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Free and immersed opposing laminar jets as viscoelastic flows with dominating extension

S. ZAHORSKI (WARSZAWA)

WE CONSIDER an application of the flows with dominating extension to the case of free and immersed laminar viscoelastic jets emerging from two opposing nozzles. An importance of the above geometries for the fluid characterization as well as for the extensional (compressive) viscosity measurements for low viscosities and high strain rates is discussed in greater detail.

1. Introduction

IN A SERIES of our previous papers (cf. [1, 2, 3]) we defined the so-called flows with dominating extension (FDEs) and discussed in greater detail some examples of possible applications. In particular, the following cases were considered: squeezing flows, flows between rotating cylinders, flows in converging slits and pipes, flows in viscoelastic boundary layers, fibre spinning and drawing processes, flows in rheometers with converging dies, etc.

In the present paper we discuss another application of the FDEs, namely the case of free and immersed laminar viscoelastic jets emerging from two opposing nozzles. An importance of the above geometries and other steady orthogonal stagnation flows for fluid characterization was stressed by WINTER *et al.* [4] already in 1979. There exist also several attempts of using various opposing jet devices, especially for immersed jets, to the extensional viscosity measurements for low viscosity fluids (< 1Pas) at high strain rates ($> 10^3 s^{-1}$) [5, 6, 7]. Moreover the above devices can be designed and used for volative and corrosive fluids in closed systems. It is noteworthy that even for purely viscous fluids there exist no exact analytical solutions of the opposing laminar jet problems. Also the exact solutions of more fundamental problems of single laminar jets, free or immersed, are known only for infinitely narrow slits and small holes, or for particular shapes of the nozzle and initial viscosity distributions (cf. [8, 9]). Some recent theoretical and experimental results on the submerged jet flows of non-Newtonian pseudoplastic fluids are presented by JORDAN *et al.* [10].

At the beginning of the paper we briefly discuss the most important relations characterizing the FDEs as applied to the opposing laminar jet flows of viscoelastic fluids. In two subsequent sections we consider separately certain approximate solutions of the case of free jets expelled from the opposing nozzles as well as the case of jets submerging into a big reservoir of the same fluid. The corresponding boundary conditions, the expressions for velocity profiles and stresses, the shapes of free and immersed jets, etc. are presented in greater detail. The case of additional mass transport from outside into the submerged jet is discussed separately. At the end some possibilities of the extensional (compressive) viscosity measurements are briefly outlined.

2. Axisymmetric flows with dominating extension

Consider the following velocity field expressed in a system of polar cylindrical coordinates (r, θ, z):

$$u^* = \frac{1}{2}\dot{\epsilon} r + u\,, \quad w^* = -\dot{\epsilon} z + w\,, \tag{2.1}$$

where the first terms describe a purely extensional flow characterized by a constant extension rate $\dot{\epsilon}$ and the second terms u, w denote the additional velocity components responsible for shearing effects. Hence, the corresponding velocity gradient takes the form:

$$[\nabla \mathbf{v}^*] = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & -1 \end{bmatrix} \dot{\epsilon} + \begin{bmatrix} \frac{\partial u}{\partial r} & 0 & \frac{\partial u}{\partial z} \\ 0 & \frac{u}{r} & 0 \\ \frac{\partial w}{\partial r} & 0 & \frac{\partial w}{\partial z} \end{bmatrix}. \tag{2.2}$$

If we assume, moreover, that the ratio of the characteristic dimension h in the z-direction to the characteristic dimension l in the r-direction is a small quantity $\varepsilon = h/l \ll 1$, we can use simplifications applied to various thin-layer flows (lubrication approximation). Introducing the following dimensionless quantities marked with overbars:

$$r = l\overline{r}, \quad z = h\overline{z}, \quad u = U\overline{u}, \quad w = \varepsilon U \overline{w}, \quad \dot{\epsilon} = \frac{U}{l}\bar{\dot{\epsilon}}, \tag{2.3}$$

where $U = \dot{\epsilon} h$ denotes the characteristic velocity for the additional field, we can express Eq. (2.2) as

$$[\nabla \mathbf{v}^*] = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & -1 \end{bmatrix} \dot{\epsilon} + \begin{bmatrix} \varepsilon\frac{\partial \overline{u}}{\partial \overline{r}} & 0 & \frac{\partial \overline{u}}{\partial \overline{z}} \\ 0 & \varepsilon\frac{\overline{u}}{r} & 0 \\ \varepsilon^2\frac{\partial \overline{w}}{\partial \overline{r}} & 0 & \varepsilon\frac{\partial \overline{w}}{\partial \overline{z}} \end{bmatrix} \dot{\epsilon}\,. \tag{2.4}$$

For relatively small vorticity components or relatively high Deborah numbers (cf. [11]), the first matrix components may be notably more meaningful than the second matrix components. A decisive role is played by the gradient $\partial\overline{u}/\partial\overline{z}$.

