})$, and from (1.57) we have $\partial(\mathbf{F}^{-1})/\partial t = -\mathbf{F}^{-1}\mathbf{L}$. Thus,

$$\begin{aligned}
\frac{d}{dt}\int_{S_t} \mathbf{u}\cdot\mathbf{n}da &= \int_{S_r} [J\mathbf{F}^{-1}\dot{\mathbf{u}} + J\mathrm{tr}\,(\mathbf{L})\mathbf{F}^{-1}\mathbf{u} - J\mathbf{F}^{-1}\mathbf{L}\mathbf{u}]\cdot\mathbf{N}dA \\
&= \int_{S_r} \{J\mathbf{F}^{-1}[\dot{\mathbf{u}} + \mathrm{tr}\,(\mathbf{L})\mathbf{u} - \mathbf{L}\mathbf{u}]\}\cdot\mathbf{N}dA \\
&= \int_{S_r} [\dot{\mathbf{u}} + \mathrm{tr}\,(\mathbf{L})\mathbf{u} - \mathbf{L}\mathbf{u}]\cdot(J\mathbf{F}^{-T}\mathbf{N})dA \\
&= \int_{S_t} [\dot{\mathbf{u}} + \mathrm{tr}\,(\mathbf{L})\mathbf{u} - \mathbf{L}\mathbf{u}]\cdot\mathbf{n}da \\
&\qquad \text{(converting back to an integral over } S_t\text{)}.
\end{aligned}$$

The other formulas are established by following the same approach.

Chapter 2

Balance laws and equations of motion

The mechanics of continuous media are described by equations which express the balance of mass, linear momentum, angular momentum and energy in a moving body. These balance equations apply to all bodies, solid or fluid, and each gives rise to field equations (differential equations for scalar, vector and tensor fields) for sufficiently smooth motions (or jump conditions across surfaces of discontinuity). The fundamental concepts are mass, force and energy.

2.1. Mass

Let B_t be an arbitrary configuration of a body $\mathcal{B}$, and let A_t be a set of points in B_t occupied by the particles in an arbitrary subset $\mathcal{A}$ of $\mathcal{B}$. If, with $\mathcal{A}$, there is associated a non-negative real number $m(\mathcal{A})$ having physical dimensions independent of time and distance, such that

(i) $m(\mathcal{A}_1 \cup \mathcal{A}_2) = m(\mathcal{A}_1) + m(\mathcal{A}_2)$
for all pairs $\mathcal{A}_1, \mathcal{A}_2$ of disjoint subsets of $\mathcal{B}$, and

(ii) $m(\mathcal{A}) \to 0$ as the volume of A tends to zero,

then $\mathcal{B}$ is said to be a body with *mass function* m. The mass of A_t is denoted $m(\mathrm{A}_t)$.

Properties (i) and (ii) imply that there exists a scalar field ρ, defined on B_t, such that

$$m(\mathrm{A}_t) = \int_{\mathrm{A}_t} \rho dv \tag{2.1}$$

(this is a result from measure theory, and a proof is not given here). We refer to ρ as the *mass density* of the material composing $\mathcal{B}$. It is a scalar field, and here it will be assumed to be continuously differentiable (although in general this need not be the case).

2.2. Mass conservation

Let R_t be an arbitrary material region in the current configuration B_t. As R_t moves it always consists of the same material, so its mass does not change, i.e.

$$\frac{d}{dt} \int_{R_t} \rho dv = 0. \tag{2.2}$$

This is one form of the *conservation of mass equation*. From the transport formula (1.69) we obtain

$$\int_{R_t} (\dot{\rho} + \rho \mathrm{div}\, \mathbf{v}) dv = 0,$$

and, since R_t is arbitrary, it follows that

$$\dot{\rho} + \rho \mathrm{div}\, \mathbf{v} = 0 \tag{2.3}$$

at each point of the body (this deduction requires that the integrand is continuous). This is the local form of the mass conservation equation and it is also known as the *continuity equation*.

Recall, from (1.51), that $\dot{J} = J \mathrm{div}\, \mathbf{v}$. Substitution for $\mathrm{div}\, \mathbf{v}$ from (2.3) then gives $\rho \dot{J} + \dot{\rho} J = 0$, i.e. $\partial(\rho J)/\partial t = 0$. Thus, ρJ is constant for any material particle. In the reference configuration $J = 1$, so that $\rho J = \rho_r$, where ρ_r is the mass density in the reference configuration. Thus,

$$\rho = J^{-1} \rho_r, \tag{2.4}$$

which is yet another form of the mass conservation equation.

An alternative way to derive (2.4) is to note that

$$\int_{R_t} \rho dv = \int_{R_r} \rho J dV = \int_{R_r} \rho_r dV,$$

where R_r is the counterpart of R_t in the reference configuration.

2.3. Force, torque and momentum

2.3.1. Body and surface forces

The concepts of *force* and *torque* describe the action on a moving material body $\mathcal{B}$ of its surroundings and the mutual actions of the parts of $\mathcal{B}$ on each other. With $R_t \subset \mathrm{B}_t$ we associate two vectors, $\mathbf{F}(R_t)$ and $\mathbf{G}(R_t; o)$, called the *force* and the *torque with respect to o* on the material in R_t. Two types of force and torque must be accounted for in general. These are *body forces* and *body torques*, which act on the particles of a body (arising from gravity or magnetic fields, for example), and *contact forces* and *contact torques* resulting from the action of one part of the body on another across a separating surface (for example, pressure or friction).

The *body* force and torque, measured *per unit mass*, are denoted $\mathbf{b}$ and $\mathbf{c}$ respectively. Their contributions to $\mathbf{F}(R_t)$ and $\mathbf{G}(R_t; o)$ are

$$\int_{R_t} \rho \mathbf{b} dv, \qquad \int_{R_t} [\mathbf{x} \times (\rho \mathbf{b}) + \rho \mathbf{c}] dv$$

respectively, where $\mathbf{x}$ is the position vector of the point at which $\mathbf{b}$ acts.

A mathematical description of contact forces (but not torques) relies on *Cauchy's stress principle*, which is regarded as an axiom. This states that

> *the action of the material occupying that part of* B_t *exterior to a closed surface* S *on the material occupying the interior part is represented by a vector field, denoted* $\mathbf{t}_{(\mathbf{n})}$, *defined on* S *and with physical dimensions of force per unit area.* This is depicted in Fig. 2.1.

We refer to $\mathbf{t}_{(\mathbf{n})}$ as the *stress vector*. It is assumed to depend continuously on $\mathbf{n}$, the unit outward normal to S.

If this stress principle gives a complete account of contact action then the material is said to be *non-polar* and does not admit contact torques. All classical theories of solids and fluids are of this type.

The contributions to $\mathbf{F}(R_t)$ and $\mathbf{G}(R_t; o)$ of the contact forces acting on the boundary ∂R_t of R_t are

$$\int_{\partial R_t} \mathbf{t}_{(\mathbf{n})} da, \qquad \int_{\partial R_t} \mathbf{x} \times \mathbf{t}_{(\mathbf{n})} da$$

respectively. We now have

$$\mathbf{F}(R_t) = \int_{R_t} \rho \mathbf{b} dv + \int_{\partial R_t} \mathbf{t}_{(\mathbf{n})} da, \tag{2.5}$$

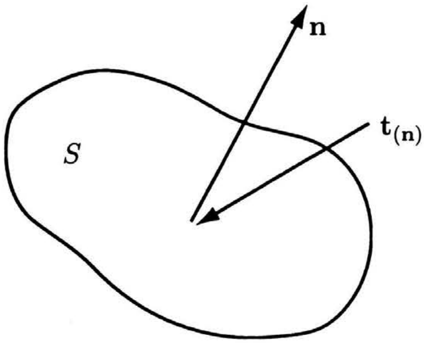

FIGURE 2.1. Stress vector $\mathbf{t}_{(\mathbf{n})}$ at a point of the surface S where the unit normal is $\mathbf{n}$.

$$\mathbf{G}(R_t; o) = \int_{R_t} \rho(\mathbf{x} \times \mathbf{b} + \mathbf{c})dv + \int_{\partial R_t} \mathbf{x} \times \mathbf{t}_{(\mathbf{n})}da \tag{2.6}$$

for the total force and total torque (sometimes referred to as couple) about o acting on R_t.

2.3.2. Momentum and angular momentum

Let $R_t \subset \mathrm{B}_t$. The *linear momentum* of the material occupying R_t is defined as

$$\mathbf{M}(R_t) = \int_{R_t} \rho\mathbf{v}dv. \tag{2.7}$$

With respect to an origin o, the *angular momentum* of R_t is defined as

$$\mathbf{H}(R_t; o) = \int_{R_t} \mathbf{x} \times (\rho\mathbf{v})dv. \tag{2.8}$$

2.4. Euler's laws of motion

Euler's laws of motion are

$$\frac{d\mathbf{M}}{dt} = \mathbf{F}, \qquad \frac{d\mathbf{H}}{dt} = \mathbf{G}, \tag{2.9}$$

and these hold independently of the choice of origin (although $\mathbf{G}$ and $\mathbf{H}$ do depend on such a choice).

They parallel Newton's laws for particles and rigid bodies. Note, however, that in classical mechanics $(2.9)_2$ is a consequence of $(2.9)_1$, whereas in continuum mechanics this is not the case and the two equations in (2.9) are *independent*.

Here we do not consider body torques, so we set $\mathbf{c} = \mathbf{0}$, and equations (2.9) are then written in full as

$$\frac{d}{dt}\int_{R_t} \rho \mathbf{v} dv = \int_{R_t} \rho \mathbf{b} dv + \int_{\partial R_t} \mathbf{t}_{(\mathbf{n})} da, \tag{2.10}$$

$$\frac{d}{dt}\int_{R_t} \rho \mathbf{x} \times \mathbf{v} dv = \int_{R_t} \rho \mathbf{x} \times \mathbf{b} dv + \int_{\partial R_t} \mathbf{x} \times \mathbf{t}_{(\mathbf{n})} da. \tag{2.11}$$

These are the equations of, respectively, *linear* and *angular momentum balance*.

Using the transport formula (1.72) with $\mathbf{u} = \rho \mathbf{v}$ and (2.3) we obtain

$$\frac{d}{dt}\int_{R_t} \rho \mathbf{v} dv = \int_{R_t} [\rho \dot{\mathbf{v}} + \dot{\rho} \mathbf{v} + \rho (\operatorname{div} \mathbf{v}) \mathbf{v}] dv = \int_{R_t} \rho \dot{\mathbf{v}} dv.$$

Hence (2.10) and (2.11) can be written

$$\int_{R_t} \rho (\mathbf{a} - \mathbf{b}) dv = \int_{\partial R_t} \mathbf{t}_{(\mathbf{n})} da, \tag{2.12}$$

$$\int_{R_t} \rho \mathbf{x} \times (\mathbf{a} - \mathbf{b}) dv = \int_{\partial R_t} \mathbf{x} \times \mathbf{t}_{(\mathbf{n})} da, \tag{2.13}$$

where $\mathbf{a} \equiv \dot{\mathbf{v}}$ is the acceleration.

2.5. The theory of stress

2.5.1. Cauchy's theorem

Let $(\mathbf{t}_{(\mathbf{n})}, \mathbf{b})$ be a system of surface (contact) and body forces for $\mathcal{B}$ during a motion. A necessary and sufficient condition for the momentum balance equations (2.12) and (2.13) to be satisfied is that there exists a second-order tensor $\boldsymbol{\sigma}$, called the *Cauchy stress tensor*, such that

(i) for each unit vector $\mathbf{n}$,

$$\mathbf{t}_{(\mathbf{n})} = \boldsymbol{\sigma}^T \mathbf{n}, \tag{2.14}$$

where $\boldsymbol{\sigma}$ is independent of $\mathbf{n}$,

(ii)

$$\boldsymbol{\sigma}^T = \boldsymbol{\sigma}, \tag{2.15}$$

(iii) $\boldsymbol{\sigma}$ satisfies the *equation of motion*

$$\operatorname{div} \boldsymbol{\sigma} + \rho \mathbf{b} = \rho \mathbf{a}. \tag{2.16}$$

Proof

Sufficiency: this can easily be checked by substituting (2.14)–(2.16) into (2.12) and (2.13). The calculations involved are similar to those required to prove necessity, so will not be given here. Necessity: assume that (2.12) and (2.13) are satisfied. The proof involves a number of steps. For this purpose, we now write $\mathbf{t}_{(\mathbf{n})}$ as $\mathbf{t}(\mathbf{n}, \mathbf{x})$ to indicate its dependence on both $\mathbf{n}$ and the position $\mathbf{x}$ on a surface.

Lemma 2.1. Given any $\mathbf{x} \in \mathrm{B}_t$, any orthonormal basis $\{\mathbf{e}_i\}$ and any vector $\mathbf{p}$ with $\mathbf{p} \cdot \mathbf{e}_i > 0, i \in \{1, 2, 3\}$, the stress vector can be written

$$\mathbf{t}(\mathbf{p}, \mathbf{x}) = -\sum_{i=1}^{3} (\mathbf{p} \cdot \mathbf{e}_i)\mathbf{t}(-\mathbf{e}_i, \mathbf{x}).$$

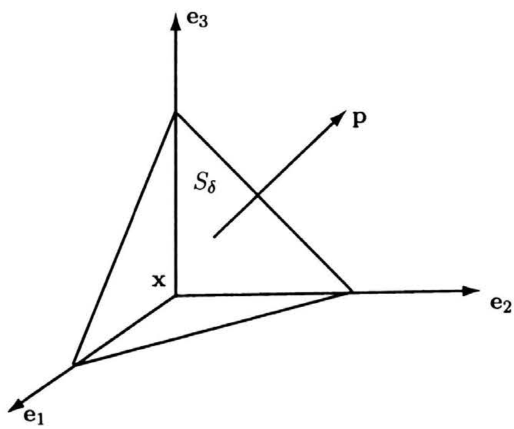

FIGURE 2.2. Tetrahedral volume bounded by the coordinate planes and the surface S_δ with unit normal $\mathbf{p}$. The point $\mathbf{x}$ is taken as the origin.

Proof

Suppose that $\mathbf{x} \in \mathrm{B}_t$, $\delta > 0$ and consider the tetrahedron shown in Fig. 2.2. The faces of the tetrahedron are denoted S_1, S_2, S_3 and S_δ, with unit (outward) normals $-\mathbf{e}_1$, $-\mathbf{e}_2$, $-\mathbf{e}_3$, $\mathbf{p}$ respectively, δ being the distance of the sloping face from $\mathbf{x}$. Since $\mathbf{a}$ and $\mathbf{b}$ are continuous in B_t they are bounded on some neighbourhood of $\mathbf{x}$ in B_t containing the tetrahedron for sufficiently small δ. Similarly, ρ is bounded, so that

$$\left|\int_{\partial R_t} \mathbf{t}_{(\mathbf{n})} da\right| = \left|\int_{R_t} \rho(\mathbf{a}-\mathbf{b}) dv\right| < k \operatorname{vol}(R_t),$$

where k is a constant independent of δ and $\operatorname{vol}(R_t)$ denotes the volume of the tetrahedron.

Let $A(\delta)$ denotes the area of the face S_δ. Then there exist positive constants c_1, c_2 such that

$$A(\delta) = c_1\delta^2, \quad \operatorname{vol}(R_t) = c_2\delta^3.$$

Hence

$$\frac{1}{A(\delta)}\int_{\partial R_t} \mathbf{t}_{(\mathbf{n})} da \to 0 \quad \text{as } \delta \to 0.$$

Let A_i denote the area of S_i. Since, by the divergence theorem, we have

$$\int_{\partial R_t} \mathbf{e}_i \cdot \mathbf{n} dS = 0 \quad i \in \{1,2,3\},$$

it follows that

$$A_i = (\mathbf{e}_i \cdot \mathbf{p}) A(\delta).$$

But

$$\int_{\partial R_t} \mathbf{t}_{(\mathbf{n})} da = \int_{S_\delta} \mathbf{t}(\mathbf{p}) da + \sum_{i=1}^{3} \int_{S_i} \mathbf{t}(-\mathbf{e}_i) da$$

and

$$\frac{1}{A(\delta)}\int_{S_\delta} \mathbf{t}_{(\mathbf{n})} da \to \mathbf{t}(\mathbf{p},\mathbf{x}) \quad \text{as } \delta \to 0,$$

$$\frac{1}{A(\delta)}\int_{S_i} \mathbf{t}(-\mathbf{e}_i) da \to (\mathbf{e}_i \cdot \mathbf{p})\mathbf{t}(-\mathbf{e}_i,\mathbf{x}) \quad \text{as } \delta \to 0.$$

Hence the stated result.

It follows that

$$\mathbf{t}(\mathbf{e}_i,\mathbf{x}) = -\mathbf{t}(-\mathbf{e}_i,\mathbf{x})$$

and hence

$$\mathbf{t}(\mathbf{p},\mathbf{x}) = \sum_{i=1}^{3} (\mathbf{e}_i \cdot \mathbf{p})\mathbf{t}(\mathbf{e}_i,\mathbf{x}) \tag{2.17}$$

for *any* vector $\mathbf{p}$.

Main result. Consider the tensor $\boldsymbol{\sigma}$ defined by

$$\boldsymbol{\sigma}^T(\mathbf{x}) = \sum_{i=1}^{3} \mathbf{t}(\mathbf{e}_i, \mathbf{x}) \otimes \mathbf{e}_i. \tag{2.18}$$

Then, by (2.17),

$$\boldsymbol{\sigma}^T \mathbf{n} = \sum_{i=1}^{3} (\mathbf{e}_i \cdot \mathbf{n}) \mathbf{t}(\mathbf{e}_i, \mathbf{x}) = \mathbf{t}(\mathbf{n}, \mathbf{x}).$$

Hence (i) is established.

On substitution of $\mathbf{t}_{(\mathbf{n})} = \boldsymbol{\sigma}^T \mathbf{n}$ into (2.12), we obtain

$$\int_{R_t} \rho(\mathbf{a} - \mathbf{b}) dv = \int_{\partial R_t} \boldsymbol{\sigma}^T \mathbf{n} da = \int_{R_t} \operatorname{div} \boldsymbol{\sigma} dv$$

by the divergence theorem (1.66). Thus,

$$\int_{R_t} [\operatorname{div} \boldsymbol{\sigma} - \rho(\mathbf{a} - \mathbf{b})] dv = \mathbf{0}.$$

Since R_t is arbitrary, (iii) follows (provided the above integrand is continuous). It remains to prove (ii).

Next, substitute (2.14) and (2.16) into (2.13) to give

$$\int_{R_t} \mathbf{x} \times (\operatorname{div} \boldsymbol{\sigma}) dv = \int_{\partial R_t} \mathbf{x} \times (\boldsymbol{\sigma}^T \mathbf{n}) da. \tag{2.19}$$

Noting that $\mathbf{u} \times \mathbf{v} = \mathbf{a} \times \mathbf{b}$, for any vectors $\mathbf{u}, \mathbf{v}, \mathbf{a}, \mathbf{b}$, is equivalent to

$$\mathbf{u} \otimes \mathbf{v} - \mathbf{v} \otimes \mathbf{u} = \mathbf{a} \otimes \mathbf{b} - \mathbf{b} \otimes \mathbf{a},$$

we write (2.19) as

$$\int_{R_t} [\mathbf{x} \otimes \operatorname{div} \boldsymbol{\sigma} - (\operatorname{div} \boldsymbol{\sigma}) \otimes \mathbf{x}] dv = \int_{\partial R_t} (\mathbf{x} \otimes \boldsymbol{\sigma}^T \mathbf{n} - \boldsymbol{\sigma}^T \mathbf{n} \otimes \mathbf{x}) da,$$

which, by application of the divergence theorem, becomes

$$\int_{R_t} [(\operatorname{grad} \mathbf{x}) \boldsymbol{\sigma} - \boldsymbol{\sigma}^T (\operatorname{grad} \mathbf{x})^T] dv.$$

Since $\operatorname{grad} \mathbf{x} = \mathbf{I}$, the identity tensor, we deduce that

$$\int_{R_t} (\boldsymbol{\sigma} - \boldsymbol{\sigma}^T) dv = \mathbf{O}.$$

Since R_t is arbitrary, (ii) follows.

2.5.2. Normal and shear stresses

Suppose an element of area da on a surface S with unit normal $\mathbf{n}$ is subjected to a contact force $\mathbf{t}(\mathbf{n})da$. The *normal component* of the stress vector, denoted σ, is defined as

$$\sigma \equiv \mathbf{n} \cdot \mathbf{t}(\mathbf{n}) = \mathbf{n} \cdot (\boldsymbol{\sigma}\mathbf{n}). \tag{2.20}$$

This is called the *normal stress* on the surface S. It is tensile (compressive) when positive (negative), and we have written $\mathbf{t}_{(\mathbf{n})} = \mathbf{t}(\mathbf{n}) = \mathbf{t}$.

The stress vector tangential to S is denoted $\boldsymbol{\tau}$, with magnitude τ, and given by

$$\boldsymbol{\tau} \equiv \mathbf{t}(\mathbf{n}) - \sigma\mathbf{n}, \quad \tau = |\mathbf{t}(\mathbf{n}) - \sigma\mathbf{n}|, \tag{2.21}$$

from which it follows that

$$\tau^2 = \mathbf{t}(\mathbf{n}) \cdot \mathbf{t}(\mathbf{n}) - [\mathbf{t}(\mathbf{n}) \cdot \mathbf{n}]^2.$$

We refer to τ as the *shear stress.*

If $\tau = 0$ and σ is independent of $\mathbf{n}$ then the stress is said to be *hydrostatic* or *isotropic*. In this case there is a scalar field p, called the *pressure*, such that

$$\mathbf{t}(\mathbf{n}) = -p\mathbf{n}, \qquad \boldsymbol{\sigma} = -p\mathbf{I}. \tag{2.22}$$

At a point $\mathbf{x}$ in the current configuration B_t let $\boldsymbol{\sigma}$ have components σ_{ij} with respect to basis vectors $\{\mathbf{e}_i\}$. Then σ_{ij} is the jth component of force per unit area in B_t acting on a surface whose normal is in the i-direction. In particular, for a surface normal to $\mathbf{e}_1$, σ_{11} is normal and σ_{12} and σ_{13} are tangential, i.e. they are shearing components. Similarly for the other components. In the case of (2.22), $\sigma_{12} = 0$, $\sigma_{13} = 0$, $\sigma_{11} = -p$, so the only component of force acting is a normal pressure.

2.6. Energy

The *kinetic energy* $K(R_t)$ of the material occupying R_t is defined as

$$K(R_t) = \int_{R_t} \frac{1}{2}\rho\mathbf{v} \cdot \mathbf{v}dv, \tag{2.23}$$

and the *rate of working*, or *power*, $P(R_t)$ of the forces acting on R_t is defined as

$$P(R_t) = \int_{R_t} \rho\mathbf{b} \cdot \mathbf{v}dv + \int_{\partial R_t} \mathbf{t} \cdot \mathbf{v}da. \tag{2.24}$$

By using $\mathbf{t} = \boldsymbol{\sigma}\mathbf{n}$ in (2.24), we obtain

$$\begin{aligned}
P(R_t) &= \int_{R_t} \rho \mathbf{b}\cdot\mathbf{v}dv + \int_{\partial R_t} (\boldsymbol{\sigma}\mathbf{n})\cdot\mathbf{v}da \\
&= \int_{R_t} \rho \mathbf{b}\cdot\mathbf{v}dv + \int_{\partial R_t} (\boldsymbol{\sigma}\mathbf{v})\cdot\mathbf{n}da \\
&\qquad \text{(since } \boldsymbol{\sigma} \text{ is symmetric)} \\
&= \int_{R_t} [\rho\mathbf{b}\cdot\mathbf{v} + \operatorname{div}(\boldsymbol{\sigma}\mathbf{v})]dv \\
&\qquad \text{(by the divergence theorem)} \\
&= \int_{R_t} [\rho\mathbf{b}\cdot\mathbf{v} + (\operatorname{div}\boldsymbol{\sigma})\cdot\mathbf{v} + \operatorname{tr}(\boldsymbol{\sigma}\mathbf{L})]dv \\
&\qquad \text{(since } (\sigma_{ij}v_j)_{,i} = \sigma_{ij,i}v_j + \sigma_{ij}v_{j,i} = \sigma_{ij,i}v_j + \sigma_{ij}L_{ji}) \\
&= \int_{R_t} [(\rho\mathbf{b} + \operatorname{div}\boldsymbol{\sigma})\cdot\mathbf{v} + \operatorname{tr}(\boldsymbol{\sigma}\mathbf{D})]dv \\
&\qquad \text{(since } \sigma_{ij}L_{ji} = \sigma_{ij}(D_{ij} + W_{ij}) = \sigma_{ij}D_{ij}) \\
&= \int_{R_t} [\rho\dot{\mathbf{v}}\cdot\mathbf{v} + \operatorname{tr}(\boldsymbol{\sigma}\mathbf{D})]dv \\
&\qquad \text{(using the equation of motion)} \\
&= \int_{R_t} \frac{1}{2}\rho\partial(\mathbf{v}\cdot\mathbf{v})/\partial t\, dv + \int_{R_t} \operatorname{tr}(\boldsymbol{\sigma}\mathbf{D})dv \\
&= \int_{R_r} \frac{1}{2}\rho_r\partial(\mathbf{v}\cdot\mathbf{v})/\partial t\, dV + \int_{R_t} \operatorname{tr}(\boldsymbol{\sigma}\mathbf{D})dv \\
&\qquad \text{(since } \rho dv = \rho_r dV) \\
&= \frac{d}{dt}\int_{R_r} \frac{1}{2}\rho_r(\mathbf{v}\cdot\mathbf{v})dV + \int_{R_t} \operatorname{tr}(\boldsymbol{\sigma}\mathbf{D})dv \\
&= \frac{d}{dt}K(R_t) + \int_{R_t} \operatorname{tr}(\boldsymbol{\sigma}\mathbf{D})dv. \qquad (2.25)
\end{aligned}$$

Thus,

$$P(R_t) = \frac{d}{dt}K(R_t) + S(R_t), \qquad (2.26)$$

where

$$S(R_t) = \int_{R_t} \operatorname{tr}(\boldsymbol{\sigma}\mathbf{D})dv. \qquad (2.27)$$

Equation (2.26) is an energy balance equation. The work done by the body and surface forces is converted into kinetic energy and $S(R_t)$. The latter may consist of stored (or potential) energy or be a measure of the amount of work dissipated in the form of heat or be a mixture of the two.

The following relations are worth noting. For a body held in equilibrium in configuration B, it may be shown, using the equations of equilibrium, that

$$\int_B \boldsymbol{\sigma} dv = \int_{\partial B} \boldsymbol{\sigma}\mathbf{n} \otimes \mathbf{x} da + \int_B \rho \mathbf{b} \otimes \mathbf{x} dv.$$

Again, if a body is in equilibrium in configuration B then, for any differentiable vector field $\mathbf{u}$,

$$\int_B \operatorname{tr}(\boldsymbol{\sigma} \operatorname{grad} \mathbf{u}) dv = \int_B \rho \mathbf{b} \cdot \mathbf{u} dv + \int_{\partial B} \mathbf{t} \cdot \mathbf{u} da.$$

2.7. Stress tensors

Using Nanson's formula (1.31) the traction on an area element $\mathbf{n}da$ in the current configuration can be written

$$\mathbf{t}da = \boldsymbol{\sigma}\mathbf{n}da = J\boldsymbol{\sigma}\mathbf{F}^{-T}\mathbf{N}dA \equiv \mathbf{S}^T\mathbf{N}dA,$$

wherein the *nominal stress tensor* $\mathbf{S}$ is defined as

$$\mathbf{S} = J\mathbf{F}^{-1}\boldsymbol{\sigma}. \tag{2.28}$$

This is also referred to as the *engineering stress*, while $\mathbf{S}^T$ is the so-called *first Piola-Kirchhoff stress tensor*, and it measures the force *per unit reference area* while $\boldsymbol{\sigma}$ measures the force *per unit deformed area*. Note that, in general, $\mathbf{S}$ is not symmetric but satisfies the connection

$$\mathbf{F}\mathbf{S} = \mathbf{S}^T\mathbf{F}^T \tag{2.29}$$

arising from symmetry of $\boldsymbol{\sigma}$.

The equation of motion

$$\operatorname{div} \boldsymbol{\sigma} + \rho\mathbf{b} = \rho\mathbf{a} \equiv \rho\dot{\mathbf{v}}$$

can be recast in terms of $\mathbf{S}$. One way to do this is to use the integral form of the balance equation, i.e.

$$\int_{\mathrm{R}_t} \rho \mathbf{b} dv + \int_{\partial \mathrm{R}_t} \boldsymbol{\sigma}\mathbf{n} da = \int_{\mathrm{R}_t} \rho \dot{\mathbf{v}} dv,$$

and convert the integrals to integrals over the reference configuration using mass conservation in the form $\rho dv = \rho_r dV$ and Nanson's formula (1.31). This leads to

$$\int_{\mathrm{R}_r} \rho_r \mathbf{b} dV + \int_{\partial \mathrm{R}_r} \mathbf{S}^T\mathbf{N} dA = \int_{\mathrm{R}_r} \rho_r \dot{\mathbf{v}} dV,$$

and hence, by the divergence theorem,

$$\operatorname{Div}\mathbf{S} + \rho_r \mathbf{b} = \rho_r \dot{\mathbf{v}}. \tag{2.30}$$

Alternatively, the identity $\operatorname{div}(J^{-1}\mathbf{F}) = \mathbf{0}$, obtained from $(1.26)_2$ by setting $\mathbf{T} = \mathbf{I}$, can be used to give

$$\operatorname{div}\boldsymbol{\sigma} = J^{-1}\operatorname{Div}\mathbf{S},$$

and then use of $J = \rho_r/\rho$ leads to (2.30).

