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Dynamic stability of plates with boundary conditions depending on membrane forces

A. TYLIKOWSKI (WARSZAWA)

IN THE PAPER the stochastic stability problem is solved for rectangular plates with elastically supported edges. The plate is compressed by in-plane wide-band Gaussian forces. It is assumed that resistances to rotation of plate edges increase linearly as the mean values of in-plane forces increase. Results are graphically presented for cylidrical bending of a plate which is simply supported and the support stiffens under compression.

1. Introduction

THE INVESTIGATION of the stochastic stability of rectangular plates under time-dependent in-plane forces has been considered in literature during the past twenty years. Most papers were concerned with idealized simply supported or clamped edges (cf. [1, 2, 3]). Based on the results obtained for simply supported edges some other more realistic stochastic dynamics problems can be treated. The plate is still assumed to have elastically supported edges against rotation. The spring constant, however, may vary as the plate rotates. If the plate is subjected to a compressive load P, the support condition may also depend on P. This behaviour has been demonstrated for compressed columns [4], where the resistance to rotation increased as P increased. This variation has a significant effect on the dynamic behavior and buckling load of structural elements [5, 6].

The effect of such support stiffening on a rectangular plate dynamic stability is examined in the present paper. The spring constant of the boundaries is assumed to increase linearly with the mean value of in-plane forces. Both isotropic and antisymmetrically laminated cross-ply plates are analyzed. Numerical results and figures are presented for cylindrical bending of a plate, the first edge of which is simply supported and the second one exhibits resistance to the rotation increasing with P.

2. Problem formulation

We introduce a probability space $(\Gamma, \mathfrak{B}, P)$ and assume that a transverse displacement of the plate and in-plane forces are measurable with respect to σ-algebra $\mathfrak{B}$. For convenience we will omit the symbol $\gamma \in \Gamma$ in a description of stochastic processes, e.g. $w(\gamma, x, t) \equiv w(x, t)$.

The flexural vibrations of a rectangular elastic thin plate compressed by in-plane time-dependent forces are described by the governing partial differential equation

$$\rho h w_{,tt} + 2\beta\rho h w_{,t} + \mathcal{K} w + (\mathcal{N}_{0x} + \mathcal{N}_x(t)) w_{,xx} + (\mathcal{N}_{0y} + \mathcal{N}_y(t)) w_{,yy} = 0, \quad (x, y) \in \Omega \equiv (0, a) \times (0, b), \tag{2.1}$$

where $\mathcal{K}$ is a linear self-adjoint operator representing elastic forces, $\mathcal{N}_x$ and $\mathcal{N}_y$ are uniformly applied in-plane forces, ρ is the plate density, β is the damping coefficient. The rectangular plate has the length a in the x direction, with b in the y direction and total thickness h in the z direction. A comma denotes a partial derivative of the main symbol with respect to the index.

The purpose of the present paper is to derive criteria for solving the following problem: will the deviations of plate surface from the unperturbed state (equilibrium state) be sufficiently small in some mathematical sense in the case when in-plane forces are time-dependent. The plate dynamically buckles when the in-plane forces get so large that the plate does not oscillate about the unperturbed plane state and a new increasing mode of oscillations occurs. To estimate the perturbed solution of Eq. (2.1) we introduce a measure of distance $\|\cdot\|$ of the solution of Eq. (2.1) with nontrivial initial conditions from the trivial one.

In the present analysis the direct Lyapunov method is proposed to establish criteria for a uniform stochastic stability of the unperturbed state (trivial solution) of rectangular plates treated as the infinite-dimensional system subjected to the in-plane wide-band Gaussian stochastic forces. The crucial point of the method is a construction of a suitable Lyapunov functional, which is positive for any motion of the analyzed system. Then the measure of distance can be chosen as the square root of functional

$$\|w(\cdot,t)\| = V^{1/2}.$$

The boundary conditions corresponding to elastically supported edges have the following form:

$$\begin{aligned}
&w(0,y,t)=0, \quad && m_x(0,y,t)+kw_{,x}(0,y,t)=0,\\
&w(a,y,t)=0, && m_x(a,y,t)-kw_{,x}(a,y,t)=0,\\
&w(x,0,t)=0, && m_y(x,0,t)+kw_{,y}(x,0,t)=0,\\
&w(x,b,t)=0, && m_y(x,b,t)-kw_{,y}(x,b,t)=0.
\end{aligned} \tag{2.2}$$