All the kinematic quantities necessary for further considerations can be presented in the form of sums consisting of fundamental and additional terms. We obtain, for instance, the first Rivlin–Ericksen kinematic tensor $\mathbf{A}_1$ in the form:

$$[\mathbf{A}_1^*] = [\mathbf{A}_1] + [\mathbf{A}_1]' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \dot{\epsilon} + \begin{bmatrix} 2\frac{\partial u}{\partial r} & 0 & \frac{\partial w}{\partial r} + \frac{\partial u}{\partial z} \\ 0 & 2\frac{u}{r} & 0 \\ \frac{\partial w}{\partial r} + \frac{\partial u}{\partial z} & 0 & 2\frac{\partial w}{\partial z} \end{bmatrix}. \tag{2.5}$$

On denoting the invariants:

$$\operatorname{tr} \mathbf{A}_1^{*2} = 6(\dot{\epsilon} + \dot{\epsilon}')^2\,, \quad \operatorname{tr} \mathbf{A}_1^{*3} = 6(\dot{\epsilon} + \dot{\epsilon}')^3 \tag{2.6}$$

we also have for the extension rate increment

$$(2.7)\qquad \dot{\epsilon}' = \frac{\partial w}{\partial z} + \frac{1}{3\dot{\epsilon}}\left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{3\dot{\epsilon}}\left(\frac{u}{r}\right)^2 + \frac{1}{3\dot{\epsilon}}\left(\frac{\partial u}{\partial z}\right)^2 + \frac{1}{6\dot{\epsilon}}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right)^2,$$

where primes refer to additional terms.

In our previous papers [1, 2, 3] we defined the flows with dominating extension (FDEs) as such thin-layer flows for which the constitutive equations describing purely extensional flows of an incompressible simple fluid, viz.

$$(2.8)\qquad \mathbf{T} = -p\mathbf{1} + \beta_1\mathbf{A}_1 + \beta_2\mathbf{A}_1^2, \quad \operatorname{tr}\mathbf{A}_1 = 0,$$

where $\mathbf{T}$ is the stress tensor, p — the hydrostatic pressure, and β_i ($i = 1, 2$) denote the material functions of the invariants $\operatorname{tr}\mathbf{A}_1^2$, $\operatorname{tr}\mathbf{A}_1^3$, can be used in a form linearly perturbed with respect to the additional velocity gradients. This definition means that

$$(2.9)\qquad \mathbf{T}^* = -p\mathbf{1} + \beta_1\mathbf{A}_1 + \beta_1\mathbf{A}_1' + \beta_2\mathbf{A}_1^2 + \beta_2(\mathbf{A}_1^2)' + \frac{\partial\beta_1}{\partial\dot{\epsilon}}\dot{\epsilon}'\mathbf{A}_1 + \frac{\partial\beta_2}{\partial\dot{\epsilon}}\dot{\epsilon}'\mathbf{A}_1^2.$$

Introducing the following dimensionless quantities:

$$(2.10)\qquad p = \frac{U\eta_0 l}{h^2}\overline{p}, \quad \beta_1 = \eta_0\overline{\beta}_1, \quad \beta_2 = \frac{\eta_0}{\dot{\epsilon}}\overline{\beta}_2,$$

and retaining in the inertialess equations of equilibrium terms of the highest order of magnitude with respect to $\varepsilon = h/l$, we arrive at the governing equations:

$$(2.11)\qquad \begin{aligned} &\frac{\partial p^*}{\partial z} = 0, \\ &\frac{dp^*}{dr} = C(r) = \beta\frac{\partial^2 u}{\partial z^2} + K\beta\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial z}\right)^2, \end{aligned}$$

where

$$(2.12)\qquad p^* = p - T_E^{*33} = -T^{*33}, \quad \beta = \beta_1 - \beta_2\dot{\epsilon}, \quad K = \frac{1}{\beta}\left(\frac{\partial\beta_1}{\partial\dot{\epsilon}} - \frac{\partial\beta_2}{\partial\dot{\epsilon}}\dot{\varepsilon}\right),$$

p^* is the modified pressure and $C(r)$ denotes an unknown function of r only.