Now recall the expression

$$S(\mathrm{R}_t) = \int_{\mathrm{R}_t} \operatorname{tr}(\boldsymbol{\sigma}\mathbf{D})dv$$

occurring in the energy balance equations (2.25)–(2.27). Over the reference configuration the integral becomes

$$\int_{\mathrm{R}_r} J \operatorname{tr}(\boldsymbol{\sigma}\mathbf{D})dV. \tag{2.31}$$

The integrand in (2.31) is the rate of working of the stresses per unit reference volume (i.e. the stress power density). Using the symmetry of $\boldsymbol{\sigma}$ together with (1.50) and (2.28) we have

$$J\operatorname{tr}(\boldsymbol{\sigma}\mathbf{D}) = J\operatorname{tr}(\boldsymbol{\sigma}\mathbf{L}) = \operatorname{tr}(\mathbf{FSL}) = \operatorname{tr}(\mathbf{SLF}) = \operatorname{tr}(\mathbf{S}\dot{\mathbf{F}}).$$

This shows that the stress power is also given by $\operatorname{tr}(\mathbf{S}\dot{\mathbf{F}})$. Because of this connection $\mathbf{S}$ and $\mathbf{F}$ are said to constitute a pair of *conjugate* stress and deformation tensors.

Furthermore, by recalling the definition (1.39) and writing

$$\mathbf{E}^{(2)} = \frac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{I}),$$

we obtain

$$\dot{\mathbf{E}}^{(2)} = \frac{1}{2}(\mathbf{F}^T\dot{\mathbf{F}} + \dot{\mathbf{F}}^T\mathbf{F}) \equiv \mathbf{F}^T\mathbf{DF}.$$

This is used to write the stress power as

$$\operatorname{tr}(\mathbf{S}\dot{\mathbf{F}}) = \operatorname{tr}(\mathbf{SF}^{-T}\mathbf{F}^T\dot{\mathbf{F}}) = \operatorname{tr}(\mathbf{SF}^{-T}\dot{\mathbf{E}}^{(2)}) = \operatorname{tr}(\mathbf{T}^{(2)}\dot{\mathbf{E}}^{(2)}) \tag{2.32}$$

using the symmetry of $\mathbf{SF}^{-T}$, which comes from the definition (2.28). We have also introduced the notation $\mathbf{T}^{(2)}$, defined through

$$\mathbf{SF}^{-T} = J\mathbf{F}^{-1}\boldsymbol{\sigma}\mathbf{F}^{-T} \equiv \mathbf{T}^{(2)}, \tag{2.33}$$

which denotes the *second Piola-Kirchhoff stress tensor*. The stress and strain pair $(\mathbf{T}^{(2)}, \mathbf{E}^{(2)})$ is a pair of conjugate stress and strain tensors.

Since, from (1.40), $\mathbf{F}^T\mathbf{F} = \mathbf{U}^2$ we also have

$$\dot{\mathbf{E}}^{(2)} = \frac{1}{2}(\mathbf{U}\dot{\mathbf{U}} + \dot{\mathbf{U}}\mathbf{U}),$$

and hence, using the symmetry of $\mathbf{T}^{(2)}$ and of $\dot{\mathbf{U}}$,

$$\mathrm{tr}\,(\mathbf{T}^{(2)}\dot{\mathbf{E}}^{(2)}) = \mathrm{tr}\,(\mathbf{T}^{(2)}\mathbf{U}\dot{\mathbf{U}}) = \mathrm{tr}\,[\frac{1}{2}(\mathbf{T}^{(2)}\mathbf{U} + \mathbf{U}\mathbf{T}^{(2)})\dot{\mathbf{U}}].$$

This motivates the definition of the *Biot stress tensor* $\mathbf{T}^{(1)}$, conjugate to the strain tensor $\mathbf{E}^{(1)} \equiv \mathbf{U} - \mathbf{I}$, as

$$\mathbf{T}^{(1)} = \frac{1}{2}(\mathbf{T}^{(2)}\mathbf{U} + \mathbf{U}\mathbf{T}^{(2)}), \tag{2.34}$$

which, by using the polar decomposition (1.35), may also be written as

$$\mathbf{T}^{(1)} = \frac{1}{2}(\mathbf{S}\mathbf{R} + \mathbf{R}^T\mathbf{S}^T). \tag{2.35}$$

We now have the connections

$$J\,\mathrm{tr}\,(\boldsymbol{\sigma}\mathbf{D}) = \mathrm{tr}\,(\mathbf{S}\dot{\mathbf{F}}) = \mathrm{tr}\,(\mathbf{T}^{(2)}\dot{\mathbf{E}}^{(2)}) = \mathrm{tr}\,(\mathbf{T}^{(1)}\dot{\mathbf{E}}^{(1)}). \tag{2.36}$$

More generally, the (symmetric) stress tensor $\mathbf{T}^{(m)}$ conjugate to the strain tensor $\mathbf{E}^{(m)} \equiv (\mathbf{U}^m - \mathbf{I})/m$ discussed in Section 1.7.1 may be defined via the identity

$$\mathrm{tr}\,(\mathbf{T}^{(m)}\dot{\mathbf{E}}^{(m)}) = \mathrm{tr}\,(\mathbf{T}^{(1)}\dot{\mathbf{E}}^{(1)}) = \mathrm{tr}\,(\mathbf{T}^{(1)}\dot{\mathbf{U}}), \tag{2.37}$$

and it should be noted that this definition is independent of any material constitutive law.

Chapter 3

Constitutive equations

3.1. Introduction

So far, we have the following equations governing the motion of a continuous body:

equation of mass conservation

$$\dot{\rho} + \rho \operatorname{div} \mathbf{v} = 0; \tag{3.1}$$

equation of motion

$$\operatorname{div} \boldsymbol{\sigma} + \rho \mathbf{b} = \rho \dot{\mathbf{v}}; \tag{3.2}$$

equation of angular momentum balance

$$\boldsymbol{\sigma}^T = \boldsymbol{\sigma}. \tag{3.3}$$

These provide 7 scalar equations for 13 scalar fields – $\rho, \mathbf{v}$ (3 components), $\boldsymbol{\sigma}$ (9 components) with the body force $\mathbf{b}$ regarded as known. Equivalently, given (3.3), equations (3.1) and (3.2) provide 4 equations for 10 scalar fields – $\rho, \mathbf{v}$ (3 components), $\boldsymbol{\sigma}$ (6 components).

The missing 6 equations are provided in the form of *constitutive equations*, which give expressions for the 6 components of $\boldsymbol{\sigma}$ in terms of kinematical quantities, as we now describe.

It is assumed that at time t the stress is uniquely determined by the motion $\boldsymbol{\chi}$, i.e. $\boldsymbol{\sigma}$ is a function (more generally functional) of $\mathbf{x}, \mathbf{v}, \mathbf{F}, \mathbf{L}, \ldots$, and possibly also higher gradients of the deformation. We then have 10 equations for 10 unknowns, and, by substituting the constitutive equations into (3.1) and (3.2) we arrive at 4 equations for ρ and $\mathbf{v}$, and (3.3) will be satisfied automatically. We now illustrate the general principles involved in the development of constitutive equations by focussing on the case of homogeneous elastic materials, for which $\boldsymbol{\sigma}$ depends only on $\mathbf{F}$. (For an inhomogeneous material there is, additionally, explicit dependence on $\mathbf{X}$.)

3.2. Elastic materials

The constitutive equation for an elastic material is written in the form

$$\boldsymbol{\sigma} = \mathbf{g}(\mathbf{F}), \tag{3.4}$$

where $\mathbf{g}$ is a *symmetric tensor-valued function*, defined on the space of deformation gradients $\mathbf{F}$. Equation (3.4) states that the stress in B_t at a point $\mathbf{X}$ depends only on the deformation gradient at $\mathbf{X}$ and not on the history of deformation, and, in particular, it is independent of the path of deformation taken to reach the point $\mathbf{F}$.

When the stress is removed the deformation returns to its original value (that in B_r), so that

$$\mathbf{g}(\mathbf{I}) = \mathbf{O}, \tag{3.5}$$

i.e. the undeformed configuration is free of stress. In some situations it will be necessary to relax this condition, but it is adopted here for the time being. We refer to $\mathbf{g}$ as the (Cauchy stress) *response function* of the material *relative* to B_r. It should be emphasized that, in general, the form of $\mathbf{g}$ depends on the choice of reference configuration.

3.3. Objectivity

Suppose that a rigid-body motion

$$\mathbf{x}^* = \mathbf{Q}(t)\mathbf{x} + \mathbf{c}(t) \tag{3.6}$$

is superimposed on the motion $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t)$. Then, the resulting deformation gradient, $\mathbf{F}^*$ say, is given by

$$\mathbf{F}^* = \mathbf{QF}. \tag{3.7}$$

In index notation, this may be proved as follows. Note first that, since

$$x_i^* = Q_{ip}x_p + c_i,$$

we obtain

$$\frac{\partial x_i^*}{\partial x_k} = Q_{ip}\frac{\partial x_p}{\partial x_k} = Q_{ip}\delta_{pk} = Q_{ik},$$

and hence

$$F_{ij}^* = \frac{\partial x_i^*}{\partial X_j} = \frac{\partial x_i^*}{\partial x_k}\frac{\partial x_k}{\partial X_j} = Q_{ik}F_{kj}.$$

For an elastic material with response function $\mathbf{g}$, the stress tensor, $\boldsymbol{\sigma}^*$ say, associated with the deformation gradient $\mathbf{F}^*$ is

$$\boldsymbol{\sigma}^* = \mathbf{g}(\mathbf{F}^*).$$

We now determine how $\boldsymbol{\sigma}^*$ is related to $\boldsymbol{\sigma}$. Under the rotation $\mathbf{Q}$ the unit normal $\mathbf{n}$ to ∂R_t becomes $\mathbf{n}^* = \mathbf{Qn}$ and the traction vector $\mathbf{t}$ becomes $\mathbf{t}^* = \mathbf{Qt}$. Since $\mathbf{t} = \boldsymbol{\sigma}\mathbf{n}$, $\mathbf{t}^* = \boldsymbol{\sigma}^*\mathbf{n}^*$ we obtain

$$\mathbf{Q}\boldsymbol{\sigma}\mathbf{n} = \boldsymbol{\sigma}^*\mathbf{Qn}.$$

This holds for arbitrary $\mathbf{n}$ and hence $\mathbf{Q}\boldsymbol{\sigma} = \boldsymbol{\sigma}^*\mathbf{Q}$, i.e.

$$\boldsymbol{\sigma}^* = \mathbf{Q}\boldsymbol{\sigma}\mathbf{Q}^T. \tag{3.8}$$

The response function $\mathbf{g}$ must therefore satisfy the *invariance requirement*

$$\mathbf{g}(\mathbf{F}^*) \equiv \mathbf{g}(\mathbf{QF}) = \mathbf{Qg}(\mathbf{F})\mathbf{Q}^T \tag{3.9}$$

for each $\mathbf{F}$ and *all* rotations $\mathbf{Q}$. This expresses the fact that the constitutive law (3.4) is *objective*. In essence, this means that material properties are independent of superimposed rigid-body motions. It can also be interpreted in terms of *observers*. In that case, rather than representing a superimposed rigid motion equation, (3.6) is treated as an *observer transformation*. We refer to Ogden [18] for discussion of this latter approach. For the elastic materials discussed here the consequences of the two interpretations are identical, but in general this may not be the case for other materials. The difference is quite subtle and has generated some controversy in the literature. For a recent account of the topic we refer to Murdoch [14].

Definition. Let $\phi, \mathbf{u}, \mathbf{T}$ be scalar, vector and (second-order) tensor fields defined on B_t, i.e. they are Eulerian in character. Let $\phi^*, \mathbf{u}^*, \mathbf{T}^*$ be the corresponding fields defined on B_t^*, where B_t^* is obtained from B_t by the rigid motion $\mathbf{x}^* = \mathbf{Q}\mathbf{x}+\mathbf{c}$. The fields are said to be *objective* if, for all such motions,

$$\phi^* = \phi, \quad \mathbf{u}^* = \mathbf{Q}\mathbf{u}, \quad \mathbf{T}^* = \mathbf{Q}\mathbf{T}\mathbf{Q}^T. \tag{3.10}$$

Example

If ϕ is an objective scalar field then $\operatorname{grad}\phi$ is an objective vector field. We note that, in components,

$$\begin{aligned}(\operatorname{grad}\phi)_i^* &= (\operatorname{grad}^*\phi^*)_i = (\operatorname{grad}^*\phi)_i \qquad (\text{since } \phi^* = \phi)\\ &= \frac{\partial\phi}{\partial x_i^*} = \frac{\partial\phi}{\partial x_k}\frac{\partial x_k}{\partial x_i^*}.\end{aligned}$$

Next, since $\mathbf{x}^* = \mathbf{Q}\mathbf{x}+\mathbf{c}$, it follows that $\mathbf{x} = \mathbf{Q}^T\mathbf{x}^* - \mathbf{Q}^T\mathbf{c}$, or, in components,

$$x_k = Q_{pk}x_p^* - Q_{pk}c_p.$$

Hence

$$\frac{\partial x_k}{\partial x_i^*} = Q_{pk}\frac{\partial x_p^*}{\partial x_i^*} = Q_{pk}\delta_{pi} = Q_{ik},$$

which leads to

$$(\operatorname{grad}\phi)_i^* = Q_{ik}(\operatorname{grad}\phi)_k,$$

i.e. $(\operatorname{grad}\phi)^* = \mathbf{Q}(\operatorname{grad}\phi)$. Thus, $\operatorname{grad}\phi$ is an objective vector.

Example

Let

$$\mathbf{x}^* \equiv \boldsymbol{\chi}^*(\mathbf{X}, t) = \mathbf{Q}(t)\boldsymbol{\chi}(\mathbf{X}, t) + \mathbf{c}(t).$$

Then

$$\mathbf{v}^* \equiv \frac{\partial\boldsymbol{\chi}^*}{\partial t} = \mathbf{Q}\mathbf{v} + \dot{\mathbf{Q}}\mathbf{x} + \dot{\mathbf{c}}.$$

Since $\mathbf{v}^* \neq \mathbf{Q}\mathbf{v}$ it follows that the velocity is *not* an objective vector. Similarly, for the acceleration,

$$\mathbf{a}^* \equiv \frac{\partial^2\boldsymbol{\chi}^*}{\partial t^2} = \mathbf{Q}\mathbf{a} + 2\dot{\mathbf{Q}}\mathbf{v} + \ddot{\mathbf{Q}}\mathbf{x} + \ddot{\mathbf{c}}.$$

3.4. Material symmetry

Let $\boldsymbol{\sigma}$ be the stress in configuration B_t, and let $\mathbf{F}, \mathbf{F}'$ be the deformation gradients in B_t relative to the reference configurations B_r, B_r' respectively, as depicted in Fig. 3.1. We denote by $\mathbf{g}$ and $\mathbf{g}'$ the response functions relative to B_r and B_r', respectively, so that

$$\boldsymbol{\sigma} = \mathbf{g}(\mathbf{F}) = \mathbf{g}'(\mathbf{F}'). \tag{3.11}$$

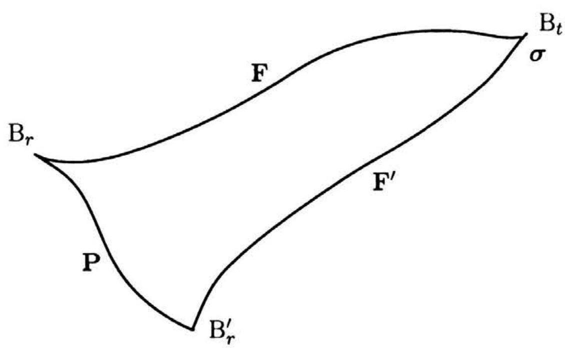

FIGURE 3.1. Paths of deformation with deformation gradients $\mathbf{F}$ and $\mathbf{F}'$ from reference configurations B_r and B_r', which are connected by deformation gradient $\mathbf{P}$.

Let $\mathbf{P} = \operatorname{Grad}\mathbf{X}'$ be the deformation gradient of B_r' relative to B_r, where $\mathbf{X}'$ is the position vector of a point in B_r'. Then

$$\mathbf{F} = \mathbf{F}'\mathbf{P}. \tag{3.12}$$

To prove (3.12), we use index notation. We have

$$F_{ij} = \frac{\partial x_i}{\partial X_j} = \frac{\partial x_i}{\partial X_k'}\frac{\partial X_k'}{\partial X_j} = F_{ik}'P_{kj}.$$

Substitution of (3.12) into (3.11) then gives

$$\mathbf{g}(\mathbf{F}'\mathbf{P}) = \mathbf{g}'(\mathbf{F}').$$

In general, the response of the material relative to B_r' differs from that relative to B_r, i.e. $\mathbf{g}' \neq \mathbf{g}$. However, for certain $\mathbf{P}$ we may have $\mathbf{g}' = \mathbf{g}$, in which case

$$\mathbf{g}(\mathbf{F}'\mathbf{P}) = \mathbf{g}(\mathbf{F}') \tag{3.13}$$

for all deformation gradients $\mathbf{F}'$ and for all such $\mathbf{P}$. Equation (3.11) then gives

$$\boldsymbol{\sigma} = \mathbf{g}(\mathbf{F}) = \mathbf{g}(\mathbf{F}'),$$

and, in order to calculate $\boldsymbol{\sigma}$, it is not necessary to distinguish between B_r and B_r'.

The set of tensors $\mathbf{P}$ for which (3.13) holds defines the *symmetry of the material relative to* B_r – the larger the set the more symmetry there is. An example is provided by the structure of a cubic crystal, which has certain rotational symmetry.

Let $\mathcal{G}$ denote the set of (invertible) second-order tensors, denoted $\mathbf{H}$, such that, in line with (3.13),

$$\mathbf{g}(\mathbf{FH}) = \mathbf{g}(\mathbf{F}) \tag{3.14}$$

for all deformation gradients $\mathbf{F}$.

Then $\mathcal{G}$ is a *multiplicative* group, called the *symmetry group of the material relative to* B_r.

To show this we note that if $\mathbf{H}_1, \mathbf{H}_2 \in \mathcal{G}$ then, by application of (3.14) for different $\mathbf{F}$ and $\mathbf{H}$,

$$\mathbf{g}(\mathbf{FH}_1\mathbf{H}_2) = \mathbf{g}(\mathbf{FH}_1) = \mathbf{g}(\mathbf{F})$$

and hence $\mathbf{H}_1\mathbf{H}_2 \in \mathcal{G}$ (closure); if $\mathbf{H} \in \mathcal{G}$ then

$$\mathbf{g}(\mathbf{FH}^{-1}) = \mathbf{g}(\mathbf{FH}^{-1}\mathbf{H}) = \mathbf{g}(\mathbf{F})$$

and hence $\mathbf{H}^{-1} \in \mathcal{G}$ (inverse); and $\mathbf{I} \in \mathcal{G}$ since $\mathbf{g}(\mathbf{FI}) = \mathbf{g}(\mathbf{F})$ (identity). Thus, the requirements of closure, inverse and identity for a group are satisfied.

3.4.1. Important example: isotropy

If $\mathcal{G}$ is the proper orthogonal group then the material is said to be *isotropic relative to* B_r, and

$$\mathbf{g}(\mathbf{FQ}) = \mathbf{g}(\mathbf{F}) \tag{3.15}$$

for all proper orthogonal $\mathbf{Q}$ (for every deformation gradient $\mathbf{F}$). Physically, this means that the response of a small specimen of material cut from B_r is independent of its orientation in B_r.

3.4.2. Noll's rule

In general, the symmetry group changes with a change in reference configuration. Let $\mathbf{P}$ be the deformation gradient $\mathrm{B}_r \to \mathrm{B}'_r$. If $\mathcal{G}$ is the symmetry group of the material relative to B_r and $\mathcal{G}'$ that relative to B'_r then

$$\mathcal{G}' = \mathbf{P}\mathcal{G}\mathbf{P}^{-1}. \tag{3.16}$$

To show this let $\mathbf{g}$ and $\mathbf{g}'$ be the response functions relative to B_r and B'_r respectively. Then

$$\begin{aligned} \mathbf{g}'(\mathbf{F}') &= \mathbf{g}(\mathbf{F}'\mathbf{P}) = \mathbf{g}(\mathbf{F}'\mathbf{P}\mathbf{H}) \qquad (\text{since } \mathbf{H} \in \mathcal{G}) \\ &= \mathbf{g}(\mathbf{F}'\mathbf{P}\mathbf{H}\mathbf{P}^{-1}\mathbf{P}) = \mathbf{g}'(\mathbf{F}'\mathbf{P}\mathbf{H}\mathbf{P}^{-1}), \end{aligned}$$

and hence $\mathbf{PHP}^{-1} \in \mathcal{G}'$, i.e. $\mathbf{H} \in \mathcal{G}$ if and only if $\mathbf{PHP}^{-1} \in \mathcal{G}'$. Equation (3.16) is known as Noll's rule.

If $\mathbf{P}$ is a rotation and $\mathcal{G}$ corresponds to isotropy, then $\mathcal{G}' = \mathcal{G}$. We now focus on isotropic materials, for which purpose we require some further results from tensor algebra.

3.5. Isotropic functions of a second-order tensor

Definition. The scalar function $\phi(\mathbf{T})$ of a symmetric second-order tensor $\mathbf{T}$ is said to be an *isotropic function* of $\mathbf{T}$ if

$$\phi(\mathbf{Q}\mathbf{T}\mathbf{Q}^T) = \phi(\mathbf{T}) \tag{3.17}$$

for all orthogonal tensors $\mathbf{Q}$. We remark that the notion of an isotropic *function* is different from that of an isotropic *tensor*.

An isotropic scalar-valued function of $\mathbf{T}$ is also called a *scalar invariant* of $\mathbf{T}$. We may check that the principal invariants I_1, I_2, I_3 of $\mathbf{T}$ are scalar invariants in accordance with (3.17), as follows. For example, with $I_1 = \mathrm{tr}\,(\mathbf{T})$ we see that, since $\mathbf{Q}$ is orthogonal,

$$\mathrm{tr}\,(\mathbf{Q}\mathbf{T}\mathbf{Q}^T) = \mathrm{tr}\,(\mathbf{Q}^T\mathbf{Q}\mathbf{T}) = \mathrm{tr}\,(\mathbf{T}),$$

while, for $I_3 = \det(\mathbf{T})$,

$$\det(\mathbf{Q}\mathbf{T}\mathbf{Q}^T) = (\det\mathbf{Q})(\det\mathbf{T})(\det\mathbf{Q}^T) = \det\mathbf{T},$$

and similarly for I_2.

Theorem 3.1. $\phi(\mathbf{T})$ is a scalar invariant if and only if it is expressible as a function of I_1, I_2, I_3.

Sketch proof

It is sufficient to consider $\mathbf{T}$ written in spectral form

$$\mathbf{T} = \sum_{i=1}^{3} t_i \mathbf{e}'_i \otimes \mathbf{e}'_i,$$

where t_i are the eigenvalues of $\mathbf{T}$. Since $\phi(\mathbf{QTQ}^T) = \phi(\mathbf{T})$ for arbitrary orthogonal $\mathbf{Q}$ we conclude that ϕ depends on $\mathbf{T}$ only through its eigenvalues, and we may write

$$\phi(\mathbf{T}) \equiv \phi(t_1, t_2, t_3)$$

with

$$\phi(t_1, t_2, t_3) = \phi(t_1, t_3, t_2) = \phi(t_2, t_1, t_3).$$

But t_1, t_2, t_3 are the (real) roots of the cubic

$$t^3 - I_1 t^2 + I_2 t - I_3 = 0$$

and are therefore functions of the coefficients I_1, I_2, I_3. Hence ϕ depends only on I_1, I_2, I_3 or, equivalently, it depends symmetrically on t_1, t_2, t_3. We refer to [18] for a more detailed proof of this and the following theorems.

Let $\mathbf{G}(\mathbf{T})$ be a symmetric second-order tensor function of $\mathbf{T}$.

Definition. $\mathbf{G}(\mathbf{T})$ is said to be an *isotropic tensor function* of $\mathbf{T}$ if

$$\mathbf{G}(\mathbf{QTQ}^T) = \mathbf{QG}(\mathbf{T})\mathbf{Q}^T \tag{3.18}$$

for all orthogonal $\mathbf{Q}$.

A specific example of a function satisfying (3.18) is as follows. Let $\phi_0, \phi_1, \ldots, \phi_N$ be scalar invariants of $\mathbf{T}$. Then

$$\mathbf{G}(\mathbf{T}) = \phi_0 \mathbf{I} + \phi_1 \mathbf{T} + \cdots + \phi_N \mathbf{T}^N$$

is an isotropic function of $\mathbf{T}$.

Theorem 3.2. If $\mathbf{G}(\mathbf{T})$ is isotropic then its eigenvalues are scalar invariants of $\mathbf{T}$.

Sketch proof

Let $\alpha(\mathbf{T})$ be a principal value of $\mathbf{G}(\mathbf{T})$, so that

$$\det[\mathbf{G}(\mathbf{T}) - \alpha(\mathbf{T})\mathbf{I}] = 0.$$

Similarly, $\alpha(\mathbf{QTQ}^T)$ is the corresponding principal value of $\mathbf{G}(\mathbf{QTQ}^T)$, so that

$$\det[\mathbf{G}(\mathbf{QTQ}^T) - \alpha(\mathbf{QTQ}^T)\mathbf{I}] = 0.$$

Using (3.18) we may re-write this as

$$\det[\mathbf{QG}(\mathbf{T})\mathbf{Q}^T - \alpha(\mathbf{QTQ}^T)\mathbf{I}] = 0,$$

and hence, noting that this may also be written

$$(\det \mathbf{Q}) \det[\mathbf{G}(\mathbf{T}) - \alpha(\mathbf{QTQ}^T)\mathbf{I}](\det \mathbf{Q}^T) = 0,$$

we deduce that

$$\det[\mathbf{G}(\mathbf{T}) - \alpha(\mathbf{QTQ}^T)\mathbf{I}] = 0.$$

Thus, $\alpha(\mathbf{QTQ}^T)$ is a principal value of $\mathbf{G}(\mathbf{T})$ *for all orthogonal* $\mathbf{Q}$ and hence

$$\alpha(\mathbf{QTQ}^T) = \alpha(\mathbf{T})$$

for all orthogonal $\mathbf{Q}$, i.e. the principal values are scalar invariants of $\mathbf{T}$.

Theorem 3.3. Every eigenvector of $\mathbf{T}$ is an eigenvector of the *isotropic function* $\mathbf{G}(\mathbf{T})$.

Proof

Let t_1, t_2, t_3 be eigenvalues of $\mathbf{T}$ corresponding to (orthonormal) eigenvectors $\mathbf{m}_1$, $\mathbf{m}_2$, $\mathbf{m}_3$. Then

$$\mathbf{T} = t_1\mathbf{m}_1 \otimes \mathbf{m}_1 + t_2\mathbf{m}_2 \otimes \mathbf{m}_2 + t_3\mathbf{m}_3 \otimes \mathbf{m}_3.$$

Suppose that

$$\mathbf{G}(\mathbf{T})\mathbf{m}_1 = \alpha\mathbf{m}_1 + \beta\mathbf{m}_2 + \gamma\mathbf{m}_3.$$

Let $\mathbf{Q}$ be a rotation about $\mathbf{m}_1$ through π, so that

$$\mathbf{Qm}_1 = \mathbf{m}_1, \quad \mathbf{Qm}_2 = -\mathbf{m}_2, \quad \mathbf{Qm}_3 = -\mathbf{m}_3.$$

Then, it follows that

$$\mathbf{Q}\mathbf{T}\mathbf{Q}^T = \mathbf{T}$$

and

$$\begin{aligned}\mathbf{G}(\mathbf{T})\mathbf{m}_1 &= \mathbf{G}(\mathbf{Q}\mathbf{T}\mathbf{Q}^T)\mathbf{m}_1 = \mathbf{Q}\mathbf{G}(\mathbf{T})\mathbf{Q}^T\mathbf{m}_1 \qquad \text{(by isotropy)}\\ &= \mathbf{Q}\mathbf{G}(\mathbf{T})\mathbf{m}_1 = \mathbf{Q}(\alpha\mathbf{m}_1 + \beta\mathbf{m}_2 + \gamma\mathbf{m}_3),\end{aligned}$$

i.e.

$$\alpha\mathbf{m}_1 + \beta\mathbf{m}_2 + \gamma\mathbf{m}_3 = \alpha\mathbf{m}_1 - \beta\mathbf{m}_2 - \gamma\mathbf{m}_3,$$

and hence, since $\mathbf{m}_1$, $\mathbf{m}_2$, $\mathbf{m}_3$ are linearly independent, $\beta = \gamma = 0$. Thus, $\mathbf{m}_1$ is an eigenvector of $\mathbf{G}(\mathbf{T})$. Similarly for $\mathbf{m}_2$ and $\mathbf{m}_3$.