We rewrite Eq. (2.1) as the Ito partial differential equation with two independent Wiener processes modeling the stochastic components of the in-plane forces

$$\begin{aligned}
dw &= w_{,t}dt,\\
dv &= -\left(2\beta w_{,t} + \frac{1}{\rho h}\mathcal{K}w + f_{0x}w_{,xx} + f_{0y}w_{,yy}\right)dt\\
&\qquad + \sigma_x w_{,xx}dW_x + \sigma_y w_{,yy}dW_y, \qquad (x,y)\in\Omega,
\end{aligned} \tag{2.3}$$

where $f_{0x} = \mathcal{N}_{0x}/\rho h$, $f_{0y} = \mathcal{N}_{0y}/\rho h$, σ_x and σ_y are the intensities of Wiener processes.

Equation (2.3) possesses the trivial solution $w = w_{,t} = 0$ and we are going to examine a uniform stochastic stability of the equilibrium state. The trivial solution is called uniformly stochastically stable if the following logic sentence is true:

$$\bigwedge_{\Delta>0}\bigwedge_{\delta>0}\bigvee_{r>0} \|w(\cdot,0)\| < r \Rightarrow P\{\gamma : \sup_{t>0}\|w(\cdot,t)\| > \delta\} < \Delta. \tag{2.4}$$

3. Isotropic plate

We start our consideration from the isotropic plates. Neglecting the in-plane inertia term and applying the Kirchhoff hypothesis, Eq. (2.1) takes the form

$$\begin{aligned}(3.1)\qquad \rho h w_{,tt} + 2\beta\rho h w_{,t} - M_{x,xx} - M_{y,yy} - 2M_{xy,xy} + (\mathcal{N}_{0x} + \mathcal{N}_x(t))w_{,xx} \\ +(\mathcal{N}_{0y} + \mathcal{N}_y(t))w_{,yy} = 0, \quad (x,y) \in \Omega .\end{aligned}$$

M_x, M_y, M_{xy} denote the bending and torsional inner moments in the plate. Dividing Eq. (3.1) by ρh and modeling the time-dependent components of in-plane forces $\mathcal{N}_x$ and $\mathcal{N}_y$ as Gaussian white noises, we have the following Ito-type system of stochastic partial differential equations of the first order with respect to time:

$$(3.2)\qquad \begin{aligned} dw &= v dt , \\ dv &= -(2\beta v - m_{x,xx} - m_{y,yy} - 2m_{xy,xy} + f_{0x}w_{,xx} + f_{0y}w_{,yy})dt \\ &\quad +\sigma_x w_{,xx} dW_x + \sigma_y w_{,yy} dW_y, \quad (x,y) \in \Omega ,\end{aligned}$$

where the reduced inner moments can be expressed by a transverse displacement in the form

$$(3.3)\qquad \begin{aligned} m_x &= -d(w_{,xx} + \nu w_{,yy}), \\ m_y &= -d(w_{,yy} + \nu w_{,xx}), \\ m_{xy} &= -d(1-\nu)w_{,xy}, \\ d &= \frac{D}{\rho h}, \end{aligned}$$

and independent standard Wiener processes are denoted by W_x, W_y. In order to examine the uniform stochastic stability of trivial solution $w(x,y,t) = v(x,y,t) = 0$ we apply the direct Lyapunov method. The Lyapunov functional suitable to the uniform stability analysis has the form [2]

$$(3.4)\qquad V = T + V_p + V_s ,$$

where T, V_p and V_s are the modified kinetic energy, potential energy and support energy, respectively

$$T = \int_\Omega (v^2 + 4\beta v w + 4\beta^2 w^2)\, d\Omega ,$$

$$V_p = \int_\Omega (-m_x w_{,xx} - m_y w_{,yy} - 2m_{xy} w_{,xy} - f_{0x} w_{,x}^2 - f_{0y} w_{,x}^2)\, d\Omega ,$$

$$V_s = \int_0^a k(w_{,y}^2(x,0,t) + w_{,y}^2(x,b,t))\, dx + \int_0^b k(w_{,x}^2(0,y,t) + w_{,x}^2(a,y,t))\, dy .$$

The functional is positive definite since the integrand of the modified kinetic energy can be rearranged as a sum of squares, and the rest is the potential energy of the plate and the edges, which is positive if the constant compressive forces are sufficiently small.

