Arch. Mech., 44, 1, pp. 117–132, Warszawa 1992

Kinematics of finite deformations

P. ROUGEE (CACHAN)

THE SIX-DIMENSIONAL nonlinear space of all possible local configurations of continuous media at unrestricted strains is built. Its Riemannian geometry is given. In this way the intrinsic Lagrangian strain rate and its associated stress tensor are defined independently of any strain measure and any reference configuration. Geometrical characterization of large change of shape is studied.

1. Introduction

DESPITE a very important research it appears, as it was for instance shown in NAGHDI [14] where a large discussion and a large bibliography on the subject may be found, that in finite plasticity, areas of disagreement are still far more important that areas of agreement. It appears in particular that important disagreements occur as soon as basic and fundamental kinematic considerations are given. Effectively, *it is the kinematics of strain in large displacements of a continuous media which is not completely elucidated*, and this is obviously a real impediment to the modelling of stresses and thus to the statement of constitutive laws. Let us see some elementary manifestations of this.

In Eulerian (or spatial) approach, the strain rate D, symmetrical part of the gradient of the Eulerian field of velocity V, and Kirchhoff's stress σ/ρ where σ is Cauchy's stress and ρ the mass density, associated in the specific internal stress power $\mathcal{P}$ by $\mathcal{P} = -(\sigma/\rho) : D$, are uncontested tools. But between Jauman's derivative, four possible convective derivatives and many other possibilities, which is the good choice when, for example, we have to take into account the stress rate in constitutive laws (TRUESDELL [25], TRUESDELL and NOLL [26], DOGUI and SIDOROFF [4])? Also, there is a question which is generally not put forward: of what configuration variable, and with what derivative rule, is D the time material derivarive?

In Lagrangian (or material) approach, the basic kinematics tools are the strain measures (GREEN and NAGHDI [5], HILL [7] and [8], CASEY [3]). But what is the influence of the chosen reference configuration and, more important, what strain measure to choose? Moreover, is the multiplicity of possible non-equivalent strain measures not the proof that this physical concept issued of small displacement theory cannot be extended to large displacements? An indirect proof of this is that the stress tensors associated to strain measures, as the second Piola–Kirchhoff stress tensor K associated to Green strain measure Σ, are always very bad models of stress. For instance, being considered a doubly-contravariant tensor, K has no eigenelements and considered a one-co-one-contravariant tensor, its eigenelements are not related to those of Cauchy stress σ (for instance, when σ is an hydrostatic pressure, eigenaxes of K are the strain eigenaxes!). *There is thus no Lagrangian way to exhibit the principal stresses and stress axes today.* The only response to those questions are some strain measure invariance properties (HILL [8], KLEIBER and RANIECKI [9]). Note also that if the coherence with the Eulerian approach is performed globally for the specific internal power $\mathcal{P}$ by means of relations such as

$(\sigma/\rho) : D = (K/\rho_1) : \dot{\Sigma}$ where ρ_1 is the mass density in the reference configuration, it is not performed for either factor by means of relations $(\sigma/\rho) = f(K/\rho_1)$ and $D = f(\dot{\Sigma})$ with f for a single push-pull rule between space and matter (such as the four classical convective transports but which would be the same for the two factors). And this explains the incompatibilities between the Lagrangian and Eulerian approaches, which disappears only by taking very general constitutive laws, but too general to be physically meaningfull (CASEY and NAGHDI [3]).

In order to overcome these difficulties we have proposed (ROUGEE [20, 21, 22]) a new intrinsic Lagrangian frame, mainly a kinematic variable and its associated stress variable. The kinematic variable is not a strain measure, always strongly dependent on the chosen reference configuration, but a variable m modeling the metric properties of the media themselves. We call it the (local) metric of the media but, and this is the key to our approach, it is not, as usual, a metric tensor. Moreover, it is not a variable lying, as a metric tensor or a strain measure, in a linear space. *It lies in a nonlinear space M whose geometry models the specific nonlinearity of the finite deformation kinematics.* The strain rate will then be the metric rate $\dot{m}$, an element of the plane Tm tangent in m to M, and on this well elucidated kinematics the modeling of stress and the statement of constitutive laws may be carried out, suitably in a Lagrangian (i.e. material) way as it has to be, without any of the difficulties previously pointed out, and in a perfect consistence with the Eulerian point of view. Note that our manifold M is the "six-dimensional configurational space" only evoked in (KLEIBER, RANIECKI [9]).

In this paper, after some algebraic reviews in Sec. 1, our purpose is first to submit our approach and to recall and explain the main results in Secs. 2 and 3. It is second to deal with finite strain "measures" in Sec. 4 and kinematics with elastic released intermediate state in Sec. 5. In order to avoid artificial problems of invariance properties our approach is entirely intrinsic. Starting from the classical material body $\mathcal{B}$ we develop the mathematical model of matter without the help of any particular space frame, any reference configuration, any coordinate system, but we give its correspondence in such classical tools. Although we call them "tensors" as usual, almost all our variables are in fact linear maps precisely defined which we do not identify with their matrix in some basis. For any physical concept we carefully distinguish its *modeling* by a point x in a space X — which implies that the mathematical structure of X is a significant part of the physical properties that we wish to postulate and express — from a simple *characterization* by a point y of a space Y — which does not imply a physical meaning for all the mathematical structure of Y. Being essential physical properties, the constitutive laws have to be first and before all expressed with variables as x.

1. Review of some algebraic results

For two finite-dimensional linear spaces E and F, $L(E, F)$ denotes the linear space of linear maps of E into F. We note: $L(E)$ for $L(E, E)$, E^* for the dual space $L(E, R)$, $\langle\ ,\ \rangle_E$ for the associated duality, $A^* \in L(F^*, E^*)$ for the adjunct of $A \in L(E, F)$ defined by $\langle A^* f, u\rangle_E = \langle f, Au\rangle_F$ for any $u \in E$ and $f \in F^*$, $L_s(E, E^*)$ for the subset of $A \in L(E, E^*)$ such as $\langle Au, v\rangle_E$ is a symmetrical bilinear form, and $L_s^+(E, E^*)$ for the subset of $A \in L_s(E, E^*)$ such as $\langle Au, u\rangle_E$ is definite positive. The last set is a six-dimensional open domain of $L_s(E, E^*)$ when E is a three-dimensional linear space.

If E is a Euclidean space it is endowed with a scalar product $\overset{E}{\cdot}$ characterized by a metric tensor $g_E \in L_s^+(E, E^*)$ such as $u \overset{E}{\cdot} v = \langle g_E u, v\rangle_E$, and its dual space E^* is also a Euclidean space with, identifying E^{**} to E, the metric tensor $g_{E^*} = g_E^{-1} \in L_s^+(E^*, E)$.

If E and F are two Euclidean linear spaces, to any $A \in L(E, F)$ is associated its Euclidean adjunct (or transposed) $A^T \in L(F, E)$, defined by $A^T v \overset{E}{\cdot} u = v \overset{F}{\cdot} Au$ for any $u \in E$ and $v \in F$, or equivalently by $A^T = g_E^{-1} A^* g_F$.

If E is Euclidean, $L(E)$ is canonically a Euclidean space with the scalar product $A : B = \mathrm{Tr}(A^T B)$, any $A \in L(E)$ is symmetrical if $A^T = A$ and skewsymmetrical if $A^T = -A$, the corresponding subsets $L_s(E)$ and $L_a(E)$ of symmetrical and skewsymmetrical A are supplementary and orthogonal subspaces of $L(E)$, the trace operator satisfies $\mathrm{Tr}\, A = 1_E : A$, and the deviatoric part A_D of A is its orthogonal projection on the subset $L_D(E)$ orthogonal to 1_E in $L(E)$.

2. Metric tensors and dual stress variables

In continuous media mechanics the vicinity of a point P_0 of the media is modeled by a linear space T_0 which is the tangent linear space in P_0 to a manifold $\mathcal{B}$ modeling the whole body [MOREAU [13], NOLL [16]]. This T_0 models the (little) "material segments" issued of P_0 *while its dual space T_0^* models the (little) "material slices"*, the slice modeled by $f_0 \in T_0^*$ being the set of points Q_0 in the vicinity of P_0 such as $0 \leq \langle f_0, \overrightarrow{P_0Q_0}\rangle_{T_0} \leq 1$.

