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# Surface stress waves in a transversely isotropic nonhomogeneous elastic semispace
# Part II. Surface stress wave in a "weakly anisotropic" semispace with "small nonhomogeneity"

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A RAYLEIGH-TYPE surface stress wave propagation is considered in a "weakly anisotropic" semispace of "small nonhomogeneity"; two elastic shear moduli are assumed to be monotone functions of depth, the ratio of Young's moduli is limited to the first two terms of a power series expansion. Waves of such type are described by the solution of an ordinary, fourth order differential equation with variable coefficients satisfying the corresponding boundary conditions (see [1], Sec. 4). In this particular case of variability of the elastic moduli, the problem has a closed-form solution expressed in terms of Bessel functions. Analysis of the dispersion equation proves the Rayleigh wave speed $C_R$ to depend on the wave-length and on the anisotropy and nonhomogeneity parameters. Using the asymptotic expansions of Bessel functions, the dispersion equation is written in an approximate form enabling a numerical analysis of the influence of the anisotropy and nonhomogeneity parameters upon the surface wave speed.

## 1. Introduction

PROPAGATION OF A RAYLEIGH-TYPE surface stress wave is considered in an elastic "weakly anisotropic" semispace of "small nonhomogeneity". The equation governing the problem was derived by the author in paper [1] from the equation of motion of a transversely isotropic body exhibiting a one-dimensional nonhomogeneity in the isotropy plane (cf. [1], the general form of Eq. (3.13), the equation describing the surface wave (3.31), (3.32), and its particular form (4.12) analyzed here). Application of the approximate equation seems to be justified in the case when certain terms in the general equation, which identically vanish when passing to the isotropic nonhomogeneous and anisotropic homogeneous cases, have a small effect on the final result (see [1], Sec. 4). Hence, it is assumed that the information on the anisotropy and nonhomogeneity contained in the leading term dominates the remaining terms.

Moreover it is assumed that the shear moduli $\mu = \mu(z)$ and $\widehat{\mu} = \widehat{\mu}(z)$ are monotone functions of variable $z$, ratio of the Young's moduli $E/\widehat{E}$ is limited to the first two terms of the power series expansion, and mass density and Poisson's ratios are constant.

The assumptions made above made it possible to obtain the solution in a closed form containing the Bessel functions. The dispersion equation implies the surface wave speed $C_R$ to depend on the wavelength and on the parameters characterizing the anisotropy and nonhomogeneity. Asymptotic expansions of the Bessel functions were used to derive the approximate form of the dispersion equation and to analyze it numerically in various cases of anisotropy and nonhomogeneity.

The solution derived may be used both for experimental determination of elastic properties of materials, and for the analysis of stress waves in anisotropic, nonhomogeneous media subject to transient surface stresses, thus indicating the applied character of the

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paper [1]; at the same time it represents a generalization of the solutions given in [3, 4] concerning isotropic nonhomogeneous media to the case of surface layer dynamics of an anisotropic nonhomogeneous body.

## 2. Formulation of the problem

Let us consider a transversely isotropic body, nonhomogeneous in the plane of isotropy with respect to the variable $z$; the body occupies the halfspace determined by the inequalities $|x| < \infty$, $0 \leq z \leq \infty$, its plane of isotropy being determined by axes $(y, z)$, the $z$-axis pointing inside the body. The stress tensor $\sigma_{ij} = \sigma_{ij}(x, z; t)$, $i, j = (1, 3)$ or $(x, z)$, $0 \leq t \leq \infty$ is sought for; its components are

$$\begin{aligned} \sigma_{xx}(x, z; t) &= \alpha(z) \exp[i(sx - pt)], \\ \sigma_{zz}(x, z; t) &= \beta(z) \exp[i(sx - pt)], \\ \sigma_{xz}(x, z; t) &= \gamma(z) \exp[i(sx - pt)]. \end{aligned} \tag{2.1}$$

Here $\alpha$, $\beta$, $\gamma$ are functions of a single variable $z$, vanishing (not necessarily exponentially) at $z \to \infty$, $2\pi/p$, $2\pi/s$, $C_R = p/s$ denote the period, length and propagation speed of the wave ($p$ — frequency, $s$ — wave number). Values of $\alpha$, $\beta$, $\gamma$ are seen to determine the amplitudes of the stress tensor harmonic components. Functions $\alpha$ and $\gamma$ are expressed in terms of $\beta$ according to the formulae

$$\begin{aligned} \alpha(z) &= -\mathcal{H}^{-1}(z)\left[f(z)(1 + B(z)g(z)) + \frac{1}{\rho p^2}D^2\right]\beta(z), \\ 2is\gamma(z) &= \mathcal{K}^{-1}(z)D\{(-\mathcal{A}(z))[D^2 - s^2(1 - \mathcal{R}(z))] + 4s^2\mathcal{E}(z)\}\beta(z), \end{aligned} \tag{2.2}$$

and function $\beta(z)$ is derived from the equation (cf. Eq. (4.12) in [1])

$$(D\mathcal{K}^{-1}(z)D - 1)\mathcal{A}(z)[D^2 - s^2(1 - \mathcal{R}(z))]\beta(z) = 0 \tag{2.3}$$

and from the boundary conditions

$$\begin{aligned} &\beta(0) = \beta(\infty) = 0, \\ &D\{\mathcal{H}^{-1}(z)B_1(z)[D^2 - s^2(1 - \mathcal{R}(z))] - 2s^2\mathcal{E}(z)\}\beta(z)\big|_{\substack{z=0 \\ z=\infty}} = 0. \end{aligned} \tag{2.4}$$

The following notations have been introduced:

$$\begin{aligned} D &= \frac{d}{dz} \text{ is differential operator,} \\ \mathcal{K}(z) &= s^2\left[1 - 2B_2(z)\rho\left(\frac{p}{s}\right)^2\right] = s^2\widetilde{\mathcal{K}}(z), \\ \mathcal{A}(z) &= 2\mathcal{H}^{-1}(z)B_1(z), \\ \mathcal{R}(z) &= \rho\left(\frac{p}{s}\right)^2[(B_1(z))^{-1}f^2(z) + B(z)f(z)g(z)], \\ \mathcal{H}(z) &= B_1(z) - f(z) + \rho^{-1}\left(\frac{s}{p}\right)^2, \\ \mathcal{E}(z) &= \mathcal{H}^{-1}(z)\left[f(z) - B_1(z) - \frac{1}{2}\rho^{-1}\left(\frac{s}{p}\right)^2\right], \end{aligned}$$

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$$
\begin{aligned}
B(z) &= E^{-1}(z), \\
B_1(z) &= l^2(z)E(z) - \widehat{E}^{-1}(z), \\
B_2(z) &= l(z) + \widehat{E}^{-1}(z), \\
f(z) &= l(z)(1 + \nu(z)), \\
g(z) &= l^{-1}(z)(1 - \nu(z)), \\
l(z) &= \widehat{\nu}(z)/\widehat{E}(z), \\
\rho &\text{ — density},
\end{aligned}
$$

$E(z)$, $\widehat{E}(z)$ — Young's modulus in tension and compression, in the direction lying in the plane of isotropy and in the plane normal to the plane of isotropy, respectively;

$\nu(z)$, $\widehat{\nu}(z)$ — Poisson's ratios determining the transversal contraction in the isotropy plane at tension applied to that plane, and the corresponding contraction at tension applied in the direction normal to the plane of isotropy, respectively.