Theorem 3.4. A symmetric second-order tensor-valued function $\mathbf{G}(\mathbf{T})$ of the second-order symmetric tensor $\mathbf{T}$ is isotropic if and only if it has the representation

$$\mathbf{G}(\mathbf{T}) = \phi_0\mathbf{I} + \phi_1\mathbf{T} + \phi_2\mathbf{T}^2, \tag{3.19}$$

where ϕ_0, ϕ_1, ϕ_2 are functions of I_1, I_2, I_3, i.e. they are scalar invariants of $\mathbf{T}$.

Sketch proof

If (3.19) holds then $\mathbf{G}(\mathbf{T})$ is clearly isotropic. On the other hand, if $\mathbf{G}(\mathbf{T})$ satisfies (3.18) then we need to show that (3.19) follows.

We know from Theorem 3.3 that $\mathbf{G}(\mathbf{T})$ is coaxial with $\mathbf{T}$, and from Theorem 3.2 that the principal values of $\mathbf{G}(\mathbf{T})$ are invariants of $\mathbf{T}$. Let t_i and g_i be the principal values of $\mathbf{T}$ and $\mathbf{G}(\mathbf{T})$ and suppose that t_1, t_2, t_3 are distinct. Consider the three equations

$$\phi_0 + \phi_1 t_i + \phi_2 t_i^2 = g_i, \qquad i \in \{1, 2, 3\}, \tag{3.20}$$

for the three unknowns ϕ_0, ϕ_1, ϕ_2. The solutions ϕ_0, ϕ_1, ϕ_2 are functions of t_i, g_i $(i = 1, 2, 3)$ which, from Theorems 3.1 and 3.2, are themselves functions only of I_1, I_2, I_3. Thus, ϕ_0, ϕ_1, ϕ_2 are uniquely defined by (3.20) as invariants of $\mathbf{T}$. Since $\mathbf{G}(\mathbf{T})$ and $\mathbf{T}$ are coaxial equation (3.20) is just (3.19) referred to principal axes. Hence, by multiplying equation (3.20) by $\mathbf{m}_i \otimes \mathbf{m}_i$ and summing over i, we obtain (3.19). If t_1, t_2, t_3 are not distinct this proof requires modification, but we omit the details.

3.6. Isotropic elasticity

► From the definition (3.15) of isotropy we have

$$\boldsymbol{\sigma} = \mathbf{g}(\mathbf{F}) = \mathbf{g}(\mathbf{FQ}) \tag{3.21}$$

for all proper orthogonal $\mathbf{Q}$ and each deformation gradient $\mathbf{F}$.

The choice $\mathbf{Q} = \mathbf{R}^T$ and use of the polar decomposition $\mathbf{F} = \mathbf{VR}$ in (3.21) gives

$$\boldsymbol{\sigma} = \mathbf{g}(\mathbf{V}). \tag{3.22}$$

Now,

$$\mathbf{Qg}(\mathbf{V})\mathbf{Q}^T = \mathbf{Qg}(\mathbf{F})\mathbf{Q}^T = \mathbf{g}(\mathbf{QF})$$

and, with $\mathbf{F}$ replaced by $\mathbf{QF}$ and $\mathbf{Q}$ by $\mathbf{P}^T$ in (3.21),

$$\mathbf{g}(\mathbf{QF}) = \mathbf{g}(\mathbf{QFP}^T)$$

for all proper orthogonal $\mathbf{P}$ and $\mathbf{Q}$. Hence

$$\mathbf{Qg}(\mathbf{V})\mathbf{Q}^T = \mathbf{g}(\mathbf{QFP}^T).$$

By choosing $\mathbf{P} = \mathbf{QR}$ and writing $\mathbf{F} = \mathbf{VR}$ we then obtain

$$\mathbf{Qg}(\mathbf{V})\mathbf{Q}^T = \mathbf{g}(\mathbf{QVQ}^T) \tag{3.23}$$

for all proper orthogonal $\mathbf{Q}$. In fact, since $\mathbf{Q}$ occurs twice on each side of (3.23), allowing $\mathbf{Q}$ to be improper orthogonal does not affect (3.23), which then states that $\mathbf{g}(\mathbf{V})$ is an isotropic function of $\mathbf{V}$ in accordance with the definition (3.18). Thus, the response function $\mathbf{g}$ has all the properties associated with the isotropic tensor function $\mathbf{G}$ discussed in Section 3.5.

In particular, for an *isotropic elastic material*, $\boldsymbol{\sigma} = \mathbf{g}(\mathbf{V})$ *is coaxial with* $\mathbf{V}$, i.e. with the Eulerian principal axes, and, from Theorem 3.4, we therefore have

$$\boldsymbol{\sigma} = \mathbf{g}(\mathbf{V}) = \phi_0\mathbf{I} + \phi_1\mathbf{V} + \phi_2\mathbf{V}^2, \tag{3.24}$$

where ϕ_0, ϕ_1, ϕ_2 are invariants of $\mathbf{V}$, i.e. functions of

$$I_1 = \lambda_1 + \lambda_2 + \lambda_3, \quad I_2 = \lambda_2\lambda_3 + \lambda_3\lambda_1 + \lambda_1\lambda_2, \quad I_3 = \lambda_1\lambda_2\lambda_3.$$

Alternatively, we may write

$$\boldsymbol{\sigma} = \sum_{i=1}^{3} \sigma_i \mathbf{v}^{(i)} \otimes \mathbf{v}^{(i)},$$

where

$$\sigma_i = \phi_0 + \phi_1\lambda_i + \phi_2\lambda_i^2 \qquad i \in \{1, 2, 3\}.$$

3.7. Hyperelastic materials

Recall, from Section 2.6, that the energy balance equation can be written in the form

$$\int_{R_t} \rho \mathbf{b} \cdot \mathbf{v} dv + \int_{\partial R_t} \mathbf{t} \cdot \mathbf{v} da = \frac{d}{dt} \int_{R_t} \frac{1}{2} \rho \mathbf{v} \cdot \mathbf{v} dv + \int_{R_t} \operatorname{tr}(\boldsymbol{\sigma} \mathbf{L}) dv. \qquad (3.25)$$

If there is no dissipation then the work done by the body and surface forces is converted into kinetic energy and stored elastic energy. In this connection an interpretation for the second term on the right-hand side of (3.25) is needed.

Write

$$\int_{R_t} \operatorname{tr}(\boldsymbol{\sigma} \mathbf{L}) dv = \int_{R_r} J \operatorname{tr}(\boldsymbol{\sigma} \mathbf{L}) dV.$$

Then, the integrand $J \operatorname{tr}(\boldsymbol{\sigma} \mathbf{L})$ is interpreted as the rate of increase of elastic energy per unit volume in B_r.

This prompts the introduction of the *elastic stored energy* $W(\mathbf{F})$ per unit volume in B_r such that

$$\frac{\partial}{\partial t} W(\mathbf{F}) = J \operatorname{tr}(\boldsymbol{\sigma} \mathbf{L}). \qquad (3.26)$$

Note that $W(\mathbf{F})$ is also referred to as the *strain energy* or *potential energy* (per unit volume in B_r). Then, we have

$$\int_{R_t} \operatorname{tr}(\boldsymbol{\sigma} \mathbf{L}) dv = \int_{R_r} \frac{\partial}{\partial t} W(\mathbf{F}) dV = \frac{d}{dt} \int_{R_r} W(\mathbf{F}) dV,$$

and

$$\int_{R_r} W(\mathbf{F}) dV$$

is the total elastic strain energy in the region R_r. The right-hand side of (3.25) can now be written as

$$\frac{d}{dt}(\text{kinetic energy} + \text{strain energy}).$$

Since W depends only on $\mathbf{F}$, we have

$$\frac{\partial}{\partial t} W(\mathbf{F}) = \frac{\partial W}{\partial F_{ij}} \frac{\partial F_{ij}}{\partial t} \equiv \operatorname{tr}\left(\frac{\partial W}{\partial \mathbf{F}} \dot{\mathbf{F}} \right),$$

where $\partial W / \partial \mathbf{F}$ is the second-order tensor with components defined by the convention

$$\left(\frac{\partial W}{\partial \mathbf{F}} \right)_{ji} = \frac{\partial W}{\partial F_{ij}}.$$

Since $\dot{\mathbf{F}} = \mathbf{LF}$, we obtain

$$\frac{\partial W}{\partial t} = \operatorname{tr}\left(\frac{\partial W}{\partial \mathbf{F}}\mathbf{LF}\right) = \operatorname{tr}\left(\mathbf{F}\frac{\partial W}{\partial \mathbf{F}}\mathbf{L}\right)$$

and comparison of this with (3.26) shows that

$$J\boldsymbol{\sigma} = \mathbf{F}\frac{\partial W}{\partial \mathbf{F}}, \tag{3.27}$$

which provides a formula for $\boldsymbol{\sigma}$ in terms of $W(\mathbf{F})$.

Since $\boldsymbol{\sigma} = \mathbf{g}(\mathbf{F})$, we deduce that

$$\mathbf{g}(\mathbf{F}) = (\det \mathbf{F})^{-1}\mathbf{F}\frac{\partial W}{\partial \mathbf{F}}, \tag{3.28}$$

and, by recalling the connection (2.28) between the Cauchy stress $\boldsymbol{\sigma}$ and the nominal stress $\mathbf{S}$, we obtain the simple formula

$$\mathbf{S} = \frac{\partial W}{\partial \mathbf{F}}, \qquad S_{ji} = \frac{\partial W}{\partial F_{ij}} \tag{3.29}$$

for the nominal stress.

We now write

$$\mathbf{S} = J\mathbf{F}^{-1}\boldsymbol{\sigma} = (\det \mathbf{F})\mathbf{F}^{-1}\mathbf{g}(\mathbf{F}) \equiv \mathbf{h}(\mathbf{F}), \tag{3.30}$$

which defines $\mathbf{h}$, the response function associated with $\mathbf{S}$ (relative to B_r).

It is easy to show that objectivity implies that

$$\mathbf{h}(\mathbf{QF}) = \mathbf{h}(\mathbf{F})\mathbf{Q}^T$$

for all proper orthogonal $\mathbf{Q}$, and that, in addition, material isotropy implies that

$$\mathbf{h}(\mathbf{FQ}^T) = \mathbf{Qh}(\mathbf{F})$$

for all orthogonal $\mathbf{Q}$. From the polar decomposition theorem it may then be deduced that for an isotropic material

$$\mathbf{h}(\mathbf{F}) = \mathbf{R}^T\mathbf{h}(\mathbf{V}) = \mathbf{h}(\mathbf{U})\mathbf{R}^T$$

and that $\mathbf{h}(\mathbf{U})$ is symmetric. We emphasize, however, that if the material is not isotropic then $\mathbf{h}(\mathbf{U})$ is not in general symmetric (although it may be for some particular deformations).

We remark that $W(\mathbf{F})$ represents the work done (per unit volume at $\mathbf{X}$) by the stress in deforming the material from B_r to B_t (i.e. from $\mathbf{I}$ to $\mathbf{F}$) and is independent of the path taken in deformation space.

An elastic material which possesses a strain-energy function W is said to be a *hyperelastic* or *Green elastic* material.

Objectivity of W. Since W is a scalar function objectivity requires that it is unaffected by a superimposed rigid-body rotation after deformation, i.e.

$$W(\mathbf{QF}) = W(\mathbf{F}) \tag{3.31}$$

for all rotations $\mathbf{Q}$ for each deformation gradient $\mathbf{F}$. This may also be expressed by referring to W as being indifferent to observer transformations.

Isotropy of W. For a hyperelastic material which is isotropic relative to B_r, $W(\mathbf{F})$ is unaffected by rotations in B_r (prior to deformation). Thus,

$$W(\mathbf{FP}^T) = W(\mathbf{F}) \tag{3.32}$$

for all rotations $\mathbf{P}$.

Setting $\mathbf{P} = \mathbf{R}, \mathbf{F} = \mathbf{VR}$ in (3.32) gives

$$W(\mathbf{F}) = W(\mathbf{V}).$$

Hence, using (3.31) and (3.32),

$$W(\mathbf{QFP}^T) = W(\mathbf{FP}^T) = W(\mathbf{F}) = W(\mathbf{V}),$$

and setting $\mathbf{P} = \mathbf{QR}$ then yields

$$W(\mathbf{QVQ}^T) = W(\mathbf{V}) \tag{3.33}$$

for all orthogonal $\mathbf{Q}$. Equation (3.33) states that W is an isotropic scalar function of $\mathbf{V}$ in accordance with the definition (3.17).

Thus, we may regard W as a function of the principal invariants I_1, I_2, I_3 of $\mathbf{V}$ or, equivalently, as a symmetric function of the principal stretches $\lambda_1, \lambda_2, \lambda_3$. In particular, we have

$$W(\lambda_1, \lambda_2, \lambda_3) = W(\lambda_1, \lambda_3, \lambda_2) = W(\lambda_3, \lambda_1, \lambda_2) \tag{3.34}$$

for all $\lambda_1, \lambda_2, \lambda_3 \in (0, \infty)$.

Mathematically, there is no restriction so far other than (3.34) on the form that the function W may take, but the predictions of material behaviour based on the form of W must make mathematical sense and must also be compatible with what is observed for real materials.

It is usual to take W to be measured from the reference configuration B_r, so that

$$W(1, 1, 1) = 0. \tag{3.35}$$

Furthermore, if the reference configuration is stress free then we also have the restriction $\mathbf{h}(\mathbf{I}) = \mathbf{O}$ or, in terms of the derivatives of W with respect to the stretches,

$$\frac{\partial W}{\partial \lambda_i}(1,1,1) = 0, \quad i \in \{1,2,3\}. \tag{3.36}$$

There are also basic restrictions required for W to reduce to the classical (quadratic) form of strain energy when the strains are small. These restrictions are discussed in Chapter 10.

In Chapters 4 and 5 we examine stress-deformation relations for isotropic elastic materials and the application of these to the characterization of the elastic properties of rubberlike solids.

Chapter 4

Stress-deformation relations for an isotropic material

4.1. Unconstrained materials

For an isotropic material the strain-energy function is expressible as a function of the principal stretches, as in (3.34). It follows that

$$\dot{W} = \sum_{i=1}^{3} \frac{\partial W}{\partial \lambda_i} \dot{\lambda}_i. \tag{4.1}$$

But, from (3.26),

$$\dot{W} = J \operatorname{tr}(\boldsymbol{\sigma} \mathbf{L}) = J \operatorname{tr}(\boldsymbol{\sigma} \mathbf{D}). \tag{4.2}$$

Also, for an isotropic material, $\boldsymbol{\sigma}$ is coaxial with $\mathbf{V}$ and can be written in the spectral form

$$\boldsymbol{\sigma} = \sum_{i=1}^{3} \sigma_i \mathbf{v}^{(i)} \otimes \mathbf{v}^{(i)}. \tag{4.3}$$

Equation (4.2) can therefore be expressed as

$$\dot{W} = J \sum_{i=1}^{3} \sigma_i D_{ii}, \tag{4.4}$$

where D_{ii} are the normal components of $\mathbf{D}$ referred to the axes $\mathbf{v}^{(i)}$. In order to obtain expressions for the principal stresses σ_i in terms of the derivatives of W with respect to the stretches we must compare (4.1) with (4.4). First, we need an expression for the components D_{ii}.

Note that by using $(1.35)_1$, (1.54) and $(1.59)_1$, $\mathbf{D}$ may be written in the form

$$\mathbf{D} = \frac{1}{2}\mathbf{R}(\dot{\mathbf{U}}\mathbf{U}^{-1} + \mathbf{U}^{-1}\dot{\mathbf{U}})\mathbf{R}^T,$$

and that $\mathbf{U}$ has the spectral decomposition

$$\mathbf{U} = \sum_{i=1}^{3} \lambda_i \mathbf{u}^{(i)} \otimes \mathbf{u}^{(i)},$$

from which it follows that

$$\dot{\mathbf{U}} = \sum_{i=1}^{3} (\dot{\lambda}_i \mathbf{u}^{(i)} \otimes \mathbf{u}^{(i)} + \lambda_i \mathbf{u}^{(i)} \otimes \dot{\mathbf{u}}^{(i)} + \lambda_i \dot{\mathbf{u}}^{(i)} \otimes \mathbf{u}^{(i)}).$$

Using the connection $\mathbf{v}^{(i)} = \mathbf{R}\mathbf{u}^{(i)}$ we calculate the components

$$\begin{aligned} D_{ii} \equiv \mathbf{v}^{(i)} \cdot (\mathbf{D}\mathbf{v}^{(i)}) &= \frac{1}{2}\mathbf{u}^{(i)} \cdot [(\dot{\mathbf{U}}\mathbf{U}^{-1} + \mathbf{U}^{-1}\dot{\mathbf{U}})\mathbf{u}^{(i)}] \\ &= \mathbf{u}^{(i)} \cdot (\dot{\mathbf{U}}\mathbf{U}^{-1}\mathbf{u}^{(i)}) = \mathbf{u}^{(i)} \cdot (\dot{\mathbf{U}}\lambda_i^{-1}\mathbf{u}^{(i)}) \\ &= \lambda_i^{-1}[\mathbf{u}^{(i)} \cdot (\dot{\mathbf{U}}\mathbf{u}^{(i)})] = \lambda_i^{-1}\dot{\lambda}_i, \end{aligned}$$

in which we have used symmetry and the fact that, since $\mathbf{u}^{(i)}$ is a unit vector, $\mathbf{u}^{(i)} \cdot \dot{\mathbf{u}}^{(i)} = 0$.

Comparison of (4.1) and (4.4) now gives

$$\sum_{i=1}^{3} \frac{\partial W}{\partial \lambda_i}\dot{\lambda}_i = \sum_{i=1}^{3} J\sigma_i \lambda_i^{-1}\dot{\lambda}_i,$$

and hence

$$J\lambda_i^{-1}\sigma_i = \frac{\partial W}{\partial \lambda_i},$$

i.e.

$$\sigma_i = J^{-1}\lambda_i \frac{\partial W}{\partial \lambda_i}, \quad i \in \{1, 2, 3\}, \tag{4.5}$$

where

$$J = \lambda_1\lambda_2\lambda_3. \tag{4.6}$$

Expressions for $\mathbf{T}^{(1)}$ and $\mathbf{S}$ analogous to (4.3) can also be obtained. First we note that since $\mathbf{S} = J\mathbf{F}^{-1}\boldsymbol{\sigma}$, $\mathbf{F}^{-1} = \mathbf{U}^{-1}\mathbf{R}^T$, $\mathbf{R}^T\mathbf{v}^{(i)} = \mathbf{u}^{(i)}$ and $\mathbf{U}^{-1}\mathbf{u}^{(i)} = \lambda_i^{-1}\mathbf{u}^{(i)}$ we may write

$$\mathbf{S} = \sum_{i=1}^{3} t_i \mathbf{u}^{(i)} \otimes \mathbf{v}^{(i)}, \tag{4.7}$$

where

$$t_i = J\lambda_i^{-1}\sigma_i = \frac{\partial W}{\partial \lambda_i}. \tag{4.8}$$

Furthermore, from Section 3.7 we see that, for an isotropic material,

$$\mathbf{S} = \mathbf{h}(\mathbf{F}) = \mathbf{h}(\mathbf{U})\mathbf{R}^T \equiv \mathbf{T}^{(1)}\mathbf{R}^T,$$

where we recall from Section 2.7 that $\mathbf{T}^{(1)}$ is the Biot stress tensor. Hence, using (4.7) and (4.8),

$$\mathbf{T}^{(1)} = \sum_{i=1}^{3} t_i \mathbf{u}^{(i)} \otimes \mathbf{u}^{(i)}, \tag{4.9}$$

and t_i are just the principal values of $\mathbf{T}^{(1)}$, i.e. the principal Biot stresses. If W is regarded as a function of $\mathbf{U}$ then we may also write

$$\mathbf{T}^{(1)} = \frac{\partial W}{\partial \mathbf{U}}. \tag{4.10}$$

More generally, for the conjugate stress and strain tensors $\mathbf{T}^{(m)}$ and $\mathbf{E}^{(m)}$, we note that

$$\mathbf{T}^{(m)} = \frac{\partial W}{\partial \mathbf{E}^{(m)}}. \tag{4.11}$$

4.2. Stress-deformation relations in terms of invariants

4.2.1. The invariants I_1, I_2, I_3

Instead of using the stretches $\lambda_1, \lambda_2, \lambda_3$ as independent measures of deformation, we now use (equivalently) the invariants I_1, I_2, I_3 defined by

$$I_1 = \operatorname{tr}(\mathbf{B}) \equiv \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \tag{4.12}$$

$$I_2 = \frac{1}{2}[I_1^2 - \operatorname{tr}(\mathbf{B}^2)] \equiv \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2 + \lambda_1^2\lambda_2^2, \tag{4.13}$$

$$I_3 = \det \mathbf{B} \equiv \lambda_1^2\lambda_2^2\lambda_3^2 \equiv J^2, \tag{4.14}$$

and we note that these are symmetric functions of the stretches. We regard the strain energy as a function of I_1, I_2, I_3 and write $\bar{W}(I_1, I_2, I_3)$ to represent this.

In order to obtain an expression for the nominal stress $\mathbf{S}$ we need the derivatives

$$\frac{\partial I_1}{\partial \mathbf{F}} = 2\mathbf{F}^T, \quad \frac{\partial I_2}{\partial \mathbf{F}} = 2I_1\mathbf{F}^T - 2\mathbf{F}^T\mathbf{F}\mathbf{F}^T, \quad \frac{\partial I_3}{\partial \mathbf{F}} = 2I_3\mathbf{F}^{-1}, \tag{4.15}$$

and hence

$$\mathbf{S} = \frac{\partial \bar{W}}{\partial \mathbf{F}} = 2\bar{W}_1 \mathbf{F}^T + 2\bar{W}_2(I_1 \mathbf{F}^T - \mathbf{F}^T \mathbf{F} \mathbf{F}^T) + 2I_3 \bar{W}_3 \mathbf{F}^{-1}, \tag{4.16}$$

where

$$\bar{W}_1 = \frac{\partial \bar{W}}{\partial I_1}, \quad \bar{W}_2 = \frac{\partial \bar{W}}{\partial I_2}, \quad \bar{W}_3 = \frac{\partial \bar{W}}{\partial I_3}. \tag{4.17}$$

The corresponding expression for the Cauchy stress is

$$\boldsymbol{\sigma} = 2I_3^{-1/2}(\bar{W}_1 + I_1 \bar{W}_2)\mathbf{B} - 2I_3^{-1/2}\bar{W}_2 \mathbf{B}^2 + 2I_3^{1/2}\bar{W}_3 \mathbf{I}. \tag{4.18}$$

4.2.2. The invariants i_1, i_2, i_3

An alternative representation for the energy function and the stresses can be obtained on the basis of the principal invariants i_1, i_2, i_3 of the stretch tensor $\mathbf{V}$. These are defined by

$$i_1 = \operatorname{tr}(\mathbf{V}) \equiv \lambda_1 + \lambda_2 + \lambda_3, \tag{4.19}$$

$$i_2 = \frac{1}{2}[i_1^2 - \operatorname{tr}(\mathbf{V}^2)] \equiv \lambda_2\lambda_3 + \lambda_3\lambda_1 + \lambda_1\lambda_2, \tag{4.20}$$

$$i_3 = \det \mathbf{V} \equiv \lambda_1\lambda_2\lambda_3 \equiv J. \tag{4.21}$$

In this case we write the strain energy as $\tilde{W}(i_1, i_2, i_3)$ and $\mathbf{S}$ may be written

$$\mathbf{S} = \frac{\partial \tilde{W}}{\partial \mathbf{F}} = \tilde{W}_1 \frac{\partial i_1}{\partial \mathbf{F}} + \tilde{W}_2 \frac{\partial i_2}{\partial \mathbf{F}} + \tilde{W}_3 \frac{\partial i_3}{\partial \mathbf{F}}, \tag{4.22}$$

where

$$\tilde{W}_1 = \frac{\partial \tilde{W}}{\partial i_1}, \quad \tilde{W}_2 = \frac{\partial \tilde{W}}{\partial i_2}, \quad \tilde{W}_3 = \frac{\partial \tilde{W}}{\partial i_3}. \tag{4.23}$$

However, expressions for $\partial i_1/\partial \mathbf{F}$ and $\partial i_2/\partial \mathbf{F}$ are not immediately forthcoming in this case. They may be obtained by making use of the connections

$$I_1 = i_1^2 - 2i_2, \quad I_2 = i_2^2 - 2i_1 i_3, \quad I_3 = i_3^2. \tag{4.24}$$

However, this requires a lengthy set of manipulations, which are not given here. We obtain the results indirectly by first calculating $\mathbf{T}^{(1)}$ and then using the connection $\mathbf{S} = \mathbf{T}^{(1)}\mathbf{R}^T$ and comparing the result with (4.22).

Using (4.8) we obtain

$$t_i = \tilde{W}_1 + (i_1 - \lambda_i)\tilde{W}_2 + i_3 \lambda_i^{-1} \tilde{W}_3.$$

Hence,

$$\mathbf{T}^{(1)} = \sum_{i=1}^{3} t_i \mathbf{u}^{(i)} \otimes \mathbf{u}^{(i)} = \tilde{W}_1 \mathbf{I} + \tilde{W}_2 (i_1 \mathbf{I} - \mathbf{U}) + i_3 \tilde{W}_3 \mathbf{U}^{-1},$$

and then

$$\mathbf{S} = \mathbf{T}^{(1)} \mathbf{R}^T = \tilde{W}_1 \mathbf{R}^T + \tilde{W}_2 (i_1 \mathbf{R}^T - \mathbf{F}^T) + i_3 \tilde{W}_3 \mathbf{F}^{-1}. \tag{4.25}$$

Comparison of (4.25) with (4.22) shows that

$$\frac{\partial i_1}{\partial \mathbf{F}} = \mathbf{R}^T, \quad \frac{\partial i_2}{\partial \mathbf{F}} = i_1 \mathbf{R}^T - \mathbf{F}^T, \quad \frac{\partial i_3}{\partial \mathbf{F}} = i_3 \mathbf{F}^{-1}. \tag{4.26}$$

In terms of i_1, i_2, i_3 the Cauchy stress tensor has the representation

$$\boldsymbol{\sigma} = i_3^{-1} (\tilde{W}_1 + i_1 \tilde{W}_2) \mathbf{V} - i_3^{-1} \tilde{W}_2 \mathbf{V}^2 + \tilde{W}_3 \mathbf{I}. \tag{4.27}$$

This may be compared directly with (3.24) to provide expressions for the coefficients ϕ_0, ϕ_1, ϕ_2 which appear in (3.24). For a discussion of these invariants and a derivation of the above formulas we refer to Steigmann [23].

Chapter 5

Constrained elastic materials

5.1. Incompressibility

If the considered material is *incompressible* then the deformation gradient must satisfy the *internal constraint*

$$J \equiv \det \mathbf{F} \equiv \det \mathbf{U} \equiv \lambda_1 \lambda_2 \lambda_3 = 1 \tag{5.1}$$

at each point of the material. It follows that

$$\log \lambda_1 + \log \lambda_2 + \log \lambda_3 = 0$$

and hence

$$\operatorname{div} \mathbf{v} \equiv \operatorname{tr}(\mathbf{D}) \equiv \frac{\dot{\lambda}_1}{\lambda_1} + \frac{\dot{\lambda}_2}{\lambda_2} + \frac{\dot{\lambda}_3}{\lambda_3} = 0. \tag{5.2}$$

Because of (5.1) the derivatives $\partial W / \partial \lambda_i$ are not now independent, and the equation

$$J \sigma_i = \lambda_i \frac{\partial W}{\partial \lambda_i}, \quad i \in \{1, 2, 3\},$$

is replaced by

$$\sigma_i = \lambda_i \frac{\partial W}{\partial \lambda_i} - p, \quad i \in \{1, 2, 3\}, \tag{5.3}$$

where p is an arbitrary scalar.

Justification for this is provided by noting that the rate of working of the stresses, namely

$$\begin{aligned}\operatorname{tr}(\boldsymbol{\sigma}\mathbf{D}) &\equiv \sum_{i=1}^{3}\sigma_i\lambda_i^{-1}\dot{\lambda}_i = \sum_{i=1}^{3}\frac{\partial W}{\partial\lambda_i}\dot{\lambda}_i - p\Big(\frac{\dot{\lambda}_1}{\lambda_1}+\frac{\dot{\lambda}_2}{\lambda_2}+\frac{\dot{\lambda}_3}{\lambda_3}\Big)\\ &= \sum_{i=1}^{3}\frac{\partial W}{\partial\lambda_i}\dot{\lambda}_i = \dot{W}\end{aligned}$$

is not affected by p. The scalar p is a Lagrange multiplier in respect of the constraint (5.1), so we replace W by $W-p(\lambda_1\lambda_2\lambda_3-1)$ and then regard this as a function of the independent variables $\lambda_1,\lambda_2,\lambda_3,p$.