Given a space frame $\mathcal{E}$ of associated Euclidean linear space E with metric tensor g, and given a global placement $p : P_0 \to P$ of $\mathcal{B}$ in $\mathcal{E}$, the associated local placement in P_0 is the linear map $a = dP/dP_0 = p'(P_0) \in L(T_0, E)$ which with correlated orientations of $\mathcal{B}$ and $\mathcal{E}$ is always positive. The position of the slice $f_0 \in T_0^*$ is the set of spatial points $Q \in \mathcal{E}$ such as $0 \leq \langle f_0, a^{-1}\overrightarrow{PQ}\rangle_{T_0} = \langle a^{-*} f_0, \overrightarrow{PQ}\rangle_E \leq 1$, i.e. the slice $a^{-*} f_0 \in E^*$. Thus a is the local placement of material segments *while $a^{-*} = (a^{-1})^*$ is the local placement of material slices*. In a classical approach a reference placement $p_1 : P_0 \to P_1$ with local placement $a_1 = dP_1/dP_0 \in L(T_0, E_1)$ is introduced (we note (E_1, g_1) for (E, g) when associated to P_1), and the classical gradient of the displacement $p_0 p_1^{-1} : P_1 \to P$ is $F = dP/dP_1 = a a_1^{-1}$.

Our first aim is to model, by a point m lying in a suitable space M, the local metric properties presented by the vicinity of P_0 in a spatial placement p. These properties are those of the material segments *but also those of the material slices*, and thus they are clasically described by *the two* scalar products obtained in T_0 and T_0^* by carrying with a and a^{-*} those of E and E^*:

$$\mathbf{U}_0 \cdot \mathbf{V}_0 = \langle g a \mathbf{U}_0, a \mathbf{V}_0\rangle_E = \langle \gamma \mathbf{U}_0, \mathbf{V}_0\rangle_{T_0} \qquad \text{in } T_0\,,$$
$$f_0 \cdot g_0 = \langle a^{-*} f_0, g^{-1} a^{-*} g_0\rangle_E = \langle f_0, \gamma^{-1} g_0\rangle_{T_0} \qquad \text{in } T_0^*$$

with, putting $\overline{\Gamma} = L_s(T_0, T_0^*)$ and $\underline{\Gamma} = L_s(T_0^*, T_0)$,

$$\gamma = a^* g a \in \overline{\Gamma}^+ \quad \text{and} \quad \gamma^{-1} \in \underline{\Gamma}^+ \tag{2.1}$$

as associated metric tensors.

By time derivation of (2.1) we obtain

$$D = \frac{1}{2}(\dot{a}a^{-1} + (\dot{a}a^{-1})^T) = g^{-1} a^{-*} \dot{\overline{m}} a^{-1} = a \dot{\underline{m}} a^* g\,, \tag{2.2}$$

with $\overline{m} = \frac{1}{2}\gamma$ and $\underline{m} = -\frac{1}{2}\gamma^{-1}$ that we take in the following as variables characterizing metric properties of T_0 and T_0^* respectively, instead of γ and γ^{-1} and for reasons which will appear later. As a consequence, $\sigma \in L_s(E)$ being Cauchy's stress tensor and ρ being the mass density, the specific internal stress power $\mathcal{P} = -\frac{\sigma}{\rho} : D$ satisfies

$$(2.3) \qquad -\mathcal{P} = \mathrm{Tr}(\underline{\theta}\dot{\overline{m}}) = \mathrm{Tr}(\overline{\theta}\dot{\underline{m}}), \quad \underline{\theta} = a^{-1}\frac{\sigma}{\rho}g^{-1}a^{-*}, \quad \overline{\theta} = a^* g\frac{\sigma}{\rho}a\,,$$

where the introduced variables $\underline{\theta} \in \underline{\Gamma}$ and $\overline{\theta} \in \overline{\Gamma}$ appear as intrinsic Lagrangian stress variables, respectively, associated to metric rate variables $\dot{\overline{m}}$ and $\dot{\underline{m}}$. We thus obtain two couples of associated Lagrangian metric and stress variables, $(\overline{m}, \underline{\theta})$ and $(\underline{m}, \overline{\theta})$, the first of which has been used before in (NOLL [15], ROUGEE [19]). They are intrinsic, independent of any reference configuration, of any coordinate system, *but unfortunately they are two and not only one*.

Before solving this problem, let us connect these tools to more classical ones, first by introducing a time-independent reference configuration p_1. Metric variables $\overline{m}$ and $\underline{m}$ being points in linear spaces $\overline{\Gamma}$ and $\underline{\Gamma}$, they lead to specific intrinsic strain measures: the finite variations $\Delta\overline{m} = \overline{m} - \overline{m}_1$ and $\Delta\underline{m} = \underline{m} - \underline{m}_1$. Let Σ and Σ' be the classical Lagrangian–Green and Almansi strain measures, and K and K' be their associated stress variables (K is the second Piola-Kirchhoff stress tensor), all elements of $L_s(E_1)$:

$$(2.4) \qquad \begin{gathered} \Sigma = \frac{1}{2}(C - 1_E), \quad \Sigma' = \frac{1}{2}(1_E - C^{-1}), \quad C = U^2 = F^T F, \quad F = aa_1^{-1}, \\ \frac{K}{\rho_1} = F^{-1}\frac{\sigma}{\rho}F^{-T}, \quad \frac{K'}{\rho_1} = F^T\frac{\sigma}{\rho}F, \quad -\mathcal{P} = \frac{K}{\rho_1} : \dot{\Sigma} = \frac{K'}{\rho_1} : \dot{\Sigma}'. \end{gathered}$$

It is easy to see that

$$(2.5) \qquad \begin{gathered} \Sigma = g_1^{-1}a_1^{-*}\Delta\overline{m}a_1^{-1}, \quad \Sigma' = a_1\Delta\underline{m}a_1^* g_1, \\ \frac{K}{\rho_1} = a_1\underline{\theta}a_1^* g_1, \quad \frac{K}{\rho_1} = g_1^{-1}a_1^{-*}\overline{\theta}a_1^{-1}, \end{gathered}$$

which shows that classical couples (Σ, K) and (Σ', K') are the representations of our couples $(\overline{m}, \underline{\theta})$ and $(\underline{m}, \overline{\theta})$ through some constant maps associated to, and thus depending on, the chosen reference configuration.

Moreover, introduce a Lagrangian coordinate system α and its associated local bases: $S_0 = \dfrac{dM_0}{d\alpha} = [S_{01}, S_{02}, S_{03}]$ with $S_{0i} = \dfrac{\partial M_0}{\partial \alpha_i}$, $S_1 = \dfrac{dM_1}{d\alpha} = a_1 S_0$, time-independent in T_0 and E_1 respectively, and $S = \dfrac{dM}{d\alpha} = aS_0$ the time-dependent convected basis in E. In *Euclidean* spaces E and $E_1 = E$, any second order tensor of any kind as g, g^{-1}, Σ, Σ', K, ... has four systems of components associated to the bases S in E and S_1 in E_1, as for instance $g_{\alpha\beta}$, $g^{\alpha\beta}$, $g^{\alpha}{}_{\beta} = g_{\beta}{}^{\alpha} = \delta^{\alpha}_{\beta}$ for g, and with rules as $\Sigma_{\alpha\beta} = g_{\alpha\gamma}\Sigma^{\gamma}_{\beta}, \ldots$ But not in T_0 which is not a Euclidean space (γ is only a time-dependent variable and not a material constant as g is a spatial constant): we have only components $x_{\alpha\beta}$ for tensors x in $\overline{\Gamma}$ as γ, $\overline{m}$, $\overline{\theta}$, and $x^{\alpha\beta}$ for γ^{-1}, $\underline{m}$, $\underline{\theta}$ in $\underline{\Gamma}$. Making use of the related bases S_0, S_1, S