In the above formulae the symbol $a^{-1}(z)$, $a = a(z)$ being a given function, defines $[a(z)]^{-1}$ provided $a(z) \neq 0$.

The problem is then formulated as follows: determine the non-vanishing function $\beta = \beta(z)$ and the constant $C_R = C_R(s) > 0$ satisfying the Eqs. (2.3), (2.4).

## 3. Solution of the problem

Equation (2.3) will be solved by the method presented in [1], Sec. 5. Introduce the operators

$$
\begin{aligned}
\mathcal{L}_1^2 &\overset{\text{df}}{=} \frac{1}{s^2} D\widetilde{\mathcal{K}}^{-1}(z)D - 1, \\
\mathcal{L}_2^2 &\overset{\text{df}}{=} D^2 - s^2(1 - \mathcal{R}(z)).
\end{aligned} \tag{3.1}
$$

Equation (2.3) assumes the form

$$\mathcal{L}_1^2 \mathcal{A}(z)\mathcal{L}_2^2\beta(z) = 0. \tag{3.2}$$

Introducing the notation

$$\mathcal{A}(z)\mathcal{L}_2^2\beta(z) = v(z), \tag{3.3}$$

we obtain two equations

$$
\begin{aligned}
\mathcal{L}_1^2 v(z) &= 0, \\
\mathcal{L}_2^2\beta(z) &= \mathcal{A}^{-1}(z)v(z).
\end{aligned} \tag{3.4}
$$

Let us prove that the differential equations $(3.4)_1$, $(3.4)_2$ are of the same type. Substitute

$$v = D\varphi \tag{3.5}$$

so that

$$
\begin{aligned}
\mathcal{L}_1^2 v(z) = \mathcal{L}_1^2 D\varphi(z) &= \left(\frac{1}{s^2} D\widetilde{\mathcal{K}}^{-1}(z)D - 1\right)D\varphi(z) \\
&= \frac{1}{s^2} D[\widetilde{\mathcal{K}}^{-1}(z)\{D^2 - s^2\widetilde{\mathcal{K}}(z)\}\varphi(z)].
\end{aligned} \tag{3.6}
$$

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Now the notations $\widetilde{\mathcal{K}}(z)$, $\mathcal{R}(z)$ and $\mathcal{A}(z)$ are written in the forms (3.7), (3.8) and (3.10):

$$\widetilde{\mathcal{K}}(z) = 1-2B_2(z)\rho\left(\frac{p}{s}\right)^2 = 1-2\frac{\widehat{\nu}(z)+1}{\widehat{E}(z)}\rho\left(\frac{p}{s}\right)^2 = 1-\frac{\rho}{\widehat{\mu}(z)}C_R^2 = 1-\Omega_n(z), \tag{3.7}$$

where

$$\begin{gathered}\Omega_n = \frac{C_R^2}{\widehat{c}_2^2(z)}, \qquad \widehat{c}_2 = \left(\frac{\widehat{\mu}(z)}{\rho}\right)^{1/2},\\ \mathcal{R}(z) = \rho\left(\frac{p}{s}\right)^2[(B_1(z))^{-1}f^2(z)+B(z)f(z)g(z)]\\ = \rho\left(\frac{p}{s}\right)^2\left[\frac{\widehat{\nu}^2(1+\nu(z))^2}{\widehat{\nu}^2E-\widehat{E}}+\frac{1}{E}(1+\nu)(1-\nu)\right]\\ = \rho\left(\frac{p}{s}\right)^2\frac{1+\nu}{2\widehat{\mu}(1+\widehat{\nu})}\left[\frac{\widehat{\nu}^2(1+\nu)}{\widehat{\nu}^2E/\widehat{E}-1}+\frac{1}{E/\widehat{E}}(1-\nu)\right]\\ = \Omega_n(z)\varkappa_n(z),\end{gathered} \tag{3.8}$$

where

$$\varkappa_n(z) = \frac{1+\nu(z)}{2(1+\widehat{\nu}(z))}\left[\frac{\widehat{\nu}^2(z)(1+\nu(z))}{\widehat{\nu}^2E(z)/\widehat{E}(z)-1}+\frac{\widehat{E}(z)}{E(z)}(1-\nu(z))\right]. \tag{3.9}$$

It is easily verified that if $\nu(z)$ is identical with $\widehat{\nu}(z)$, a magnitude known from the description of isotropic nonhomogeneous medium will be obtained,

$$\varkappa_n(z) = \frac{1}{2}\left[\frac{\nu^2(1+\nu)}{\nu^2-1}+(1-\nu)\right] = \frac{1}{2}\frac{1-2\nu}{1-\nu} = \varkappa,$$

$$\mathcal{A}(z) = \mathcal{A}_1(z)\cdot\mathcal{A}_2(z), \tag{3.10}$$

where

$$\begin{gathered}\mathcal{A}_1(z) = 2\frac{\widehat{\nu}^2\frac{E}{\widehat{E}}-1}{1+\widehat{\nu}},\\ \mathcal{A}_2(z) = \frac{\Omega_n}{2-\left[1-\frac{\widehat{\nu}}{1+\widehat{\nu}}\left(\widehat{\nu}\frac{E}{\widehat{E}}-\nu\right)\right]\Omega_n}.\end{gathered} \tag{3.11}$$

Equations (3.6), (3.7), (3.8), (3.9), $(3.1)_2$, (3.10) may be used to replace the system (3.4) with the set of equations

$$\begin{gathered}[D^2-s^2(1-\Omega_n(z))]\varphi(z) = 0,\\ [D^2-s^2(1-\varkappa_n(z)\Omega_n(z))]\beta(z) = \mathcal{A}_1^{-1}(z)\mathcal{A}_2^{-1}(z)D\varphi(z).\end{gathered} \tag{3.12}$$

From Eq. (3.7) it follows that $\Omega_n$ depends solely on parameters $p$ and $s$ (see Eq. (2.1)), and on the variable shear modulus $\widehat{\mu}(z)$; Eq. (3.9), however, manifests the complex dependence of $\varkappa_n$ on all the material parameters assumed, $\nu(z)$, $\mu(z)$, $\widehat{\nu}(z)$, $\widehat{\mu}(z)$. Hence, the variable coefficients of the system (3.12) contain all the anisotropy and nonhomogeneity parameters of the problem considered; qualitative, and the quantitative analysis of the system in particular, makes it possible to determine the principal properties of surface

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waves propagating in a transversely isotropic halfspace and exhibiting elastic nonhomogeneity in the vertical direction.

We are looking for a solution of the system (3.12) in a closed form. To this end let us assume the functions $\mu(z)$ and $\widehat{\mu}(z)$ to be monotone functions of $z$,

$$\mu(z) = \frac{\mu_0\mu_\infty}{\mu_0 - (\mu_0 - \mu_\infty)\exp[-2\varepsilon z]}, \tag{3.13}$$
$$\widehat{\mu}(z) = \frac{\widehat{\mu}_0\widehat{\mu}_\infty}{\widehat{\mu}_0 - (\widehat{\mu}_0 - \widehat{\mu}_\infty)\exp[-2\widehat{\varepsilon} z]},$$

where $\mu_0$, $\mu_\infty$; $\widehat{\mu}_0$, $\widehat{\mu}_\infty$ denote the elastic shear moduli $\mu(z)$ and $\widehat{\mu}(z)$ calculated at the surface of the halfspace $z = 0$ and at infinity, $z = \infty$, respectively; $\varepsilon$, $\widehat{\varepsilon}$ — nonhomogeneity parameters in the plane of isotropy and in every plane normal to it, $\varepsilon > 0$, $\widehat{\varepsilon} > 0$.