Using the Ito calculus we evaluate the differential of functional

$$
\begin{aligned}
(3.5)\quad dV = \int_{\Omega} (2vdv + 4\beta w dv + 4\beta v^2 dt + 8\beta^2 wv dt + dm_x w_{,xx} - dm_y w_{,yy} - 2dm_{xy} w_{,xy} \\
- m_x dw_{,xx} - m_y dw_{,yy} - 2m_{xy} dw_{,xy} - 2f_{0x} w_{,x} v_{,x} dt - 2f_{0y} w_{,y} v_{,y} dt)\, d\Omega \\
+ 2\int_0^a k(w_{,y}(x,0,t)v_{,y}(x,0,t) + w_{,y}(x,b,t)v_{,y}(x,b,t))\, dx \\
+ 2\int_0^b k(w_{,x}(0,y,t)v_{,x}(0,y,t) + w_{,x}(a,y,t)v_{,x}(a,y,t))\, dy .
\end{aligned}
$$

Remembering that

$$
\begin{aligned}
dm_x &= -d(v_{,xx} + \nu v_{,yy})dt = m_{x,t} dt , \\
dm_y &= m_{y,t} dt, \qquad dm_{xy} = m_{xy,t} dt
\end{aligned}
$$

and substituting equations of motion (3.2), we rewrite the differential of the functional in the form

$$
\begin{aligned}
(3.6)\quad dV = \int_{\Omega} (2v(m_{x,xx} + m_{y,yy} + 2m_{xy,xy}) - 2f_{0x} v w_{,xx} - 2f_{0y} v w_{,yy} \\
+ 4\beta w(m_{x,xx} - m_{y,yy} - 2m_{xy,xy}) - 4\beta f_{0x} w w_{,xx} - 4\beta f_{0x} w w_{,xx} \\
- m_{x,t} w_{,xx} - m_{y,t} w_{,yy} - 2m_{xy,t} w_{,xy} - m_x v_{,xx} - m_y v_{,yy} - 2m_{xy} v_{,xy} \\
- 2f_{0x} w_{,x} v_{,x} - 2f_{0y} w_{,y} v_{,y} + \sigma_x^2 w_{,xx}^2 + \sigma_y^2 w_{,yy}^2)\, d\Omega\, dt \\
+ 2\int_0^a k(w_{,y}(x,0,t)v_{,y}(x,0,t) + w_{,y}(x,b,t)v_{,y}(x,b,t))\, dx \\
+ 2\int_0^b k(w_{,x}(0,y,t)v_{,x}(0,y,t) + w_{,x}(a,y,t)v_{,x}(a,y,t))\, dy \\
+ \int_{\Omega} (2v + 4\beta w)(\sigma_x w_{,xx} dW_x + \sigma_y w_{,yy} dW_y)\, d\Omega .
\end{aligned}
$$

Integrating by parts and using boundary conditions (2.2) we can prove the following equality:

$$
\begin{aligned}
\int_{\Omega} 4\beta w m_{x,xx}\, d\Omega = \int_0^b 4\beta w m_{x,x}\Big|_0^a dy - \int_{\Omega} 4\beta w_{,x} m_{x,x}\, dy = - \int_0^b 4\beta w_{,x} m_x \Big|_0^a dy \\
+ \int_{\Omega} 4\beta w_{,xx} m_x\, d\Omega = -4\beta \int_0^b k(w_{,x}^2(0,y,t) + w_{,x}^2(a,y,t))\, dy + \int_{\Omega} 4\beta w m_{x,xx}\, d\Omega .
\end{aligned}
$$

Finally, we can rewrite the differential (3.6) in the form

$$
\begin{aligned}
(3.7)\quad dV = -4\beta \Big\{ V_p - \int_{\Omega} ((\sigma_x^2 w_{,xx}^2 + \sigma_y^2 w_{,yy}^2)/4\beta)\, d\Omega + V_s \Big\}\, dt \\
+ \int_{\Omega} (2v + 4\beta w)(\sigma_x w_{,xx} dW_x + \sigma_y w_{,yy} dW_y)\, d\Omega .
\end{aligned}
$$