we can see that

$$
(2.6)\quad
\begin{aligned}
&\gamma_{\alpha\beta} = g_{\alpha\beta}, \quad (\Delta\overline{m})_{\alpha\beta} = \Sigma_{\alpha\beta} = \frac{1}{2}(g_{\alpha\beta} - g_{1\alpha\beta}), \quad (\dot{\overline{m}})_{\alpha\beta} = \frac{1}{2}\dot{g_{\alpha\beta}} = D_{\alpha\beta},\\
&\gamma^{-1\alpha\beta} = g^{\alpha\beta}, \quad (\Delta\underline{m})^{\alpha\beta} = \Sigma'^{\alpha\beta} = \frac{1}{2}(g^{\alpha\beta} - g_1^{\alpha\beta}), \quad (\dot{\underline{m}})^{\alpha\beta} = -\frac{1}{2}\dot{g^{\alpha\beta}} = D^{\alpha\beta},\\
&\underline{\theta}^{\alpha\beta} = \left(\frac{K}{\rho_1}\right)^{\alpha\beta} = \left(\frac{\sigma}{\rho}\right)^{\alpha\beta}, \quad \overline{\theta}_{\alpha\beta} = \left(\frac{K'}{\rho_1}\right)_{\alpha\beta} = \left(\frac{\sigma}{\rho}\right)_{\alpha\beta},
\end{aligned}
$$

which presents a beginning of unity for both Lagrangian and Eulerian approaches as regards matricial calculus, and also emerges as some usual formalism making use of components $g_{\alpha\beta}$ and $g^{\alpha\beta}$ of the spatial metric. This formalism is common, but its physical meaning is obtained only after a toilsome byroad: g is a physical constant, thus variations of $g_{\alpha\beta}$ and $g^{\alpha\beta}$ reflect only variations of the convected basis S (note that $\frac{d}{dt}(g_{\alpha\beta}) \neq \left(\frac{dg}{dt}\right)_{\alpha\beta} = 0$) which themselves reflect variations of the studied metric properties. On the contrary, variations of $\gamma_{\alpha\beta}$ reflect only variations of γ because S_0 is time-independent. And to say for instance (MARSDEN [12], SIMO [23]) that a strain energy which in Lagrangian coordinates appears as function of $\gamma_{\alpha\beta} = g_{\alpha\beta}$ is a "function of space metric tensor" is physically incomprehensible.

A last remark is that although Σ and K, lying in $L_s(E_1)$, have four kinds of components, one often uses only $\Sigma_{\alpha\beta}$ and $K^{\alpha\beta}$. This is a way to work in fact with our more intrinsic variables $(\overline{m}, \underline{\theta})$ not depending on the reference configuration.

3. The manifold M of metrics m and associated stress tensor

3.1. Definition of M

We have obtained intrinsic couples for associated metric and stress variables, $(\overline{m}, \underline{\theta})$ and $(\underline{m}, \overline{\theta})$, but first they are two and second *any of them is completely satisfying*. The main proof of this is that, as elements of spaces $\overline{\Gamma}$ and $\underline{\Gamma}$, $\dot{\overline{m}}$, $\underline{\theta}$, $\dot{\underline{m}}$ and $\overline{\theta}$ have not eigenelements. Note that this is nevertheless an improvement because when working with classical couples as (Σ, K), $\dot{\Sigma}$ and K have eigenelements *but these are not related to those of their Eulerian analogue D and σ/ρ*. For instance, the eigenaxes of the Piola–Kirchhoff stress tensor K when σ is an hydrostatic pressure $\sigma = -p1_E$, and also those of $\dot{\Sigma}$ when $D = \alpha 1_E$, are exactly the strain principal axes! *There is thus no direct Lagrangian way today to obtain principal values and axes of strain rate and stress.*

Another remark is that in (2.6) right members are components $x^{\alpha\beta}$ or $x_{\alpha\beta}$ of some geometrical spatial elements g, D and σ/ρ, while left members are the only component systems of two different material elements γ and γ^{-1}, $\dot{\overline{m}}$ and $\dot{\underline{m}}$, $\underline{\theta}$ and $\overline{\theta}$. This denotes a non-intrinsic character of these Lagrangian elements and also puts the question of what happens for the mixed components $x^{\alpha}{}_{\beta}$ and $x_{\alpha}{}^{\beta}$ of g, D and σ/ρ.

To overcome these difficulties we start from the remark that $\overline{m} \in \overline{\Gamma}^+$ and $\underline{m} \in \underline{\Gamma}^-$ give two intrinsic *linear representations* of the set M of all a priori possible local metrics, because $\overline{\Gamma}^+$ and $\underline{\Gamma}^-$ are open subsets of linear spaces $\overline{\Gamma}$ and $\underline{\Gamma}$, but *which are not compatible* because the map $\overline{m} \to \underline{m} = -\frac{1}{4}\overline{m}^{-1}$ is not linear. In order to save such a linear modelization, may we choose one of them? This is implicitly stated when, as it is usual, we decide to work exclusively with (Σ, K'), and is sometime argumented by a supposed tensorial nature of stresses (MARSDEN, HUGHES [12]), supposition which can be discussed: are forces vectors or covectors when making power with velocities? *Our opinion is, on the contrary, that we may not choose between* $\overline{m}$ *and* $\underline{m}$. The first describes angles and lengths of material segments while the second describes angles (of dihedrons) and thickness of material slices. To be noted, in classical approach with a reference configuration the essential and symmetrical parts respectively played by Green and Almansi strain measures Σ and Σ' have already been pointed out HILL [8], HAVNER [6] but not fully exploited. Note also that if many other strain measures between two configurations may be defined, as the important logarithmic measure $\frac{1}{2}\log C$ in some way middle point between Σ and Σ', this does not happen with metric parameters not making use of a reference configuration (note in particular that $\overline{m}$ and $\underline{m}$ have not eigenelements, $\overline{m}^2$ or $e^{\overline{m}}$ have no sense, ...).

Thus, *keeping these two incompatible linear representations on equal terms, they appear as being only two privileged charts of a necessarily curved space M whose particular geometry they state*. The two metric parameters $\overline{m} = \frac{1}{2}\gamma$ and $\underline{m} = -\frac{1}{2}\gamma^{-1}$ have then to be considered representatives in these two charts of a point of M that we shall denote m and call a metric of T_0. And starting only from the known change of intrinsic charts $\underline{m} = -\frac{1}{4}\overline{m}^{-1}$, the geometry of M, which appears first as a six-dimensional manifold, has been studied in ROUGEE [20, 21, 22]. We only recall, and explain, in this section, some essential results.

3.2. Tangent linear space Tm and stress variable

As a manifold, M admits in each of its points m a six-dimensional tangential linear space Tm. Denoting μ or dm an element of Tm, and for any map $f : M \to X$ denoting dx for $f'(m)dm$ if $f(m)$ is denoted x, *this tangent plane admits the four linear intrinsic representations*:

$$
(3.1)\qquad \mu = dm \quad
\begin{cases}
\overline{\mu} = \overline{dm} = d(\overline{m}) & \in \overline{\Gamma} = L_s(T_0, T_0^*),\\
\underline{\mu} = \underline{dm} = d(\underline{m}) = \gamma^{-1}\overline{\mu}\gamma^{-1} & \in \underline{\Gamma} = L_s(T_0^*, T_0),\\
\widetilde{\mu} = \widetilde{dm} = \gamma^{-1}\overline{\mu} = \underline{\mu}\gamma & \in \widetilde{\Gamma} = L_s(T_0, m),\\
\underset{\sim}{\mu} = \underset{\sim}{dm} = \gamma\underline{\mu} = \overline{\mu}\gamma^{-1} & \in \underset{\sim}{\Gamma} = L_s(T_0^*, m),
\end{cases}
$$

where γ and γ^{-1} are those of the considered point m and where (T_0, m) and (T_0^*, m) are the Euclidean spaces obtained by endowing T_0 and T_0^* with metric tensors γ and γ^{-1}. The two first ones are classically obtained by differentiating the two intrinsic charts. They give two m-independent images $\overline{\Gamma}$ and $\underline{\Gamma}$ of Tm while the images $\widetilde{\Gamma}$ and $\underset{\sim}{\Gamma}$ given by the two last are m-dependent six-dimensional linear subspaces of $L(T_0)$ and $L(T_0^*)$. Let L_s denote the algebraic structure of category of spaces $L_s(E)$ with for E a three-dimensional Euclidean space (allowing all classical algebraic calculus with symmetrical Euclidean tensors: scalar product, unity $\mathbf{1}_E$, trace, invariants, eigenelements, ...). *Because the map $\widetilde{\mu} \to \underset{\sim}{\mu}$ is a L_s-isomorphism, Tm is a L_s-space, which for convenience may for instance be identified to the m-dependent space $L_s(T_0, m)$ by means of map $\mu \to \widetilde{\mu}$.*