Relations (3.9)–(3.11) contain also the ratios of the corresponding elastic moduli in tension, denoted by $m(z)$. We have

$$m(z) = \frac{E(z)}{\widehat{E}(z)} = \frac{\mu(z)(1+\nu(z))}{\widehat{\mu}(z)(1+\widehat{\nu}(z))}. \tag{3.14}$$

Expand the expression (3.14) into a Taylor series in the neighbourhood of $z = 0$ assuming that the Poisson ratios are constant. We obtain, after reductions,

$$m(z) \simeq \frac{1+\nu}{1+\widehat{\nu}}\left\{\frac{\mu_0}{\widehat{\mu}_0} - 2\frac{\mu_0}{\widehat{\mu}_0}\left[\varepsilon\left(\frac{\mu_0}{\mu_\infty} - 1\right) - \widehat{\varepsilon}\left(\frac{\widehat{\mu}_0}{\widehat{\mu}_\infty} - 1\right)\right]z + \ldots\right\}. \tag{3.15}$$

Since the surface wave is analyzed at the surface $z = 0$, and the first two terms contain the nonhomogeneity and anisotropy parameters, confine our considerations to the particular case in which the coefficient multiplying $z$ vanishes, i.e.

$$\frac{\varepsilon}{\widehat{\varepsilon}} = \frac{\mu_\infty(\widehat{\mu}_0 - \widehat{\mu}_\infty)}{\widehat{\mu}_\infty(\mu_0 - \mu_\infty)}, \tag{3.16}$$

and we are left with the simple relation

$$\frac{E}{\widehat{E}} \simeq \frac{1+\nu}{1+\widehat{\nu}}\frac{\mu_0}{\widehat{\mu}_0}, \tag{3.17}$$

which will be used in further considerations. In this particular case also $\varkappa_n$ is a constant which may by expressed by the formula (cf. (3.9))

$$\varkappa_n = \varkappa_{n0} = \frac{1}{2}\frac{1+\nu}{1+\widehat{\nu}}\left[\frac{\widehat{\nu}^2(1+\nu)}{\widehat{\nu}^2\frac{1+\nu}{1+\widehat{\nu}}\frac{\mu_0}{\widehat{\mu}_0} - 1} + \frac{(1-\nu)(1+\widehat{\nu})}{1+\nu}\frac{\widehat{\mu}_0}{\mu_0}\right]. \tag{3.18}$$

As a result of such sumplifications we obtain the relations

$$\mathcal{A}_1(z) = \mathcal{A}_{10} = 2\frac{\widehat{\nu}^2\frac{1+\nu}{1+\widehat{\nu}}\frac{\mu_0}{\widehat{\mu}_0} - 1}{1+\widehat{\nu}} = \text{const}, \tag{3.19}$$
$$\mathcal{A}_2(z) = \mathcal{A}_{20}(z) = \frac{\Omega_n(z)}{2 - \left[1 - \frac{\widehat{\nu}}{1+\widehat{\nu}}\left(\widehat{\nu}\frac{1+\nu}{1+\widehat{\nu}}\frac{\mu_0}{\widehat{\mu}_0} - \nu\right)\right]\Omega_n(z)}.$$

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Substitution of Eqs. (3.18) and (3.19) into Eqs. (3.12) yields

$$(3.20)\qquad \begin{aligned} &L_1^2\varphi(z) = [D^2 - s^2(1 - \Omega_n(z))]\varphi(z) = 0\,,\\ &L_2^2\beta(z) = [D^2 - s^2(1 - \varkappa_{n0}\Omega_n(z))]\beta(z) = \mathcal{A}_{10}^{-1}\mathcal{A}_{20}^{-1}(z)D\varphi(z)\,, \end{aligned}$$

where, in accordance with Eqs. (3.7) and $(3.13)_2$,

$$(3.21)\qquad \Omega_n(z) = C_R^2/\widehat{c}_2^2 - C_R^2(1/\widehat{c}_{2\infty}^2 - 1/\widehat{c}_{20}^2)\exp[-2\widehat{\varepsilon}z]\,.$$

Solution of the system (3.20) will be sought in the form satisfying the conditions (2.4), in which the function $\mathcal{E}(z)$ is expressed by the formula

$$(3.22)\qquad \mathcal{E}(z) = -\frac{1 - \left[1 - \dfrac{\widehat{\nu}}{1+\widehat{\nu}}\left(\widehat{\nu}\dfrac{1+\nu}{1+\widehat{\nu}}\dfrac{\mu_0}{\widehat{\mu}_0} - \nu\right)\right]\Omega_n}{2 - \left[1 - \dfrac{\widehat{\nu}}{1+\widehat{\nu}}\left(\widehat{\nu}\dfrac{1+\nu}{1+\widehat{\nu}}\dfrac{\mu_0}{\widehat{\mu}_0} - \nu\right)\right]\Omega_n}\,.$$

Let us approach the solution of Eq. $(3.20)_1$. On substituting for $\Omega_n$ the expression (3.21) and performing the necessary transformations, the differential equation is obtained

$$(3.23)\qquad \varphi'' + \frac{1}{\tau}\varphi' + \left(1 - \frac{\eta^2}{\tau^2}\right)\varphi = 0;$$

its solution, satisfying the condition of vanishing at infinity, has the form

$$(3.24)\qquad \varphi(\tau) = \varphi(\widehat{k}_n t) = \varphi(\widehat{k}_n e^{-\widehat{\varepsilon}z}) = CJ_{\widehat{\eta}}(\widehat{k}_n e^{-\widehat{\varepsilon}z})\,,$$

with the notations

$J_\eta(z)$ — Bessel function of first kind and order $\eta$,

$$(3.25)\qquad \begin{aligned} \widehat{\eta} &= \frac{s}{\widehat{\varepsilon}}(1 - C_R^2/\widehat{c}_{2\infty}^2)^{1/2} > 0\,,\\ \widehat{k}_n &= \frac{s}{\widehat{\varepsilon}}C_R(1/\widehat{c}_{20}^2 - 1/\widehat{c}_{2\infty}^2)^{1/2} > 0\,,\\ \tau &= \widehat{k}_n t, \qquad t = e^{-\widehat{\varepsilon}z}\,, \end{aligned}$$

$C$ — arbitrary constant.

It is seen that in the definition of function $\varphi$ the material parameters appear, characterizing the deformation occurring in the plane perpendicular to the plane of isotropy.

Passing now to the solution of Eq. $(3.20)_2$, let us use the formula (3.24) defining the function $\varphi$ and proceed similarly to the case of Eq. $(3.20)_1$. The solution obtained in this manner is written in the form

$$(3.26)\qquad \beta(z) = \overline{\beta}_n(z) + \beta_{n0}(z)\,,$$

where

$$(3.27)\qquad \begin{aligned} &L_2^2\overline{\beta}_n(z) = 0\,,\\ &L_2^2\beta_{n0}(z) = \mathcal{A}_{10}^{-1}\mathcal{A}_{20}^{-1}(z)D\varphi(z)\,. \end{aligned}$$

Conditions of the problem imply that $\overline{\beta}_n$ and $\beta_{n0}$ should be selected so as tc make the function $\beta$ and its derivatives vanish at $z \to \infty$.