Integrating Eq. (3.7) with respect to time from s to $\tau_\delta(t)$, where $\tau_\delta(t)$ is the first exit time of the system trajectory from the domain $V^{1/2} \leq \delta$, we obtain $V(\tau_\delta(t))$. Remembering that the exit time is a random variable we conditionally average it with respect to a σ-algebra generated by events earlier that s. Therefore the last term in Eq. (3.7) vanishes and we have

$$EV(\tau_\delta(t)) = V(s) - E \int_0^{\tau_\delta(t)} 4\beta\Big\{V_p - \int_\Omega ((\sigma_x^2 w_{,xx}^2 + \sigma_y^2 w_{,yy}^2)/4\beta)\, d\Omega + V_s\Big\}\, dt. \tag{3.8}$$

The functional $V(\tau_\delta(t))$ is a supermartingale [7] if the expression in cubic brackets is positive. Proceeding similarly to the proof of Chebyshev's inequality we can prove that the supermartingality is a sufficient condition for the uniform stochastic stability of the trivial solution. Thus the positive definiteness of the following functional is the main condition for the uniform stochastic stability,

$$\begin{aligned} &\int_\Omega (w_{,xx}^2 + w_{,yy}^2 + 2w_{,xy}^2 - (f_{0x}w_{,x}^2 - f_{0y}w_{,x}^2)/d - (\sigma_x^2 w_{,xx}^2 + \sigma_y^2 w_{,yy}^2)/4\beta d)\, d\Omega \\ &+ \int_0^b \frac{k}{d}(w_{,x}^2(0,y,t) + w_{,x}^2(a,y,t))\, dy + \int_0^a \frac{k}{d}(w_{,y}^2(x,b,t) + w_{,y}^2(x,0,t))\, dx \geq 0. \end{aligned} \tag{3.9}$$

The stability condition can be expressed by the potential energy of plate and the energy of support as follows:

$$V_p - \int_\Omega ((\sigma_x^2 w_{,xx}^2 + \sigma_y^2 w_{,yy}^2)/4\beta)\, d\Omega + V_s > 0. \tag{3.10}$$

4. Antisymmetrically laminated cross-ply plate

Let us consider a thin cross-ply laminated rectangular plate consisting of an odd number of elastic orthotropic layers antisymmetrically laminated about its middle surface from both the geometric and a material standpoint. The Kirchhoff hypothesis on nondeformable normal element is taken into account. Neglecting the in-plane, rotary, and coupling inertias, linear vibrations obey the following set of partial equations in terms of the in-plane displacement w [8]

$$N_{x,x} + N_{xy,y} = 0, \tag{4.1}$$

$$N_{xy,x} + N_{y,y} = 0, \tag{4.2}$$

$$\begin{aligned} &\rho h w_{,tt} + 2\beta\rho h w_{,t} + \mathcal{N}_x w_{,xx} + \mathcal{N}_y w_{,yy} + D_{11} w_{,xxxx} \\ &+2(D_{12} + 2D_{66}) w_{,xxyy} + D_{22} w_{,yyyy} - B_{11}(u_{,xxx} - v_{,yyy}) = 0, \qquad (x,y) \in \Omega, \end{aligned} \tag{4.3}$$

where the membrane forces are defined as follows:

$$\begin{aligned} N_x &= A_{11} u_{,x} + A_{12} v_{,y} - B_{11} w_{,xx}, \\ N_y &= A_{12} u_{,x} + A_{22} v_{,y} + B_{11} w_{,yy}, \\ N_{xy} &= A_{66}(u_{,y} + v_{,x}). \end{aligned} \tag{4.4}$$

The cross-ply laminate has unidirectionally reinforces (orthotropic) layers with principal material directions alternatively oriented at 0 and $\Pi/2$ to the laminate coordinate axes. The in-plane stiffnesses A_{ij}, coupling stiffnesses B_{ij}, and the bending stiffnesses D_{ij} are defined as follows:

$$(A_{ij}, B_{ij}, D_{ij}) = \int_{-\frac{h}{2}}^{\frac{h}{2}} (1, z, z^2) Q_{ij}\, dz\,.$$

The reduced in-plane stiffnesses of an individual lamina are expressed in terms of the lamina principal material properties [8], E_1, E_2, G_{12} and ν_{12} are major Young's modulus, minor Young's modulus, the shear modulus and major Poisson ratio, respectively.