If m is the metric induced by a spatial placement a, Tm *appears as the Lagrangian homologue of the Eulerian space of the so-called symmetrical tensors on the configuration* a (the same m and thus the same Tm for two placements a_1 and a_2 such as the displacement $a_1a_2^{-1}$ is a spatial isometry), *the corresponding transport rule being the L_s-isomorphism*

$A : Tm \to L_s(E)$ *defined by*

(3.2) $$\boldsymbol{\mu} = A(\mu) = a\widetilde{\mu}a^{-1} = g^{-1}a^{-*}\overline{\mu}a^{-1} = g^{-1}a^{-*}\underset{\sim}{\mu}a^{*}g = a\underline{\mu}a^{*}g\,.$$

Taking a reference configuration a_1, and putting $a = Fa_1$ we have also

$$\overline{\mu} = a_1^{*}g_1(F^T\boldsymbol{\mu}F)a_1^{-1}, \quad \underline{\mu} = a_1^{-1}(F^{-1}\boldsymbol{\mu}F^{-T})g_1^{-1}a_1^{-*}\,,$$
$$\widetilde{\mu} = a_1^{-1}(F^{-1}\boldsymbol{\mu}F)a_1, \quad \underset{\sim}{\mu} = a_1^{*}g_1(F^T\boldsymbol{\mu}F^{-T})g_1^{-1}a_1^{-*}\,,$$

which shows that the four Lagrangian tensors $F^T\boldsymbol{\mu}F$, $F^{-1}\boldsymbol{\mu}F^{-T}$, $F^{-1}\boldsymbol{\mu}F$, $F^T\boldsymbol{\mu}F^{-T}$, classically obtained on the reference configuration by classical convective transports of $\boldsymbol{\mu} \in L_s(E)$, are the representations of our four images $\overline{\mu}$, $\underline{\mu}$, $\widetilde{\mu}$, $\underset{\sim}{\mu}$ of $\mu \in Tm$, through some constant maps a_1 and $a_1^{*}g_1$ associated to the chosen reference configuration as in (2.5). And (3.2) implies that, prolonged by the inverse of the four maps (3.1), these four classical convective transports lead to the one and only image μ: *A is thus, for symmetrical second order tensors on the configuration a, and from our intrinsic point of view, the one and only convective transport from the Lagrangian* (*or material*) *representation to the Eulerian* (*or spatial*) *representation* (Fig. 1).

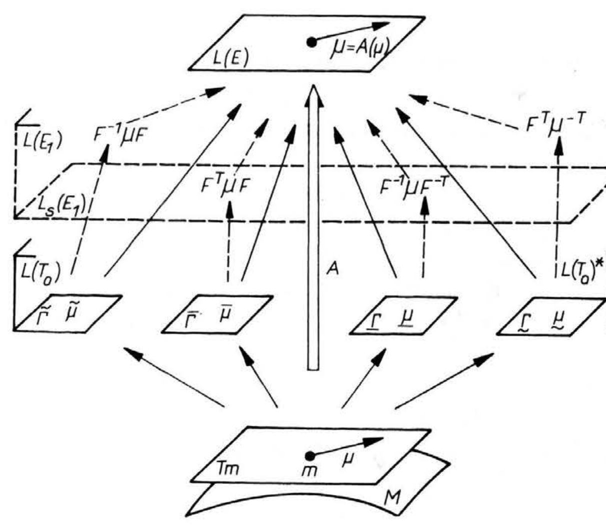

FIG. 1. Linear representations of Tm.

Taking a Lagrangian coordinate system, each space $\overline{\Gamma}$, $\underline{\Gamma}$, $\widetilde{\Gamma}$ and $\underset{\sim}{\Gamma}$ has its specific basis which, carried by (3.1), give four bases for Tm, as for $L_s(E)$ with the convected basis in E. Moreover, with these related bases, μ and $\boldsymbol{\mu} = A(\mu)$ have exactly the same four component systems

$$\mu_{ij} = \overline{\mu}_{ij} = \boldsymbol{\mu}_{ij}, \quad \mu^{ij} = \underline{\mu}^{ij} = \boldsymbol{\mu}^{ij}, \quad \mu^i_j = \widetilde{\mu}^i{}_j = \underset{\sim}{\mu}{}_j{}^i = \boldsymbol{\mu}^i_j$$

thus achieving identity for matricial calculus, between our Lagrangian and the Eulerian approaches, initiated in (2.6).

During a deformation process, m follows some trajectory curve on M. The metric rate $v = \dot{m} = \dfrac{dm}{dt} \in Tm$ is such as, from (3.1):

$$\overline{v} = \frac{d\overline{m}}{dt} = \frac{1}{2}\dot{\gamma}, \quad \underline{v} = \frac{1}{2}\gamma^{-1}\dot{\gamma}\gamma^{-1}, \quad \widetilde{v} = \frac{1}{2}\gamma^{-1}\dot{\gamma}, \quad \underset{\sim}{v} = \frac{1}{2}\dot{\gamma}\gamma^{-1}.$$

The associated stress variable is the element θ of Tm such as, for any $\dot{m} \in Tm$, the specific internal power $\mathcal{P} = -\frac{\sigma}{\rho} : D = -\mathrm{Tr}\left(\frac{\sigma}{\rho}A(\dot{m})\right)$ is equal to $-\theta : \dot{m}$. Because the scalar product in Tm is

$$\mu_1 : \mu_2 = \widetilde{\mu}_1 : \widetilde{\mu}_2 = \mathrm{Tr}(\widetilde{\mu}_1\widetilde{\mu}_2) = \mathrm{Tr}(\overline{\mu}_1\underline{\mu}_2),$$

it is easy to verify that the images of this θ by the two first maps (3.1) are precisely the two elements $\overline{\theta}$ and $\underline{\theta}$ introduced in (2.3). Thus:

$$\overline{\theta} = a^*g\frac{\sigma}{\rho}a, \quad \underline{\theta} = a^{-1}\frac{\sigma}{\rho}g^{-1}a^{-*}, \quad \widetilde{\theta} = a^{-1}\frac{\sigma}{\rho}a, \quad \underset{\sim}{\theta} = a^*g\frac{\sigma}{\rho}g^{-1}a^{-*}.$$

The Eulerian images by the generalized convective map A of these coupled variables $(\dot{m}, \theta)$ are exactly, as results of (2.2), (2.3), (3.2), the classical Eulerian couple $(D, \frac{\sigma}{\rho})$. *We have thus, between the Eulerian and our Lagrangian approachs, the expected total coherence*:

$$(3.3) \qquad -\mathcal{P} = \frac{\sigma}{\rho} : D = \theta : \dot{m}, \quad A(\dot{m}) = D, \quad A(\theta) = \frac{\sigma}{\rho}.$$

Note first that A being a L_s-isomorphism, $\dot{m}$ and D and also θ and $\frac{\sigma}{\rho}$ have the same eigenvalues and eigenvectors homologous by a. *The problem of a Lagrangian approach of principal values and axes for strain rate and stress is thus solved.* Second, the two factors of $\mathcal{P}$ in (2.3), (2.4) were also some Lagrangian images of D and σ/ρ but not by the same convective map for the two factors as in (3.3). Third, *as expected, D appears as the Eulerian image of the derivative of a configuration variable*, but which is our $m \in M$ and not a more classical tensorial variable.