The general solution of Eq. $(3.27)_1$ has the form

$$(3.28)\qquad \overline{\beta}_n(z) = \overline{C}_1 J_{\overline{\eta}}(\overline{k}_n t) + \overline{C}_2 Y_{\overline{\eta}}(\overline{k}_n t)\,,$$

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where $Y_\eta(z)$ — Bessel function of second kind and order $\eta$,

$$\overline{\eta} = \frac{s}{\widehat{\varepsilon}}\left(1 - \varkappa_{n0}\frac{C_R^2}{\widehat{c}_{2\infty}^2}\right)^{1/2} > 0\,,$$

$$\overline{k}_n = \frac{s}{\widehat{\varepsilon}}\varkappa_{n0}C_R(1/\widehat{c}_{20}^2 - 1/\widehat{c}_{2\infty}^2)^{1/2} > 0\,,$$

$$\widehat{\eta} > 0 \Rightarrow \overline{\eta} > 0$$

and $\overline{C}_1$, $\overline{C}_2$ are arbitrary constants.

Solution $\beta_{n0}(z)$ of Eq. $(3.27)_2$ is assumed in the form

$$\beta_{n0}(z) = \mathcal{A}_{10}^{-1}\beta_{n2}(z)\int\limits_0^z \frac{\beta_{n1}h_n}{w}d\widehat{z} - \mathcal{A}_{10}^{-1}\beta_{n1}(z)\int\limits_0^z \frac{\beta_{n2}h_n}{w}\,d\widehat{z}\,, \tag{3.29}$$

where

$$\begin{aligned}
\beta_{n1} &= J_{\overline{\eta}}(\overline{k}_n t)\,,\\
\beta_{n2} &= Y_{\overline{\eta}}(\overline{k}_n t)\,,\\
w &= \beta_{n1}\beta_{n2}' - \beta_{n1}'\beta_{n2}\,,\\
h_n &= \mathcal{A}_{20}^{-1}(z)D\varphi\,.
\end{aligned}$$

It is easily verified that function $\beta(z)$ given by the formulae (3.26), (3.28), (3.29) satisfies Eq. $(3.20)_2$. To this end we must perform the necessary substitutions, operations and reductions. Function $\beta(z)$ evidently contains the information on the anisotropy and nonhomogeneity of the semispaces via the magnitudes $\overline{\eta}$ and $\overline{k}_n$ defining $\overline{\beta}_n$ and, moreover, through the parameters $h_n$ and $\mathcal{A}_{10}$ appearing in the definition of $\beta_{n0}$.

Obviously, this function (called in what follows the stress function) is defined by the expressions (3.26), (3.28), (3.29), (3.24) in which three arbitrary constants $C$, $\overline{C}_1$, $\overline{C}_2$ appear; the constants are determined by using the conditions of vanishing of $\beta(z)$ at $z = 0$ and $z = \infty$. According to Eqs. (3.26), (3.28), (3.29), (3.24) we obtain

$$\beta(z) = \overline{C}_1 J_{\overline{\eta}}(\overline{k}_n t) + \overline{C}_2 Y_{\overline{\eta}}(\overline{k}_n t) + \overset{*}{C}\mathcal{A}_{10}^{-1}\{Y_{\overline{\eta}}(\overline{k}_n t)\mathcal{J}_{n2}(z) - J_{\overline{\eta}}(\overline{k}_n t)\mathcal{J}_{n4}(z)\}\,, \tag{3.30}$$

where

$$\begin{aligned}
\mathcal{J}_{n2}(z) &= \int\limits_0^z u_n(\xi)J_{\overline{\eta}}(\overline{k}_n t)D_\xi J_{\widehat{\eta}}(\widehat{k}_n t)\,d\xi\,,\\
\mathcal{J}_{n4}(z) &= \int\limits_0^z u_n(\xi)Y_{\overline{\eta}}(\overline{k}_n t)D_\xi J_{\widehat{\eta}}(\widehat{k}_n t)\,d\xi\,,\\
u_n &= \mathcal{A}_{20}^{-1}(z) = \frac{\left[1 + \frac{\widehat{\nu}}{1+\widehat{\nu}}\left(\widehat{\nu}\frac{1+\nu}{1+\widehat{\nu}}\frac{\mu_0}{\widehat{\mu}_0} - \nu\right)\right]}{1 - (\widehat{\varepsilon}\widehat{\eta}/s)^2 + (\widehat{\varepsilon}\widehat{k}_n/s)^2\cdot\exp[-2\widehat{\varepsilon}z]}\\
&\quad + \frac{\left[1 - \frac{\widehat{\nu}}{1+\widehat{\nu}}\left(\widehat{\nu}\frac{1+\nu}{1+\widehat{\nu}}\frac{\mu_0}{\widehat{\mu}_0} - \nu\right)\right]\left[(\widehat{\varepsilon}\widehat{\eta})^2 - (\widehat{\varepsilon}\widehat{k}_n)^2\cdot\exp[-2\widehat{\varepsilon}z]\right]}{s^2 - (\widehat{\varepsilon}\widehat{\eta})^2 + (\widehat{\varepsilon}\widehat{k}_n)^2\cdot\exp[-2\widehat{\varepsilon}z]}\,,\\
\overset{*}{C} &= -\frac{\pi}{2\widehat{\varepsilon}}C\,.
\end{aligned} \tag{3.31}$$

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From the condition $\beta(0) = 0$ it follows that

$$\overline{C}_2 = -\overline{C}\frac{J_{\overline{\eta}}(\overline{k}_n)}{Y_{\overline{\eta}}(\overline{k}_n)}, \qquad \overline{C}_1 \equiv C\,, \tag{3.32}$$

and from the condition $\beta(\infty) = 0$ we obtain

$$\overline{C} = \overset{*}{C}\frac{1+\widehat{\nu}}{2\left(\widehat{\nu}^2\frac{1+\nu}{1+\widehat{\nu}}\frac{\mu_0}{\widehat{\mu}_0} - 1\right)}\frac{Y_{\overline{\eta}}(\overline{k}_n)}{J_{\overline{\eta}}(\overline{k}_n)}\mathcal{J}_{n2}^{\infty}\,, \tag{3.33}$$

where

$$\mathcal{J}_{n2}^{\infty} = \lim_{z\to\infty}\mathcal{J}_{n2}(z) = \int\limits_0^{\infty} u_n(\xi)J_{\overline{\eta}}(\overline{k}_n t)D_\xi J_{\widehat{\eta}}(\widehat{k}_n t)\,d\xi\,.$$