It is worth noting that the above nonlinear differential equations can be obtained immediately from the equations of equilibrium after using the following simplified relations:

$$(2.13)\qquad \begin{aligned} T^{*11} = T^{*22} &= -p + \beta_1\dot{\epsilon} + \beta_2\dot{\epsilon}^2 + \beta_2\left(\frac{\partial u}{\partial z}\right)^2 + \frac{1}{6}\frac{\partial\beta_1}{\partial\dot{\epsilon}}\left(\frac{\partial u}{\partial z}\right)^2 + \frac{1}{6}\frac{\partial\beta_2}{\partial\dot{\epsilon}}\dot{\epsilon}\left(\frac{\partial u}{\partial z}\right)^2, \\ T^{*33} &= -p - 2\beta_1\dot{\epsilon} + 4\beta_2\dot{\epsilon}^2 + \beta_2\left(\frac{\partial u}{\partial z}\right)^2 - \frac{1}{3}\frac{\partial\beta_1}{\partial\dot{\epsilon}}\left(\frac{\partial u}{\partial z}\right)^2 + \frac{2}{3}\frac{\partial\beta_2}{\partial\dot{\epsilon}}\dot{\epsilon}\left(\frac{\partial u}{\partial z}\right)^2, \\ T^{*13} &= \beta\frac{\partial u}{\partial z}. \end{aligned}$$

As a consequence of the above formulae we also have

$$T^{*11} - T^{*33} = 3\beta\dot{\epsilon} + \frac{1}{2}K\beta\left(\frac{\partial u}{\partial z}\right)^2. \tag{2.14}$$

It is noteworthy that the material function β (cf. Eq. $(2.12)_2$) is simply related to the corresponding elongational viscosity defined for steady flows as

$$\eta^*(\dot{\epsilon}) = \frac{1}{\dot{\epsilon}}(T^{*11} - T^{*33}) \simeq 3\beta\dot{\epsilon}, \tag{2.15}$$

this fact justifies for β the name "extensional viscosity".

Equations (2.12) can be solved with appropriate boundary conditions expressed either in velocities or stresses. The surface boundary conditions for free jets are very similar to those derived in [2] and result from the fact that all the forces acting on a free surface have to be mutually balanced.

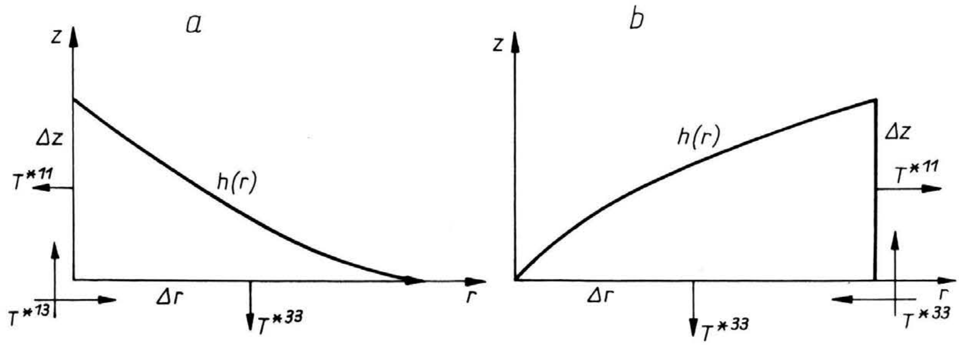

FIG. 1.

Considering a concave surface element based on the coordinates Δr, $r\Delta\theta$, Δz, whose projection is shown in Fig. 1a, we arrive at

$$T_s^{*11}\xi = T_s^{*13}, \quad T_s^{*33} = T_s^{*13}\xi, \tag{2.16}$$

where $\xi = dh/dr$ characterizes the inclination of the surface and s marks the values on the surface. This leads to the condition

$$(T_s^{*11} - T_s^{*33})\frac{dh}{dr} = T_s^{*13}, \tag{2.17}$$

if $\xi \ll 1$ is a small quantity and its squares may be neglected as compared with ξ.

For a convex surface element shown in Fig. 1b, we obtain

$$T_s^{*11}\xi = -T_s^{*13}, \quad T_s^{*33} = -T_s^{*13}\xi, \tag{2.18}$$

and finally

$$(T_s^{*11} - T_s^{*33})\frac{dh}{dr} = -T_s^{*13} \tag{2.19}$$

if ξ^2 are disregarded.

It is worth noting that Eqs. (2.18) and (2.19) remain valid, if the free surface is subjected to arbitrary hydrostatic pressure.

3. The case of free jets emerging from the opposing nozzles

A solution of Eqs. (2.11) together with appropriate boundary conditions can be achieved either in a numerical or in an analytical way. A simple perturbation method can be applied in the case of slightly viscoelastic fluids (cf. [1, 2]), if the following quantity:

$$k = K\dot{\epsilon} = \frac{1}{\beta}\left(\frac{\partial\beta_1}{\partial\dot{\epsilon}} - \frac{\partial\beta_2}{\partial\dot{\epsilon}}\dot{\epsilon}\right)\dot{\epsilon} \tag{3.1}$$

is small enough, i.e. if k^2 are small as compared with k. Since $k \equiv 0$ corresponds to the case of Newtonian fluids, perturbed solutions are approximately valid for a weak variability of β_1 and β_2 (or β) with the extensiòn rate $\dot{\epsilon}$.