After these algebraic relations between all kinds of tensors, Eulerian and Lagrangian, classical and in our new space Tm, let us introduce differential calculus.

3.3. Differential calculus on M

The scalar product in Tm makes M a Riemannian manifold whose metric tensor is given by

$$(3.4) \quad ds^2 = dm : dm = \mathrm{Tr}(d\overline{m}d\underline{m}) = \frac{1}{4}\mathrm{Tr}(d\gamma\gamma^{-1}d\gamma\gamma^{-1}) = (C^{-1}d\Sigma) : (C^{-1}d\Sigma)$$
$$= (Cd\Sigma') : (Cd\Sigma') = d\Sigma : d\Sigma'.$$

The associated length of the trajectory of m, because $\dot{m} : \dot{m} = D : D$, is exactly equal to the classical cumulated scalar strain $\int_{t_1}^{t_2} ||D||dt$. The associated covariant derivative of a vector field μ in a direction dm, $\nu = \nabla_{dm}\mu$, is characterized by any one of the following relations giving its images by (3.1) (the first is obtained by working in the chart $m \to \overline{m}$ and applying classical formula [PHAM MAU QUAN [18], and leads to the other):

$$(3.5) \quad \begin{aligned} &\overline{\nu} = d\overline{\mu} + 2(d\overline{m}\underline{m}\overline{\mu} + \overline{\mu}\underline{m}d\overline{m}), && \underline{\nu} = d\underline{\mu} + 2(d\underline{m}\overline{m}\underline{\mu} + \underline{\mu}\overline{m}d\underline{m}), \\ &\widetilde{\nu} = d\widetilde{\mu} + \widetilde{dm}\widetilde{\mu} - \widetilde{\mu}\widetilde{dm} = (d\widetilde{\mu})_{s-m}, && \underset{\sim}{\nu} = d\underset{\sim}{\mu} - \underset{\sim}{dm}\underset{\sim}{\mu} + \underset{\sim}{\mu}\underset{\sim}{dm} = (d\underset{\sim}{\mu})_{s-m}, \end{aligned}$$

where the index s-m indicates the m-symmetrical part. *The associated time derivative $\nabla\mu/dt$ of a variable μ lying, as $\dot{m}$ or θ, in the time-dependent space Tm during the deformation process, is closely related to the Jauman derivative $\boldsymbol{\mu}^J$ of its Eulerian image by*

$$(3.6)\qquad \boldsymbol{\mu} = A(\mu) \;\Rightarrow\; \boldsymbol{\mu}^J = A\left(\frac{\nabla\mu}{dt}\right).$$

This indicates that, for symmetrical Eulerian tensors $\boldsymbol{\mu} \in L_s(E)$, Jauman's derivative is exactly the convective derivative associated to our single convective transport A. This gives a meaningful Lagrangian sense to Jauman's derivative of symmetrical second order Eulerian tensors.

These results, and other joining horizontality above M and isotropic maps, lead to a physically meaningful intrinsic Lagrangian statement of some constitutive laws usually stated in a Eulerian way, from perfect gas law to elastoplastic laws without released configuration (BOEHLER [1]). For instance, an isotropic hypoelastic law will be stated by $\dfrac{\nabla\theta}{dt} = \lambda \operatorname{Tr}\dot{m}\mathbf{1} + 2\mu\dot{m}$, and the variable $\varepsilon \in Tm$ defined by $\theta = \lambda \operatorname{Tr}\varepsilon\mathbf{1} + 2\mu\varepsilon$ appears as satisfying the flow rule $\nabla\varepsilon/dt = \dot{m}$, which makes it a *cumulated* (in Tm) *tensorial deformation variable* (ROUGEE [22]).

3.4. Height and shape of a metric

Manifold M is intrinsically structured by integral curves S of the field of unity elements $\mathbf{1}_m$ of Tm, and by their orthogonal five-dimensional submanifolds H. Submanifolds H are subsets of metrics in which the media have the same specific volume. Processes of deformation in such a submanifold H are isovolumic processes. We denoted by $H(\tau)$ the submanifold corresponding to specific volume $\tau = \rho^{-1}$. The curves S are subsets of metrics in which the media have the same angular properties. During processes of deformation following such a curve S, the variation of metric properties is only a uniform lengthening of material segments and thickening of slices without change of angles. Thus it is a change of "height" without change of "shape". We denoted by $S(m)$ and $H(m)$ the curve S and manifold H containing the metric m. The tangent spaces in m to $S(m)$ and $H(m)$ are subspaces of spherical and deviatoric elements of Tm.

Choosing some specific volume τ_0 as a unity and thus $H(\tau_0)$ as the origin of H manifolds, any $m \in M$ is characterized by its coordinates $f = S(m) \cap H(\tau_0)$ which models its shape (the angular properties) and $\nu = \widehat{fm}$, the curvilinear abscissa of m on $S(m)$ with f as an origin, which models its height. These intrinsic coordinates are given by:

$$(3.7)\qquad \nu = \frac{1}{\sqrt{3}} \operatorname{Log}\frac{\tau}{\tau_0}, \qquad \overline{m} = \overline{f}\tau^{2/3} = \overline{f}e^{2/\sqrt{3}\nu}.$$

By logarithmic derivation this gives $\widetilde{dm} = \widetilde{df} + d\nu\dfrac{1_{Tm}}{\sqrt{3}}$ which implies that the spherical part of dm is $d\nu\dfrac{\mathbf{1}_m}{\sqrt{3}}$ and its deviatoric part dm_D is such as $\widetilde{dm_D} = \widetilde{df}$. We have also

$$(3.8)\qquad d^1m : d^2m = d^1f : d^2f + d^1\nu d^2\nu$$

which implies that *the coordinate map $m \to (\nu, f)$ is in fact an intrinsic isomorphism of the Riemannian manifold M on the Cartesian product $\mathbf{R} \times H(\tau_0)$.*

As a remark, note that for a purely mechanical perfect gas the constitutive law $\sigma = -k\rho 1_E$ gives, in our material model,

$$\theta = -k\mathbf{1}_m, \quad \mathcal{P} = -\theta : \dot{m} = k\mathbf{1}_m : \dot{m} = k\dot{\nu}\,.$$

The vector field $m \to \theta \in Tm$ is thus the only horizontal vector field on M (k is a constant), and *the internal energy is equal to $k\nu$.*

4. Geodesics of M and strain measures

In this section our purpose is to study the way to model or characterize the finite deformation when passing from a first metric m_1 to a second m_2. If M was a linear space it would be associated to m_1 and m_2, a vector $\Delta m = m_2 - m_1$ which would be an intrinsic modeling of the change of metric properties, allowing for some algebraic operations such as addition of two successive finite deformations, $(m_3 - m_1) = (m_2 - m_1) + (m_3 - m_2)$, or some comparison as $(m_2 - m_1) = 2(m_4 - m_3)$. Note that in this case $m_2 - m_1$ also characterizes the oriented arc of geodesic $G(m_1, m_2)$ joining m_1 to m_2. But M is a *curved space*, $m_2 - m_1$ *has no sense*, and strain measures as $\Delta\overline{m} = \overline{m}_2 - \overline{m}_1$, $\Delta\underline{m}$, Σ, Σ', ... are only obtained through non-intrinsic procedures giving incompatible linear representations: to add the $\Delta\overline{m}$ or to add the $\Delta\underline{m}$ in two successive finite deformations are not equivalent procedures. *The only concept which remains, because M is a Riemanian manifold, is that of oriented geodesic joining m_1 to m_2, but without any way of comparing two geodesics $G(m_1, m_2)$ and $G(m_3, m_4)$* (except their lengths).