This results in a closed-form solution, proportional to a single arbitrary constant

$$\beta(z) = \widehat{C}^*\left\{Y_{\overline{\eta}}(\overline{k}_n t)[\mathcal{J}_{n2}(z) - \mathcal{J}_{n2}^{\infty}] - J_{\overline{\eta}}(\overline{k}_n t)\left[\mathcal{J}_{n4}(z) - \frac{Y_{\overline{\eta}}(\overline{k}_n)}{J_{\overline{\eta}}(\overline{k}_n)}\mathcal{J}_{n2}^{\infty}\right]\right\}, \tag{3.34}$$

where

$$\widehat{C}^* = \overset{*}{C}\frac{1+\widehat{\nu}}{2\left(\widehat{\nu}^2\frac{1+\nu}{1+\widehat{\nu}}\frac{\mu_0}{\widehat{\mu}_0} - 1\right)},$$

or

$$\begin{aligned}\beta(z) = \widehat{C}^*\Big\{&Y_{\overline{\eta}}(\overline{k}_n t)\mathcal{J}_{n2}(z) - J_{\overline{\eta}}(\overline{k}_n t)\mathcal{J}_{n4}(z)\\ &- \frac{1}{J_{\overline{\eta}}(\overline{k}_n)}[J_{\overline{\eta}}(\overline{k}_n)Y_{\overline{\eta}}(\overline{k}_n t) - J_{\overline{\eta}}(\overline{k}_n t)Y_{\overline{\eta}}(\overline{k}_n)]\mathcal{J}_{n2}^{\infty}\Big\}.\end{aligned} \tag{3.35}$$

The dispersion equation is derived from the condition $\gamma(0) = 0$. To obtain the equation, the following relations must be used:

$$\begin{aligned}D\beta(z) = \widehat{C}^*\Big\{&D_z Y_{\overline{\eta}}(\overline{k}_n t)[\mathcal{J}_{n2}(z) - \mathcal{J}_{n2}^{\infty}]\\ &- D_z J_{\overline{\eta}}(\overline{k}_n t)\left[\mathcal{J}_{n4}(z) - \frac{Y_{\overline{\eta}}(\overline{k}_n)}{J_{\overline{\eta}}(\overline{k}_n)}\mathcal{J}_{n2}^{\infty}\right]\Big\},\end{aligned} \tag{3.36}$$

where

$$D_z Y_{\overline{\eta}}(\overline{k}_n t) = -\widehat{\varepsilon}[\overline{\eta}Y_{\overline{\eta}}(\overline{k}_n t) - \overline{k}_n t Y_{\overline{\eta}+1}(\overline{k}_n t)]\,,$$

$$\lim_{z\to 0}[J_{\overline{\eta}}(\overline{k}_n e^{-\widehat{\varepsilon}z})DY_{\overline{\eta}}(\overline{k}_n e^{-\widehat{\varepsilon}z}) - Y_{\overline{\eta}}(\overline{k}_n e^{-\widehat{\varepsilon}z})DJ_{\overline{\eta}}(\overline{k}_n e^{-\widehat{\varepsilon}z})] = -\frac{2\widehat{\varepsilon}}{\pi}\,,$$

$$\beta(0) = 0\,,$$

$$\begin{aligned}D\beta(z)|_{z=0} &= -\widehat{C}^*\left\{D_z Y_{\overline{\eta}}(\overline{k}_n e^{-\widehat{\varepsilon}z}) - D_z J_{\overline{\eta}}(\overline{k}_n e^{-\widehat{\varepsilon}z})\frac{Y_{\overline{\eta}}(\overline{k}_n)}{J_{\overline{\eta}}(\overline{k}_n)}\right\}\mathcal{J}_{n2}^{\infty}|_{z=0}\\ &= \widehat{C}^*\frac{2\widehat{\varepsilon}}{\pi J_{\overline{\eta}}(\overline{k}_n)}\mathcal{J}_{n2}^{\infty} = -C\mathcal{A}_{10}^{-1}\frac{1}{J_{\overline{\eta}}(\overline{k}_n)}\mathcal{J}_{n2}^{\infty}\,.\end{aligned}$$

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The amplitude of shear stresses $\gamma(z)$ is given by $(2.2)_2$ which now has the form

$$(3.37)\qquad 2is\gamma(z) = \frac{1}{s^2(1-\Omega_n(z))}D\Bigg\{ -\mathcal{A}_{10}\mathcal{A}_{20}(z)[D^2 - s^2(1-\varkappa_{n0}\Omega_n(z))] \\ -4s^2\frac{1-\left[1-\frac{\widehat{\nu}}{1+\widehat{\nu}}\left(\widehat{\nu}\frac{1+\nu}{1+\widehat{\nu}}\frac{\mu_0}{\widehat{\mu}_0}-\nu\right)\right]\Omega_n(z)}{2-\left[1-\frac{\widehat{\nu}}{1+\widehat{\nu}}\left(\widehat{\nu}\frac{1+\nu}{1+\widehat{\nu}}\frac{\mu_0}{\widehat{\mu}_0}-\nu\right)\right]\Omega_n(z)}\Bigg\}\beta(z).$$

Denoting

$$(3.38)\qquad q = 1-\frac{\widehat{\nu}}{1+\widehat{\nu}}\left(\widehat{\nu}\frac{1+\nu}{1+\widehat{\nu}}\frac{\mu_0}{\widehat{\mu}_0}-\nu\right)$$

and making use of the fact that

$$D\left\{ -\mathcal{A}_{10}\mathcal{A}_{20}(z)[D^2 - s^2(1-\varkappa_{n0}\Omega_n(z))]\beta(z) - 4s^2\frac{1-q\Omega_n(z)}{2-q\Omega_n(z)}\beta(z)\right\} \\ = -s^2(1-\Omega_n(z))\varphi(z) + 4s^2\left[\frac{q\Omega_n'(z)}{(2-q\Omega(z))^2}\beta(z) - \frac{1-q\Omega_n(z)}{2-q\Omega_n(z)}D\beta(z)\right],$$

equation (3.37) is reduced to the form

$$(3.39)\qquad 2is\gamma(z) = \frac{1}{s^2(1-\Omega_n(z))}\Bigg\{ -s^2(1-\Omega_n(z))CJ_{\widehat{\eta}}(\widehat{k}_n t) \\ +4s^2\left[\frac{q\Omega_n'(z)}{(2-q\Omega_n(z))^2}\beta(z) - \frac{1-q\Omega_n(z)}{2-q\Omega_n(z)}D\beta(z)\right]\Bigg\}.$$

Hence, the dispersion equation $\gamma(0)=0$, called also the period equation, may be represented in the form

$$(3.40)\qquad (2-q\Omega_{n0})J_{\widehat{\eta}}(\widehat{k}_n)J_{\overline{\eta}}(\overline{k}_n) - 4\frac{(1+\widehat{\nu})(1-q\Omega_{n0})}{2\left(\widehat{\nu}^2\frac{1+\nu}{1+\widehat{\nu}}\frac{\mu_0}{\widehat{\mu}_0}-1\right)(1-\Omega_{n0})}\mathcal{J}_{n2}^{\infty} = 0.$$

At $\widehat{\nu}=\nu$ and $\widehat{\mu}=\mu$ the above result leads to the equation of dispersion in an isotropic semispace of "small nonhomogeneity" derived by the present author in [3] (1977).

$$(3.41)\qquad (2-\Omega_0)J_{\widehat{\eta}}(\widehat{k})J_{\overline{\eta}}(\overline{k}) + 4(1-\varkappa_0)\mathcal{J}_2^{\infty} = 0.$$