Thus,

$$J\sigma_i = \lambda_i\frac{\partial W}{\partial\lambda_i}$$

becomes

$$\sigma_i = \lambda_i\frac{\partial}{\partial\lambda_i}[W-p(\lambda_1\lambda_2\lambda_3-1)] = \lambda_i\frac{\partial W}{\partial\lambda_i}-p,$$

with $\lambda_1\lambda_2\lambda_3$ having been set equal to 1 on the right-hand side after the differentiation has been carried out.

More generally, for a material which is not necessarily isotropic, consider the strain energy $W(\mathbf{F})$ modified to

$$W(\mathbf{F})-p(\det\mathbf{F}-1)$$

to accommodate the constraint $\det\mathbf{F}=1$. Then the nominal stress tensor defined by (3.29) for a compressible material is modified to

$$\mathbf{S} = \frac{\partial W}{\partial\mathbf{F}} - p\mathbf{F}^{-1}, \tag{5.4}$$

and, from (3.30) with $J=1$, the Cauchy stress $\boldsymbol{\sigma}$ is given by

$$\boldsymbol{\sigma} = \mathbf{F}\frac{\partial W}{\partial\mathbf{F}} - p\mathbf{I}. \tag{5.5}$$

This shows that p may be interpreted as a hydrostatic pressure.

The corresponding expression for the Biot stress tensor, with $\det\mathbf{U}=1$, is

$$\mathbf{T}^{(1)} = \frac{\partial W}{\partial\mathbf{U}} - p\mathbf{U}^{-1}. \tag{5.6}$$

5.2. Stress-deformation relations

5.2.1. Invariants I_1, I_2

For an incompressible material $I_3 \equiv 1$. Thus, for an incompressible isotropic material the dependence of the strain energy on the invariants discussed in Section 4.1.1 now reduces to a representation in terms of the two independent invariants I_1 and I_2 alone, and we write $\bar{W}(I_1, I_2)$. It follows from (5.5), on use of (4.15), that

$$\boldsymbol{\sigma} = 2(\bar{W}_1 + I_1\bar{W}_2)\mathbf{B} - 2\bar{W}_2\mathbf{B}^2 - p\mathbf{I}. \tag{5.7}$$

5.2.2. Invariants i_1, i_2

The counterpart of (5.7) in respect of $\tilde{W}(i_1, i_2)$ (with $i_3 \equiv 1$) is

$$\boldsymbol{\sigma} = (\tilde{W}_1 + i_1\tilde{W}_2)\mathbf{V} - \tilde{W}_2\mathbf{V}^2 - p\mathbf{I}, \tag{5.8}$$

although p in (5.8) differs from that in (5.7).

5.3. Other constraints

Any single internal constraint on the deformation can be written in the form

$$C(\mathbf{F}) = 0 \tag{5.9}$$

for all deformation gradients $\mathbf{F}$, where C (for constraint) is a scalar function. Since a constraint (such as incompressibility) is unaffected by a superposed rigid motion, C must be an objective scalar function, so that

$$C(\mathbf{QF}) = C(\mathbf{F}) \tag{5.10}$$

for all rotations $\mathbf{Q}$. In particular, the choice $\mathbf{Q} = \mathbf{R}^T$ yields

$$C(\mathbf{F}) = C(\mathbf{U}). \tag{5.11}$$

Note that in general, however, $C(\mathbf{U})$ is *not* a scalar invariant of $\mathbf{U}$.

To accommodate the constraint in the stress-deformation relation we consider

$$W(\mathbf{F}) + qC(\mathbf{F}),$$

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5.4.1. Use of the invariants I_1, I_2

A basic strain-energy function, known as the *neo-Hookean* material, has the form

$$\bar{W} = \frac{1}{2}\mu(I_1 - 3), \tag{5.16}$$

where $\mu\,(> 0)$ is a material constant referred to as the *shear modulus* of the material in the natural configuration. This is a prototype model for rubber elasticity. The associated Cauchy and nominal stresses are given by

$$\boldsymbol{\sigma} = \mu\mathbf{B} - p\mathbf{I}, \quad \mathbf{S} = \mu\mathbf{F}^T - p\mathbf{F}^{-1}, \tag{5.17}$$

respectively.

Another such model is the *Mooney-Rivlin* material, defined by

$$\bar{W} = \frac{1}{2}\mu_1(I_1 - 3) - \frac{1}{2}\mu_2(I_2 - 3), \tag{5.18}$$

where $\mu_1\,(\geq 0)$ and $\mu_2\,(\leq 0)$ are constants such that $\mu_1 - \mu_2 = \mu\,(> 0)$. The Cauchy stress can be calculated from (5.7).

5.4.2. Use of the invariants i_1, i_2

The *Varga* material has the form

$$\tilde{W} = 2\mu(i_1 - 3), \tag{5.19}$$

while, analogously to (5.18), we could also consider

$$\tilde{W} = \mu_1(i_1 - 3) - \mu_2(i_2 - 3), \tag{5.20}$$

where again $\mu_1\,(\geq 0)$ and $\mu_2\,(\leq 0)$ are constants (not the same as in (5.18)), this time satisfying $\mu_1 - \mu_2 = 2\mu\,(> 0)$. These two strain-energy functions are useful in circumstances when the strains are of moderate magnitude. In respect of (5.20) the Cauchy stress may be obtained from (5.8).

5.4.3. Use of the stretches

An example of a strain-energy function for incompressible materials is that given by

$$W = \sum_{n=1}^{N} \frac{\mu_n}{\alpha_n}(\lambda_1^{\alpha_n} + \lambda_2^{\alpha_n} + \lambda_3^{\alpha_n} - 3), \quad \lambda_1\lambda_2\lambda_3 = 1, \tag{5.21}$$

where N is a positive integer and μ_n and α_n are material constants such that

$$\mu_n\alpha_n > 0, \quad n = 1, 2, \dots, N, \quad \sum_{n=1}^{N} \mu_n\alpha_n = 2\mu. \tag{5.22}$$

► From (5.3) the principal Cauchy stresses are calculated as

$$\sigma_i = \sum_{n=1}^{N} \mu_n \lambda_i^{\alpha_n} - p, \quad i \in \{1, 2, 3\}. \tag{5.23}$$

Note that since, on use of the incompressibility condition, I_2 and i_2 may be written as

$$I_2 = \lambda_1^{-2} + \lambda_2^{-2} + \lambda_3^{-2}, \quad i_2 = \lambda_1^{-1} + \lambda_2^{-1} + \lambda_3^{-1},$$

the energy function (5.21) includes (5.16) and (5.18)–(5.20) as special cases.

For more details of strain-energy functions in terms of the stretches we refer to Ogden [15, 16], for example.

5.5. Application to homogeneous deformations

We recall that for a homogeneous deformation the deformation gradient $\mathbf{F}$ is constant, i.e. independent of position $\mathbf{X}$.

A *pure homogeneous strain* is a deformation of the form

$$x_1 = \lambda_1 X_1, \quad x_2 = \lambda_2 X_2, \quad x_3 = \lambda_3 X_3, \tag{5.24}$$

where the principal stretches $\lambda_1, \lambda_2, \lambda_3$ are constants and (X_1, X_2, X_3) and (x_1, x_2, x_3) are Cartesian coordinates. For this deformation $\mathbf{F} = \mathbf{U} = \mathbf{V}$, $\mathbf{R} = \mathbf{I}$ and the principal axes of the deformation coincide with the Cartesian coordinate directions, i.e. they do not change their orientation as the values of the stretches change. For an unconstrained isotropic elastic material the associated principal Biot stresses are given by (4.8). These equations serve as a basis for determining the form of W from *triaxial* experimental tests in which $\lambda_1, \lambda_2, \lambda_3$ and t_1, t_2, t_3 are measured. If *biaxial* tests are conducted on a thin sheet of material which lies in the (X_1, X_2)-plane with no force applied to the faces of the sheet (plane stress) then, when written in full, equations (4.8) are

$$t_1 = \frac{\partial W}{\partial \lambda_1}(\lambda_1, \lambda_2, \lambda_3), \quad t_2 = \frac{\partial W}{\partial \lambda_2}(\lambda_1, \lambda_2, \lambda_3), \quad t_3 = \frac{\partial W}{\partial \lambda_3}(\lambda_1, \lambda_2, \lambda_3) = 0, \tag{5.25}$$

and the third equation gives λ_3 implicitly in terms of λ_1 and λ_2 when W is known. In this situation the stretches λ_1 and λ_2 can be varied independently, but such a test is not sufficient to enable a complete characterization of W to be achieved since λ_3 is not varied independently of λ_1 and λ_2. The situation is different for an incompressible material and we now focus on materials subject to the constraint

$$\lambda_1\lambda_2\lambda_3 = 1. \tag{5.26}$$

In this case the biaxial test is important since, in principle, it affords the possibility of determining the stress-stretch characteristics of the material for all valid values of the stretches. The counterpart of (4.8) for the incompressible case is given by

$$t_i = \frac{\partial W}{\partial \lambda_i} - p\lambda_i^{-1}, \tag{5.27}$$

or, in terms of the principal Cauchy stresses,

$$\sigma_i = \lambda_i \frac{\partial W}{\partial \lambda_i} - p, \tag{5.28}$$

At this point we use (5.26) to express the strain energy as a function of two independent stretches and for this purpose we define

$$\hat{W}(\lambda_1, \lambda_2) = W(\lambda_1, \lambda_2, \lambda_1^{-1}\lambda_2^{-1}). \tag{5.29}$$

This enables p to be eliminated from equations (5.28) and leads to

$$\sigma_1 - \sigma_3 = \lambda_1 \frac{\partial \hat{W}}{\partial \lambda_1}, \quad \sigma_2 - \sigma_3 = \lambda_2 \frac{\partial \hat{W}}{\partial \lambda_2}. \tag{5.30}$$

It is important to observe that, because of the incompressibility constraint, equation (5.30) is unaffected by the superposition of an arbitrary hydrostatic stress. Thus, without loss of generality, we may set $\sigma_3 = 0$ in (5.30). In terms of the principal Biot stresses we then have simply

$$t_1 = \frac{\partial \hat{W}}{\partial \lambda_1}, \quad t_2 = \frac{\partial \hat{W}}{\partial \lambda_2}. \tag{5.31}$$

These equations are important since they provide two equations relating the two independent stretches λ_1, λ_2 to the stresses t_1, t_2 and therefore a basis for characterizing $\hat{W}$ from measured biaxial data.

We note here that in the undeformed stress-free (natural) configuration $\hat{W}(\lambda_1, \lambda_2)$ should satisfy the conditions

$$\hat{W}(1,1) = 0, \quad \hat{W}_1 = \hat{W}_2 = 0, \tag{5.32}$$

$$\hat{W}_{12} = 2\mu, \quad \hat{W}_{11} = \hat{W}_{22} = 4\mu, \tag{5.33}$$

where the subscripts denote differentiation with respect to λ_1 and λ_2 and μ again is the shear modulus.

5.6. Comparison of theory and experiment for rubber

In order to relate the theory to experimental data on rubber it is convenient to write the strain energy (5.21) in the Valanis-Landel separable form

$$W = w(\lambda_1) + w(\lambda_2) + w(\lambda_3), \tag{5.34}$$

where

$$w(\lambda_i) = \sum_{n=1}^{N} (\lambda_i^{\alpha_n} - 1)/\alpha_n. \tag{5.35}$$

► From (5.30) the stress difference $\sigma_1 - \sigma_2$ is then written

$$\sigma_1 - \sigma_2 = \lambda_1 w'(\lambda_1) - \lambda_2 w'(\lambda_2). \tag{5.36}$$

It turns out that this is a very useful representation since, for fixed λ_2, the shape of the curve of $\sigma_1 - \sigma_2$ plotted against λ_1 for certain rubbers is essentially independent of the value of λ_2. This means that the shape of the curve is determined by taking $\lambda_2 = 1$, in which case (5.36) reduces to

$$\sigma_1 - \sigma_2 = \lambda_1 w'(\lambda_1) - w'(1). \tag{5.37}$$

For $\lambda_2 \neq 1$ the corresponding curve is obtained by a vertical shift defined by

$$w'(1) - \lambda_2 w'(\lambda_2), \tag{5.38}$$

which, when added to (5.37), reproduces (5.36). Typical data for a vulcanized natural rubber, taken from biaxial experiments of Jones and Treloar [12], are shown in Fig. 5.1(a)–(d) with $\sigma_1 - \sigma_2$ plotted against λ_1 for four different values of λ_2: (a) 1, (b) 1.502, (c) 1.984, (d) 2.295. The experimental re-

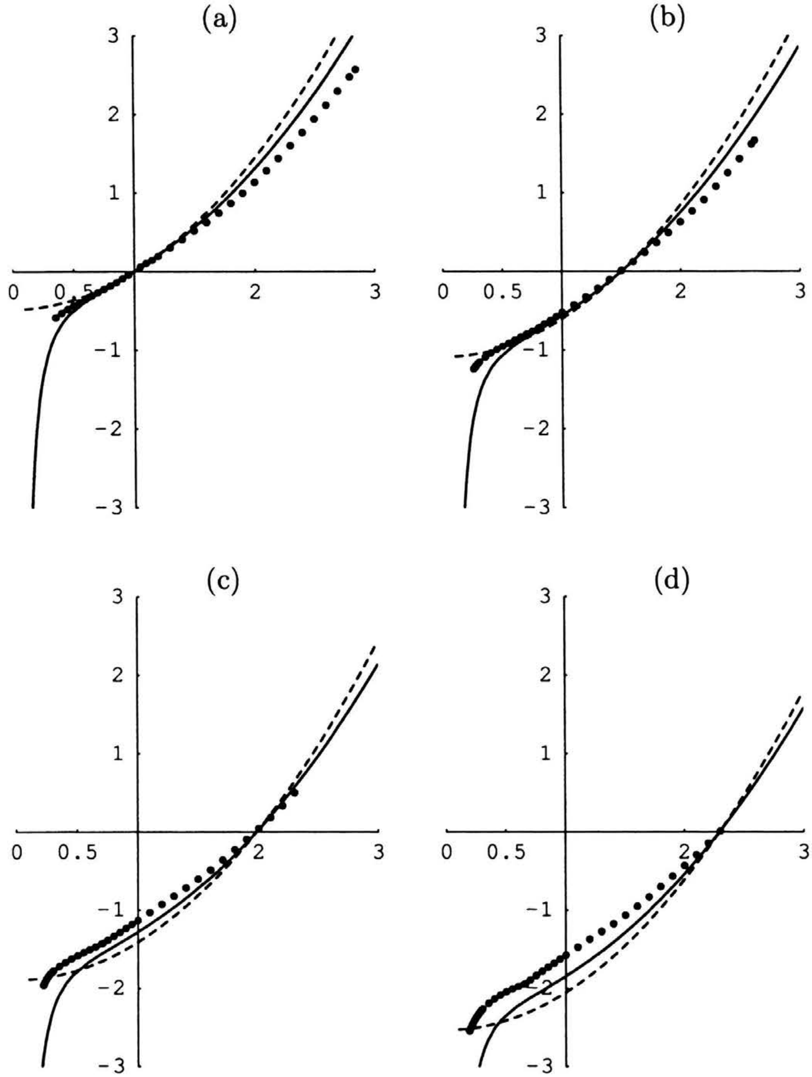

FIGURE 5.1. Plot of $\sigma_1 - \sigma_2$ (vertical axes) against λ_1 for (a) $\lambda_2 = 1$, (b) $\lambda_2 = 1.502$, (c) $\lambda_2 = 1.984$, (d) $\lambda_2 = 2.295$. The data (circles) are compared with the theoretical curves corresponding to the Mooney-Rivlin material (continuous curves) and the neo-Hookean material (dashed curves).

sults (circles) are compared with the predictions of the neo-Hookean material (dashed curves), with $\mu = 0.4807$ Nmm^{-2} and the Mooney-Rivlin material (continuous curves), with $\mu_1 = 0.4206, \mu_2 = 0.0601$ Nmm^{-2}.

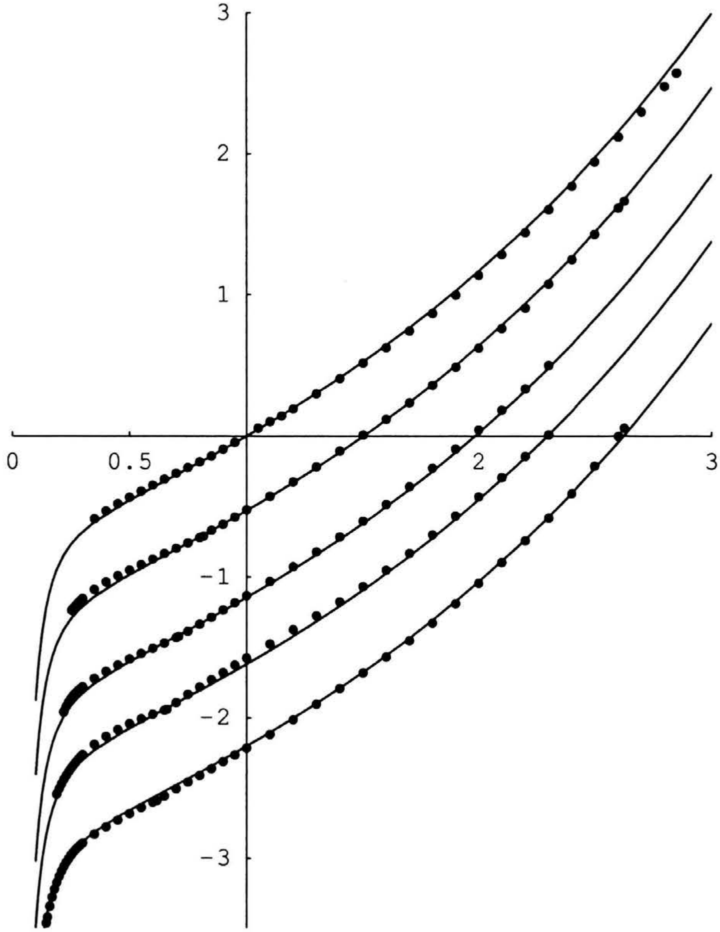

FIGURE 5.2. Plot of $\sigma_1 - \sigma_2$ against λ_1 with the data in Figures 1(a)–(d) superimposed and, additionally, data for $\lambda_2 = 2.623$. The continuous curves are based on the strain-energy function (5.21) with constants given by (5.39).

Figure 5.2 shows the data from Fig. 5.1 for the four values of λ_2 superimposed, together with corresponding data for $\lambda_2 = 2.623$. This plot shows clearly that the shape of the curves is independent of λ_2. The data have been fitted with a strain-energy function of the form (5.21) with $N = 3$ and the following values of the material constants:

$$\begin{aligned} \alpha_1 &= 1.3, \quad \alpha_2 = 4.0, \alpha_3 = -2.0 \\ \mu_1 &= 0.69, \quad \mu_2 = 0.01, \mu_3 = -0.0122\text{Nmm}^{-2}. \end{aligned} \tag{5.39}$$

The theoretical curves are shown as continuous curves in Fig. 5.2.

There are several special cases of the biaxial test which are of interest, but we just give the details for *simple tension*, for which we set $t_2 = 0$. This has the advantage that relatively large values of the stretches can be achieved. By symmetry, the incompressibility constraint yields $\lambda_2 = \lambda_3 = \lambda_1^{-1/2}$. The strain energy may now be treated as a function of just $\lambda = \lambda_1$, and we write

$$W_{st}(\lambda) = \hat{W}(\lambda, \lambda^{-1/2}), \tag{5.40}$$

and (5.31) reduces to

$$t \equiv t_1 = W'_{st}(\lambda), \tag{5.41}$$

where the prime indicates differentiation with respect to λ and the subscript $_{st}$ signifies simple tension.

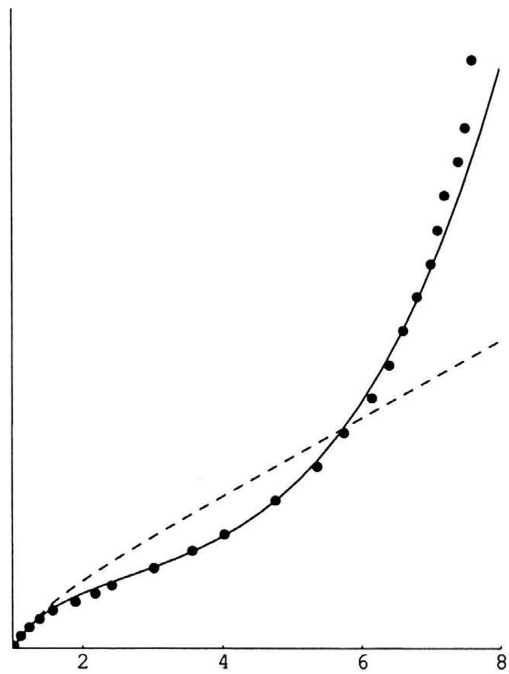

FIGURE 5.3. Simple tension data with the nominal stress t (dimensionless) plotted on the vertical axis against the stretch λ for a vulcanized natural rubber (circles) compared with the predictions of the neo-Hookean material (dashed curve) and a strain-energy function of the form (5.21) with $N = 3$ (continuous curve).

Representative simple tension data are shown in Fig. 5.3 for a vulcanized natural rubber [24]. The data are compared with the theory based on the neo-Hookean material (dashed curves) and a three-term energy function of the form (5.21) (continuous curve).

5.6.1. Simple shear

Experimental tests such as biaxial deformation and simple tension are such that the principal axes of strain do not change as the magnitude of the strain is varied. We now consider the predictions of the theory for a deformation for which the orientation of the principal axes of strain *does* change. This is the simple shear deformation discussed in Section 1.7.2. We recall from Section 1.7.2 that in a simple shear deformation the Eulerian principal axes $\mathbf{v}^{(1)}$ and $\mathbf{v}^{(2)}$ are given by

$$\mathbf{v}^{(1)} = \cos\phi\,\mathbf{e}_1 + \sin\phi\,\mathbf{e}_2, \quad \mathbf{v}^{(2)} = -\sin\phi\,\mathbf{e}_1 + \cos\phi\,\mathbf{e}_2,$$

where

$$\tan 2\phi = \frac{2}{\gamma}, \quad \gamma = \lambda - \lambda^{-1},$$

and the stretches λ, λ^{-1} correspond to $\mathbf{v}^{(1)}, \mathbf{v}^{(2)}$ respectively. Note that it also follows that $\tan\phi = \lambda^{-1}$.

Since the material is isotropic we must have

$$\boldsymbol{\sigma} = \sigma_1 \mathbf{v}^{(1)} \otimes \mathbf{v}^{(1)} + \sigma_2 \mathbf{v}^{(2)} \otimes \mathbf{v}^{(2)} + \sigma_3 \mathbf{v}^{(3)} \otimes \mathbf{v}^{(3)}$$

with $\mathbf{v}^{(3)} = \mathbf{e}_3$, so that the Cartesian components of $\boldsymbol{\sigma}$ are

$$\sigma_{11} = \sigma_1 \cos^2\phi + \sigma_2 \sin^2\phi, \quad \sigma_{12} = (\sigma_1 - \sigma_2)\sin\phi\cos\phi,$$
$$\sigma_{22} = \sigma_1 \sin^2\phi + \sigma_2 \cos^2\phi, \quad \sigma_{33} = \sigma_3, \quad \sigma_{13} = \sigma_{23} = 0.$$

By substituting for the various expressions involving ϕ in favour of λ we obtain the normal stresses in the form

$$\begin{aligned}\sigma_{11} &= \frac{1}{2}(\sigma_1 + \sigma_2) + \frac{1}{2}(\sigma_1 - \sigma_2)\frac{\lambda - \lambda^{-1}}{\lambda + \lambda^{-1}},\\ \sigma_{22} &= \frac{1}{2}(\sigma_1 + \sigma_2) - \frac{1}{2}(\sigma_1 - \sigma_2)\frac{\lambda - \lambda^{-1}}{\lambda + \lambda^{-1}},\end{aligned}$$

and the shear stress as

$$\sigma_{12} = \frac{\sigma_1 - \sigma_2}{\lambda + \lambda^{-1}}.$$

The connection

$$\sigma_{11} - \sigma_{22} = (\lambda - \lambda^{-1})\sigma_{12} \equiv \gamma\sigma_{12} \tag{5.42}$$

then follows. This is important to note since it is an example of a *universal relation*, i.e. a connection between the stress components that is independent of the form of constitutive law (in this case, the class of incompressible isotropic elastic solids). For a recent discussion of universal relations we refer to the article by Saccomandi in [4].

Instead of regarding W as a function of I_1 and I_2 or of the stretches we may (for this specific deformation) take it to be a function of γ and define

$$W_{\mathrm{ss}}(\gamma) = \hat{W}(\lambda, \lambda^{-1}), \tag{5.43}$$

the subscript $_{\mathrm{ss}}$ signifying simple shear. Then, we have simply

$$\sigma_{12} = W'_{\mathrm{ss}}(\gamma), \tag{5.44}$$

Note that for the neo-Hookean form of strain-energy function this gives $\sigma_{12} = \mu\gamma$, i.e. the shear stress is *linear* in the amount of shear γ. Note also that in general *normal* stresses are required in addition to shear stresses in order to maintain the shape of the material. The necessity for normal forces is an example of the *Kelvin effect*.

For the considered simple shear deformation we record here for later reference that the invariants I_1, I_2, I_3 are given by

$$I_1 = I_2 = 3 + \gamma^2, \quad I_3 = 1 \tag{5.45}$$

emphasizing that simple shear is an *isochoric* deformation. Simple shear is an important deformation since it arises locally in many problems of practical and theoretical interest, such as the problem of azimuthal shear to be considered in Chapter 10.

Chapter 6

Boundary-value problems

We now consider the formulation of (equilibrium) boundary-value problems. Specifically, we consider the equilibrium equation in the absence of body forces. The appropriate specialization of the equation of motion (3.2) is then

$$\operatorname{div} \boldsymbol{\sigma} = \mathbf{0}, \tag{6.1}$$

or, in terms of nominal stress,

$$\operatorname{Div} \mathbf{S} = \mathbf{0}. \tag{6.2}$$

Equations (6.1) and (6.2) have to be taken in conjunction with the stress-deformation relations

$$\boldsymbol{\sigma} = J^{-1}\mathbf{F}\frac{\partial W}{\partial \mathbf{F}}, \quad \mathbf{S} = \frac{\partial W}{\partial \mathbf{F}}, \tag{6.3}$$

respectively, in the case of an unconstrained material, with the deformation gradient $\mathbf{F}$ given by $\mathbf{F} = \operatorname{Grad} \mathbf{x}$ with $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X})$. For an incompressible material the stress-deformation relations (6.3) are replaced by

$$\boldsymbol{\sigma} = \mathbf{F}\frac{\partial W}{\partial \mathbf{F}} - p\mathbf{I}, \quad \mathbf{S} = \frac{\partial W}{\partial \mathbf{F}} - p\mathbf{F}^{-1}, \quad \det \mathbf{F} \equiv 1. \tag{6.4}$$

Appropriate boundary conditions are required in order to formulate a boundary-value problem. Typical boundary conditions arising in problems of nonlinear elasticity are those in which $\mathbf{x}$ is specified on part of the boundary,

$\partial B_r^x \subset \partial B_r$ say, and the stress vector on the remainder, ∂B_r^τ, so that $\partial B_r^x \cup \partial B_r^\tau = \partial B_r$ and $\partial B_r^x \cap \partial B_r^\tau = \emptyset$. We write

$$\mathbf{x} = \boldsymbol{\xi}(\mathbf{X}) \qquad \text{on } \partial B_r^x, \tag{6.5}$$

$$\mathbf{S}^T\mathbf{N} = \boldsymbol{\tau}(\mathbf{F}, \mathbf{X}) \qquad \text{on } \partial B_r^\tau, \tag{6.6}$$

where $\boldsymbol{\xi}$ and $\boldsymbol{\tau}$ are specified functions. In general, $\boldsymbol{\tau}$ may depend on the deformation and this is indicated in (6.6) by showing the dependence of $\boldsymbol{\tau}$ on the deformation gradient $\mathbf{F}$. (Note that $\boldsymbol{\tau}$ differs from the stress vector $\boldsymbol{\tau}$ defined in Section 2.5.) If the surface traction defined by (6.6) is independent of $\mathbf{F}$ it is referred to as a *dead-load traction*. In the particular case in which the boundary traction in (6.6) is associated with a hydrostatic pressure, P say, so that $\boldsymbol{\sigma}\mathbf{n} = -P\mathbf{n}$, then $\boldsymbol{\tau}$ depends on the deformation in the form

$$\boldsymbol{\tau} = -JP\mathbf{F}^{-T}\mathbf{N} \quad \text{on } \partial B_r^\tau. \tag{6.7}$$

When coupled with suitable boundary conditions, either of the equations (6.1) or (6.2) in conjunction with (6.3) or (6.4), as appropriate, forms a coupled system of three highly nonlinear second-order partial differential equations for the components of $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X})$.

For homogeneous deformations, of course, the equilibrium equations are satisfied automatically and such deformations can be maintained by the application of suitable boundary tractions. For non-homogeneous deformations, it is necessary to solve the equilibrium equations. In the case of *unconstrained materials* very few explicit solutions have been obtained for boundary-value problems involving non-homogeneous deformations, and these arise for very special choices of the form of W and for relatively simple geometries. For *incompressible materials*, on the other hand, many more explicit solutions are available. In the following section we describe a simple example of a boundary-value problem for an incompressible isotropic elastic material in which the deformation is non-homogeneous. A different nonhomogeneous deformation, namely azimuthal shear, is discussed in Chapter 10.