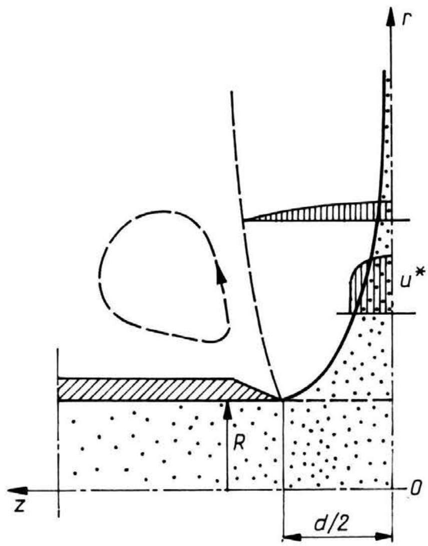

FIG. 2.

Bearing in mind that the whole procedure concerned with the concept of FDEs can be used, if there exists a small parameter $\varepsilon = h/l \ll 1$, we assume, moreover, that (Fig. 2)

$$h = d/2, \quad l = nR, \quad \varepsilon = d/2nR, \tag{3.2}$$

where d denotes the distance between the nozzles, R — the hole radius, and n is some number depending on the geometry of a system. An idea of applying perturbation methods to slender jets is also presented in [12].

3.1. Solution for Netwonian fluids

In the case considered $\beta = \beta_1 = \beta_0 = \text{const}$, $\beta_2 = 0$, and the straightforward integration of Eq. $(2.11)_2$ leads to the following velocity field for $r \geq R$:

$$u_0 = \frac{C_0(r)}{2\beta_0}z^2 + E_0(r), \tag{3.3}$$

where the subscripts 0 refer to Newtonian quantities and E_0 is a function of r only. Taking into account Eq. $(2.1)_1$ and the condition that the flow rate Q is constant in all cross-sections of impinging jets, i.e.

$$\frac{Q}{2} = \int_0^h u_0^* 2\pi r dz = \pi r^2 \dot{\epsilon} h + \int_0^h u_0 2\pi r dz, \tag{3.4}$$

where $h(r)$ denotes the outer surface of a jet, we arrive at

$$u_0^* = \frac{Q}{4\pi r h} + \frac{C_0(r)}{2\beta_0}\left(z^2 - \frac{h^2}{3}\right). \tag{3.5}$$

Since we assume $u_0^* = 0$ at the nozzle tip, i.e. for $r = R$ and $h = d/2$ (cf. Fig. 2), it turns out that

$$C_0(R) = -\frac{6Q\beta_0}{\pi R d^3}. \tag{3.6}$$

The unknown function $C_0(r)$ must be consistent with Eq. (3.6), with the boundary condition on a free surface (2.17) as well as with the assumption that $C_0(r) \to 0$ for $r \to \infty$. Without any loss of generality we assume at the moment that

$$C_0(r) \simeq -\frac{6Q\beta_0}{\pi r d^3}. \tag{3.7}$$

This specific form of the function $C_0(r)$ will be verified a little later.

The second unknown function $h(r)$ determines the outer surface of impinging jets (cf. Fig. 2). Since all the forces acting on that surface are mutually balanced (including presence of an atmospheric pressure), the corresponding impulse (momentum) is conserved in the flow considered. Thus, we have the following integral condition:

$$J = \int_0^h \varrho u^{*2} 2\pi r dz = \text{const}. \tag{3.8}$$

Introducing Eqs. (3.5) and (3.7), and comparing the values of J calculated for arbitrary $h(r)$ as well as for $h = d/2$, we arrive at

$$h = \frac{R}{r}\frac{d}{2}\left[1 + \frac{1}{5}\left(h/\frac{d}{2}\right)^6\right]\frac{5}{6}. \tag{3.9}$$

It is seen that for h close to $d/2$ (close to the nozzle tip) the above expression simplifies to

$$h \simeq \frac{R}{r}\frac{d}{2}, \tag{3.10}$$

while for $r \to \infty$, we obtain the value 0.83 times smaller.

Differentiating Eq. (3.10) with respect to r, viz.