Parametrized by its curvilinear abscissa s taking values 0 in m_1 and 1 in m_2, $G(m_1, m_2)$ admits as an equation the map $s \to m$ solution of

$$\frac{\nabla}{ds}\left(\frac{dm}{ds}\right) = 0 \Leftrightarrow \frac{\nabla}{ds}\left(\widetilde{\frac{dm}{ds}}\right) \equiv \frac{d}{ds}\left(\widetilde{\frac{dm}{ds}}\right) = 0 \Leftrightarrow \widetilde{\frac{dm}{ds}} \equiv \overline{m}^{-1}\frac{d\overline{m}}{ds} = cste$$

and satisfying the boundary conditions $m(0) = m_1$, $m(1) = m_2$. Putting $\ell = \frac{dm}{ds}(0)$ and $\lambda = \frac{dm}{ds}(1)$ this leads to the equation

$$\overline{m} = \overline{m}_1 e^{2s\widetilde{\ell}} \quad \text{with} \quad \widetilde{\ell} = \frac{1}{2}\,\mathrm{Log}(\gamma_1^{-1}\gamma_2) = \widetilde{\lambda}\,. \tag{4.1}$$

As it was introduced, ℓ is the "geodesic coordinate of pole m_1" of m_2, $\mathcal{G}_{m_1}(m_2)$, i.e. the vector of Tm_1 which is tangent in m_1 to $G(m_1, m_2)$ oriented from m_1 to m_2, and whose norm is equal to the geodesic distance between m_1 and m_2. *It is the best tool when lying in m_1 and making use of geometrical elements associated to m_1, we have to characterize the position of m_2 relatively to m_1 and thus in some way the finite deformation from m_1 to m_2.* In the same way, λ is equal to $-\mathcal{G}_{m_2}(m_1)$: it has the same norm and orientation as ℓ but it is tangent to $G(m_1, m_2)$ in m_2. *It is the best tool to characterize the finite deformation from the final point m_2 as point of view* (Fig. 2).

Let now m_1 and m_2 be the metrics associated to two spatial placements a_1 and a_2, and $A_i : Tm_i \to L_s(E_i)$ be the associated intrinsic convective maps. We have:

$$\begin{aligned} A_1(\ell) = a_1\widetilde{\ell}a_1^{-1} &= a_1\left[\frac{1}{2}\,\mathrm{Log}(\gamma_1^{-1}\gamma_2)\right]a_1^{-1} = \frac{1}{2}\,\mathrm{Log}(a_1\gamma_1^{-1}\gamma_2 a_1^{-1}) \\ &= \frac{1}{2}\,\mathrm{Log}(g^{-1}a_1^{-*}a_2^{*}g a_2 a_1^{-1}) = \frac{1}{2}\,\mathrm{Log}((a_2a_1^{-1})^T a_2 a_1^{-1}) \end{aligned}$$

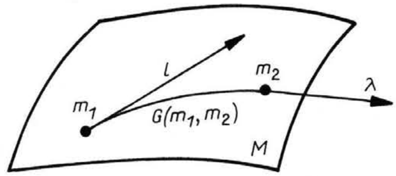

FIG. 2. Strain parameters ℓ and λ.

and thus, with notations of (2.4) with $a_2 = a$,

$$A_1(\ell) = \frac{1}{2}\operatorname{Log} C = \operatorname{Log} U \quad \text{and} \quad A_2(\lambda) = \frac{1}{2}\operatorname{Log} B = \operatorname{Log} V . \tag{4.2}$$

for λ, obtained in the same way, with $B = V^2 = FF^T$.

Parameters ℓ and λ are thus intrinsic Lagrangian homologue of classical logarithmic strain measures $\operatorname{Log} U$ *on initial configuration a_1 and* $\operatorname{Log} V$ *on final configuration a_2.* This explains the success of these measures (HILL [7], HAVNER [6], PERIC and OWEN [17]). In finite deformations strain measures are only particular *characterizations* of deformations, and not intrinsic *modeling* as in small displacement. But used as modeling, *the logarithmic measures are the least bad.* Note also that if M was a linear space we would have $\mathcal{G}_{m_1}(m_2) = -\mathcal{G}_{m_2}(m_1) = m_2 - m_1$.

By putting

$$u = e^{\ell}, \quad c = u^2, \quad e_n = \frac{1}{2n}(c^n - \mathbf{1}_m), \ldots$$

in Tm_1 (defined by analogous relations $\widetilde{u} = e^{\widetilde{\ell}}, \ldots$ in $L_s(T_0, m_1)$), and

$$v = e^{\lambda}, \quad b = v^2, \ldots$$

in Tm_2, we obtain other Lagrangian parameters characterising the deformation from one or the other of the two points of view, whose images by A_1 or A_2 are the classical strain parameters U, C, $\Sigma_n = \frac{1}{2n}(C^n - 1_{E_1}), \ldots$ in $L_s(E_1)$ and V, B, $\ldots$, in $L_s(E_2)$. These parameters are also defined by

$$\widetilde{\ell} = \widetilde{\lambda} = \frac{1}{2}\operatorname{Log}(\gamma_1^{-1}\gamma_2), \quad \widetilde{u} = \widetilde{v} = (\gamma_1^{-1}\gamma_2)^{1/2}, \quad \widetilde{c} = \widetilde{b} = \gamma_1^{-1}\gamma^2, \ldots$$

and it may be seen that $\widetilde{u} = \widetilde{v}$ is an isometry of (T_0, m_1) on (T_0, m_2). All these elements lie in $L_s(T_0, m_1) \cap L_s(T_0, m_2)$. They have, and thus ℓ, u, c, e_n, $\ldots$ in Tm_1 and λ, v, b, $\ldots$ in Tm_2 too, the same eigenvectors defining the Lagrangian principal deformation axes. *These are thus intrinsic while principal deformation values, depending on the choosen strain measure, are not.*

A path above $G(m_1, m_2)$ in tangent bundle TM, $\mu(s) \in Tm(s)$, is horizontal if it satisfies $\frac{\nabla\mu}{ds} = 0$, i.e. if $\frac{d\widetilde{\mu}}{ds} + \widetilde{\frac{dm}{ds}}\widetilde{\mu} - \widetilde{\mu}\widetilde{\frac{dm}{ds}} = 0$ which with (4.1) gives $\widetilde{\mu}(s) = e^{-s\widetilde{\ell}}\widetilde{\mu}(0)e^{s\widetilde{\ell}}$. As a consequence, *the parallel transport above $G(m_1, m_2)$ is the L_s-isomorphism G : $Tm_1 \to Tm_2$ defined by*

$$\mu_2 = G(\mu_1) \Leftrightarrow \widetilde{\mu}_2 = \widetilde{u}^{-1}\widetilde{\mu}_1\widetilde{u} \tag{4.3}$$

and we note that $\lambda = G(\ell)$, $v = G(u)$, $b = G(c)$, $\ldots$ Putting $F = a_2 a_1^{-1} = RU = VR$

and $\boldsymbol{\mu}_i = A_i(\mu_i)$ it may also be seen that

$$(4.4) \qquad \mu_2 = G(\mu_1) \;\Leftrightarrow\; \boldsymbol{\mu}_2 = R\boldsymbol{\mu}_1 R^{-1}.$$

The Eulerian image of the parallel transport above $G(m_2, m_1)$ is thus the rotation of the symmetrical tensors by the rotation R (while, as seen in [ROUGEE [22], it is by the rotation of the corotational space frame for the parallel transport above the trajectory of a deformation process).

Let us now see what occurs with heights and shapes of m_1 and m_2. The field $m \to \mathbf{1}_m$ is horizontal: $\nabla \mathbf{1} = 0$. It follows that curves S are geodesics of M. This also results from the fact that the coordinate map $m \to (\nu, f)$ is a canonical isomorphism of the Riemannian manifold M on $\mathbb{R} \times H(\tau_0)$. Another result is that the projection, by curves S, of the geodesic $G(m_1, m_2)$ on $H(\tau_0)$ is the geodesic $G(f_1, f_2)$ joining in $H(\tau_0)$ the shapes f_1 and f_2 of m_1 and m_2. Also, the decomposition of ℓ in spherical and deviatoric parts, $\ell = \ell_s + \ell_D$, is such as $\ell_s = (\nu_1 - \nu_2)\mathbf{1}_{m_1}$ and $\widetilde{\ell}_D = \widetilde{\ell}_f$ where $\ell_f = \mathcal{G}_{f_1}(f_2)$ is the geodesic coordinates of pole f_1 of f_2. An analogous decomposition may be written for λ. A finite deformation from m_1 to m_2 presents thus two additional and independent components. First, a change of height, i.e. a change of surface H from $H(m_1)$ to $H(m_2)$, whose measure is the distance $\Delta\nu = \nu_2 - \nu_1$ between these surfaces, independent of the choice of τ_0. This measure has the good additivity properties, when composing the deformations, resulting from the linearity of the factor $\mathbb{R}$ in $\mathbb{R} \times H(\tau_0)$. Second, a change of shape, passing from curve $S(m_1)$ to curve $S(m_2)$, which after choice of τ_0 as a specific volume unity, may be characterized by a deviatoric element ℓ_f of Tf_1, or by its analogue λ_f in Tf_2, or by any other strain parameter built of them as previously.