The plate is assumed to be elastically supported. Denoting the solutions of the first two Eqs. (4.1) and (4.2) as

$$u = \mathcal{L}_u(w)\,, \qquad v = \mathcal{L}_v(w)\,,$$

we substitute them into the third Eq. (4.3), divide by ρh and preserve for convenience the same notations

$$\begin{aligned}(4.5)\qquad w_{,tt} + 2\beta w_{,t} + (f_{0x} + f_x(t))w_{,xx} + (f_{0y} + f_x(t))w_{,yy} + D_{11}w_{,xxxx} \\ +2(D_{12} + D_{66})w_{,xxyy} + D_{22}w_{,yyyy} \\ -B_{11}(\mathcal{L}_u(w)_{,xxx} + \mathcal{L}_v(w)_{,yyy}) = 0, \quad (x, y) \in \Omega.\end{aligned}$$

We choose the functional in the form (3.4), but it should be remembered that the inner moments are expressed by modified expressions

$$\begin{aligned} M_x &= -B_{11}u_{,x} - D_{11}w_{,xx} - D_{12}w_{,yy}\,, \\ M_y &= -B_{11}v_{,y} - D_{12}w_{,xx} - D_{22}w_{,yy}\,, \\ M_{xy} &= -2D_{66}w_{,xy}\,. \end{aligned}$$

Proceeding similarly to the stability analysis of isotropic plates we apply the Ito lemma and calculate the functional differential. It can be shown that the main stability condition for antisymmetrically laminated cross-ply plate has the same form as that for the isotropic ones (3.10).

5. Cylindrical bending

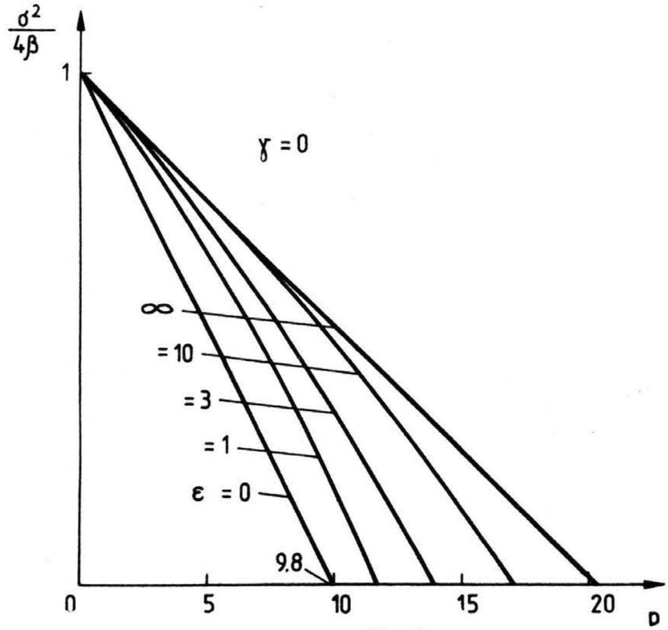
Effective stability conditions in terms of force intensity σ, critical axial force $P_{\rm cr}$, damping coefficient β and edge coefficients can be easily calculated for a plate in cylindrical bending, where results of static buckling calculated by PLAUT [5] can be used. Dimensionless notations are introduced in the following way:

$$\frac{x}{a} \Rightarrow x\,, \quad \frac{w}{a} \Rightarrow w\,, \quad \frac{ak}{D} \Rightarrow k\,, \quad \frac{Pa^2}{D} \Rightarrow p\,,$$