$$\frac{dh}{dr} \simeq -\frac{R}{r^2}\frac{d}{2} = -\frac{h}{r}, \tag{3.11}$$

and putting the above result into the boundary condition (2.17), we have

$$-3\beta_0 \dot{\epsilon}\frac{h}{r} = C_0(r) h, \tag{3.12}$$

and finally

$$C_0(r) = -\frac{3\beta_0\dot{\epsilon}}{r}. \tag{3.13}$$

This form of the function $C_0(r)$ is also consistent with our previous assumption expressed through Eq. (3.7), if the constant extension rate amounts to

$$\dot{\epsilon} = \frac{2Q}{\pi d^3}. \tag{3.14}$$

3.2. Solution for slightly viscoelastic fluids

Under the assumption that the parameter k defined in Eq. (3.1) is small enough to reject terms proportional to k^2 and to higher powers of k, the straightforward integration of Eq. (2.11) leads to the following velocity field:

$$u = \frac{C(r)}{2\beta}z^3 + \frac{3}{2}k\frac{Q}{\pi d^3 r^3}z^4 + E(r), \tag{3.15}$$

where $E(r)$ is a function of r only. Taking into account Eq. $(2.1)_1$ and the condition of constant flow rate in the form (3.4), we arrive at

$$u^* = \frac{Q}{4\pi r h} + \frac{C(r)}{2\beta}\left(z^2 - \frac{h^2}{3}\right) + \frac{3}{2}k\frac{Q}{\pi d^3 r^3}\left(z^4 - \frac{h^4}{5}\right), \tag{3.16}$$

where we have used Eq. (3.14).

Since we assume $u^* = 0$ at the nozzle tip, i.e. for $r = R$ and $h = d/2$ (cf. Fig. 2), we obtain

$$C(R) = -\frac{6Q\beta}{\pi R d^3} - \frac{9}{10}k\beta\frac{Q}{\pi R^3 d}. \tag{3.17}$$

Without any loss of generality we may write the function $C(r)$ as

$$C(r) = -\frac{6Q\beta}{\pi r d^3} - Ak\beta\frac{Qh^2}{\pi r^3 d^3}, \tag{3.18}$$

where A is a numerical constant. The above form of the function $C(r)$ will be verified a little later.

Calculating the impulse efflux determined by Eq. (3.8) for arbitrary $h(r)$ as well as for $h = d/2$, we arrive at

$$h = \frac{R}{r}\frac{d}{2}\left[1 + \frac{1}{5}\left(h/\frac{d}{2}\right)^6 + Bk\left(h/\frac{d}{2}\right)^6\frac{h^2}{r^2}\right]\left[\frac{6}{5} + \frac{1}{4}Bk\frac{d^2}{R^2}\right]^{-1}, \tag{3.19}$$

where B is another numerical constant.

It is seen that for h close to $d/2$ (close to the nozzle tip), we rediscover Eq. (3.10), while for $r \to \infty$, we have

$$h \simeq \frac{R}{r}\frac{d}{2}\left[\frac{6}{5} + Bk\frac{d^2}{4R^2}\right]^{-1}. \tag{3.20}$$

Substituting from Eqs. (2.13) into the boundary condition (2.17), we obtain

$$-\left(3\beta\dot{\epsilon} + 9k\beta\frac{Q}{\pi d^3}\frac{h^2}{r^2}\right)\frac{h}{r} = \beta\left(\frac{C_0}{\beta}h + 6k\beta\frac{Q}{\pi d^3}\frac{h^3}{r^3}\right), \tag{3.21}$$

and finally

$$C(r) = -\frac{3\beta\dot{\epsilon}}{r} - 15k\beta\frac{Q}{\pi d^3}\frac{h^2}{r^3}. \tag{3.22}$$

This form of the function $C(r)$ is therefore consistent with the previously postulated Eq. (3.18). To this end the constant rate of extension amounts to

$$\dot{\epsilon} = \frac{20}{\pi d^3}\left[1 - \frac{19}{10}k\left(\frac{d}{2}/R\right)^2\right]. \tag{3.23}$$

Hence, the constant B present in Eqs. (3.19) and (3.20) must be equal to 29/35.

The above considerations are based on the simplified expression (3.10) valid for $h(r)$ close to $d/2$. On the other hand, the value of $h(r)$ valid for $r \to \infty$ leads to

$$\frac{dh}{dr} = -\frac{5}{6}\frac{R}{r^2}\frac{d}{2} = -\frac{5}{6}\frac{h}{r}, \tag{3.24}$$

and under the Newtonian approximation

$$C_0(r) = -\frac{5\beta_0\dot{\epsilon}}{2r}, \quad \dot{\epsilon} = \frac{12Q}{5\pi d^3}. \tag{3.25}$$

The latter quantity is 40% greater than the extension rate described by Eq. (3.14).

Since the effect of k is opposite (the value of $\dot{\epsilon}$ decreases with k), the total effect of r tending to infinity and viscoelastic properties of a fluid may be meaningless.