Here $\Omega_0$, $\widehat{k}$, $\overline{k}$, $\varkappa_0$, $\mathcal{J}_2^{\infty}$ are the limits of $\Omega_{n0}$, $\widehat{k}_n$, $\overline{k}_n$, $\varkappa_{n0}$, $\mathcal{J}_{n2}^{\infty}$ for $\nu\to\widehat{\nu}$ and $\mu\to\widehat{\mu}$. If the material coefficients, both those appearing explicitly in Eq. (3.40) or through the parameters $\widehat{\eta}$, $\overline{\eta}$, $\widehat{k}_n$, $\overline{k}_n$, $q$, $\Omega_{n0}$, are taken into account, and if the left-hand side of that equation is denoted by $R$, then Eq. (3.40) may be written in the form

$$(3.42)\qquad R\left(C_R, s, \frac{\varepsilon}{\widehat{\varepsilon}}, \widehat{\varepsilon}, \nu, \mu_0, \mu_\infty, \widehat{\nu}, \widehat{\mu}_0, \widehat{\mu}_\infty, \varkappa_{n0}\right) = 0.$$

Evidently, speed $C_R$ of the surface wave sought for depends on the wavelength $\lambda = 2\pi/s$ and on the nonhomogeneity and anisotropy parameters: $\varepsilon$, $\nu$, $\mu_0$, $\mu_\infty$, $\widehat{\varepsilon}$, $\widehat{\nu}$, $\widehat{\mu}_0$, $\widehat{\mu}_\infty$, $\varkappa_{n0}$. Finally, let us present the formula for the amplitude $\alpha(z)$ of the normal stresses $\sigma_{11}=\sigma_{xx}$ defined by Eq. $(2.2)_1$, use being made of Eq. (3.34) and of the notations concerning the

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material parameters,

$$(3.43)\qquad \alpha(z) = -s^2\left[2 - \left(1 - \widehat{\nu}\frac{\widehat{\nu}\dfrac{1+\nu}{1+\widehat{\nu}}\dfrac{\mu_0}{\widehat{\mu}_0} - \nu}{1+\widehat{\nu}}\right)\Omega_n\right]^{-1} \times\left[s^2\frac{\widehat{\mu}_0}{\mu_0}\left(\frac{1+\nu}{1+\widehat{\nu}}\frac{\mu_0}{\widehat{\mu}_0}\widehat{\nu} + 1 - \nu\right)\Omega_n + 2D^2\right]\beta(z).$$

Once the speed $C_R$ is calculated from Eq. (3.40) and the amplitudes $\alpha(z)$, $\beta(z)$, $\gamma(z)$ are evaluated from Eqs. (3.43), (3.34), (3.39), the stresses are determined on the basis of Eqs. (2.1).

## 4. Analysis of the dispersion equation

Let us consider the equation of dispersion (3.40) in the particular case when the nonhomogeneity parameter $\widehat{\varepsilon}$ is large enough. To this end analyze first the case of passing to the limit $\widehat{\varepsilon} \to \infty$. Using the asymptotic expansions of the Bessel function

$$(4.1)\qquad \lim_{\widehat{\varepsilon}\to\infty} J_{\overline{\eta}}(\overline{k}_n e^{-\widehat{\varepsilon}z})/J_{\overline{\eta}}(\overline{k}_n) \to \exp[-\widehat{\varepsilon}\overline{\eta}z]$$

Eq. (3.40) is reduced to the form

$$(4.2)\qquad (2 - q\Omega_{n0}) - 4\frac{1+\widehat{\nu}}{2\chi_2}\frac{1-q\Omega_{n0}}{1-\Omega_{n0}}\int_0^\infty u_n(\xi)\frac{J_{\overline{\eta}}(\overline{k}_n t)D_\xi J_{\widehat{\eta}}(\widehat{k}_n t)}{J_{\overline{\eta}}(\overline{k}_n)J_{\widehat{\eta}}(\widehat{k}_n)}\,d\xi$$

$$\equiv 2 - q\Omega_{n0} + 4\frac{1+\widehat{\nu}}{2\chi_2}\frac{1-q\Omega_{n0}}{1-\Omega_{n0}}\frac{2-\left(1-\dfrac{1}{1+\widehat{\nu}}\chi_1\right)\Omega_{n\infty}}{\Omega_{n\infty}} \times h_{n1}\int_0^\infty \exp[-z(h_{n1}+h_{n2})]\,dz = 0,$$

where

$$\lim_{\widehat{\varepsilon}\to\infty} u_n(z) = \lim_{\widehat{\varepsilon}\to\infty}\frac{2-\left(1-\dfrac{1}{1+\widehat{\nu}}\chi_1\right)\Omega_n(z)}{\Omega_n(z)} \sim \frac{2-\left(1-\dfrac{1}{1+\widehat{\nu}}\chi_1\right)\Omega_{n\infty}}{\Omega_{n\infty}},$$

$$\chi_1 = \widehat{\nu}\left(\widehat{\nu}\frac{1+\nu}{1+\widehat{\nu}}\frac{\mu_0}{\widehat{\mu}_0} - \nu\right),\qquad \chi_2 = \widehat{\nu}^2\frac{1+\nu}{1+\widehat{\nu}}\frac{\mu_0}{\widehat{\mu}_0} - 1,$$

$$h_{n1} = s\left(1 - \frac{C_R^2}{\widehat{c}_{2\infty}^2}\right)^{1/2} = \widehat{\varepsilon}\widehat{\eta},\qquad h_{n2} = s\left(1 - \varkappa_{n0}\frac{C_R^2}{\widehat{c}_{2\infty}^2}\right)^{1/2} = \widehat{\varepsilon}\overline{\eta},$$

$$q = 1 - \chi_1 .$$

Performing the prescribed operations and using the relations

$$(4.3)\qquad h_{n1}^2 - h_{n2}^2 = -s^2C_R^2(1-\varkappa_{n0})/\widehat{c}_{2\infty}^2 = -s^2\Omega_{n\infty}(1-\varkappa_{n0}),$$

we arrive at the equation

$$\Omega_{n\infty}^2(2-q\Omega_{n0}) - 4\frac{1+\widehat{\nu}}{2\chi_2}\frac{1}{1-\varkappa_{n0}}\frac{1-q\Omega_{n0}}{1-\Omega_{n0}}\left(2-\left(1-\frac{1}{1+\widehat{\nu}}\chi_1\right)\Omega_{n\infty}\right) \times(1-\Omega_{n\infty})^{1/2}[(1-\Omega_{n\infty})^{1/2} - (1-\varkappa_{n0}\Omega_{n\infty})^{1/2}] = 0.$$

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Observe that for an isotropic and homogeneous medium

$$2\chi_2/(1+\widehat{\nu}) = 2(\nu^2-1)/(1+\nu) = -2(1-\nu) = (\varkappa_0-1)^{-1}, \qquad \chi_1 = 0,$$

so that assumption of $\Omega_0 = \Omega_{n\infty}$ makes it possible to reduce Eq. (4.4) to the form

$$(2-\Omega_0)^2 - 4(1-\Omega_0)^{1/2}(1-\varkappa_0\Omega_0)^{1/2} = 0\,.$$

This is the classical form of the Rayleigh equation, cf. [2].