6.1. Extension and inflation of a thick-walled tube

We consider a thick-walled circular cylindrical tube whose initial geometry is defined by

$$A \le R \le B, \quad 0 \le \Theta \le 2\pi, \quad 0 \le Z \le L, \tag{6.8}$$

where A, B, L are positive constants and R, Θ, Z are cylindrical polar coordinates associated with basis vectors $(\mathbf{E}_R, \mathbf{E}_\Theta, \mathbf{E}_Z)$. The deformed configuration is specified in terms of cylindrical polar coordinates (r, θ, z), with basis vectors $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z)$ and the position vector in the deformed configuration may be written

$$\mathbf{x} = r\mathbf{e}_r + z\mathbf{e}_z. \tag{6.9}$$

The tube is deformed so that the circular cylindrical shape is maintained. Since the material is incompressible the deformation is described by the equations

$$r = f(R) \equiv [a^2 + \lambda_z^{-1}(R^2 - A^2)]^{1/2}, \quad \theta = \Theta, \quad z = \lambda_z Z, \tag{6.10}$$

where λ_z is the (uniform) axial stretch and a is the internal radius of the deformed tube.

Since, for this deformation, $\mathbf{e}_r = \mathbf{E}_R, \mathbf{e}_\theta = \mathbf{E}_\Theta, \mathbf{e}_z = \mathbf{E}_Z$, the deformation gradient is then calculated as

$$\begin{aligned} \mathbf{F} = \operatorname{Grad} \mathbf{x} &= \frac{\partial \mathbf{x}}{\partial R} \otimes \mathbf{e}_r + \frac{1}{R}\frac{\partial \mathbf{x}}{\partial \Theta} \otimes \mathbf{e}_\theta + \frac{\partial \mathbf{x}}{\partial Z} \otimes \mathbf{e}_z \\ &= f'(R)\mathbf{e}_r \otimes \mathbf{e}_r + \frac{r}{R}\mathbf{e}_\theta \otimes \mathbf{e}_\theta + \lambda_z \mathbf{e}_z \otimes \mathbf{e}_z \\ &= \lambda_1 \mathbf{e}_r \otimes \mathbf{e}_r + \lambda_2 \mathbf{e}_\theta \otimes \mathbf{e}_\theta + \lambda_3 \mathbf{e}_z \otimes \mathbf{e}_z. \end{aligned} \tag{6.11}$$

Thus, $\mathbf{F}$ is symmetric and in spectral form with respect to the cylindrical polar axes. The principal stretches $\lambda_1, \lambda_2, \lambda_3$, which are associated respectively with the radial, azimuthal and axial directions, are therefore identified. Thus,

$$\lambda_1 = \lambda^{-1}\lambda_z^{-1}, \quad \lambda_2 = \frac{r}{R} = \lambda, \quad \lambda_3 = \lambda_z, \tag{6.12}$$

where the notation λ has been introduced. It follows from (6.10) and (6.12) that

$$\lambda_a^2\lambda_z - 1 = \frac{R^2}{A^2}(\lambda^2\lambda_z - 1) = \frac{B^2}{A^2}(\lambda_b^2\lambda_z - 1), \tag{6.13}$$

where

$$\lambda_a = a/A, \quad \lambda_b = b/B, \quad b = f(B). \tag{6.14}$$

For a fixed value of λ_z the inequalities

$$\lambda_a^2\lambda_z \geq 1, \quad \lambda_a \geq \lambda \geq \lambda_b \tag{6.15}$$

hold during inflation of the tube, with equality holding if and only if $\lambda = \lambda_z^{-1/2}$ for $A \le R \le B$. Note that when this latter equality holds the deformation corresponds to simple tension.

We use the notation (5.29) for the strain energy but with $\lambda_2 = \lambda$ and $\lambda_3 = \lambda_z$ as the independent stretches (instead of λ_1 and λ_2), so that

$$\hat{W}(\lambda, \lambda_z) = W(\lambda^{-1}\lambda_z^{-1}, \lambda, \lambda_z). \tag{6.16}$$

Hence

$$\sigma_2 - \sigma_1 = \lambda \hat{W}_\lambda, \quad \sigma_3 - \sigma_1 = \lambda_z \hat{W}_{\lambda_z}, \tag{6.17}$$

where the subscripts indicate partial derivatives, and, because the material is isotropic,

$$\boldsymbol{\sigma} = \sigma_1 \mathbf{e}_r \otimes \mathbf{e}_r + \sigma_2 \mathbf{e}_\theta \otimes \mathbf{e}_\theta + \sigma_3 \mathbf{e}_z \otimes \mathbf{e}_z. \tag{6.18}$$

Since the deformation depends only on the radial coordinate, it follows from (6.18) that

$$\operatorname{div} \boldsymbol{\sigma} \equiv \left[\frac{\partial \sigma_1}{\partial r} + \frac{1}{r}(\sigma_1 - \sigma_2)\right] \mathbf{e}_r,$$

and the equilibrium equation (6.1) therefore reduces to the radial equation

$$\frac{\mathrm{d}\sigma_1}{\mathrm{d}r} + \frac{1}{r}(\sigma_1 - \sigma_2) = 0 \tag{6.19}$$

in terms of the principal Cauchy stresses. Associated with this equation we have the (radial) boundary conditions

$$\sigma_1 = \begin{cases} -P & \text{on } r = a \\ 0 & \text{on } r = b \end{cases} \tag{6.20}$$

corresponding to pressure $P\,(\ge 0)$ on the inside of the tube and zero traction on the outside.

By making use of (6.10) and (6.12)–(6.14) we obtain (after some rearrangement)

$$r\frac{d\lambda}{dr} = -\lambda(\lambda^2\lambda_z - 1),$$

and it is convenient to use this to change the independent variable from r to λ. Then, integration of (6.19) and application of the boundary conditions (6.20) leads to

$$P = \int_{\lambda_b}^{\lambda_a} (\lambda^2\lambda_z - 1)^{-1} \frac{\partial \hat{W}}{\partial \lambda} \mathrm{d}\lambda. \tag{6.21}$$

► From (6.13) we recall that λ_b depends on λ_a. Equation (6.21) therefore provides an expression for P as a function of λ_a (equivalently of the deformed radius) when λ_z is fixed.

In order to hold λ_z fixed an axial load, N say, must be applied to the ends of the tube. This is given by

$$N = 2\pi \int_a^b \sigma_3 r dr. \tag{6.22}$$

After some rearrangements and use of (6.17) and (6.19) equation (6.22) can be expressed in the form

$$N/\pi A^2 = (\lambda_a^2\lambda_z - 1)\int_{\lambda_b}^{\lambda_a} (\lambda^2\lambda_z - 1)^{-2}\Big(2\lambda_z\frac{\partial \hat{W}}{\partial \lambda_z} - \lambda\frac{\partial \hat{W}}{\partial \lambda}\Big)\lambda \mathrm{d}\lambda + P\lambda_a^2. \tag{6.23}$$

We note that πA^2 times the integral in (6.23), i.e. $N - P\pi a^2$, is referred to as the *reduced axial load* since it accounts for the effect of the pressure on the ends of the cylinder, it being assumed that the cylinder has closed ends. For a more detailed discussion of this problem, including an analysis of bifurcation into non-circular cylindrical modes of deformation, we refer to Haughton and Ogden [5, 6].

Representative results for the pressure P calculated from (6.21) are shown in Fig. 6.1 in dimensionless form. This demonstrates the very different behaviour of biological soft tissues and rubberlike materials.

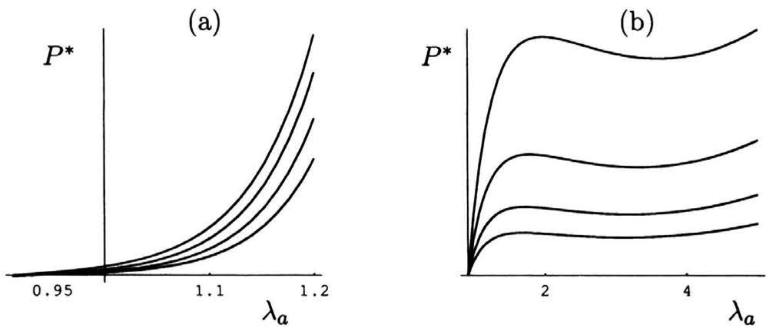

FIGURE 6.1. Plot of the dimensionless pressure P^* against the stretch λ_a for different wall thicknesses and an axial pre-stretch $\lambda_z = 1.2$ in respect of (a) a typical soft tissue, and (b) a typical rubberlike material.

In the special case in which the wall thickness of the tube is small compared with the radius the integral (6.21) may be approximated in the following way (this corresponds to the membrane approximation). Let $\epsilon \equiv (B - A)/A$ be a dimensionless measure of the wall thickness in the reference configuration. Then, from (6.13), we may obtain the approximation

$$\lambda_a \approx \lambda_b + \epsilon\lambda^{-1}\lambda_z^{-1}(\lambda^2\lambda_z - 1), \tag{6.24}$$

where, to the first order in ϵ, λ may be taken as either λ_a or λ_b. On use of (6.24) we may then approximate P as

$$P \approx \epsilon\lambda_z^{-1}\lambda^{-1}\hat{W}_\lambda(\lambda, \lambda_z), \tag{6.25}$$

so that, at fixed λ_z, the behaviour of P as a function of λ is that of $\lambda^{-1}\hat{W}_\lambda$.

Chapter 7

Anisotropic elastic materials

The elastic response of some rubberlike materials is in essence isotropic. This is also true to a limited extent for some biological soft tissues. However, when subjected to tensile stresses of sufficient magnitude soft tissues exhibit anisotropy in their mechanical response. This is associated with distributions of collagen fibres that endow the material locally with preferred directions. In ligaments and tendons, for example, the material can be regarded as having a single preferred direction (on average). The material can then be treated as *transversely isotropic*. Other soft tissues have two distinct distributions of collagen fibre directions and these can be associated with two preferred directions. This is the situation for the layers of an artery wall, for example. Also, in many industrial applications of rubber the material is rendered anisotropic by the inclusion of layers of steel wires (in high pressure hoses and car tyres, for example) and/or fabric (also in car tyres). The elastic response of such composite materials can be regarded as that of a homogeneous material with anisotropic properties associated with the preferred directions generated by the fibres.

In this chapter we illustrate the structure of the strain-energy function of an anisotropic elastic solid for two important examples: (i) transverse isotropy (characterized by a single family of fibres), and (ii) the anisotropy associated with two families of fibres, and, in particular, orthotropy. The work in this chapter owes much to the theory of invariants developed by Spencer (see, for example, [21, 22]).

7.1. Transverse isotropy

Let the unit vector $\mathbf{M}$ be a preferred direction in the reference configuration B_r of the material. Without the preferred direction the material would be isotropic relative to B_r. In general $\mathbf{M}$ varies with position $\mathbf{X}$ and is a unit-vector field which, when the strain-energy function is endowed with suitable properties, can be regarded as modelling the fibres as a continuous distribution. The material response is therefore indifferent to arbitrary rotations about the direction $\mathbf{M}$. Also, no physical distinction can be made between the directions $\mathbf{M}$ and $-\mathbf{M}$. Thus, the response must also be unaffected by interchange of $\mathbf{M}$ and $-\mathbf{M}$.

The strain energy $W(\mathbf{F})$ must therefore satisfy $W(\mathbf{FQ}) = W(\mathbf{F})$ for all proper orthogonal $\mathbf{Q}$ such that $\mathbf{QM} = \pm\mathbf{M}$. Note that the direction of $\mathbf{M}$ is reversed by a rotation of π about any axis perpendicular to $\mathbf{M}$. Equivalently, such a material can be characterized by a strain energy that is an isotropic function of $\mathbf{F}$ and the tensor $\mathbf{M}\otimes\mathbf{M}$ jointly. Since, by objectivity, W depends on $\mathbf{F}$ only through the right stretch tensor $\mathbf{U}$ (or, equivalently, $\mathbf{C} = \mathbf{U}^2$), this means that, on writing the dependence as $W(\mathbf{C}, \mathbf{M}\otimes\mathbf{M})$, we must have

$$W(\mathbf{QCQ}^T, \mathbf{QM}\otimes\mathbf{QM}) = W(\mathbf{C}, \mathbf{M}\otimes\mathbf{M}) \quad \text{for all proper orthogonal } \mathbf{Q}. \tag{7.1}$$

For an unconstrained material, the requirement (7.1) implies that W depends on five invariants, namely the principal invariants I_1, I_2, I_3 of $\mathbf{C}$, defined by (4.12)–(4.14) with $\mathbf{B}$ replaced by $\mathbf{C}$, together with two invariants, denoted I_4 and I_5, that depend on $\mathbf{M}$ and are defined by

$$I_4 = \mathbf{M}\cdot(\mathbf{CM}), \quad I_5 = \mathbf{M}\cdot(\mathbf{C}^2\mathbf{M}). \tag{7.2}$$

Note that I_4 has a direct kinematical interpretation since, in accordance with (1.37), $\sqrt{I_4}$ represents the stretch in the direction $\mathbf{M}$. In general, however, there is no immediate simple interpretation for I_5. We use the notation

$$\bar{W}(I_1, I_2, I_3, I_4, I_5) \tag{7.3}$$

to represent the strain energy when treated as a function of the invariants based on $\mathbf{C}$, extending the notation used in the isotropic case to include I_4 and I_5.

In order to calculate the stresses we require the derivatives

$$\frac{\partial I_4}{\partial \mathbf{F}} = 2\mathbf{M}\otimes\mathbf{FM}, \quad \frac{\partial I_5}{\partial \mathbf{F}} = 2(\mathbf{M}\otimes\mathbf{FCM} + \mathbf{CM}\otimes\mathbf{FM}), \tag{7.4}$$

together with the derivatives of I_1, I_2, I_3 given by (4.15). The resulting nominal stress tensor is given by

$$\mathbf{S} = 2\bar{W}_1\mathbf{F}^T + 2\bar{W}_2(I_1\mathbf{I} - \mathbf{C})\mathbf{F}^T + 2I_3\bar{W}_3\mathbf{F}^{-1} + 2\bar{W}_4\mathbf{M}\otimes\mathbf{FM} \\ + 2\bar{W}_5(\mathbf{M}\otimes\mathbf{FCM} + \mathbf{CM}\otimes\mathbf{FM}), \tag{7.5}$$

where $\bar{W}_i = \partial\bar{W}/\partial I_i, i = 1, \dots, 5$. The result for an isotropic material is recovered by omitting the terms in $\bar{W}_4$ and $\bar{W}_5$. Equation (7.5) gives the stress in a fibre-reinforced material for which the fibre direction corresponds to $\mathbf{M}$ locally in the reference configuration. The Cauchy stress can be calculated from (7.5) using the general formula $J\boldsymbol{\sigma} = \mathbf{FS}$.

Henceforth, we restrict attention to incompressible materials, so that $I_3 \equiv 1$. The notation (7.3) is retained but with I_3 omitted. Equation (7.5) is then replaced by

$$\mathbf{S} = 2\bar{W}_1\mathbf{F}^T + 2\bar{W}_2(I_1\mathbf{I} - \mathbf{C})\mathbf{F}^T - p\mathbf{F}^{-1} + 2\bar{W}_4\mathbf{M}\otimes\mathbf{FM} \\ + 2\bar{W}_5(\mathbf{M}\otimes\mathbf{FCM} + \mathbf{CM}\otimes\mathbf{FM}), \tag{7.6}$$

and Cauchy stress tensor is given by

$$\boldsymbol{\sigma} = \mathbf{FS} = -p\mathbf{I} + 2\bar{W}_1\mathbf{B} + 2\bar{W}_2(I_1\mathbf{B} - \mathbf{B}^2) + 2\bar{W}_4\mathbf{FM}\otimes\mathbf{FM} \\ + 2\bar{W}_5(\mathbf{FM}\otimes\mathbf{BFM} + \mathbf{BFM}\otimes\mathbf{FM}), \tag{7.7}$$

where $\mathbf{B}$ is the left Cauchy-Green deformation tensor and we have used the connection $\mathbf{FCM} = \mathbf{BFM}$. The symmetry of $\boldsymbol{\sigma}$ is apparent from (7.7). Note that (7.7) reduces to the corresponding result (5.7) for an isotropic material when the dependence on I_4 and I_5 is omitted.

7.1.1. Application to pure homogeneous deformation

In Section 5.5 we examined the pure homogeneous strain defined by (5.24) in the case of an isotropic material. Here we obtain, for comparison, the corresponding results derived from (7.7). Let $\mathbf{M}$ lie in the (X_1, X_2)-plane and suppose it has components $(\cos\varphi, \sin\varphi, 0)$. Then, we calculate

$$I_4 = \lambda_1^2\cos^2\varphi + \lambda_2^2\sin^2\varphi, \quad I_5 = \lambda_1^4\cos^2\varphi + \lambda_2^4\sin^2\varphi, \tag{7.8}$$

while, in terms of the (independent) stretches λ_1 and λ_2, we have

$$I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_1^{-2}\lambda_2^{-2}, \quad I_2 = \lambda_1^2\lambda_2^2 + \lambda_1^{-2} + \lambda_2^{-2}. \tag{7.9}$$

► From (7.7) the components of $\boldsymbol{\sigma}$ are calculated as

$$\begin{aligned}\sigma_{11} &= -p + 2\bar{W}_1\lambda_1^2 + 2\bar{W}_2\lambda_1^2(\lambda_2^2 + \lambda_3^2) \\ &\quad + 2\bar{W}_4\lambda_1^2\cos^2\varphi + 4\bar{W}_5\lambda_1^4\cos^2\varphi, \qquad (7.10)\\ \sigma_{22} &= -p + 2\bar{W}_1\lambda_2^2 + 2\bar{W}_2\lambda_2^2(\lambda_1^2 + \lambda_3^2) \\ &\quad + 2\bar{W}_4\lambda_2^2\sin^2\varphi + 4\bar{W}_5\lambda_2^4\sin^2\varphi, \qquad (7.11)\end{aligned}$$

$$\sigma_{12} = 2[\bar{W}_4 + \bar{W}_5(\lambda_1^2 + \lambda_2^2)]\lambda_1\lambda_2\sin\varphi\cos\varphi, \qquad (7.12)$$

$$\sigma_{33} = -p + 2\bar{W}_1\lambda_3^2 + 2\bar{W}_2\lambda_3^2(\lambda_1^2 + \lambda_2^2), \quad \sigma_{13} = \sigma_{23} = 0. \qquad (7.13)$$

Note that σ_{12} does not in general vanish, unlike the situation for an isotropic material. This means that (as a result of lack of symmetry) shear stress is required to maintain the pure homogeneous strain in this case, and it vanishes only if the preferred direction is along one of the coordinate axes. This illustrates the fact that the principal axes of $\boldsymbol{\sigma}$ do not in general coincide with the Eulerian principal axes (which, here, are the coordinate axes).

► From (7.10), (7.11) and (7.13) we obtain

$$\begin{aligned}\sigma_{11} - \sigma_{33} &= 2\lambda_1^{-2}\lambda_2^{-2}(\lambda_1^4\lambda_2^2 - 1)(\check{W}_1 + \lambda_2^2\bar{W}_2) \\ &\quad + 2\bar{W}_4\lambda_1^2\cos^2\varphi + 4\bar{W}_5\lambda_1^4\cos^2\varphi, \qquad (7.14)\\ \sigma_{22} - \sigma_{33} &= 2\lambda_1^{-2}\lambda_2^{-2}(\lambda_1^2\lambda_2^4 - 1)(\bar{W}_1 + \lambda_1^2\bar{W}_2) \\ &\quad + 2\bar{W}_4\lambda_2^2\sin^2\varphi + 4\bar{W}_5\lambda_2^4\sin^2\varphi. \qquad (7.15)\end{aligned}$$

Equations (7.8) and (7.9) show that I_1, I_2, I_4, I_5, and hence the strain energy, depend only on λ_1, λ_2 and the angle φ. We express this dependence by extending the notation $\hat{W}$ defined in (5.29) to the present situation. Thus, we define

$$\hat{W}(\lambda_1, \lambda_2, \varphi) = \bar{W}(I_1, I_2, I_4, I_5). \qquad (7.16)$$

It is important to note, however, that, in general, in contrast to the isotropic situation, $\hat{W}(\lambda_1, \lambda_2, \varphi)$ is *not symmetric* in λ_1 and λ_2. It is then easy to show that (7.14) and (7.15) may be written in the simple forms

$$\sigma_{11} - \sigma_{33} = \lambda_1\frac{\partial\hat{W}}{\partial\lambda_1}, \quad \sigma_{22} - \sigma_{33} = \lambda_2\frac{\partial\hat{W}}{\partial\lambda_2}. \qquad (7.17)$$

Equations (7.17) are identical in form to the corresponding equations (5.30) in the isotropic case, except that here σ_{11} and σ_{22} are *not* principal stresses since the shear stress σ_{12} does not in general vanish and, it should be emphasized, that $\hat{W}(\lambda_1, \lambda_2, \varphi)$ is not symmetric in λ_1 and λ_2.

We recall that for incompressible isotropic materials homogeneous biaxial deformations in which two independent stretches (or I_1 and I_2) are varied independently are sufficient to characterize the material properties (i.e. the strain-energy function). This is clearly not the case for an incompressible transversely isotropic material, for which there are four independent invariants. Characterization of the properties of a transversely isotropic material requires experiments in which (in principle) these invariants are varied independently.

We note here that for the considered pure homogeneous strain the components of $\mathbf{FM}$ are $(\lambda_1 \cos\phi, \lambda_2 \sin\phi, 0)$. Let $\mathbf{m}$ denote the unit vector in the direction $\mathbf{FM}$ and suppose $\mathbf{m}$ has components $(\cos\varphi^*, \sin\varphi^*, 0)$. Then, we have

$$\tan\varphi^* = \lambda_2 \lambda_1^{-1} \tan\varphi. \tag{7.18}$$

7.1.2. Plane strain

It is interesting to examine the simplifications that arise in the case of a plane deformation. We consider a plane deformation in which $\lambda_3 = 1$. It then follows that $\lambda_1\lambda_2 = 1$ and from (7.8) and (7.9) that

$$I_2 = I_1, \quad I_5 = (I_1 - 1)I_4 - 1. \tag{7.19}$$

Thus, we may regard the energy as a function of just two independent invariants, such as I_1 and I_4, and we write

$$\bar{\bar{W}}(I_1, I_4) = \bar{W}\left(I_1, I_1, I_4, (I_1 - 1)I_4 - 1\right). \tag{7.20}$$

It follows that the Cauchy stress is given simply by

$$\boldsymbol{\sigma} = 2\bar{\bar{W}}_1 \mathbf{B} + 2\bar{\bar{W}}_4 \mathbf{FM} \otimes \mathbf{FM} - p\mathbf{I}, \tag{7.21}$$

which should be compared with (7.7).

For pure homogeneous strain, the in-plane components of $\boldsymbol{\sigma}$ are obtained from (7.21) as

$$\sigma_{11} = 2\bar{\bar{W}}_1 \lambda_1^2 + 2\bar{\bar{W}}_4 \lambda_1^2 \cos^2\varphi - p, \tag{7.22}$$

$$\sigma_{22} = 2\bar{\bar{W}}_1 \lambda_2^2 + 2\bar{\bar{W}}_4 \lambda_2^2 \sin^2\varphi - p, \tag{7.23}$$

$$\sigma_{12} = 2\bar{\bar{W}}_4 \sin\varphi\cos\varphi, \tag{7.24}$$

with $\lambda_1\lambda_2 = 1$.

For the simple shear deformation discussed in Sections 1.7.2 and 5.6.1, we obtain from (7.21)

$$\sigma_{11} = 2\bar{\bar{W}}_1(1+\gamma^2) + 2\bar{\bar{W}}_4(\cos\varphi + \gamma\sin\varphi)^2 - p, \tag{7.25}$$

$$\sigma_{22} = 2\bar{\bar{W}}_1 + 2\bar{\bar{W}}_4\sin^2\varphi - p, \tag{7.26}$$

$$\sigma_{12} = 2\gamma\bar{\bar{W}}_1 + 2\bar{\bar{W}}_4\sin\varphi(\cos\varphi + \gamma\sin\varphi), \tag{7.27}$$

with

$$I_1 = 3+\gamma^2, \quad I_4 = 1 + \gamma\sin 2\varphi + \gamma^2\sin^2\varphi. \tag{7.28}$$

Note that

$$\sigma_{11} - \sigma_{22} - \gamma\sigma_{12} = \bar{\bar{W}}_4(2\cos 2\varphi + \gamma\sin 2\varphi), \tag{7.29}$$

so that the universal relation (5.42) obtained in the isotropic case does not carry over to transverse isotropy, except in the very special case in which $\mathbf{M}$ coincides with the Lagrangian principal direction $\mathbf{u}^{(1)}$ (which can only happen for an isolated value of γ). On the other hand, the formula (5.44) does apply, as can be shown by differentiating $\bar{\bar{W}}(I_1, I_4)$ with respect to γ and making use of (7.28).

7.2. Two preferred directions

We now consider the situation in which there are two distinct preferred directions in the reference configuration. Let $\mathbf{M}$ and $\mathbf{M}'$ denote the associated unit vectors. Then, in addition to I_1, I_2, I_4, I_5, the strain energy depends on the invariants

$$I_6 = \mathbf{M}'\cdot(\mathbf{C}\mathbf{M}'), \quad I_7 = \mathbf{M}'\cdot(\mathbf{C}^2\mathbf{M}'), \quad I_8 = \mathbf{M}\cdot(\mathbf{C}\mathbf{M}'). \tag{7.30}$$

Note that I_6 and I_7 are the counterparts for $\mathbf{M}'$ of I_4 and I_5, respectively, and that there is now a coupling term I_8. The energy also depends explicitly on the angle between the directions, as determined by the product $\mathbf{M}\cdot\mathbf{M}'$ (which does not depend on the deformation). There is no term $\mathbf{M}\cdot(\mathbf{C}^2\mathbf{M}')$ since it can be shown that it depends on the other invariants and on $\mathbf{M}\cdot\mathbf{M}'$. The invariant I_8 as defined above is not unchanged with respect to reversal of $\mathbf{M}$ or $\mathbf{M}'$ separately, but it can be made so by multiplying by $\mathbf{M}\cdot\mathbf{M}'$. For simplicity, however, we retain I_8 as given above and note that, by symmetry, this 'correction' is unnecessary for the problem considered in Section 7.2.1 below.

We now use the notation $\bar{W}$ to represent W for an incompressible material when regarded as a function of $I_1, I_2, I_4, I_5, I_6, I_7, I_8$, and $\mathbf{M}\cdot\mathbf{M}'$. The Cauchy stress tensor is then written

$$\begin{aligned}\boldsymbol{\sigma} = &-p\mathbf{I} + 2\bar{W}_1\mathbf{B} + 2\bar{W}_2(I_1\mathbf{B} - \mathbf{B}^2) + 2\bar{W}_4\mathbf{FM}\otimes\mathbf{FM} \\ &+ 2\bar{W}_5(\mathbf{FM}\otimes\mathbf{BFM} + \mathbf{BFM}\otimes\mathbf{FM}) + 2\bar{W}_6\mathbf{FM}'\otimes\mathbf{FM}' \\ &+ 2\bar{W}_7(\mathbf{FM}'\otimes\mathbf{BFM}' + \mathbf{BFM}'\otimes\mathbf{FM}') \\ &+ \bar{W}_8(\mathbf{FM}\otimes\mathbf{FM}' + \mathbf{FM}'\otimes\mathbf{FM}),\end{aligned} \tag{7.31}$$

where the notation $W_i = \partial W/\partial I_i$ now applies for $i = 1, 2, 4, \ldots, 8$.

Although (7.31) is in general very complicated some useful information can be obtained by restricting attention again to pure homogeneous strains and simple shear. The extension and inflation of a tube discussed in Section 6.1 for an isotropic material will also be examined in respect of (7.31) appropriately specialized.

7.2.1. Pure homogeneous strain

Again we consider the pure homogeneous strain defined by (5.24) and now we include two preferred directions, symmetrically disposed in the (X_1, X_2)-plane and given by

$$\mathbf{M} = \cos\varphi\,\mathbf{e}_1 + \sin\varphi\,\mathbf{e}_2, \quad \mathbf{M}' = \cos\varphi\,\mathbf{e}_1 - \sin\varphi\,\mathbf{e}_2, \tag{7.32}$$

where the angle φ is constant and $\mathbf{e}_1, \mathbf{e}_2$ denote the Cartesian coordinate directions, see Fig. 7.1. Let the corresponding unit vectors in the deformed configuration be denoted

$$\mathbf{m} = \cos\varphi^*\,\mathbf{e}_1 + \sin\varphi^*\,\mathbf{e}_2, \quad \mathbf{m}' = \cos\varphi^*\,\mathbf{e}_1 - \sin\varphi^*\,\mathbf{e}_2, \tag{7.33}$$

with φ^* given by (7.18).