The final approximate expression for the velocity profile in the cross-section of two opposing jets can be presented in the form:

$$u^* = \frac{Q}{4\pi rh} + \frac{\dot{\epsilon}h^2}{2r}\left(1 - 3\frac{z^2}{h^2}\right) + 5k\frac{\dot{\epsilon}h^4}{r^3}\left(1 - 3\frac{z^2}{h^2}\right) - \frac{3}{20}k\frac{\dot{\epsilon}h^4}{r^3}\left(1 - 5\frac{z^4}{h^4}\right), \tag{3.26}$$

where the extension rate $\dot{\epsilon}$ results from Eq. (3.14) or more precisely from Eq. (3.23). Similarly, the normal stress difference defined by Eq. (2.14) amounts to

$$T^{*11} - T^{*33} = 3\beta\dot{\epsilon}\left(1 + \frac{3}{2}k\frac{z^2}{r^2}\right) \tag{3.27}$$

and tends to $3\beta\dot{\epsilon}$ for $r \to \infty$.

4. The case of jets emerging from the opposing nozzles immersed in a fluid

Although this case seems to be quite different as compared with the previous one, the applied procedure is performed in a similar way. We also use the previously described perturbation method, if the quantity (3.1) is small enough and the assumption expressed in Eq. (3.2) is furthermore valid.

4.1. Solution for Newtonian fluids

In the case considered an additional velocity field defined in Eq. (2.1) should be written in the following form:

$$u = -\frac{1}{2}\dot{\epsilon}r + \widetilde{u}, \quad w = \dot{\epsilon}z + \widetilde{w}, \tag{4.1}$$

under the assumption that the concept of the FDEs is justified. The straightforward integration of Eq. $(2.11)_2$ for $\beta = \beta_1 = \beta_0 = \text{const}$, $\beta_2 = 0$ leads again to Eq. (3.3).

If we assume, moreover, that $h(r)$ denotes an interface between the submerged jet and the stagnant (or weakly recirculating) region of a fluid (cf. Fig. 2), the condition that $u_0^* = 0$ for $z \geq h(r)$ gives

$$u_0^* = \widetilde{u}_0 = \frac{C_0(r)}{2\beta_0}(z^2 - h^2), \tag{4.2}$$

where the subscript refers to Newtonian quantities. The assumption that the fluid is at rest for $z \geq h(r)$ has been made intentionally for the sake of simplicity, since in reality there is always a mass transport between a jet and a surrounding fluid.

At the exit of impinging jets or, more precisely, on the surface $r = R$, $0 < z \leq d/2$, we have for the flow rate

$$\frac{Q}{2} = \int_0^{h/2} u^* 2\pi R dz. \tag{4.3}$$

Since also $u_0^* = 0$ at the nozzle tip, i.e. for $r = R$ and $h = d/2$ (cf. Fig. 2), we arrive at the same relation as that described by Eq. (3.6). Reasoning in a way similar to that presented in Sec. 3.1, we postulate the unknown function $C_0(r)$ in the form of Eq. (3.7). A verification of this assumption is also necessary.

If the fluid outside a jet is at rest under the action of at most hydrostatic pressure, all the forces acting on the interface $h(r)$ have to be mutually balanced. This means that the boundary condition (2.19) valid for a convex interface $h(r)$ must be satisfied. If, moreover, there is no remarkable mass transfer through the interface $h(r)$, the impulse efflux in the moving part of a jet is conserved. Thus, on the basis of Eq. (3.8) we arrive at

$$h^5 = \frac{R}{r}\frac{C_0^2(R)}{C_0^2(r)}\frac{d^5}{32}. \tag{4.4}$$

After taking into account Eqs. (3.6) and (3.7), we obtain the following approximate formula:

$$h \simeq \left(\frac{r}{R}\right)^{\frac{1}{5}}\frac{d}{2}, \tag{4.5}$$

what confirms that the interface $h(r)$ is convex.

Differentiating Eq. (4.5) with respect to r, viz.

$$\frac{dh}{dr} \simeq \frac{1}{5}\left(\frac{r}{R}\right)^{-\frac{4}{5}} R^{-1}\frac{d}{2} = \frac{1}{5}\frac{h}{r} \tag{4.6}$$

and introducing the above result into Eq. (2.19), we have

$$\frac{-3}{5}\frac{\beta_0\dot{\epsilon}}{r}\frac{h}{r} = C_0(r)h; \tag{4.7}$$

what implies that

$$C_0(r) = -\frac{3\beta_0\dot{\epsilon}}{5r}. \tag{4.8}$$

This form of the function $C_0(r)$ is consistent with our previous assumption (cf. (3.7)), if the constant extension rate amounts to

$$\dot{\epsilon} = \frac{10Q}{\pi d^3}. \tag{4.9}$$

4.2. Solution for slightly viscoelastic fluids

On applying the previous perturbation scheme, after the straightforward integration of Eq. $(2.11)_2$, we obtain

$$u_0^* = \frac{C(r)}{2\beta}(z^2 - h^2) + 3k\frac{Q}{\pi d^3 r^3}(z^4 - h^4), \tag{4.10}$$

where it is assumed that $u^* = 0$ for $z = h(r)$, and the parameter k is defined by Eq. (3.1). Thus for $r = R$, $h = d/2$ we have

$$C(R) = -\frac{6Q\beta}{\pi R d^3} - \frac{39}{200}k\beta\frac{Q}{\pi R^3 d}. \tag{4.11}$$

Without any loss of generality we may write that

$$C(r) = -\frac{6Q\beta}{\pi r d^3} - Ak\beta\frac{Qh^2}{\pi r^3 d^3}, \tag{4.12}$$

where A is a numerical constant. This form of the function $C(r)$ will be verified a little later.