Let us now see what happens when, choosing a reference configuration a_1, we work with a particular measure of the deformation between this reference configuration and the studied variable configuration $m = m_2$, for instance $\Sigma_n \in L_s(E_1)$. The map a_1 is a constant isometry between (T_0, m_1) and E_1, and thus A_1 is a constant L_s-isomorphism between the tangent space Tm_1 in m_1 and the space $L_s(E_1)$ of symmetrical tensors on the reference configuration, which may be used to identify these two spaces. To work with Σ_n as a representative of the studied metric m, and consequently with $\dot{\Sigma}_n$ as strain rate, is thus equivalent to substituting e_n and $\dot{e}_n$ in Tm_1 to $m \in M$ and $\dot{m} \in Tm$. *Thus it is equivalent to substituting to M its tangent plane in m_1 by means of the projection map $m \to e_n$.* For the set of all possible local metrics, a non-intrinsic linear model is then substituted to the actual nonlinear model M. This is acceptable only in case of small deformation. First, because for the logarithmic measure $\ell = e_0$ we have from (4.1) $\overline{m} = \overline{m}_1 e^{\overline{m}_1^{-1}\overline{\ell}}$ which by differentiating at point $\ell = 0$ gives $d\overline{m} = d\overline{\ell}$, and thus $\left(\dfrac{dl}{dm}\right)_{m=m_1} = \mathbf{1}_{Tm_1}$ for the derivative of the projection map $m \to \ell$. Second, because it is well known that in Tm_1 identified with $L_s(E_1)$ the usual precautions in defining strain measures make these measures equivalent in case of small strain: $\left(\dfrac{de_n}{de_p}\right)_{e_p=0} = \mathbf{1}_{Tm_1}$ (and equivalent to the classical strain tensor of small displacement theory).

5. Kinematics with intermediate state

In elastoplasticity with an intermediate elastically released state leading to the well known decomposition $F = F_e F_p$, an indeterminate rotation enters the definition of the released state. It is thus only its metric properties, modeled by a point m_p of M, which are useful in the concept of released state. We have thus for these media two points of M as time-dependent parameters: the actual metric m and the metric m_p. The latter would be obtained if, stopping the process, the stress was locally and elastically released. Such a release of stress is only an abstract idea and is never performed, except for some instants and some points where eventually stress happens to be zero. The points m and m_p go thus over two trajectories on M, parametrized by time t. The first one describes the total deformation process while the second describes what we shall call the plastic deformation process. But there is no trajectory joining $m_p(t)$ to $m(t)$. And because, as it was seen previously, there is no variable susceptible to model the elastic difference between m_p and m, *the physical concept of elastic part of the total deformation may not be constituted by giving only the two metric states m_p and m.* Another aspect of this difficulty is that there is no intrinsic way to decide if, during the process and considering the metric rates $\dot{m}$ and $\dot{m}_p$, the elastic deformation stays or does not stay constant. In particular, denoting by ℓ_e and λ_e the parameters ℓ and λ associated to $m_1 = m_p$ and $m_2 = m$, $\dfrac{\nabla \ell_e}{dt} = 0$ does not imply that $\dfrac{\nabla \lambda_e}{dt} = 0$. *There is not an elastic strain rate intrinsically associated to the total and plastic rates $\dot{m}$ and $\dot{m}_p$.* Note also that $\dot{m}_p$ and $\dfrac{\nabla \ell}{dt}$ are in Tm_p while $\dot{m}$ and $\dfrac{\nabla \lambda}{dt}$ are in Tm, which does not make the composition of strain rates easy.

Numerous supplementary particular choices, as non-equivalent choices between ℓ_e and λ_e as elastic variable, have been done to overcome this difficulty, leading to various theories. Unfortunately they are not physically justified and generally not recognized as being optional choices and thus questionable. We have given in ROUGEE [21] two such choices, geometrically argumented. We present here the case where an elastic microstructure (as a crystalline structure) brings additional physics which allow for overcoming the difficulty.

For such media we introduce two local three-dimensional linear spaces. First T_0 as before, with its space of metrics M, for the continuous media itself, and second T_{0s} with its space of metrics M_s for the microstructure. This second space is a Euclidean space: it is endowed with a particular metric $m_{0s} \in M_s$, time-independent, which, with eventual geometrical elements as those describing the slide systems, models the crystalline structure in released state. Two parameters appear as essential for the statement of constitutive laws of these media. First, the actual metric $m \in M$ of the media as previously, and second a map $p \in L(T_0, T_{0s})$ modeling the actual placement of the media with regard to the microstructure. This map p allows to carry on T_{0s} any scalar product defined in T_0 and thus induces a map $\mathbf{P}$ of M on M_s characterited by

$$m_s = \mathbf{P}(m) \quad \Leftrightarrow \quad \overline{m}_s = p^{-*}\overline{m}p^{-1} \tag{5.1}$$

which by logarithmic derivative gives the following characterization of the tangential linear map $\mathbf{P}'(m) : Tm \to Tm_s$

$$dm_s = \mathbf{P}'(m)dm \quad \Leftrightarrow \quad \widetilde{dm}_s = p\widetilde{dm}p^{-1}\,. \tag{5.2}$$

This last map is a L_s-isomorphism because p is an isometry of (T_0, m) on (T_{0s}, m_s). As a result $\mathbf{P}$ is an isomorphism of M on M_s for all their structure. The map $\mathbf{P}$ transforms a geodesic of M into a geodesic of M_s of the same length. The curves S and their orthogonal surfaces H of M, and those S_s and H_s of M_s, are homologous. Polar geodesic coordinates in two homologous poles m_1 and $m_{s1} = \mathcal{P}(m_1)$ of two homologous metrics m_2 and $m_{s2} = \mathbf{P}(m_2)$ are homologous by $\mathbf{P}'(m_1)$.

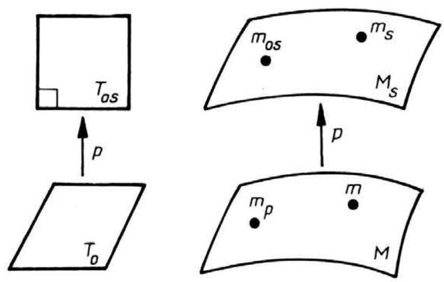

FIG. 3. Maps p and $\mathbf{P}$.

If $m \in M$ is the actual metric of the media and $m_{0s} \in M_s$ the (time-independent) released metric of the microstructure, $m_s = \mathbf{P}(m)$ is the actual metric of the microstructure and $m_p = \mathbf{P}^{-1}(m_{0s})$ is the released metric of the media (Fig. 3). *The elastic part of the process is now well defined*: *it is the deformation process of the elastic microstructure, modeled by m_s moving on M_s.* The plastic part is modeled by p moving in the nine-dimensional linear space $L(T_0, T_{0s})$. This is a more general variable than the six-dimensional metric variable $m_p = \mathbf{P}^{-1}(m_{0s})$ that it induces. The global deformation process is the product of these two basic processes by means of (5.1) which, by time derivative, leads to the following rate composition (written in Tm_s)

$$
\begin{aligned}
&d = d_e + d_p \quad \text{with} \quad d_e = \dot{m}_s\,,\\
(5.3)\qquad &d_p = -\dot{\mathbf{P}}(m) = (\dot{p}p^{-1})_{s-m_s} \;\Leftrightarrow\; \tilde{d}_p = \frac{1}{2}(\gamma_s^{-1}(\dot{p}p^{-1})^*\gamma_s + \dot{p}p^{-1})\,,\\
&d = \mathbf{P}'(m)\dot{m} \;\Leftrightarrow\; \tilde{d} = p\overline{m}^{-1}\dot{\overline{m}}p^{-1}\,,
\end{aligned}
$$

where d is the total metric rate $\dot{m}$ carried in Tm_s by $\mathcal{P}'(m)$, d_e is the elastic strain rate and d_p the plastic strain rate. The elastic deformation may be characterized by any of the deformation parameters ℓ_{se}, u_{se}, $\ldots$, λ_{se}, $\ldots$ associated to the couple (m_{s0}, m_s) but $\frac{\nabla \ell_{se}}{dt}$, $\frac{\nabla \lambda_{se}}{dt}$, $\ldots$ are not the elastic strain rate $d_e = \dot{m}_s$.