When expressed in terms of λ_1 and λ_2 the invariants I_1, I_2 are given by (7.9) and the other invariants are calculated as

$$I_4 = I_6 = \lambda_1^2\cos^2\varphi + \lambda_2^2\sin^2\varphi, \quad I_5 = I_7 = \lambda_1^4\cos^2\varphi + \lambda_2^4\sin^2\varphi, \tag{7.34}$$

$$I_8 = \lambda_1^2\cos^2\varphi - \lambda_2^2\sin^2\varphi. \tag{7.35}$$

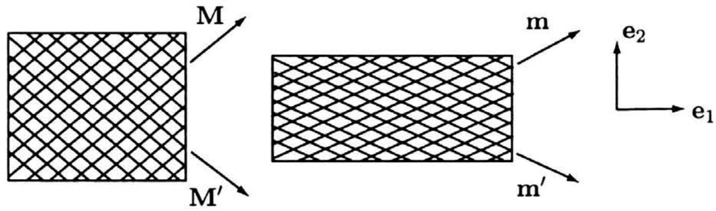

FIGURE 7.1. Depiction of pure homogeneous strain with two symmetrically disposed families of fibres in the (X_1, X_2) plane.

The components of $\boldsymbol{\sigma}$ are obtained from (7.31) as

$$\sigma_{11} = -p + 2\bar{W}_1\lambda_1^2 + 2\bar{W}_2(I_1\lambda_1^2 - \lambda_1^4) + 2(\bar{W}_4 + \bar{W}_6 + \bar{W}_8)\lambda_1^2\cos^2\varphi + 4(\bar{W}_5 + \bar{W}_7)\lambda_1^4\cos^2\varphi, \tag{7.36}$$

$$\sigma_{22} = -p + 2\bar{W}_1\lambda_2^2 + 2\bar{W}_2(I_1\lambda_2^2 - \lambda_2^4) + 2(\bar{W}_4 + \bar{W}_6 - \bar{W}_8)\lambda_2^2\sin^2\varphi + 4(\bar{W}_5 + \bar{W}_7)\lambda_2^4\sin^2\varphi, \tag{7.37}$$

$$\sigma_{12} = 2[\bar{W}_4 - \bar{W}_6 + (\bar{W}_5 - \bar{W}_7)(\lambda_1^2 + \lambda_2^2)]\lambda_1\lambda_2\sin\varphi\cos\varphi, \tag{7.38}$$

$$\sigma_{33} = -p + 2\bar{W}_1\lambda_3^2 + 2\bar{W}_2(I_1\lambda_3^2 - \lambda_3^4), \quad \sigma_{13} = \sigma_{23} = 0. \tag{7.39}$$

Note that $(7.39)_1$ is identical in form to $(7.13)_1$ but is different in content since $\bar{W}$ now depends on I_6, I_7, I_8.

As for the case of transverse isotropy, $\sigma_{12} \neq 0$ in general. Thus, shear stresses are required to maintain the pure homogeneous strain and the principal axes of stress do not coincide with the Cartesian axes. However, in the special case in which *the two preferred directions are mechanically equivalent* the strain energy must be symmetric with respect to interchange of I_4 and I_6 and of I_5 and I_7. For the considered deformation, we have $I_4 = I_6, I_5 = I_7$ and it then follows that $W_4 = W_6, W_5 = W_7$ and hence, from (7.38), that $\sigma_{12} = 0$. In this special case the principal axes of stress coincide with the Cartesian axes (i.e. with the Eulerian principal axes), and $\sigma_{11}, \sigma_{22}, \sigma_{33}$ are therefore precisely the principal Cauchy stresses $\sigma_1, \sigma_2, \sigma_3$.

► From (7.9), (7.34) and (7.35) we see that, just as in the case of transverse isotropy, the invariants collectively depend only on λ_1, λ_2 and φ and we may therefore write

$$\hat{W}(\lambda_1, \lambda_2, \varphi) = \bar{W}(I_1, I_2, I_4, I_5, I_6, I_7, I_8, \mathbf{M}\cdot\mathbf{M}'). \tag{7.40}$$

Again, as in the transversely isotropic case, $\hat{W}$ is *not* symmetric with respect to interchange of λ_1 and λ_2. It is straightforward to show that

$$\sigma_1 - \sigma_3 = \lambda_1 \frac{\partial \hat{W}}{\partial \lambda_1}, \quad \sigma_2 - \sigma_3 = \lambda_2 \frac{\partial \hat{W}}{\partial \lambda_2}, \tag{7.41}$$

which are identical *in form* to equations (5.30) except that here $\hat{W}$ depends on φ and is not (in general) symmetric in (λ_1, λ_2). These equations describe an *orthotropic* material with the axes of orthotropy coinciding with the Cartesian axes.

7.2.2. Simple shear

We now extend the discussion of simple shear in Section 7.1.2 to the present material. For the simple shear deformation the invariants are given by

$$I_1 = I_2 = 3 + \gamma^2, \tag{7.42}$$

$$I_4 = 1 + \gamma \sin 2\varphi + \gamma^2 \sin^2 \varphi, \quad I_6 = 1 - \gamma \sin 2\varphi + \gamma^2 \sin^2 \varphi, \tag{7.43}$$

$$I_5 = (1 + \gamma^2)\cos^2 \varphi + 2\gamma(2 + \gamma^2)\sin\varphi\cos\varphi + (\gamma^4 + 3\gamma^2 + 1)\sin^2\varphi, \tag{7.44}$$

$$I_7 = (1 + \gamma^2)\cos^2 \varphi - 2\gamma(2 + \gamma^2)\sin\varphi\cos\varphi + (\gamma^4 + 3\gamma^2 + 1)\sin^2\varphi, \tag{7.45}$$

$$I_8 = \cos^2\varphi - (1 + \gamma^2)\sin^2\varphi. \tag{7.46}$$

The components of the Cauchy stress tensor are now calculated as

$$\begin{aligned}\sigma_{11} = {} & -p + 2\bar{W}_1(1 + \gamma^2) + 2\bar{W}_2(2 + \gamma^2) \\ & + 2[\bar{W}_4 + \bar{W}_6 + \bar{W}_8 + 2(\bar{W}_5 + \bar{W}_7)(1 + \gamma^2)]\cos^2\varphi \\ & + 4[\bar{W}_4 - \bar{W}_6 + (\bar{W}_5 - \bar{W}_7)(3 + \gamma^2)]\gamma\sin\varphi\cos\varphi \\ & + 2[\bar{W}_4 + \bar{W}_6 - \bar{W}_8 + 2(\bar{W}_5 + \bar{W}_7)(2 + \gamma^2)]\gamma^2\sin^2\varphi,\end{aligned} \tag{7.47}$$

$$\begin{aligned}\sigma_{22} = {} & -p + 2\bar{W}_1 + 4\bar{W}_2 + 2(\bar{W}_4 + \bar{W}_6 - \bar{W}_8)\sin^2\varphi \\ & + 4(\bar{W}_5 - \bar{W}_7)\gamma\sin\varphi\cos\varphi + 4(\bar{W}_5 + \bar{W}_7)(1 + \gamma^2)\sin^2\varphi,\end{aligned} \tag{7.48}$$

$$\begin{aligned}\sigma_{12} = {} & 2(\bar{W}_1 + \bar{W}_2)\gamma + 2(\bar{W}_4 - \bar{W}_6)\sin\varphi\cos\varphi \\ & + 2(\bar{W}_4 + \bar{W}_6 - \bar{W}_8)\gamma\sin^2\varphi \\ & + 2(\bar{W}_5 + \bar{W}_7)\gamma[\cos^2\varphi + (3 + \gamma^2)\sin^2\varphi],\end{aligned} \tag{7.49}$$

$$\sigma_{33} = -p + 2\bar{W}_1\lambda_3^2 + 2\bar{W}_2(I_1\lambda_3^2 - \lambda_3^4), \quad \sigma_{13} = \sigma_{23} = 0. \tag{7.50}$$

Since the invariants (7.42)–(7.46) depend only on γ and φ we may treat the strain energy as a function of these two quantities and write $W_{ss}(\gamma, \varphi)$ to represent this, where, as in (5.43), the subscript $_{ss}$ stands for simple shear. It is then straightforward to show, using (7.42)–(7.46) and (7.49) that

$$\sigma_{12} = \frac{\partial W_{ss}}{\partial \gamma}, \tag{7.51}$$

exactly as in the isotropic and transversely isotropic cases.

It is interesting to note that while the orientation of the Eulerian principal axes, in the $(1,2)$-plane, is given in terms of the angle ϕ through the formula

$$\tan 2\phi = \frac{2}{\gamma}, \tag{7.52}$$

the corresponding orientation of the principal axes of $\boldsymbol{\sigma}$ is defined by an angle, ϕ^* say, which is given by

$$\tan 2\phi^* = \frac{2\sigma_{12}}{\sigma_{11} - \sigma_{22}}. \tag{7.53}$$

In respect of (7.47)–(7.49) the right-hand side of (7.53) is not equal to that of (7.52), and hence $\phi^* \neq \phi$. We observe that $\phi^* = \phi$ if and only if the universal relation (5.42) holds.

7.2.3. Extension and inflation of a thick-walled tube

We now revisit the problem of extension and inflation of a thick-walled tube which was discussed in Section 6.1 for an isotropic material. Since, locally, the deformation corresponds to a pure homogeneous strain certain formulas obtained in Section 6.1 carry over to the anisotropic material considered here. We suppose that the preferred directions $\mathbf{M}$ and $\mathbf{M}'$ are locally in the (Θ, Z)-plane and symmetrically distributed with respect to the axial direction. The cylindrical polar directions are then the principal directions of strain (and stress) and the strain energy may be written in the form

$$\hat{W}(\lambda, \lambda_z, \varphi), \tag{7.54}$$

where, as in Section 6.1, $\lambda = \lambda_2$ and $\lambda_z = \lambda_3$ respectively are the azimuthal and axial stretches. The formulas (6.21) and (6.23) also apply here. We repeat equation (6.21) here in the form

$$P = \int_{\lambda_b}^{\lambda_a} (\lambda^2\lambda_z - 1)^{-1} \frac{\partial \hat{W}}{\partial \lambda}(\lambda, \lambda_z, \varphi)\mathrm{d}\lambda \tag{7.55}$$

with the arguments of $\hat{W}$ made explicit.

It is easy to evaluate the integral in (7.55) for particular choices of energy function, as was indicated in the case of isotropy in Section 6.1. It turns out that the qualitative nature of the results based on equation (7.55) does not depend significantly on the thickness of the wall of the tube wall. Here, therefore, it suffices to consider the thin-wall (membrane) approximation of (7.55), which has the form

$$P = \epsilon\lambda^{-1}\lambda_z^{-1}\frac{\partial \hat{W}}{\partial \lambda}(\lambda, \lambda_z, \varphi), \tag{7.56}$$

where $\epsilon = (B - A)/A$, as in Section 6.1, and λ represents any value of the azimuthal stretch through the wall.

We now illustrate the dependence of the pressure-stretch response on the degree of anisotropy by using (7.56). For this purpose we consider an energy function that is a natural extension to the type of anisotropy considered here of the isotropic law (5.21). With just a single term this has the form (see [19])

$$\begin{aligned}\hat{W}(\lambda, \lambda_z, \varphi) = [&\mu_1(\varphi)(\lambda^n - 1 - n\ln\lambda) + \mu_2(\varphi)(\lambda_z^n - 1 - n\ln\lambda_z)\\ &+ \mu_3(\lambda^{-n}\lambda_z^{-n} - 1 + n\ln(\lambda\lambda_z))]/n,\end{aligned} \tag{7.57}$$

where the logarithmic terms are needed to ensure that the stresses vanish in the undeformed configuration, μ_3 is a material constant and $\mu_1(\varphi)$ and $\mu_2(\varphi)$ are material parameters dependent of the angle φ. Note that the single-term version of (5.21) is recovered by setting $\mu_1 = \mu_2 = \mu_3 = 2\mu/n$ and $n = \alpha_1$ since, by incompressibility, the logarithmic terms cancel.

On substitution of (7.57) into equation (7.56) we obtain, in dimensionless form,

$$P^* \equiv \lambda_z P/\epsilon\mu_3 = \mu_1^*\lambda^{n-2} - (\mu_1^* - 1)\lambda^{-2} - \lambda^{-n-2}\lambda_z^{-n}, \tag{7.58}$$

where $\mu_1^* = \mu_1/\mu_3$. It should be noted that (7.58) is independent of μ_2. The results for isotropy are recovered by setting $\mu_1^* = 1$. The material can be regarded as reinforced in the circumferential direction (relative to the radial direction) if $\mu_1^* > 1$ and weakened if $\mu_1^* < 1$. Results for $\mu_1^* = 0.5, 1, 2$ are plotted in Fig. 7.2 for comparison, with λ_z set to the value 1.2, as for Fig. 6.1.

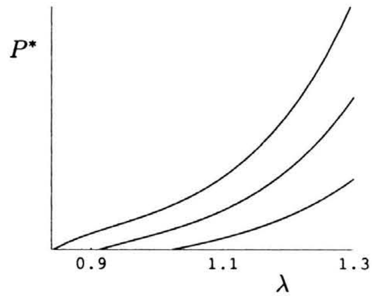

FIGURE 7.2. Plot of the dimensionless pressure P^* against the azimuthal stretch λ for fixed λ_z and for values $\mu_1^* = 2, 1, 0.5$ of the anisotropy parameter, corresponding to the upper, middle and lower curves respectively.

We recall that for isotropy the inequality $\lambda^2\lambda_z \geq 1$ must hold for inflation following an initial axial stretch. For the considered anisotropic material this must be replaced by an inequality on λ^n whose lower limit is determined by setting $P = 0$ in (7.58). This is reflected in the curves in Fig. 7.2, which cut the λ axis at different points. The upper, middle and lower curves in Fig. 7.2 correspond to $\mu_1^* = 2, 1, 0.5$ respectively. For illustrative purposes only the value $n = 10$ has been used for the above calculations.

The membrane counterpart of (7.58) for equation (6.23) has the form

$$F/\pi A^2 \equiv N/\pi A^2 - P\lambda^2 = \epsilon\Big[2\frac{\partial \hat{W}}{\partial \lambda_z}(\lambda, \lambda_z, \varphi) - \lambda\lambda_z^{-1}\frac{\partial \hat{W}}{\partial \lambda}(\lambda, \lambda_z, \varphi)\Big], \quad (7.59)$$

where F is the *reduced axial load* on the ends of the tube.

The combination of equations (7.56) and (7.59) with an appropriate form of strain-energy function can be used to fit data from experiments in which the reduced axial load F is held constant. A representative set of data from a human iliac artery is shown in Fig. 7.3. The pressure P is plotted against the circumferential stretch λ for a range of fixed values of the reduced axial load F. These curves show the characteristic stiffening of the material as the radius increases.

In Fig. 7.4 the same data as in Fig. 7.3 are plotted with the pressure against the axial stretch λ_z. This reveals a so-called *inversion effect* at the value of λ_z corresponding to the change from positive to negative gradients of the curves. This critical value of λ_z is determined by solution of the equation

$$\lambda\frac{\partial^2 \hat{W}}{\partial \lambda^2}(\lambda, \lambda_z, \varphi) - 2\lambda_z\frac{\partial^2 \hat{W}}{\partial \lambda \partial \lambda_z}(\lambda, \lambda_z, \varphi) + \lambda\frac{\partial \hat{W}}{\partial \lambda}(\lambda, \lambda_z, \varphi) = 0 \qquad (7.60)$$

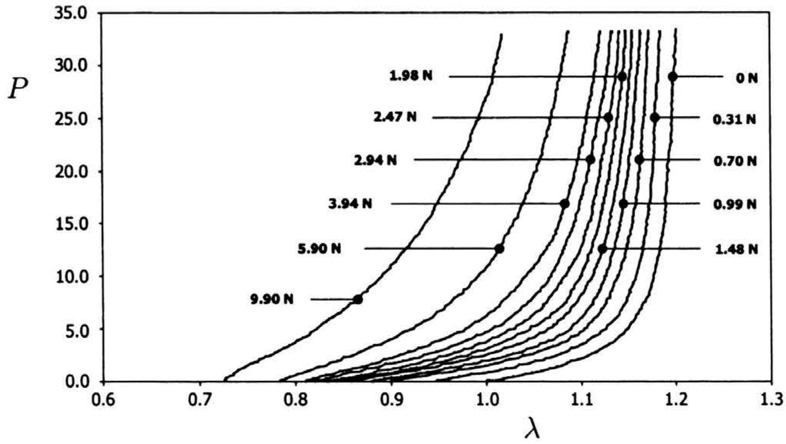

FIGURE 7.3. Typical characteristics of the response of a human iliac artery under pressure and axial load. Dependence of the internal pressure P (kPa) on the circumferential stretch λ at a series of fixed values of the reduced axial load.

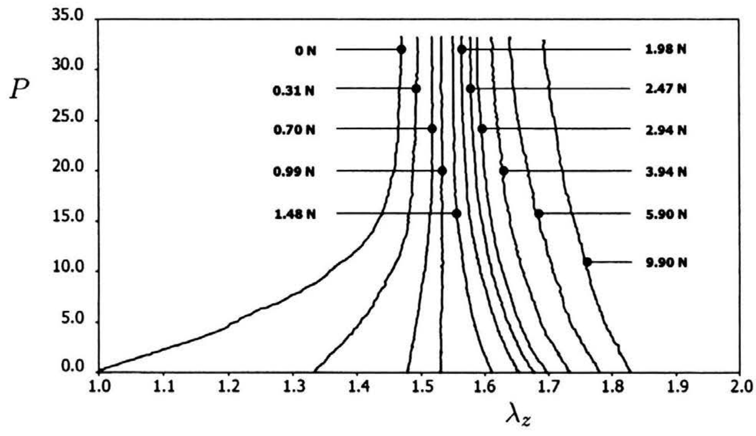

FIGURE 7.4. Typical characteristics of the response of a human iliac artery under pressure and axial load. Dependence of the internal pressure P (kPa) on the axial stretch λ_z at a series of fixed values of the reduced axial load.

in conjunction with (7.59) for constant F, where $F^* = F/\pi A^2\epsilon$. Equation (7.60) is obtained from (7.56) and (7.59) by setting $\mathrm{d}\lambda_z/\mathrm{d}P = 0$ at constant F. For further details we refer to Ogden and Schulze-Bauer [19].

Although the membrane approximation gives a good qualitative picture of the pressure-stretch behaviour it should be used with caution. For example, membrane theory is not able to account for the through-thickness stress distribution in arterial walls or the important influence of residual stresses which are present in arterial wall components. To account for these influences it is necessary to use a 'thick-wall' model. Both residual stresses and stress distributions through the wall thickness will be discussed in Chapter 8 following a treatment of some particular aspects of the influence of residual stress on the formulation of elastic constitutive laws.

Chapter 8

The effect of residual stress on elastic response

Thus far we have assumed that the reference configuration B_r is stress free. However, there are many situations in which a (global) stress-free reference configuration does not exist and there are so-called *residual stresses* not associated with a deformation and not given by a constitutive law. They may, for example, be induced by some manufacturing process or, in the case of biological tissues, be generated by the processes of growth, remodelling or adaptation. In this chapter we examine some basic aspects of the effect of residual stress on the constitutive law of a nonlinearly elastic solid.

8.1. Elastic response in the presence of residual stress

We suppose that the reference configuration B_r is not stress free and denote by $\boldsymbol{\sigma}^{(r)}$ the residual stress in B_r. Since this is the reference configuration there is no distinction between the Cauchy stress in B_r and the nominal stress $\mathbf{S}^{(r)}$ relative to B_r. In general, the residual stress is not obtained from a strain-energy function, and we may take the strain-energy function W to be measured from B_r and to vanish in B_r. The stress calculated from this energy function must reduce to the residual stress when evaluated in B_r. We shall discuss this further in Section 8.2.

The residual stress must satisfy the equilibrium equation

$$\operatorname{Div} \mathbf{S}^{(r)} = \mathbf{0} \quad \text{in } B_r, \tag{8.1}$$

where the Div operator refers to the position vector $\mathbf{X}$ in B_r.

If the boundary $\partial \mathrm{B}_r$ is traction free (unloaded) then, additionally, the residual stress must satisfy the boundary conditions

$$\mathbf{S}^{(r)T}\mathbf{N} = \mathbf{0} \quad \text{on } \partial \mathrm{B}_r. \tag{8.2}$$

Now, since

$$\mathrm{Div}\left(\mathbf{S}^{(r)} \otimes \mathbf{X}\right) = \left(\mathrm{Div}\,\mathbf{S}^{(r)}\right) \otimes \mathbf{X} + \mathbf{S}^{(r)}, \tag{8.3}$$

it follows from (8.1), (8.2) and by use of the divergence theorem that

$$\int_{\mathrm{B}_r} \mathbf{S}^{(r)} \mathrm{d}V = \mathbf{0}. \tag{8.4}$$

An immediate consequence of (8.4) is that *residual stress cannot be uniform*. In other words, if, in a residually-stressed configuration, the boundary $\partial \mathrm{B}_r$ is load free then *the residual stress distribution is necessarily inhomogeneous* and is therefore geometry dependent. A further consequence is that the material response of a residually-stressed body relative to the residually-stressed configuration, and hence the constitutive law, is geometry dependent and inhomogeneous. If, however, $\partial \mathrm{B}_r$ is not traction free and all or part of the boundary is fixed spatially then the above conclusion requires modification. We do not pursue this here.

Residual stress places restrictions on the material symmetry in B_r and, in view of the above remarks, the material symmetry may therefore vary from point to point within the considered material body. The constitutive laws resulting from these restrictions are, in general, very complicated, and we shall not discuss the associated analysis in detail. We remark, however, that, in the presence of a residual stress and without any preferred directions, the elastic strain energy relative to B_r depends on the independent invariants of $\boldsymbol{\sigma}^{(r)}$ and the Cauchy-Green deformation tensor $\mathbf{C}$ and their combinations. Moreover, if there are also preferred directions in B_r, such as $\mathbf{M}$, then further independent invariants involving $\boldsymbol{\sigma}^{(r)}$, $\mathbf{C}$ and $\mathbf{M} \otimes \mathbf{M}$ are needed. It is left as an exercise to determine the number of independent invariants of (a) $\mathbf{C}$ and $\boldsymbol{\sigma}^{(r)}$ for the cases in which $\boldsymbol{\sigma}^{(r)}$ has one, two or three distinct principal values, and (b) $\mathbf{C}$, $\boldsymbol{\sigma}^{(r)}$ and $\mathbf{M} \otimes \mathbf{M}$ for the cases in which $\boldsymbol{\sigma}^{(r)}$ has one, two or three distinct principal values.

Here we shall adopt a simpler approach and examine what restrictions are imposed on the residual stress by specific material symmetries. In this we follow the work of Coleman and Noll [3], Hoger [7] and the article by Ogden in [10].

Suppose that $\mathbf{Q}$ is a rotation tensor belonging to a symmetry group relative to B_r. Then, by combining the stress-deformation relation (3.30) for the nominal stress (relative to B_r) with the objectivity and material symmetry requirements, we obtain

$$\mathbf{h}(\mathbf{QF}) = \mathbf{h}(\mathbf{F})\mathbf{Q}^T, \tag{8.5}$$

for all proper orthogonal $\mathbf{Q}$, and

$$\mathbf{h}(\mathbf{FQ}) = \mathbf{Q}^T\mathbf{h}(\mathbf{F}), \tag{8.6}$$

for all members $\mathbf{Q}$ of the symmetry group. By setting $\mathbf{F} = \mathbf{I}$ and $\mathbf{S}^{(r)} = \mathbf{h}(\mathbf{I})$ and using (8.5) and (8.6), we then obtain

$$\mathbf{S}^{(r)}\mathbf{Q} = \mathbf{Q}\mathbf{S}^{(r)}, \tag{8.7}$$

or, equivalently,

$$\mathbf{Q}\boldsymbol{\sigma}^{(r)}\mathbf{Q}^T = \boldsymbol{\sigma}^{(r)}, \tag{8.8}$$

for every member $\mathbf{Q}$ *of the symmetry group.* Thus, equation (8.7) imposes restrictions on the form of $\mathbf{S}^{(r)}$. We now examine three specific material symmetries in order to determine the nature of these restrictions.

8.1.1. Isotropy

For *isotropic response* equation (8.8) must hold for *all* rotations $\mathbf{Q}$. This implies that the residual stress has the form $\boldsymbol{\sigma}^{(r)} = \sigma^{(r)}\mathbf{I}$, where $\sigma^{(r)}$ is a scalar. The equilibrium equation (8.1) reduces to $\operatorname{Grad}\sigma^{(r)} = \mathbf{0}$, so that $\sigma^{(r)}$ is constant. Application of the boundary condition (8.2) then shows that $\sigma^{(r)} \equiv 0$.

Thus, *residual stress cannot be supported by an isotropic body* whatever the geometry of the body if the boundary is traction free. This is an important result in the context of soft tissues, for some of which residual stress contributes to their effective function. It therefore emphasizes the need to consider soft tissues as anisotropic materials.

8.1.2. Transverse isotropy

If the material response is *transversely isotropic* relative to B_r then there is a preferred direction, defined by a unit vector, denoted $\mathbf{k}$, which will in general depend on position in the material. The symmetry group consists of all rotations $\mathbf{Q}$ that preserve or reverse $\mathbf{k}$. It may be shown, by following the procedure outlined for the isotropic case, that $\boldsymbol{\sigma}^{(r)}$ must have two equal principal values and is expressible in the form

$$\boldsymbol{\sigma}^{(r)} = \sigma_1^{(r)}(\mathbf{I} - \mathbf{k} \otimes \mathbf{k}) + \sigma_3^{(r)}\mathbf{k} \otimes \mathbf{k}, \tag{8.9}$$

where $\sigma_1^{(r)} = \sigma_2^{(r)}$ and $\sigma_3^{(r)}$ are the principal values, in general dependent on position.

8.1.3. Orthotropy

In the case of *orthotropic response* the material symmetry identifies three mutually orthogonal directions, here specified by the unit vectors $\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3$. The symmetry group consists of rotations through π about each $\mathbf{k}_i$, $i \in \{1, 2, 3\}$ together with reversal of each $\mathbf{k}_i$. The resulting form of $\boldsymbol{\sigma}^{(r)}$, obtained using (8.8), is

$$\boldsymbol{\sigma}^{(r)} = \sigma_1^{(r)}\mathbf{k}_1 \otimes \mathbf{k}_1 + \sigma_2^{(r)}\mathbf{k}_2 \otimes \mathbf{k}_2 + \sigma_3^{(r)}\mathbf{k}_3 \otimes \mathbf{k}_3, \tag{8.10}$$

the principal values of $\boldsymbol{\sigma}^{(r)}$ being distinct and associated with principal directions $\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3$. In general, $\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3$ and $\sigma_1^{(r)}, \sigma_2^{(r)}, \sigma_3^{(r)}$ depend on position. Of course, $\boldsymbol{\sigma}^{(r)}$ can always be put in the form (8.10) for *some* orthonormal basis $\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3$ whatever the material symmetry, but here $\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3$ are specifically determined by the symmetry.

An important special case is that in which one of the principal directions, $\mathbf{k}_3$ say, is independent of position. It follows on substitution of (8.10) into the equilibrium equation (8.1) that $\sigma_3^{(r)}$ is independent of the Cartesian coordinate associated with $\mathbf{k}_3$. If we identify this direction with the axis of a right circular cylindrical tube then application of the boundary condition (8.2) on the ends of the tube leads to $\sigma_3^{(r)} \equiv 0$. In terms of cylindrical polar coordinates (R, Θ, Z) in B_r, this means that there is no dependence on Z. If, further, there is no dependence on Θ then the equilibrium equation (8.1) reduces to the radial equation

$$\frac{\mathrm{d}\sigma_{RR}^{(r)}}{\mathrm{d}R} + \frac{\sigma_{RR}^{(r)} - \sigma_{\Theta\Theta}^{(r)}}{R} = 0, \tag{8.11}$$

together with the azimuthal equation

$$\frac{\mathrm{d}\sigma^{(r)}_{R\Theta}}{\mathrm{d}R} + \frac{2}{R}\sigma^{(r)}_{R\Theta} = 0, \tag{8.12}$$

where $\sigma^{(r)}_{RR}, \sigma^{(r)}_{R\Theta}, \sigma^{(r)}_{\Theta\Theta}$ are the relevant components of $\boldsymbol{\sigma}^{(r)}$.

It follows from (8.12) and the zero traction boundary conditions on the cylindrical surfaces that $\sigma^{(r)}_{R\Theta} \equiv 0$ and hence that $\mathbf{k}_1$ and $\mathbf{k}_2$ coincide with the polar coordinate axes and $\sigma^{(r)}_{RR} = \sigma^{(r)}_1, \sigma^{(r)}_{\Theta\Theta} = \sigma^{(r)}_2$. Equation (8.11) then remains and is coupled with the boundary conditions

$$\sigma^{(r)}_1 = 0 \quad \text{on } R = A, B. \tag{8.13}$$

Equation (8.11) and the boundary conditions (8.13) are important in connection with the analysis of the effect of residual stress on the elastic response of an artery treated as a circular cylindrical tube subject to extension and inflation, which will be discussed in Chapter 9.