Calculating the impulse efflux determined by Eq. (3.8) for arbitrary $h(r)$ as well as for $h = d/2$, we arrive at

$$h = \left(\frac{r}{R}\right)^{\frac{1}{5}}\frac{d}{2}\left[1 + Bk\left(\frac{d}{2}/R\right)^2\right]^{\frac{1}{5}}\left[1 + Bk\left(\frac{h}{r}\right)^2\right]^{-\frac{1}{5}}, \tag{4.13}$$

where B is another constant.

It is seen that for h close to $d/2$, we rediscover Eq. (4.5), while for $r \to \infty$, we obtain

$$h \simeq \left(\frac{r}{R}\right)^{\frac{1}{5}}\frac{d}{2}\left[1 + Bk\frac{d^2}{4R^2}\right]^{\frac{1}{5}}. \tag{4.14}$$

Application Eq. (4.6) to the boundary condition (2.19) leads to

$$\frac{h}{5r}\left(3\beta\dot{\epsilon} + \frac{9}{5}k\beta\frac{Q}{\pi d^3}\frac{h^2}{r^2}\right) = -\beta\left(\frac{C}{\beta}h + \frac{6}{5}k\beta\frac{Q}{\pi d^3}\frac{h^3}{r^3}\right), \tag{4.15}$$

and finally to

$$C(r) = -\frac{3\beta\dot{\epsilon}}{5r} - \frac{39}{25}k\beta\frac{Q}{\pi R^3 d}\frac{h^2}{r^3}. \tag{4.16}$$

This is consistent with the postulate (3.18), if the constant rate of extension is equal to

$$\dot{\epsilon} = \frac{10Q}{\pi d^3}\left[1 - \frac{28}{125}k\left(\frac{d}{2}/R\right)^2\right]^{-1}. \tag{4.17}$$

Hence, the constant B in Eq. (4.13) amounts to 51/175.

The velocity profile in an arbitrary cross-section of two opposing jets can be presented in the form:

$$u^* = \frac{3}{10}\frac{\dot{\epsilon}h^2}{r}\left(1-\frac{z^2}{h^2}\right) + \frac{39}{500}k\frac{\dot{\epsilon}h^4}{r^3}\left(1-\frac{z^2}{h^2}\right) - \frac{3}{100}k\frac{\dot{\epsilon}h^4}{r^3}\left(1-\frac{z^4}{h^4}\right), \tag{4.18}$$

where the extension rate $\dot{\epsilon}$ results from Eq. (4.9) or, more precisely, from Eq. (4.17). Similarly, the normal stress difference defined by Eq. (2.14) amounts to

$$T^{*11} - T^{*33} = 3\beta\dot{\epsilon}\left(1 + \frac{3}{50}k\frac{z^2}{r^2}\right) \tag{4.19}$$

and tends to $3\beta\dot{\epsilon}$ for $r \to \infty$.

4.3. Improvements taking into account a mass exchange

When dealing with the submerged jets we assume as a first approximation that the outer fluid is at rest. In more realistic approach a stagnant region must be replaced by a weakly recirculating region schematically shown in Fig. 2 (cf. [6, 7]). This means that, depending on the fluid properties and the boundary conditions in a reservoir, true velocities disappear only in the centers of occurring eddies. As a second approximation we assume that $z = h(r)$ denotes the interface where all the forces are practically balanced (what is equivalent, to some extent, to the constant pressure assumption for Newtonian fluids) but u^* is not zero, since there exists some mass transport with the mean velocity u_m. Of course, the corresponding impulse efflux must be conserved too.

Bearing in mind the impulse balance for an element based on the coordinates $r + dr$ and r, $\theta + \Delta\theta$ and θ, 0 and $h(r)$, respectively, we obtain in the case considered

$$\frac{dJ}{dr} = \frac{dM}{dr}u_m, \tag{4.20}$$

where J denotes the corresponding impulse and M is the mass transported through the interface $h(r)$, viz.

$$M = \varrho Q = \varrho\int_0^h u^* 2\pi r\, dz = \frac{2}{5}\varrho\pi\dot{\epsilon}h^3. \tag{4.21}$$

Expressing the velocity u_m in the form:

$$u_m = m\frac{\dot{\epsilon}h^2}{r}, \tag{4.22}$$

where m is a number greater than zero, we see that $m = 0$ corresponds exactly to the previous case with no mass exchange. Under the above notations the straightforward integration of Eq. (4.10) leads to

$$h = \left(\frac{r}{R}\right)^{\frac{1}{5}\frac{1}{1-5/2m}}\frac{d}{2}. \tag{4.23}$$

It is seen that for $m \to 2/5$, the function $h(r)$ as well as its derivative with respect to r tend to infinity; this means a possibility of recirculation or eddies in the outer fluid. Such a picture of flow with stream lines deviating suddenly from the direction almost parallel to the r-coordinate is presented in [6, 7].