Consider now another asymptotic form of Eq. (3.40) for large values of $\widehat{\varepsilon}$. Substituting the asymptotic Bessel function expansions into Eq. (3.40) we obtain

$$(4.5)\qquad (2-q\Omega_{n0}) + 4\frac{1+\widehat{\nu}}{2\chi_2}\frac{1-q\Omega_{n0}}{1-\Omega_{n0}}\frac{2-\left(1-\frac{1}{1+\widehat{\nu}}\chi_1\right)\Omega_{n\infty}}{\Omega_{n\infty}} \times \int\limits_0^\infty e^{-h_{n2}z}\left[\widehat{\eta}e^{-h_{n1}z} - \frac{\widehat{k}_n^2 e^{-\widehat{\varepsilon}(\widehat{\eta}+2)z}}{2(\widehat{\eta}+1)}\right]dz = 0,$$

or

$$(4.6)\qquad (2-q\Omega_{n0}) + 4\frac{1+\widehat{\nu}}{2\chi_2}\frac{1-q\Omega_{n0}}{1-\Omega_{n0}}\frac{2-\left(1-\frac{1}{1+\widehat{\nu}}\chi_1\right)\Omega_{n\infty}}{\Omega_{n\infty}} \times\left(\frac{h_{n1}}{h_{n1}+h_{n2}} - \frac{k_n^2}{2(h_{n1}+\widehat{\varepsilon})(h_{n1}+h_{n2}+2\widehat{\varepsilon})}\right) = 0,$$

where

$$k_n = \widehat{\varepsilon}\widehat{k}_n = sC_R(1/\widehat{c}_{20}^2 + 1/\widehat{c}_{2\infty}^2)^{1/2}.$$

This equation may also be represented in the form

$$(4.7)\qquad R_1(C_R) + R_2\left(C_R, \lambda\frac{\widehat{\varepsilon}}{2\pi}\right) = 0,$$

with the notations

$$(4.8)\qquad R_1(C_R) = (2-q\Omega_{n0}) + 4\frac{1+\widehat{\nu}}{2\chi_2}\frac{1-q\Omega_{n0}}{1-\Omega_{n0}}\frac{2-\left(1-\frac{1}{1-\widehat{\nu}}\chi_1\right)\Omega_{n\infty}}{\Omega_{n\infty}} \times\frac{(1-\Omega_{n\infty})^{1/2}}{(1-\Omega_{n\infty})^{1/2}+(1-\varkappa_{n0}\Omega_{n\infty})^{1/2}},$$

$$(4.9)\qquad R_2\left(C_R, \lambda\frac{\widehat{\varepsilon}}{2\pi}\right) = -4\frac{1+\widehat{\nu}}{2\chi_2}\frac{1-q\Omega_{n0}}{1-\Omega_{n0}}\frac{2-\left(1-\frac{1}{1+\widehat{\nu}}\chi_1\right)\Omega_{n\infty}}{\Omega_{n\infty}} \times\frac{\Omega_{n0}-\Omega_{n\infty}}{2\left[\frac{\widehat{\varepsilon}}{s}+(1-\Omega_{n\infty})^{1/2}\right]\left[2\frac{\widehat{\varepsilon}}{s}+(1-\Omega_{n\infty})^{1/2}+(1-\varkappa_{n0}\Omega_{n\infty})^{1/2}\right]}.$$

In this particular approximation of the dispersion equation the surface wave speed $C_R$ is evidently dependent on the wavelength $\lambda = 2\pi/s$ and on the nonhomogeneity and anisotropy parameters, so that we obtain the relation

$$(4.10)\qquad C_R = C_R\left(\lambda, \frac{\varepsilon}{\widehat{\varepsilon}}, \widehat{\varepsilon}, \nu, \widehat{\nu}, \varkappa_{n0}, \frac{\mu_0}{\widehat{\mu}_0}, \widehat{c}_{20}, \widehat{c}_{2\infty}\right).$$

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To facilitate the numerical calculations, introduce the notations

$$R_2(C_R,\psi)=\frac{A(C_R)}{R(C_R,\psi)},\qquad \psi=\frac{\widehat{\varepsilon}}{s},\ R(C_R,\psi)\neq 0\,, \tag{4.11}$$

$$\begin{aligned}A(C_R)&=-2\frac{1+\widehat{\nu}}{2\chi_2}\frac{1-q\Omega_{n0}}{1-\Omega_{n0}}\frac{2-\left(1-\frac{1}{1+\widehat{\nu}}\chi_1\right)\Omega_{n\infty}}{\Omega_{n\infty}}(\Omega_{n0}-\Omega_{n\infty})\,,\\ R(C_R,\psi)&=[\psi+(1-\Omega_{n\infty})^{1/2}][2\psi+(1-\Omega_{n\infty})^{1/2}+(1-\varkappa_{n0}\Omega_{n\infty})^{1/2}]\\ &=2\psi^2+[3(1-\Omega_{n\infty})^{1/2}+(1-\varkappa_{n0}\Omega_{n\infty})^{1/2}]\psi\\ &\quad+(1-\Omega_{n\infty})^{1/2}[(1-\Omega_{n\infty})^{1/2}+(1-\varkappa_{n0}\Omega_{n\infty})^{1/2}]\,,\end{aligned} \tag{4.12}$$

and reduce Eq. (4.7) to the form

$$2\psi^2+B\psi+C=0\,, \tag{4.13}$$

where

$$\begin{aligned}B&=3(1-\Omega_{n\infty})^{1/2}+(1-\varkappa_{n0}\Omega_{n\infty})^{1/2}\,,\\ C&=(1-\Omega_{n\infty})^{1/2}[(1-\Omega_{n\infty})^{1/2}+(1-\varkappa_{n0}\Omega_{n\infty})^{1/2}]+\frac{A(C_R)}{R_1(C_R)}\,.\end{aligned} \tag{4.14}$$

Equation (4.13) yields directly the quantitative results. Let us determine the speed of the Rayleigh wave as a function of parameter $\psi=\widehat{\varepsilon}/s$ assuming the particular "weakly anisotropic" semispace with a "small nonhomogeneity" specified as follows.

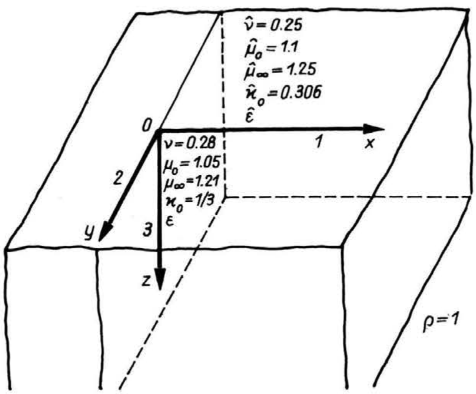


FIG. 1. Notation of material parameters.

In the plane of isotropy (cf. Fig. 1)

$$\nu=0.28,\qquad \mu_0=1.05,\qquad \mu_\infty=1.21\qquad (\varkappa_0=1/3) \tag{4.15}$$

and in the plane normal to the plane of isotropy

$$\widehat{\nu}=0.25,\qquad \widehat{\mu}_0=1.1,\qquad \widehat{\mu}_\infty=1.25\qquad (\widehat{\varkappa}_0=0.306)\,. \tag{4.16}$$

Several values of the function $C_R(\psi)$ given in Table 1 were used to prepare the diagram (curve A) in Fig. 2. The curve has an asymptote at the value $C_R$ slightly greater than

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1.017. In the assumed medium the possible surface wave speeds are distributed in the interval 0.98 to 1.017, parameter $\psi$ varying between 0.00087 and 3.9471.