8.2. Change in reference configuration and strain energy

Let B_r be a residually-stressed configuration, $\mathbf{F}$ the deformation gradient in a deformed configuration B_t measured relative to B_r and $W(\mathbf{F})$ the strain energy per unit volume in B_r. Suppose that there exists a reference configuration, denoted $\bar{\mathrm{B}}_r$, that is stress free and let $\mathbf{P}$ be the deformation gradient of B_r relative to $\bar{\mathrm{B}}_r$, as depicted in Fig. 8.1. Then, the deformation gradient in B_t relative to $\bar{\mathrm{B}}_r$, denoted $\bar{\mathbf{F}}$, is given by

$$\bar{\mathbf{F}} = \mathbf{FP}. \tag{8.14}$$

Let $\bar{W}(\bar{\mathbf{F}})$ denote the strain energy in B_t relative to $\bar{\mathrm{B}}_r$ per unit volume in $\bar{\mathrm{B}}_r$. Then, for an incompressible material, we must have the connection

$$W(\mathbf{F}) = \bar{W}(\bar{\mathbf{F}}) - \bar{W}(\mathbf{P}). \tag{8.15}$$

The corresponding formula for an unconstrained material is similar but involves factors related to the determinants of the deformation gradients.

According to the discussion in Section 3.4, if $\mathcal{G}$ denotes the symmetry group relative to B_r and $\bar{\mathcal{G}}$ that relative to $\bar{\mathrm{B}}_r$ then $\mathcal{G} = \mathbf{P}\bar{\mathcal{G}}\mathbf{P}^{-1}$. Thus, in the situation where a global stress-free configuration exists the material symmetry in a residually-stressed configuration can be determined directly from

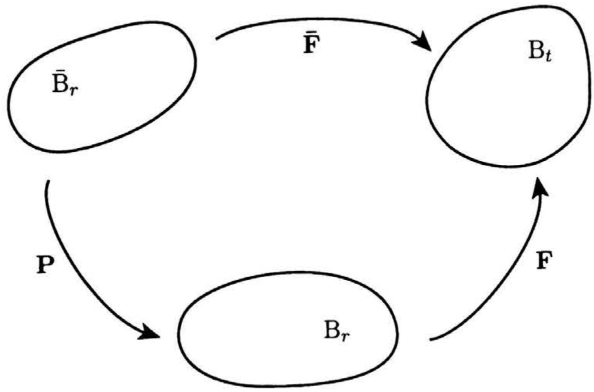

FIGURE 8.1. Schematic of the stress-free reference configuration $\bar{\mathrm{B}}_r$, the residually-stressed reference configuration B_r and the deformed configuration B_t showing the connecting deformation gradients $\mathbf{P}$, $\mathbf{F}$ and $\bar{\mathbf{F}}$.

that in the stress-free configuration without the need to consider invariants associated with the residual stress. The existence of such a stress-free configuration is problematic in general, but in some circumstances a configuration that is approximately stress free can be considered useful (and is the basis of part of the analysis in Chapter 9). The stresses associated with the different configurations are related in the following way.

For an incompressible material the nominal stresses, denoted $\mathbf{S}$ and $\bar{\mathbf{S}}$, relative to B_r and $\bar{\mathrm{B}}_r$ respectively are given by

$$\mathbf{S} = \frac{\partial W}{\partial \mathbf{F}}(\mathbf{F}) - p\mathbf{F}^{-1}, \quad \bar{\mathbf{S}} = \frac{\partial \bar{W}}{\partial \bar{\mathbf{F}}}(\bar{\mathbf{F}}) - \bar{p}\bar{\mathbf{F}}^{-1}, \tag{8.16}$$

while the Cauchy stress $\boldsymbol{\sigma} = \mathbf{FS} = \bar{\mathbf{F}}\bar{\mathbf{S}}$ in B_t is independent of the choice of reference configuration. Thus, we must have the connections

$$\mathbf{S} = \mathbf{P}\bar{\mathbf{S}}, \quad \bar{p} = p, \tag{8.17}$$

which may also be deduced by differentiation of (8.15) and use of (8.14).

When evaluated in B_r these give

$$\mathbf{S}^{(r)} = \boldsymbol{\sigma}^{(r)} = \frac{\partial W}{\partial \mathbf{F}}(\mathbf{I}) - p^{(r)}\mathbf{I} = \mathbf{P}\bar{\mathbf{S}}^{(r)}, \tag{8.18}$$

where $p^{(r)}$ is the value of p in B_r, $\bar{\mathbf{S}}^{(r)}$ is the corresponding value of the residual stress and $\mathbf{I}$ is the identity in B_r.

Chapter 9

Application to arterial tissue

When a length of artery is excised from a body it contracts. Thus, *in vivo* arteries are stretched (i.e. subject to a large axial deformation) and tethered (i.e. held in place) by the surrounding tissue. However, an excised artery, although in an *unloaded* configuration, i.e. it is not subject to any axial load or to any tractions on its inner and outer surfaces, is not unstressed. In fact, there is a *residual stress distribution* through the artery wall, and this has a very important influence on the mechanical response of the artery under physiological conditions. The existence of the residual stresses is demonstrated by the so-called 'opening angle experiment' in which a short length of artery in the form of a ring is cut radially. The ring springs open to form an open sector, thus indicating the presence of a compressive circumferential stress in the inner part of the wall of the ring and a tensile circumferential stress in the outer part. The magnitude of the opening angle gives a rough estimate of the residual stress (at least the circumferential residual stress, but it should be noted that there will in general also be residual axial and radial stresses). However, even such an open sector is not stress free since the opening angles of circumferentially separated layers are different.

In most analyses in the literature to date, however, the opened-up sector is assumed, for simplicity, to be stress free in order to facilitate calculation of the (residual) stress required to re-form the intact ring (the unloaded configuration). It is normally assumed that the ring is a circular annulus, that the opened-up sector is also circular and that the deformation required to re-form the ring depends only on the radius. Any assumptions that are less simple than these would almost certainly require a purely numerical treatment. Some aspects of the opening angle approach are discussed in Section 9.2. The opening angle experiment gives only a very rough estimate of the residual stress, and a detailed understanding of the mechanical influence of residual stress therefore remains to be developed. Influences that need to be accounted for are, for example, growth, remodelling and adaptation since these are clearly candidates for generating residual stresses. Analysis of such effects is at an early stage of development and much more needs to be done in this area.

The residual stresses have an influence on the overall behaviour of an artery under extension and internal pressure and, more significantly, on the stress and strain distributions through the arterial wall. It has been suggested in the literature that in the physiological state a healthy artery has an essentially constant circumferential stress in each layer of its wall (note that because of different material properties in different layers of the artery wall there is a discontinuity in the circumferential stress across a layer boundary, and also, in general, in the axial stress). This can only be the case if there is residual stress present. Some consequences of the assumption of uniform circumferential stress will be examined in Section 9.3. It is interesting to note that the residual stress distributions calculated on the basis of the opening angle method and the uniform circumferential stress assumption are very similar in character.

We begin by extending our previous analysis of the extension and inflation of a thick-walled circular cylindrical tube to allow for residual stresses.

9.1. Extension and inflation of a thick-walled tube

In Section 7.2.3 the problem of extension and inflation of a thick-walled tube was analyzed for the case of an orthotropic material. Here we adapt that theory so as to incorporate residual stresses. The strain energy may again be written in the form (7.54) and is again denoted by $\hat{W}(\lambda, \lambda_z, \varphi)$, with λ and λ_z being the azimuthal and axial stretches. We emphasize, once more, that $\hat{W}(\lambda, \lambda_z, \varphi)$ is not in general symmetric in λ and λ_z and that the angle φ may depend on R.

The principal Cauchy stress differences are given (locally) by

$$\sigma_3 - \sigma_1 = \lambda_z \hat{W}_{\lambda_z}, \quad \sigma_2 - \sigma_1 = \lambda \hat{W}_{\lambda}. \tag{9.1}$$

Residual stresses associated with the unloaded (traction-free) configuration may be incorporated through $\hat{W}$, in which case the residual stress differences are given by (9.1) evaluated for $\lambda = \lambda_z = 1$ and subject to $\sigma_3^{(r)} = 0$, as discussed for the case of orthotropy in Section 8.1. Alternatively, it may be convenient to separate out from the residual stresses the additional stresses required to deform the material from the unloaded configuration. It is then these additional stresses that are accounted for through $\hat{W}$ via (9.1), but the residual stresses (which, in general, are unknown) then have to be incorporated separately. This approach enables the separate contribution of the residual stresses to be highlighted, and is therefore adopted here. Accordingly, and consistently with the assumed cylindrical orthotropy, we replace (9.1) by

$$\sigma_3 - \sigma_1 = \lambda_z \hat{W}_{\lambda_z} + \sigma_3^{(r)} - \sigma_1^{(r)}, \quad \sigma_2 - \sigma_1 = \lambda \hat{W}_\lambda + \sigma_2^{(r)} - \sigma_1^{(r)}, \tag{9.2}$$

where $\sigma_1^{(r)}$, $\sigma_2^{(r)}$ and $\sigma_3^{(r)} = 0$ denote the residual principal Cauchy stresses in the unloaded configuration, in which the terms in $\hat{W}$ in (9.2) vanish. Note that $\sigma_1^{(r)}$ and $\sigma_2^{(r)}$ are independent of the deformation from the unloaded configuration (i.e. they depend only on R).

For the considered cylindrically symmetric deformation the (radial) equilibrium equation for the deformed configurations is

$$\frac{\mathrm{d}\sigma_1}{\mathrm{d}r} + \frac{1}{r}(\sigma_1 - \sigma_2) = 0, \tag{9.3}$$

in terms of the principal Cauchy stresses. The solution of equation (9.3) should satisfy the boundary conditions

$$\sigma_1 = \begin{cases} -P & \text{on} \quad r = a \\ 0 & \text{on} \quad r = b, \end{cases} \tag{9.4}$$

corresponding to pressure $P\,(\geq 0)$ on the inside of the tube and zero traction on the outside. We do not include the effect of tethering and surrounding material here.

In the unloaded configuration the residual stresses must satisfy the equation

$$\frac{\mathrm{d}\sigma_1^{(r)}}{\mathrm{d}R} + \frac{1}{R}(\sigma_1^{(r)} - \sigma_2^{(r)}) = 0, \tag{9.5}$$

and this is coupled with the boundary conditions

$$\sigma_1^{(r)} = 0 \quad \text{on} \quad R = A \text{ and } R = B. \tag{9.6}$$

By making use of (6.10) and (6.12)–(6.14) together with equations (9.2)–(9.6), we obtain

$$P = \int_{\lambda_b}^{\lambda_a} (\lambda^2\lambda_z - 1)^{-1} \frac{\partial \hat{W}}{\partial \lambda} \mathrm{d}\lambda + \lambda_z^{-1} \int_A^B \frac{R^2}{r^2} \frac{\mathrm{d}\sigma_1^{(r)}}{\mathrm{d}R} \mathrm{d}R, \tag{9.7}$$

where, as in (6.21), the independent variable has been changed from r to λ in the first integral, while in the second integral

$$r^2 = a^2 + \lambda_z^{-1}(R^2 - A^2). \tag{9.8}$$

When the residual stress is unknown the latter term in (9.7) cannot be determined. When the residual stress is absent the formula (7.55) is recovered.

Since, from (6.13), λ_b depends on λ_a, equation (9.7) provides an expression for P as a function of λ_a when λ_z is fixed provided that the distribution of residual stress is known. In order to hold λ_z fixed an axial load, N say, must be applied to the ends of the tube. Recalling that $\sigma_3^{(r)} = 0$, this can be expressed, after some rearrangements, in the form

$$\begin{aligned} N/\pi A^2 = (\lambda_a^2\lambda_z - 1) \int_{\lambda_b}^{\lambda_a} (\lambda^2\lambda_z - 1)^{-2} \Big(2\lambda_z \frac{\partial \hat{W}}{\partial \lambda_z} - \lambda \frac{\partial \hat{W}}{\partial \lambda}\Big) \lambda \mathrm{d}\lambda + P\lambda_a^2 \\ - \lambda_z^{-1} \int_A^B (\sigma_1^{(r)} + \sigma_2^{(r)}) R \mathrm{d}R / A^2, \end{aligned} \tag{9.9}$$

and, as for P, this can only be calculated if the residual stress is known.

The formulas (9.7) and (9.9) are valid for a tube with any number of concentric layers and for a general strain energy with the specified symmetry. In general, $\hat{W}$ will be different for each layer, or, at least, the angle φ will be different in each layer. The radial stress is continuous across the boundary between two layers but, as noted above, the circumferential stress is in general discontinuous at such a boundary.

At this point we *emphasize* that the residual stress distribution is unknown, and, therefore, to proceed further we require some means of determining or estimating it. For this purpose some additional information is needed. One possible approach is to take the opened-up sector of an arterial ring after a radial cut to correspond to the unstressed configuration and to investigate the consequences of this assumption. For a thin layer this can be regarded as a reasonable approximation. We now examine some aspects of this 'opening angle experiment'.

9.2. The opening angle method

In Fig. 9.1 an arterial ring in three different configurations is depicted. Figure 9.1(b) shows the cross-section of an intact artery in the unloaded configuration, while (c) corresponds to an artery subject to internal pressure P. The deformation from (b) to (c) has already been discussed. Here, we focus on the deformation from the opened-up configuration, shown in Fig. 9.1(a), to the unloaded configuration (b). For reference, we recall that the strain energy associated with the deformation from (b) to (c) is given by $\hat{W}(\lambda, \lambda_z, \varphi)$, where λ_z (constant) is the axial stretch and $\lambda = r/R$ is the circumferential stretch. The fibre angle in (b) is φ.

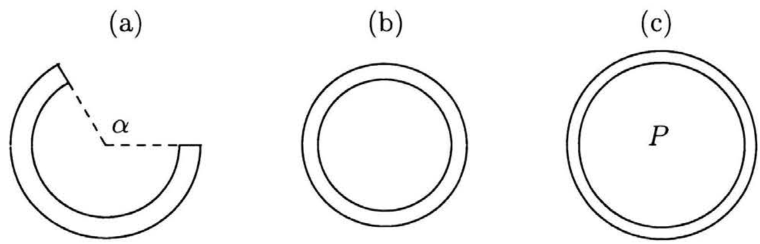

FIGURE 9.1. Arterial ring: (a) opened-up configuration; (b) unloaded intact ring; (c) deformed configuration under pressure P.

We assume that the sector in (a) is circular and has an opening angle α, as indicated in the figure. It should be noted, however, that a different definition of opening angle is often used in the literature. For convenience we introduce the notation

$$k = 2\pi/(2\pi - \alpha), \quad 1 \leq k < \infty, \tag{9.10}$$

as a measure of the opening angle. In the deformation from (a) to (b) we assume that there is a uniform stretch λ_{zo} induced in the axial direction. The radial part of the deformation is then given by

$$R^2 = A^2 + k^{-1}\lambda_{zo}^{-1}(R_o^2 - A_o^2), \tag{9.11}$$

where R_o is the radial coordinate in (a) and A_o is the inner radius. The associated circumferential stretch, denoted λ_o, is

$$\lambda_o = kR/R_o, \tag{9.12}$$

and we denote by φ_o the fibre angle in (a). In the deformation from (a) to (b) the fibre angles are related by

$$\tan\varphi = \lambda_o\lambda_{zo}^{-1}\tan\varphi_o. \tag{9.13}$$

Next, we assume that the deformation from (a) to (b) is an elastic deformation and described by the strain energy $\hat{W}_o(\lambda_o, \lambda_{zo}, \varphi_o)$, where the subscript o is attached to $\hat{W}$ since, in general, the material response relative to (a) will be different from that relative to (b) even after accounting for the change in fibre angle because, in general, the deformation induces anisotropy in the response relative to (b) separate from the anisotropy associated with the fibres.

In most analyses it is assumed that the configuration (a) is stress free. We now show that (under the restrictions adopted) this assumption is valid since *the choice of geometry* necessarily leads to (a) being stress free. Suppose that (a) is not stress free. The geometry ensures that the principal axes of strain are radial and circumferential. Since the deformation is independent of the polar coordinate angle, denoted Θ_o, it follows that the principal axes of stress coincide with those of strain and that the only equilibrium equation not satisfied trivially in (a) is the radial equation

$$\frac{\mathrm{d}\sigma_{o1}^{(r)}}{\mathrm{d}R_o} + \frac{1}{R_o}(\sigma_{o1}^{(r)} - \sigma_{o2}^{(r)}) = 0, \tag{9.14}$$

where $\sigma_{o1}^{(r)}$ and $\sigma_{o2}^{(r)}$ are, respectively, the radial and circumferential (residual) principal stresses in (a). Since the load must vanish pointwise on the (flat) ends of the opened-up ring we must have $\sigma_{o2}^{(r)} = 0$ on those ends (on which Θ_o is constant). It follows from (9.14) that $\mathrm{d}(R_o\sigma_{o1}^{(r)})/\mathrm{d}R_o = 0$ on the ends, and hence for all Θ_o. Integration of this and application of the zero traction condition $\sigma_{o1}^{(r)} = 0$ on $R_o = A_o$ shows that $\sigma_{o1}^{(r)} \equiv 0$ and hence, by (9.14), $\sigma_{o2}^{(r)} \equiv 0$.

This result applies for one layer or for two or more concentric layers, and hence, in particular, for the case of two layers, the interface must form a perfect geometrical match in the configuration (a). In practice this is unlikely to happen, and experiments have shown that this not the case. The length of the outer boundary of the middle layer of an arterial wall (the *media*) is not in general the same as the length of the inner boundary of the outer layer (the *adventitia*) in the opened-up configuration. Moreover, the curvatures of these boundaries are not in general the same. For the media and adventitia to fit together in the opened-up configuration there will necessarily be residual stresses in that configuration. In view of the above analysis such a configuration cannot be described by the geometry discussed above and the deformation from (a) to (b) must depend on Θ_o, and possibly also on the axial coordinate Z. The analysis associated with this more general geometry is, of course, more complicated than described above and will undoubtedly require numerical treatment. In particular, the plane strain assumption is unlikely to be a good approximation to the real situation for a short length of artery. Specifically, the assumption that λ_{zo} is uniform is untenable without the application of an axial load, which we are omitting from consideration here. A further comment on this is made below. The analysis here is based on (9.11) with λ_{zo} constant.

The residual stress distribution in (b) is governed by equation (9.5), which, on integration, gives

$$\sigma_1^{(r)} = \int_A^R (\sigma_2^{(r)} - \sigma_1^{(r)}) \frac{\mathrm{d}R}{R}, \tag{9.15}$$

but now the integrand in (9.15) is given by

$$\sigma_2^{(r)} - \sigma_1^{(r)} = \lambda_o \hat{W}_{o\lambda_o}(\lambda_o, \lambda_{zo}, \varphi_o). \tag{9.16}$$

Thus, in principle, the residual stress can be calculated. However, this requires some additional information.

First, we note that if B_o denotes the outer radius in (a) then the geometrical quantities in (a) and (b) are related by

$$B^2 = A^2 + k^{-1}\lambda_{zo}^{-1}(B_o^2 - A_o^2). \tag{9.17}$$

Secondly, by applying the boundary condition $\sigma_1^{(r)} = 0$ on $R = B$ to (9.15) we obtain

$$\int_A^B \lambda_o \hat{W}_{o\lambda_o}(\lambda_o, \lambda_{zo}, \varphi_o) \frac{\mathrm{d}R}{R} = 0, \tag{9.18}$$

or, equivalently, by changing the integration variable from R to λ_o using (9.11) and (9.12),

$$\int_{\lambda_{ob}}^{\lambda_{oa}} \frac{\hat{W}_{o\lambda_o}(\lambda_o, \lambda_{zo}, \varphi_o)}{\lambda_o^2 \lambda_{zo} - k} \mathrm{d}\lambda_o = 0, \tag{9.19}$$

where λ_{oa} and λ_{ob} are the values of λ_o on the boundaries $R_o = A_o$ and $R_o = B_o$ respectively.

Since our objective is to calculate the residual stress distribution, we suppose that k, A_o, B_o and λ_{zo} are known. Equations (9.17) and (9.19) are then two equations from which to determine A and B, the latter equation depending on the material properties through $\hat{W}$ and φ_o. Note that A, B, A_o, B_o occur in (9.19) only through the limits. Once A and B are determined the residual stresses can be calculated from (9.15) and (9.16). In this way the residual stresses in the unloaded configuration can be determined as functions of the opening angle.

In the above considerations we have not made use of the equation

$$\sigma_3^{(r)} - \sigma_1^{(r)} = \lambda_{zo}\hat{W}_{o\lambda_{zo}}(\lambda_o, \lambda_{zo}, \varphi_o). \tag{9.20}$$

This is important to note since the zero axial load condition $\sigma_3^{(r)} = 0$ in (b) is not in general compatible with the assumed geometrical transformation from (a) to (b). Thus, (9.20) must be regarded as giving the stress distribution $\sigma_3^{(r)}$ needed to maintain the cylindrical geometry in (b), in particular uniform λ_{zo}. As is done in some treatments, this problem can be circumvented by setting to zero the total axial load

$$2\pi \int_A^B \sigma_3^{(r)} R \mathrm{d}R \tag{9.21}$$

so as to determine the value of λ_{zo}. Alternatively, λ_{zo} can be prescribed and $\sigma_3^{(r)}$ calculated from (9.20) once $\sigma_1^{(r)}$ has been determined by the procedure outlined above.

Some results based on the latter approach are shown in Fig. 9.2 with λ_{zo} set to 1. In Fig. 9.2(a) dimensionless radial and circumferential residual

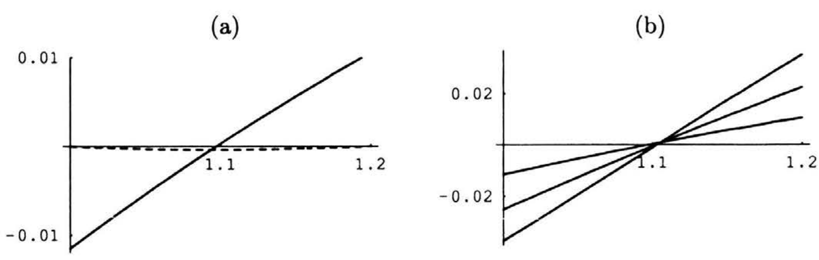

FIGURE 9.2. (a) Plot, in dimensionless form, of the residual radial stress (dashed curve) and residual circumferential stress (continuous curve); $k = 1.5$: (b) comparison of the residual circumferential stresses for $k = 1.5, 1.6, 1.7$.

stresses are plotted against the dimensionless radius R_o/A_o with $B_o/A_o = 1.2$. An opening angle of $2\pi/3$, corresponding to $k = 1.5$, has been selected. The calculations are based on use of the energy function (7.57) and the non-dimensionalization is through division by the constant $|\mu_3|$. Figure 9.2(b) shows a comparison of the circumferential stresses for three different opening angles, corresponding to $k = 1.5, 1.6, 1.7$. Two features should be noted. First, the circumferential stress is compressive on the inner boundary and tensile on the outer boundary; second, the maximum magnitudes of the stresses increase with the value of k. The point at which the stress vanishes is slightly different for the three curves although this is not apparent on the scale used here. The radial stress likewise increases with k but remains very small compared with the circumferential stress and hence the corresponding comparison is not shown.

If it is not assumed that λ_{zo} is uniform then the problem becomes more difficult because the deformation from (a) to (b) then necessarily involves shearing through the wall thickness.

9.3. Uniform circumferential stress

For simplicity of illustration we restrict attention here to a tube with a single layer, but the analysis (although somewhat more complicated) can easily be carried over to a tube with two or more layers. If the circumferential stress $\sigma_2 = \sigma_{20}$ is assumed to be constant then it follows from the equilibrium equation (9.3) and the boundary conditions (9.4) that

$$\sigma_{20} = \frac{P_0 a_0}{b_0 - a_0}, \qquad \sigma_{10} = \sigma_{20}\left(1 - \frac{b_0}{r_0}\right), \tag{9.22}$$

where the zero subscript indicates evaluation at the normal physiological pressure (denoted P_0) and

$$r_0^2 = a_0^2 + \lambda_{z0}^{-1}(R^2 - A^2). \tag{9.23}$$

Note that the zero subscript $_0$ should be distinguished from the 'oh' subscript o used earlier.

Use of equations (8.13), $(9.2)_2$, (9.3) and (9.5) then enables the residual radial stress to be calculated explicitly as

$$\sigma_1^{(r)} = \frac{P_0 a_0 b_0}{b_0 - a_0}\frac{1}{2c_0}\log\left(\frac{(r_0 - c_0)(a_0 + c_0)}{(r_0 + c_0)(a_0 - c_0)}\right) - \int_A^R \lambda_0 \hat{W}_\lambda(\lambda_0, \lambda_{z0}, \varphi)\frac{\mathrm{d}R}{R}, \tag{9.24}$$

where

$$c_0 = (a_0^2 - \lambda_{z0}^{-1} A^2)^{1/2}. \tag{9.25}$$

The corresponding residual circumferential stress is then obtained using (9.5). This leads to

$$\sigma_2^{(r)} = \sigma_1^{(r)} - \lambda_0 \hat{W}_\lambda(\lambda_0, \lambda_{z0}, \varphi) + \frac{a_0 b_0 P_0}{(b_0 - a_0) r_0}. \tag{9.26}$$

Once the residual stresses have been calculated for any given form of $\hat{W}$, the pressure P in a general (cylindrically symmetric) configuration can be calculated from (9.7) and the corresponding stresses from (9.2) and (9.3). The axial load N can be obtained from (9.9).

By applying the boundary condition (9.6) at $R = B$ to (9.24) we obtain

$$\frac{P_0 a_0 b_0}{b_0 - a_0} \frac{1}{2c_0} \log\left(\frac{(b_0 - c_0)(a_0 + c_0)}{(b_0 + c_0)(a_0 - c_0)}\right) = \int_A^B \lambda_0 \hat{W}_\lambda(\lambda_0, \lambda_{z0}, \varphi) \frac{\mathrm{d}R}{R}. \tag{9.27}$$

Since, from (9.23), $b_0^2 = a_0^2 + \lambda_{z0}^{-1}(B^2 - A^2)$, equation (9.27) provides a connection between the pressure P_0 and the internal radius a_0 (equivalently, $\lambda_{0a} = a_0/A$) for any given value of the axial stretch λ_{z0} and aspect ratio B/A.

A representative plot of the residual stresses is shown in Fig. 9.3 in dimensionless form with the dimensionless stresses defined by

$$\sigma_1^{(r)*} = \sigma_1^{(r)} l/\mu_3, \quad \sigma_2^{(r)*} = \sigma_2^{(r)} l/\mu_3, \tag{9.28}$$

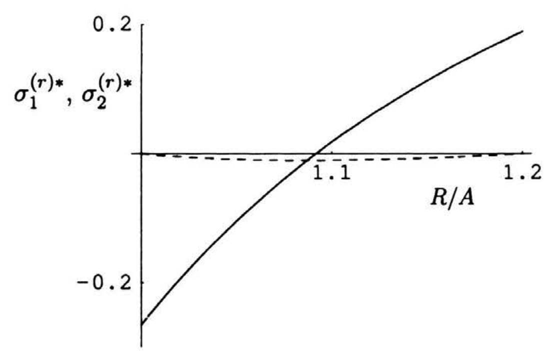

FIGURE 9.3. Plot of the dimensionless residual stress distribution for a typical member of the class of anisotropic strain-energy functions (7.57) based on equations (9.24) (radial stress – dashed curve) and (9.26) (circumferential stress – continuous curve).

where $l > 0$ is defined by

$$l = \log\left(\frac{(b_0 - c_0)(a_0 + c_0)}{(b_0 + c_0)(a_0 - c_0)}\right) \tag{9.29}$$

and μ_3 is the material constant appearing in the strain-energy function (7.57), which has been used in this calculation with $n = 12$ and $\mu_1^* \equiv \mu_1/\mu_3 = 2$. The axial stretch λ_{z0} has been set to 1.2 and the aspect ratio to $B/A = 1.2$. The general qualitative character of the results in Fig. 9.3 is not significantly affected by using different values of the material parameters over quite a large range of values.

We observe that the residual radial stress is quite small and is negative except at the boundaries (where it vanishes). The circumferential stress is compressive at the inner boundary and tensile at the outer boundary, as anticipated on the basis of the opening-angle experiment. It is also much larger in magnitude than the radial stress.

Chapter 10

Boundary-value problems: some exact solutions

As mentioned earlier, there are relatively few exact solutions known for boundary-value problems in nonlinear elasticity when the constraint of incompressibility is not imposed. Here we provide a method of generating forms of strain-energy function for which such solutions can be found. It is based on the idea of considering the possibility that *isochoric* deformations can be maintained in a compressible material. For an incompressible material, the incompressibility constraint is compensated for by including an additional variable – the Lagrange multiplier p – and the governing equations and boundary conditions determine this function. By contrast, if the deformation is assumed to be isochoric in a compressible material there is no such compensation and the equations, in general, over-determine the deformation in such a way that they are incompatible. However, in some problems the equations can be made compatible if suitable restrictions are placed on the form of strain-energy function. In this chapter we illustrate this procedure for a representative problem, namely azimuthal shear of a circular cylindrical tube.

10.1. The azimuthal shear problem

The discussion in this chapter follows closely that in Jiang and Ogden [11], to which we refer for more details. We consider a compressible nonlinearly elastic thick-walled circular cylindrical tube whose cross-section in its natural (unstressed) configuration is defined by

$$0 < A \leq R \leq B, \quad 0 \leq \Theta \leq 2\pi, \tag{10.1}$$

where (R, Θ) are polar coordinates. Attention is restricted to plane deformations in which there is no extension along the axis of the cylinder and the deformation of a cross-section is independent of the axial coordinate, Z say. To maintain plane-strain conditions appropriate axial loading is required on the ends of the tube, but this will not be needed explicitly for our purposes here.