Alternatively, taking into account the balance of impulses for $z = h$ and $z = d/2$ (at the nozzle tip), we arrive at

$$J|_{z=h} - J|_{z=d/2} = (M|_{z=h} - M|_{z=d/2})\overline{u}_m\,, \tag{4.24}$$

where $\overline{u}_m$ denotes an average velocity across the interface $h(r)$. Hence, we obtain

$$\overline{u}_m = \frac{6}{25}\frac{\dot{\epsilon}h^2}{r}\left[1 - \frac{r}{R}\left(\frac{d}{2}/h\right)^5\right]\left[1 - \left(\frac{d}{2}/h\right)^3\right]^{-1}. \tag{4.25}$$

It is seen from the above relation that for $r \to \infty$, we have $\overline{u}_m \to \frac{6}{25}\dot{\epsilon}h/r$. Therefore, a recirculating region is very probable for some increasing values of h.

5. Determination of the extensional (compressive) viscosity

Application of the impinging jet geometries to determination of the extensional (compressive) viscosities of Newtonian as well as non-Newtonian fluids has been discussed in several papers (cf. [5, 6, 7]). Omitting a detailed description of the corresponding devices, we want only to emphasize that in all cases any determination of the extensional (compressive) viscosity is performed by means of force (or torque) measurements. These forces (torques) balance tensile stress forces, as well as pressure and momentum forces. In many situations a simple momentum balance indicates that the pressure and momentum forces will cancel each other (cf. [5]) and that the gap separation should be comparable to the nozzle diameter in order to obtain optimal results. In the paper [6] some possibilities of inferring nearly extensional stress over some internal area from a measured force (torque) are widely discussed for Newtonian fluids. On the other hand, some instrument design optimization problems are also presented. In what follows we shall stop at a possibility of force determination in the case of free impinging jets (cf. [4]).

The net thrust force F, i.e. the excess of the momentum efflux over the force exerted by the axial stress amounts to

$$F = \int_0^{r_s} \varrho w^{*2} 2\pi r dr - \int_0^{r_s} T^{*33} 2\pi r dr\,, \tag{5.1}$$

where w^* denotes the axial velocity and r_s — the radius of a jet surface. The above condition is satisfied, in particular, at the nozzle exit, i.e. for $r_s = R$ and $z = d/2$. Hence, after taking into account Eq. (3.27) and disregarding smaller terms proportional to k (Newtonian approximation), we arrive at

$$F = \int_0^{R} \varrho w^{*2} 2\pi r dr - \int_0^{R} T^{*33} 2\pi r dr = \varrho\frac{\dot{\epsilon}^2 d^2}{4}\pi R^2 + 3\beta\epsilon\pi R^2 - \int_0^{R} T^{*11} 2\pi r dr\,, \tag{5.2}$$

where we have introduced $w^* = -\frac{1}{2}\dot{\epsilon}d$ as for purely extensional flows. It can easily be shown that the first term in the above relation, viz.

$$\varrho\frac{\dot{\epsilon}^2 d^2}{4}\pi R^2 = (\mathrm{Re})^2\frac{\pi}{\varrho}\beta^2\frac{R^4}{d^4}\,, \quad \mathrm{Re} = \frac{\varrho w_0 R}{\beta_0}\,, \tag{5.3}$$

is relatively unimportant for laminar flows at small Reynolds numbers. If, moreover, the radial stress T^{*11} distribution over the exit cross-section can be disregarded, we obtain the following simplified expression for the extensional (compressive) viscosity:

$$\eta_E = 3\beta \simeq \frac{F}{\pi R^2 \dot{\epsilon}} . \tag{5.4}$$

Both extensional and compressive viscosities can be determined by reversing the flow direction for the opposing jet geometry. This means that the case of fluid expelled from the nozzles should, in principle, be replaced by the case of fluid sucked into the nozzles. One would anticipate the compressive viscosity to be lower for most fluids because of different residence times required for proper alignments of molecules. Taking into account any inertia effects may also lead to different results.

An essential disadvantage of the opposing jet instrument is the fact that for relatively short residence times steady-state conformations of less or more flexible molecules cannot be achieved at all. A steady-state behaviour of the flow considered is the necessary condition for reliable viscosity measurements.

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