**Table 1. Data for the material defined by Eqs. (4.15), (4.16).**

| $\psi$ | 0.00087 | 0.1038 | 0.2525 | 0.3803 | 0.5909 | 1.5448 | 3.3174 | 3.9471 |
|---|---|---|---|---|---|---|---|---|
| $C_R$ | 0.98 | 0.99 | 1.00 | 1.005 | 1.01 | 1.015 | 1.0165 | 1.017 |

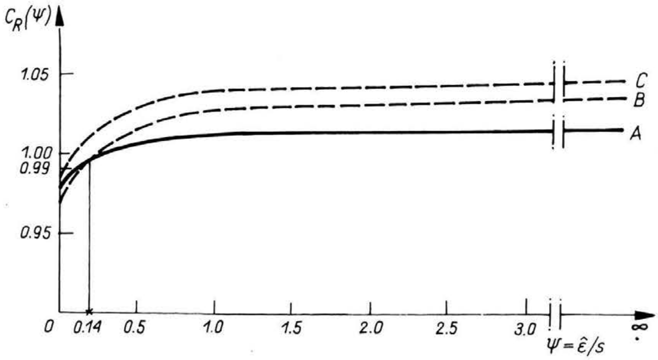


FIG. 2. Propagation speed $C_R = C_R(\psi)$ of a Rayleigh surface wave as a function of $\psi = \hat{\varepsilon}/s$, $s$ — wave number, $\hat{\varepsilon}$ — nonhomogeneity parameter. Curve $A$: "weakly anisotropic" semispace with "small nonhomogeneity", according to the data given in Table 1. Ratio $\varepsilon/\hat{\varepsilon}$ in accordance with condition (3.16). Curve $B$ — the case of an isotropic material with "small nonhomogeneity", data given in Table 2. Curve $C$ — data according to Table 3.

In order to compare the information contained in Eq. (4.6) with that following from the equation of dispersion in the isotropic medium of small nonhomogeneity (cf. [3], Eqs. (4.6), (4.7)) curves $B$ and $C$ are also shown in Fig. 2; they express the relation $C_R = C_R(\psi)$ for two isotropic materials with "small nonhomogeneity" characterized by the material constants (4.15), (4.16) (see Fig. 1). Curve $B$ should be referred to the material with constants $\nu = 0.28$, $\mu_0 = 1.05$, $\mu_\infty = 1.21$, $\varkappa_0 = 0.333$, and curve $C$ — to the material with constants $\nu = 0.25$, $\mu_0 = 1.1$, $\mu_\infty = 1.25$, $\varkappa_0 = 0.306$. Curves $B$ and $C$ are prepared on the basis of data given in Tabs. 2, 3.

**Table 2. Data for the isotropic nonhomogeneous semispace with parameters $\nu = 0.28$, $\mu_0 = 1.05$, $\mu_\infty = 1.21$, $\varkappa_0 = 0.333$.**

| $\psi = \frac{\hat{\varepsilon}}{s}$ | 0.0174 | 0.0445 | 0.218 | 0.3402 | 0.5647 | 0.7685 | 1.0955 | 2,5584 |
|---|---|---|---|---|---|---|---|---|
| $C_R$ | 0.975 | 0.98 | 1.00 | 1.01 | 1.02 | 1.025 | 1.03 | 1.035 |

**Table 3. Data for the isotropic nonhomogeneous semispace with parameters $\nu = 0.25$, $\mu_0 = 1.1$, $\mu_\infty = 1.25$, $\varkappa_0 = 0.306$**

| $\psi = \frac{\hat{\varepsilon}}{s}$ | 0.0191 | 0.0816 | 0.1705 | 0.2997 | 0.5129 | 1.0381 | 2.2105 | $-\infty$ |
|---|---|---|---|---|---|---|---|---|
| $C_R$ | 0.99 | 1.0 | 1.01 | 1.02 | 1.03 | 1.04 | 1.045 | -1.048 |

Curves $A$ and $B$ intersect at the point $\psi = 0.14$, $C_R = 0.993$. If we assume $\hat{\varepsilon} = 1.5$, what means that $\varepsilon = 1.36$ (see Eq. (3.16)), then it may be stated that at the wave number $s = 10.7$ (wavelength $\lambda = 0.587$), the speed of the wave in a "weakly anisotropic"

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material with "small nonhomogeneity" determined by parameters (4.15), (4.16) equals the speed in an isotropic material with "small nonhomogeneity" (parameters (4.15)) and i.e. $C_R = 0.993$. Further analysis of the curves enables us to observe that at $\psi = 0.1$ and $\psi = 0.2$, assuming $\widehat{\varepsilon} = 1.5$ (i.e. $s = 15$ and $s = 7.5$) we obtain the respective results $C_R = 0.99$ and $C_R = 0.9967$ in case of the material represented by curve $A$, and $C_R = 0.887$ and $C_R = 0.998$ for the material represented by curve $B$. This is why it seems probable that variation of the wave number in the interval 0.7–1.4 makes it possible to differentiate the materials from each other. If the nonhomogeneity parameter $\widehat{\varepsilon} = 2.0$, the same surface wave speed in both materials occurs at the wave number $s = 14.29$.

Equation (4.6) is seen to yield new quantitative information concerning the effect of anisotropy and nonhomogeneity upon the Rayleigh surface stress wave velocity. Analysis of various material parameters appearing in Eqs. (4.7)–(4.14) allows for a proper evaluation of their influence upon the propagating surface wave. This confirms the opinion that the solution to the problem formulated by Eqs. (2.3), (2.4) makes it possible to analyze the surface wave propagation processes occurring in anisotropic and nonhomogeneous media.

## 5. Conclusions

1. The problem formulated in [1] and concerning the propagation of surface waves in transversely isotropic nonhomogeneous elastic semispace is shown to be of importance from the point of view of both the theory and applications; it makes it possible to formulate the simplified problem and to perform its qualitative and quantitative analysis, thus furnishing new important information on the properties of such waves.

2. The accurate solutions of the simplified problem described by Eqs. (2.3)–(2.4) governing the Rayleigh surface wave propagation are derived in the case of "weakly anisotropic" elastic semispace with "small nonhomogeneity"; the elastic shear moduli are monotone functions of $z$, ratio of the Young's moduli is confined to the first two terms of the power series expansion, the remaining elastic properties of the medium are constant.

3. Numerical analysis of the dispersion equation demonstrates that the curves illustrating the dependence of the wave speed on the wave number in a "weakly anisotropic" medium of "small nonhomogeneity", and in an isotropic medium with "small nonhomogeneity", may have some points in common at certain values of the wave number.

4. Properties of a certain class of "weakly anisotropic" structural materials exhibiting "small nonhomogeneity" may be determined by measuring the length and propagation speed of the surface waves, and by comparing the experimental data with the theoretical solution. The experiments should be repeated for several wave-lengths in view of the reason mentioned in Conclusion 3.

## References

1. T. Rożnowski, *Surface stress waves in a transversely isotropic nonhomogeneous elastic semispace. Part I. Equations of motion and equations of a Rayleigh-type surface wave*, Arch. Mech., **44**, 4, 417–435, 1992.
2. J. Ignaczak, *Rayleigh waves in a nonhomogeneous isotropic elastic semi-space, Part I*, Arch. Mech., **15**, 3, 341–345, 1963.

http://rcin.org.pl

3. T. ROŻNOWSKI, *Surface waves in an isotropic elastic semispace with small nonhomogeneity*, Bull. Acad. Polon. Sci., Série Sci. tech., **25**, 2, 67–77, 1977.
4. T. ROŻNOWSKI, *Surface waves in an isotropic elastic non-homogeneous semi-space*, Arch. Mech., **39**, 4, 337–358, 1987.

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