Let $a \in L(T_0, E)$ be the actual local spatial placement of the media, e the product $ap^{-1} \in L(T_{0s,E})$, describing the spatial placement of the microstructure, and A and E the associated intrinsic convective maps for symmetrical tensors. We have

$$
\begin{gathered}
a = ep, \quad \overline{m} = \frac{1}{2}a^*ga, \quad \overline{m}_s = \frac{1}{2}e^*ge\,,\\
(\forall \mu \in Tm) \quad \mu_s = \mathbf{P}'(m)\mu \Rightarrow A(\mu) = E(\mu_s)
\end{gathered}
$$

and by convective transport of (5.3) the Eulerian rate composition is:

$$
(5.4)\qquad
\begin{aligned}
&D = D_e + D_p \quad \text{with} \quad D = A(\dot{m}) = (\dot{a}a^{-1})_s, \quad D_e = E(\dot{m}_s) = (\dot{e}e^{-1})_s\\
&\qquad\qquad D_p = E(d_p) = (e\dot{p}p^{-1}e^{-1})_s\,.
\end{aligned}
$$

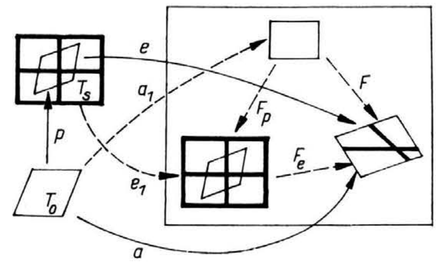

FIG. 4. Composition of placements.

If a_1 and e_1 are two time-independent reference placements for the media and the microstructure (Fig. 4), e_1 being an isometry of (T_{0s}, m_{0s}) on (E, g), and putting $F = aa_1^{-1}$, $F_e = ee_1^{-1}$, $F_p = e_1pa_1^{-1}$, we obtain the classical relations:

$$F = F_eF_p, \quad D = (\dot{F}F^{-1})_s, \quad D_e = (\dot{F}_eF_e^{-1})_s, \quad D_p = (F_e\dot{F}_pF_p^{-1}F_e^{-1})_s .$$

In the case where T_{0s} is chosen equal to $\mathbb{R}^3$, $p \in L(T_0, \mathbb{R}^3)$ is a basis operator. The parameter p and the reference configuration e_1 are then the trihedron director and the isocline released configuration introduced and used in (MANDEL [10, 11], STOLZ [24]).

References

1. J. P. BOEHLER, *Lois de comportement anisotrope des milieux continus*, J. Méc., **17**, 2, pp. 153–190, 1978.
2. J. CASEY, *On finitely deforming rigid-plastic materials*, Int. J. Plast., **2**, pp. 247–277, 1986.
3. J. CASEY and P. M. MAGHDI, *On the relationship between the Eulerian and Lagrangian descriptions of finite rigid plasticity*, Arch. Rat. Mech. Anal., **102**, 4, 351–375, 1988.
4. A. DOGUI and F. SIDOROFF, *Large strain formulation of anisotropic elasto-plasticity for metal forming*, in: Computational Methods for Predicting Materials Processing Defects, M. PREDELEANU [Ed.], Elsevier Sci., Amsterdam, 1987.
5. A. E. GREEN and P. M. NAGHDI, *A general theory of an elastic-plastic continuum*, Arch. Rat. Mech., Anal., **18**, pp. 251–281, 1965.
6. K. S. HAVNER, *The theory of finite plastic deformation of crystalline solids* in: Mechanics of Solids, Rodney Hill Anniversary Volume, H. G. HOPKINS and M. J. SEWELL [Eds.]; Pergamon Press, 1982.
7. R. HILL, *On constitutive inequalities for simple material. I and II*, J. Mech. Phys. Solids, **16**, pp. 229–242 and 315–322, 1968.
8. R. HILL, *Aspects of invariance in solid mechanics*, Adv. Appl. Mech., **18**, pp. 1–75, 1978.
9. M. KLEIBER and B. RANIECKI, *Elastic-plastic materials at finite strains*, in: Plasticity Today, A. SAWCZUK [ed.], 1983.
10. J. MANDEL, *Plasticité et viscoplasticité*, Course CISM 97 Udine, Springer, New York 1971.
11. J. MANDEL, *Director vectors and constitutive equations for plastic and viscoplastic media*, Proc. Symp. Problems of Plasticity, [Ed.] A. SAWCZUK, p. 125, Nordhoff, 1972.
12. J. E. MARSDEN and J. HUGHES, *Mathematical foundation of elasticity*, Prentice-Hall, 1983.
13. J. J. MOREAU, *Sur la notion de systeme de référence fluide et ses applications en aérodynamique*, Congrès National de l'Aviation Francaise, Rapport 368, 1945.
14. P. M. NAGHDI, *A critical review of the state of finite plasticity*, J. Appl. Math. Phys., **41**, pp. 315–394, 1990.
15. W. NOLL, *A new mathematical theory of simple material*, Arch. Rat. Mech. Anal., **48**, 1–50, 1972.
16. W. NOLL, *On the continuity of the solid and fluid states*, J. Rat. Mech. Anal., **4**, pp. 3–81, 1955.
17. DJ. PERIC and D. R. J. OWEN, *A model for finite strain elastoplasticity*, Proc. 2nd Int. Conf. on Comput. Plasticity, Barcelona, [Eds.] OWEN, HINTON, ONATE, Pineridge Press, Swansea U.K. 1989.
18. PHAM MAU QUAN, *Introduction a la géométric des variétés différentielles*, Ed. Dunod, Paris 1968.
19. P. ROUGEE, *Formulation lagrangienne intrinsèque en mécanique des milieux continus*, J. Méc., **18**, 1, pp. 7–32, 1980.

20. P. ROUGEE, *La variete des formes d'un élément simple et son utilisation dans la formulation lagrangienne intrinsèque du comprortement*, C.R.A.S., 306, série II, pp. 947–952, 1988.
21. P. ROUGEE, *Analyse lagrangienne de la déformation — modèles élastoplastiques*, Rapport interne L.M.T. Cachan, 84, 1988.
22. P. ROUGEE, *A new Lagrangian intrinsic approach of continuous media in large deformation*, European J. Mech. A. Solids, **10**, 1, pp. 15–39, 1991.
23. J. C. SIMO, J. E. MARSDEN and P. S. KRISHNAPRASAD, *The Hamiltonian structure of nonlinear elasticity: the material and convective representations of solids, rods, plates*, Arch. Rat. Mech. Anal., vol. 104-2, pp. 125–182, 1988.
24. J. STOLZ, *Anelasticité et stabilité*, Thèse Université Paris VI, Paris 1987.
25. C. TRUESDELL, *Remarks on hypoelasticity*, J. Res. Nall. Bur. Standards 67-B, pp. 141–143, 1963.
26. C. TRUESDELL and W. NOLL, *The nonlinear field theories of mechanics*, Handbuch der Physik, [Ed.] S. Flügge, vol. III/3 sect. 36, Springer Verlag, 1987.

LABORATOIRE DE MECANIQUE ET TECHNOLOGIE
ENS DE CACHAN, UNIVERSITE P. ET M. CURIE, CACHAN, FRANCE.

Received September 3, 1991.