where

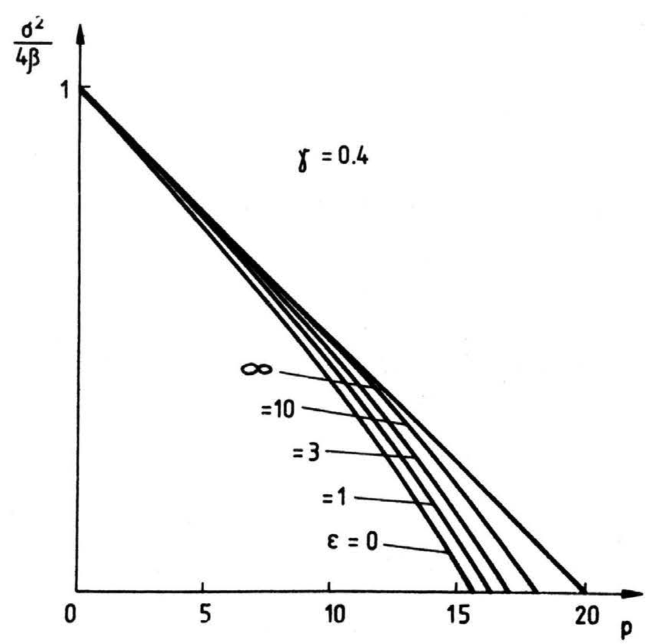
$$D = \begin{cases} D & \text{for isotropic plate,} \\ D_{11} - \dfrac{B_{11}^2}{A_{11}} & \text{for antisymmetrically laminated cross-ply plate,} \end{cases}$$

Assuming that the plate is infinitely long in y direction, neglecting the compression in y direction as $w(x, y) = w(x)$, and omitting partial derivatives with respect to y, we obtain



$\frac{\sigma^2}{4\beta}$
1
γ = 0
∞
= 10
= 3
= 1
ε = 0
9.8
0
5
10
15
20
p

FIG. 1.



$\frac{\sigma^2}{4\beta}$
1
γ = 0.4
∞
= 10
= 3
= 1
ε = 0
0
5
10
15
20
p

FIG. 2.

the basic inequality in the form

$$\int_0^1 ((1-\sigma_x^2/4\beta)w_{,xx}^2 - pw_{,x}^2)\,dx + k(w_{,x}^2(0,t) + w_{,x}^2(1,t)) \geq 0\,, \tag{5.1}$$

where the simplified boundary conditions have the form

$$\begin{aligned} w(0,t) &= 0\,, \quad w_{,xx}(0,t) - kw_{,x}(0,t) = 0 \\ w(1,t) &= 0\,, \quad w_{,xx}(1,t) - kw_{,x}(1,t) = 0\,. \end{aligned} \tag{5.2}$$

If the edge $x = 1$ is a classical simple support independent of the axial force and the characteristic of boundary condition for $x = 0$ depends on x, the buckling force can be determined by minimizing the potential energy of the plate

$$\Pi = \int_0^1 w_{,xx}^2\,dx - P_{\text{cr}} \int_0^1 w_{,x}^2\,dx + kw_{,x}^2(0,t)\,. \tag{5.3}$$

Since the critical force P_{cr} can be calculated from the Rayleigh quotient

$$P_{\text{cr}} = \min_w \frac{\int_0^1 w_{,xx}^2\,dx + kw_{,x}^2(0,t)}{\int_0^1 w_{,x}^2\,dx}\,,$$

it satisfies the inequality

$$\Pi \geq 0 \tag{5.4}$$

for any function $w(x,t)$ satisfying boundary conditions (5.2). The appropriate Euler equation derived from condition $\delta\Pi = 0$ has the classical form

$$w_{,xxxx} - Pw_{,xx} = 0\,, \tag{5.5}$$

for which the critical force is determined [5] for a linearly stiffening spring constant of the support

$$k = \varepsilon_1 + P\varepsilon_2\,, \tag{5.6}$$

where ε_1 is the value of the spring constant, and ε_2 is a proportionality coefficient.

We see that the uniform stability condition (3.7) can be rewritten in the form similar to inequality (5.4) with (5.3) as follows

$$\int_0^1 w_{,xx}^2\,dx - \frac{P_{\text{cr}}}{1-\dfrac{\sigma^2}{4\beta}} \int_0^1 w_{,x}^2\,dx + \frac{k}{1-\dfrac{\sigma^2}{4\beta}} w_{,x}^2(0,t) \geq 0\,. \tag{5.7}$$

Using numerical results of paper [5] we can calculate $\sigma_x^2/4\beta$ as a function of the constant component of axial force p and boundary coefficients ε and γ. Figures 1 and 2 present uniform stability regions for $\gamma = 0.0$ and 0.4, respectively. Stable domains are situated under curves shown in the figures. These results demonstrate how the stability domains are affected by supports which stiffen when the axial force is increased. It is seen that if the constant component ε of spring support is increased, the stability domain increases.

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WARSAW UNIVERSITY OF TECHNOLOGY, WARSZAWA.

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