An azimuthal shear deformation is defined by

$$r = r(R), \quad \theta = \Theta + g(R), \quad z = Z, \tag{10.2}$$

where (r, θ, z) are cylindrical polar coordinates associated with the deformed configuration.

We take the boundary conditions as

$$a \equiv r(A) = A, \quad b \equiv r(B) = B, \quad g(A) = 0, \quad g(B) = \psi \tag{10.3}$$

in the cross-section of the tube, ψ being the angle through which the boundary $R = B$ is rotated.

Referred to cylindrical polar coordinates the deformation gradient tensor $\mathbf{F}$ has components

$$\mathsf{F} = \begin{bmatrix} r' & 0 & 0 \\ rg' & r/R & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{10.4}$$

where the prime indicates differentiation with respect to R, and its inverse is

$$\mathsf{F}^{-1} = \begin{bmatrix} 1/r' & 0 & 0 \\ -Rg'/r' & R/r & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{10.5}$$

The principal invariants I_1, I_2, I_3 of the deformation tensor $\mathbf{B} = \mathbf{F}\mathbf{F}^T$ are given by

$$\begin{aligned} I_1 &= r'^2 + r^2 g'^2 + r^2/R^2 + 1, \\ I_2 &= r^2 g'^2 + r^2/R^2 + r'^2 + r^2 r'^2/R^2, \\ I_3 &= r^2 r'^2/R^2, \end{aligned} \tag{10.6}$$

and it follows immediately that

$$I_2 = I_1 + I_3 - 1. \tag{10.7}$$

Note that (10.7) holds in general for plane strain deformations, not only for the deformation (10.2).

With the restriction to plane strain only two of the invariants I_1, I_2, I_3 are independent, and the strain energy $\bar{W}(I_1, I_2, I_3)$ per unit reference volume of a compressible isotropic elastic material may then be regarded as a function of two invariants. Accordingly, we define $\hat{W}(I_1, I_3)$ by

$$\hat{W}(I_1, I_3) = \bar{W}(I_1, I_1 + I_3 - 1, I_3) \tag{10.8}$$

when (10.7) holds identically. Note that $\hat{W}$ here is different from that used earlier.

The in-plane restriction of the nominal stress tensor $\mathbf{S}$ is then calculated as

$$\mathbf{S} = \frac{\partial \hat{W}}{\partial \mathbf{A}} = 2\hat{W}_1 \mathbf{A}^T + 2I_3 \hat{W}_3 \mathbf{A}^{-1}, \tag{10.9}$$

where $\hat{W}_1 = \partial \hat{W}/\partial I_1, \hat{W}_3 = \partial \hat{W}/\partial I_3$ and $\mathbf{A}$ is the in-plane restriction of $\mathbf{F}$, with components given by the leading 2×2 matrix in (10.4) and similarly for $\mathbf{F}^{-1}$. The corresponding (in-plane) Cauchy stress tensor $\boldsymbol{\sigma} = I_3^{-1/2} \mathbf{AS}$ is

$$\boldsymbol{\sigma} = 2I_3^{-1/2} \hat{W}_1 \mathbf{B} + 2I_3^{1/2} \hat{W}_3 \mathbf{I}, \tag{10.10}$$

where $\mathbf{I}$ is the (in-plane) identity tensor and $\mathbf{B}$ is now taken as $\mathbf{AA}^T$.

For the strain energy and the stress to vanish in the natural configuration and for compatibility with the classical (linear) theory of isotropic elasticity we require

$$\begin{gathered} \hat{W}(3,1) = 0, \quad \hat{W}_1(3,1) + \hat{W}_3(3,1) = 0, \\ \hat{W}_1(3,1) = -\hat{W}_3(3,1) = \frac{1}{2}\mu, \end{gathered} \tag{10.11}$$

and

$$\hat{W}_{11}(3,1) + 2\hat{W}_{13}(3,1) + \hat{W}_{33}(3,1) = \frac{1}{4}\kappa + \frac{1}{3}\mu, \tag{10.12}$$

where μ is the shear modulus and κ the bulk modulus in the natural configuration.

After substitution of the components of **S** from (10.9) with (10.4) and (10.5) into the (in-plane) equilibrium equations Div $\mathbf{S} = \mathbf{0}$ two equations are obtained. The radial equation may be written

$$\frac{d}{dR}(Rr'\hat{W}_1) + r\frac{d}{dR}\Big(\frac{rr'}{R}\hat{W}_3\Big) - \frac{r}{R}\hat{W}_1 - rRg'^2\hat{W}_1 = 0, \tag{10.13}$$

while, after integration, the azimuthal equation yields

$$rRS_{R\theta} \equiv r^2\sigma_{r\theta} = 2r^2Rg'\hat{W}_1 = b^2\tau, \tag{10.14}$$

where the constant τ is the value of the azimuthal shear stress $\sigma_{r\theta}$ (or $S_{R\theta}$) at the outer boundary $r = b = B$.

10.2. Pure azimuthal shear

Pure azimuthal shear is the isochoric specialization of the deformation (10.2) corresponding to $r = R$. With this specialization, equations (10.6) reduce to

$$I_2 = I_1 = 3 + r^2g'^2, \quad I_3 = 1, \tag{10.15}$$

and, locally, the deformation is a simple shear with amount of shear rg', the azimuthal direction being the direction of shear.

When the restrictions (10.15) apply equations (10.13) and (10.14) reduce to

$$\frac{d}{dr}(\hat{W}_1 + \hat{W}_3) - rg'^2\hat{W}_1 = 0, \tag{10.16}$$

$$2r^3g'\hat{W}_1 = b^2\tau. \tag{10.17}$$

Let $\gamma = rg'$ denote the amount of shear. Then $\gamma > 0$ is associated with $\tau > 0$ (shearing in the positive θ direction with $g(r) > 0$ for $r > a$) and $\gamma < 0$ corresponds to $\tau < 0$. Thus, we now have $I_1 = 3 + \gamma^2$, as in the case of simple shear discussed in Section 5.6.1. By defining

$$w(\gamma) = \hat{W}(3 + \gamma^2, 1), \tag{10.18}$$

we can rewrite (10.17) as

$$\sigma_{r\theta} \equiv w'(\gamma) = b^2\tau/r^2 \tag{10.19}$$

with $w'(\gamma) > 0\,(< 0)$ for $\gamma > 0\,(< 0)$.

Increasing shear γ corresponds to increasing shearing stress $\sigma_{r\theta}$ provided

$$w''(\gamma) > 0, \tag{10.20}$$

and we therefore impose (10.20) for all γ. The monotonicity of $w'(\gamma)$ implied by (10.20) ensures that, in principle, (10.19) can be inverted to give $\gamma\,(= rg')$ uniquely as a function of r and hence g is determined by integration. Note that from (10.19) and (10.20) it follows that $rg'' + g' < 0\,(> 0)$ when $\gamma > 0\,(< 0)$.

► From (10.18)–(10.20) it is easy to show that the above requirements on $w'(\gamma)$ and $w''(\gamma)$ are equivalent to

$$\hat{W}_1(I_1, 1) > 0, \quad 2(I_1 - 3)\hat{W}_{11}(I_1, 1) + \hat{W}_1(I_1, 1) > 0. \tag{10.21}$$

With these conditions holding we may replace r by I_1 as the independent variable in (10.16) by using (10.15) and (10.17). First, we rewrite (10.16) as

$$r\frac{d}{dr}(\hat{W}_1 + \hat{W}_3) = (I_1 - 3)\hat{W}_1$$

and then differentiate (10.17) with respect to r and use (10.17) again to obtain

$$r\frac{d}{dr}\left(\sqrt{I_1 - 3}\,\hat{W}_1\right) = -2\sqrt{I_1 - 3}\,\hat{W}_1.$$

On elimination of differentiation with respect to r in favour of that with respect to I_1 the combination of the latter two equations leads to the key condition

$$2(I_1 - 1)\hat{W}_{11}(I_1, 1) + 4\hat{W}_{13}(I_1, 1) + \hat{W}_1(I_1, 1) = 0 \tag{10.22}$$

on the strain-energy function.

It is emphasized that equations (10.21) and (10.22) together are *sufficient conditions* for the strain-energy function $\hat{W}$ to admit a pure azimuthal shear deformation *for all* τ (provided $w'(\gamma) \to \infty$ as $\gamma \to \infty$). On the other hand, whilst (10.22) is also a *necessary condition* the inequality $(10.21)_2$ *is not in general necessary* since the latter can be relaxed, if need be, to allow for shear softening effects in which the shear stress exhibits a maximum as a function of γ (with consequent loss of ellipticity). In these circumstances non-uniqueness of solution arises. Existence and uniqueness of solution is guaranteed if (10.20) holds. To ensure existence of solution *for all* τ when the strain-energy satisfies (10.22) and when (10.20) does not hold, the (weaker) requirement is that $w'(\gamma)$ be continuous and unbounded. If the latter has a finite global maximum then there will be values of τ for which solutions do not exist, and this point is illustrated by one of the examples considered below.

We may integrate (10.22) with respect to I_1 to obtain

$$2(I_1 - 1)\hat{W}_1(I_1, 1) + 4\hat{W}_3(I_1, 1) - \hat{W}(I_1, 1) = 0, \qquad (10.23)$$

where the conditions (10.11) have been used to eliminate the constant of integration. Thus, (10.23) is equivalent to but, since it only involves first derivatives, slightly simpler than (10.22).

In the following section we apply (10.23) to a particular class of strain-energy functions. Together with the inequalities (10.21) equation (10.23) then determines a subclass of materials for which the pure azimuthal shear deformation is possible. We then use (10.17) to determine $g(r)$ subject to the boundary conditions (10.3) for several members of the subclass.

10.3. Solutions for a class of strain-energy functions

We now consider the class of strain-energy function for which $\hat{W}$ is given in the form

$$\hat{W}(I_1, I_3) = f(I_1)h_1(I_3) + h_2(I_3), \qquad (10.24)$$

where $f(I_1)$ is to be determined using (10.23) while the functions h_1, h_2 are to be consistent with (10.11) and (10.12). The motivation for considering (10.24) is that for strain-energy functions considered previously for which pure azimuthal shear solutions have been found $W(I_1, I_2, I_3)$ is *linear* in I_1 and I_2 and hence, by (10.7), $\hat{W}(I_1, I_3)$ is linear in I_1, while $h_1(I_3)$, $h_2(I_3)$ have very specific forms. Thus, (10.24) provides a more general class of strain-energy functions for which pure azimuthal shear might be possible.

Without loss of generality we take $h_1(1) = 1$, and we define the constant k by

$$2k = 1 - 4h_1'(1). \qquad (10.25)$$

Then, on use of (10.24), equations (10.11) and (10.12) give

$$f(3) + h_2(1) = 0, \quad f'(3) = \frac{1}{2}\mu, \quad f(3)h_1'(1) + h_2'(1) = -\frac{1}{2}\mu, \qquad (10.26)$$

$$f''(3) + \mu h_1'(1) + f(3)h_1''(1) + h_2''(1) = \frac{1}{4}\kappa + \frac{1}{3}\mu, \qquad (10.27)$$

while (10.23) yields

$$2(I_1-1)f'(I_1)-2kf(I_1)+4h_2'(1)-h_2(1)=0. \tag{10.28}$$

Provided $k \neq 0$ we may, without loss of generality, set

$$4h_2'(1)-h_2(1)=0 \tag{10.29}$$

since the constant particular solution of (10.28) for $f(I_1)$ times $h_1(I_3)$ may be absorbed into $h_2(I_3)$.

It then follows that, in order to satisfy (10.28) and $(10.26)_1$, $f(I_1)$ must have the form

$$f(I_1)=\frac{\mu}{2^k k}(I_1-1)^k, \tag{10.30}$$

from which the values of $f(3)$, $f''(3)$ appearing in (10.26) and (10.27) may be read off. From (10.26), (10.29) and (10.27) we then have

$$h_2(1)=-\mu/k, \quad h_2'(1)=-\mu/4k, \tag{10.31}$$

$$h_2''(1)+\frac{\mu}{k}h_1''(1)=\frac{1}{4}\kappa+\frac{1}{3}\mu+\frac{1}{4}\mu k, \tag{10.32}$$

in terms of the parameter k.

Equation (10.19) may now be written as

$$w'(\gamma)\equiv\mu 2^{1-k}\gamma(2+\gamma^2)^{k-1}=b^2\tau/r^2. \tag{10.33}$$

It is easy to show that (10.20) holds for all γ if and only if

$$k\geq\frac{1}{2}. \tag{10.34}$$

Here we restrict attention to values of k satisfying (10.34) so that γ is determined uniquely as a function of r. Note that in the limiting case $k=1/2$, $w'(\gamma)\to\mu\sqrt{2}$ as $\gamma\to\infty$, which puts an upper bound on admissible values of τ in this case. In what follows we obtain solutions for $g(r)$ from (10.33) with $\gamma=rg'(r)$ for specific values of k.

Case (i): $k=1/2$. With $h_1'(1)=0$ this case yields

$$\hat{W}(I_1,I_3)=\sqrt{2}\mu(I_1-1)^{1/2}h_1(I_3)+h_2(I_3), \tag{10.35}$$

where

$$h_2(1)=-2\mu, \quad h_2'(1)=-\frac{1}{2}\mu, \quad h_2''(1)+2\mu h_1''(1)=\frac{1}{4}\kappa+\frac{11\mu}{24}. \tag{10.36}$$

Equation (10.33) simplifies to

$$\gamma(2+\gamma^2)^{-1/2} = b^2 s/r^2, \tag{10.37}$$

where the notation

$$s = \tau/\mu\sqrt{2} \tag{10.38}$$

is introduced as a dimensionless measure of the shearing stress on $r = b$.

It is easily shown that the solution of (10.37) for $g(r)$ satisfying the boundary conditions (10.3) is

$$g(r) = \frac{1}{\sqrt{2}}\left[\sin^{-1}(\eta s) - \sin^{-1}\left(\frac{b^2 s}{r^2}\right)\right] \tag{10.39}$$

and hence

$$\psi \equiv g(b) = \frac{1}{\sqrt{2}}\left[\sin^{-1}(\eta s) - \sin^{-1} s\right], \tag{10.40}$$

where $\eta = b^2/a^2$. We note that (10.40) has limited validity in that it yields a real value of ψ only if $|s| \leq \eta^{-1}$, and hence an upper bound is placed on the permissible values of the shearing stress, i.e.

$$|\tau| \leq \sqrt{2}\mu a^2/b^2. \tag{10.41}$$

In this sense the applicability of the strain-energy function (10.35) is limited.

Case (ii): $k = 1$**.** In this case we have

$$\hat{W}(I_1, I_3) = \frac{1}{2}\mu(I_1 - 1)h_1(I_3) + h_2(I_3) \tag{10.42}$$

with

$$h_1'(1) = -\frac{1}{4}, \quad h_2(1) = -\mu, \quad h_2'(1) = -\frac{1}{4}\mu, \tag{10.43}$$

$$\mu h_1''(1) + h_2''(1) = \frac{1}{4}\kappa + \frac{7\mu}{12}. \tag{10.44}$$

Equation (10.33) yields

$$\mu\gamma = b^2\tau/r^2 \tag{10.45}$$

and the solution for $g(r)$ satisfying (10.3) is then simply

$$g(r) = \frac{s}{\sqrt{2}}\left(\eta - \frac{b^2}{r^2}\right), \tag{10.46}$$

so that

$$\psi \equiv g(b) = \frac{s}{\sqrt{2}}(\eta - 1), \tag{10.47}$$

s again being defined by (10.38).

Equation (10.47) is precisely the result obtained in the incompressible theory for the neo-Hookean (or Mooney-Rivlin) form of strain-energy function, ψ being linear in s. Equation (10.42) is a compressible version of the neo-Hookean strain-energy for plane deformations. The solution (10.46) is valid for any functions $h_1(I_3), h_2(I_3)$ satisfying $h_1(1) = 1$ and the conditions (10.43) and (10.44).

Case (iii): $k = 3/2$**.** Here we have

$$\hat{W}(I_1, I_3) = \frac{\mu}{3\sqrt{2}}(I_1 - 1)^{3/2} h_1(I_3) + h_2(I_3), \tag{10.48}$$

with (10.31) and (10.32) appropriately specialized, and (10.33) becomes

$$\gamma(2 + \gamma^2)^{1/2} = 2b^2 s / r^2. \tag{10.49}$$

Equation (10.49) can be solved to give

$$g(r) = \sqrt{2}\, m(a) - \sqrt{2}\, m(r) + \frac{1}{\sqrt{2}} \tan^{-1} m(r) - \frac{1}{\sqrt{2}} \tan^{-1} m(a), \tag{10.50}$$

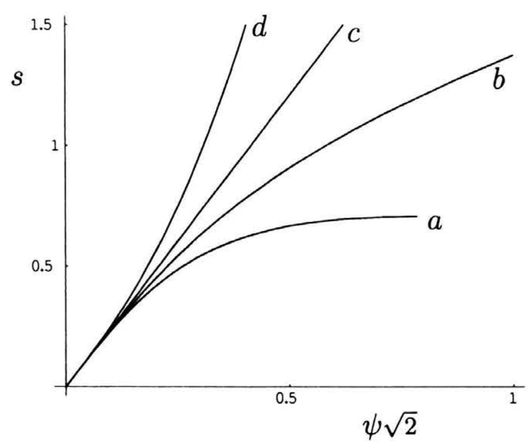

FIGURE 10.1. Plot of the dimensionless shear stress s against $\psi\sqrt{2}$ for $\eta = \sqrt{2}$ and the following values of k: (a) 1/2, (b) 3/4, (c) 1, (d) 3/2.

where $m(r)$ is defined as

$$m(r) = \frac{1}{\sqrt{2}}\left[\left(1 + \frac{4b^4s^2}{r^4}\right)^{1/2} - 1\right]^{1/2}. \tag{10.51}$$

For illustration, numerical results in which s is plotted against $\psi\sqrt{2}$ are shown in Fig. 10.1 for the above three cases for a representative value of $\eta = \sqrt{2}$ together with the corresponding result for $k = 3/4$.

10.4. Strain-energy functions in terms of i_1, i_2, i_3

We now consider an alternative representation for the strain-energy function W in terms of the principal invariants i_1, i_2, i_3 of the stretch tensor $\mathbf{V}$. In general, I_1, I_2, I_3 are given in terms of i_1, i_2, i_3 by

$$I_1 = i_1^2 - 2i_2, \quad I_2 = i_2^2 - 2i_1 i_3, \quad I_3 = i_3^2, \tag{10.52}$$

but for the plane deformation considered here these may be reduced using (10.7) and, analogously to (10.7), we have

$$i_2 = i_1 + i_3 - 1. \tag{10.53}$$

Thus, we may write

$$\hat{W}(I_1, I_3) = \tilde{W}(i_1, i_3), \tag{10.54}$$

where $\tilde{W}$ satisfies

$$\tilde{W}(3,1) = 0, \quad \tilde{W}_1(3,1) = -\tilde{W}_3(3,1) = 2\mu, \tag{10.55}$$

$$\tilde{W}_{11}(3,1) + 2\tilde{W}_{13}(3,1) + \tilde{W}_{33}(3,1) = \kappa + \frac{4}{3}\mu. \tag{10.56}$$

In terms of $\tilde{W}$ it is easy to show by using (10.52) and (10.53) that the condition (10.23) becomes

$$(i_1 - 1)\tilde{W}_1(i_1, 1) + 2\tilde{W}_3(i_1, 1) - \tilde{W}(i_1, 1) = 0. \tag{10.57}$$

This prompts consideration of the class of strain-energy functions defined by

$$\tilde{W} = \tilde{f}(i_1)\tilde{h}_1(i_3) + \tilde{h}_2(i_3), \tag{10.58}$$

in parallel with (10.24).

Following the procedure used in Section 10.3 we obtain

$$\tilde{f}(i_1) = \frac{4\mu}{2^{\tilde{k}}\tilde{k}}(i_1 - 1)^{\tilde{k}}, \tag{10.59}$$

with

$$\tilde{h}_1(1) = 1, \quad \tilde{h}_1'(1) = \frac{1}{2}(1 - \tilde{k}), \tag{10.60}$$

$$\tilde{h}_2(1) = -\frac{4\mu}{\tilde{k}}, \quad \tilde{h}_2'(1) = -\frac{2\mu}{\tilde{k}}, \tag{10.61}$$

$$-\frac{4\mu}{\tilde{k}}\tilde{h}_1''(1) + \tilde{h}_2''(1) = \kappa + \frac{4}{3}\mu, \tag{10.62}$$

where $(10.60)_2$ defines the parameter $\tilde{k}$.

The equilibrium equation (10.19) becomes

$$w'(\gamma) \equiv \mu 2^{2-\tilde{k}}\gamma(4 + \gamma^2)^{(\tilde{k}-2)/2} = b^2\tau/r^2, \tag{10.63}$$

and (10.20) is satisfied for all γ if and only if

$$\tilde{k} \geq 1. \tag{10.64}$$

If $\tilde{k} = 2$ equation (10.63) yields, apart from differences in notation, the same solution as in Case (ii) in Section 10.3, and (10.24) and (10.58) represent the same (compressible neo-Hookean) form of strain-energy function.

If $\tilde{k} = 1$ then (10.58) becomes

$$\tilde{W}(i_1, i_3) = 2\mu(i_1 - 1)\tilde{h}_1(i_3) + \tilde{h}_2(i_3), \tag{10.65}$$

with (10.60)–(10.62) appropriately specialized, and (10.63) reduces to

$$\gamma(4 + \gamma^2)^{-1/2} = b^2 s/\sqrt{2}r^2, \tag{10.66}$$

where s is again defined by (10.38).

Equation (10.66) is solved for $g(r)$ using $\gamma = rg'(r)$ to give

$$g(r) = \sin^{-1}\left(\frac{\eta s}{\sqrt{2}}\right) - \sin^{-1}\left(\frac{b^2 s}{\sqrt{2}r^2}\right), \tag{10.67}$$

and the twist-shearing stress relationship is therefore

$$\psi \equiv g(b) = \sin^{-1}\left(\frac{\eta s}{\sqrt{2}}\right) - \sin^{-1}\left(\frac{s}{\sqrt{2}}\right). \tag{10.68}$$

Comparison of (10.66) with (10.37) shows that (10.67) may be obtained from (10.39) by making the transformations $g \to g/\sqrt{2}, s \to s/\sqrt{2}$ in (10.39).

Equation (10.67) should be compared with (10.39) arising in Case (i) in Section 10.3. As with (10.39) the solution (10.67) is restricted to a finite range of values of s, in this case $|s| \leq \sqrt{2}\eta^{-1}$.

Solutions for certain other values of $\tilde{k}$ may also be obtained explicitly. It suffices here to mention briefly one such solution.

For $\tilde{k} = 3$ we have

$$\tilde{f}(i_1) = \frac{1}{6}\mu(i_1 - 1)^3, \tag{10.69}$$

and (10.19) becomes

$$\gamma(4 + \gamma^2)^{1/2} = 2\sqrt{2}sb^2/r^2. \tag{10.70}$$

Again the transformations $g \to g/\sqrt{2}, s \to s/\sqrt{2}$ enable the solution of (10.70) to be read off from (10.50) with (10.51). More generally, the transformations $k \to \tilde{k}/2, \gamma \to \gamma/\sqrt{2}, \tau \to \tau/\sqrt{2}$ take (10.33) into (10.63), thus establishing a direct relationship between the solutions of (10.33) and (10.63) for all k in the considered range of values. Curves of s against $\psi\sqrt{2}$ for these strain energy functions are very similar to those shown in Fig. 10.1 but coincide only for $\tilde{k} = 2$, which corresponds to $k = 1$. Further details are given by Jiang and Ogden [11].

10.5. Use of the stretches

In Sections 10.3 and 10.4 we have obtained solutions by considering two distinct pairs of deformation invariants, namely (in plane strain) (I_1, I_3) and (i_1, i_3). Further solutions may be obtained by using different pairs of invariants or the stretches, λ_1, λ_2 say, with $\lambda_3 = 1$. In the latter case we note that (10.23), or equivalently (10.57), may be expressed in terms of the stretches to give

$$\lambda_1 \frac{\partial W}{\partial \lambda_1} + \lambda_2 \frac{\partial W}{\partial \lambda_2} - W = 0, \tag{10.71}$$

evaluated for $\lambda_1\lambda_2 = 1$.

We illustrate this by considering a strain energy of the form

$$W = f(I^{(\alpha)})h_1(J) + f(I^{(-\alpha)})h_2(J) + h_3(J), \tag{10.72}$$

where

$$I^{(\alpha)} = \lambda_1^{(\alpha)} + \lambda_2^{(\alpha)} + \lambda_3^{(\alpha)}, \quad J = \lambda_1\lambda_2\lambda_3, \tag{10.73}$$

as discussed by Kirkinis and Ogden [13]. It follows that for $J = 1$ with $\lambda_3 = 1$ we have $I^{(-\alpha)} = I^{(\alpha)}$. By taking $h_1(1) = h_2(1)$ it is then easy to show that (10.71) is satisfied for *arbitrary* functions f and arbitrary values of α provided the mild restrictions

$$h_1'(1) + h_2'(1) = 1, \quad 2h_3'(1) = h_3(1) \tag{10.74}$$

are imposed.

Then,

$$w(\gamma) = 2f(I) + h_3(1), \quad 2f(3) + h_3(1) = 0, \tag{10.75}$$

where $I = I^{(\alpha)} = I^{(-\alpha)} = \lambda^\alpha + \lambda^\alpha + 1$, $\gamma = \lambda - \lambda^{-1}$ and we have set $\lambda = \lambda_1, \lambda_2 = \lambda^{-1}$. Equation (10.19) now gives

$$w'(\gamma) \equiv 2\alpha f'(I)\frac{\lambda^\alpha - \lambda^{-\alpha}}{\lambda + \lambda^{-1}} = \frac{\tau b^2}{r^2}. \tag{10.76}$$

For particular choices of α and the function f this equation can be solved in a similar way to Sections 10.3 and 10.4. For further details see [13].

10.6. The incompressible problem

Results for incompressible materials may be deduced from those for pure azimuthal shear in a compressible material by considering the incompressible material to have strain-energy function (in plane strain), $\hat{W}_{\text{inc}}$ say, defined by

$$\hat{W}_{\text{inc}}(I_1) = \hat{W}(I_1, 1), \tag{10.77}$$

where $\hat{W}(I_1, I_3)$ is defined by (10.8).

Correspondingly, in terms of the amount of shear γ, we define $w_{\text{inc}}(\gamma)$ by

$$w_{\text{inc}}(\gamma) = \hat{W}_{\text{inc}}(3 + \gamma^2). \tag{10.78}$$

Then, equation (10.19), which serves to determine γ, is unchanged but now written

$$w_{\text{inc}}'(\gamma) = b^2\tau/r^2. \tag{10.79}$$

For an incompressible material the Cauchy stress tensor $\boldsymbol{\sigma}$ in (10.10) is replaced by the plane-strain specialization of (5.7), namely

$$\boldsymbol{\sigma} = 2\hat{W}_{\text{inc}}'(I_1)\mathbf{B} - p\,\mathbf{I}, \tag{10.80}$$

where p is the arbitrary hydrostatic pressure associated with the incompressibility constraint. The radial equation of equilibrium may be written

$$r\frac{d\sigma_{rr}}{dr} = \sigma_{\theta\theta} - \sigma_{rr} \equiv \gamma w'_{\text{inc}}(\gamma). \tag{10.81}$$

The role of this equation in the incompressible theory is different from that of its counterpart (10.16) in the compressible theory. For any given incompressible isotropic strain-energy function equation (10.81) serves to determine σ_{rr} (or, equivalently, p) once γ is found using (10.79).

Equations (10.79) and (10.81) involve no restriction on the strain-energy function other than that imposed by the incompressibility constraint. But, by adapting the strain-energy functions discussed in Sections 10.3 and 10.4 to the incompressible situation the solutions obtained there are seen to be equally valid for incompressible materials. For example, by taking

$$\hat{W}_{\text{inc}}(I_1) = f(I_1) - f(3) \tag{10.82}$$

with $f(I_1)$ given by (10.30) we have

$$w_{\text{inc}} = \frac{\mu}{2^k k}\left[(2+\gamma^2)^k - 2^k\right]. \tag{10.83}$$

Similarly, in view of the connection $I_1 = i_1^2 - 2i_1$ obtained from (10.52) and (10.53) with $i_3 = 1$, we may consider

$$\hat{W}_{\text{inc}}(I_1) = \tilde{f}(i_1) - \tilde{f}(3) \tag{10.84}$$

with (10.59) and obtain

$$w_{\text{inc}} = \frac{4\mu}{2^{\tilde{k}}\tilde{k}}\left[(4+\gamma^2)^{\tilde{k}/2} - 2^{\tilde{k}}\right]. \tag{10.85}$$

The solutions given in Sections 10.3 and 10.4 for specific values of k and $\tilde{k}$ can now be applied in the incompressible situation and equation (10.81) may be used to calculate the complete stress distribution.

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