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Application of dynamical system methods to flow processes of dissipative solids

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THE PURPOSE of this paper is to apply dynamical system methods to the investigation of stability of a flow process of dissipative solids. The material of a body is described within the framework of the modified material structure with internal state variables. The internal imperfections are also taken into consideration. To describe the effects of transport phenomena (kinetics of imperfections) the evolution equations for the internal state variables are postulated in the form of partial differential equations. An isothermal flow process of dissipative solids with internal imperfections is described by means of the dissipative dynamical system methods. The criteria of discontinuous solution and criteria of bifurcation are discussed. The problem of stability of an equilibrium intrinsic state is also investigated. The first and second Liapounov methods are used. The particular forms of the Liapounov function are assumed. It has been proved that the criteria of asymptotic stability of an equilibrium intrinsic state are stronger than those required by the thermodynamic inequality.

Celem pracy jest zastosowanie metod układów dynamicznych do zbadania stabilności procesu płynięcia ciał dyssypatywnych. Materiał ciała jest opisany w ramach zmodyfikowanej struktury materialnej z parametrami wewnętrznymi. Uwzględniono również wewnętrzne imperfekcje materiału. W celu opisu efektów zjawisk transportu (kinetyki imperfekcji) równania ewolucji dla parametrów wewnętrznych są postulowane w postaci równań różniczkowych cząstkowych. Izotermiczny proces płynięcia ciał dyssypatywnych z wewnętrznymi imperfekcjami został opisany za pomocą metod dyssypatywnych układów dynamicznych. Przedyskutowano kryteria rozwiązań nieciągłych oraz zagadnienie bifurkacji. Zbadano zagadnienie stabilności stanu równowagi wewnętrznej. Wykorzystano pierwszą i drugą metodę Lapunowa. Przyjęto szczególne postacie funkcji Lapunowa. Wykazano, że kryteria asymptotycznej stabilności stanu równowagi wewnętrznej są silniejsze od warunków wynikających z postulatu termodynamicznego.

Целью работы является применение метод динамических систем для исследования устойчивости процесса течения диссипативных тел. Материал тела описывается в рамках модифицированной материальной структуры с внутренними параметрами. Учитываются также внутренние несовершенства (дефекты) материала. Для описания явлений переноса (кинетики дефектов) уравнения эволюции внутренних параметров предполагаются в виде уравнений в частных производных. Изотермический процесс течения диссипативных тел с внутренними дефектами описывается с использованием методов диссипативных динамических систем. Обсуждаются критерии разрывных решений и вопросы бифуркации. Изследуются вопросы устойчивости состояния внутреннего равновесия. Используются первый и второй методы Ляпунова. Принимаются специальные виды функций Ляпунова. Доказывается, что критерии асимптотической устойчивости равновесного состояния — сильнее условий, вытекающих из термодинамического постулата.

1. Introduction

THE MAIN objective of the present paper is to apply the methods of a dynamical system to the investigation of stability of a flow process of dissipative solids.

The application of dynamical system methods to problems of continuum mechanics has been given in many papers (cf. A. A. MOVCHAN [15], V. I. ZUBOV [20], J. E. GILBERT and R. J. KNOPS [4], J. K. HALE [6], J. C. WILLEMS [19], M. E. GURTIN [5] and R. J. KNOPS and E. W. WILKES [12]). In our considerations we shall follow mainly the presentation of J. E. GILBERT and R. J. KNOPS [4] and R. J. KNOPS and E. W. WILKES [12].

The material of a body is described within the framework of the modified material structure with internal state variables (cf. Refs. [17, 18]). The aim is to include in the description the imperfection effects generated by migration, nucleation and growth of voids during an inelastic flow process. To describe effects of transport phenomena we postulate the partial differential evolution equations for the internal state variables assumed.

In Sect. 2 the description of an isothermal flow process for dissipative solids with internal imperfections is presented. Section 3 focusses on the discussion of the modified material structure with internal state variables. The formal mathematical definition of the internal state variables is given. A physical interpretation of the definition proposed is presented and the difference between the modified and classical formulations of the internal state variable material structure is discussed.

In Sect. 4 an isothermal flow process for dissipative solids with internal imperfections is described within the framework of dissipative dynamical system methods.

In Sect. 5 the criteria of discontinuous solution and criteria of bifurcation (branching of solution) are discussed. The criteria of discontinuous solution are investigated by means of an auxiliary function (Liapounov function). The Liapounov function introduced is interpreted by using a notion of the storage function defined for a dissipative dynamical system. An equilibrium solution and an equilibrium intrinsic state are defined. The criteria of bifurcation are discussed by means of the uniqueness theorems for the process considered.

Section 6 is devoted to the problem of stability of equilibrium intrinsic states. The first Liapounov method is used to examine the evolution of the intrinsic states. A topological concept associated with continuity is emphasised.

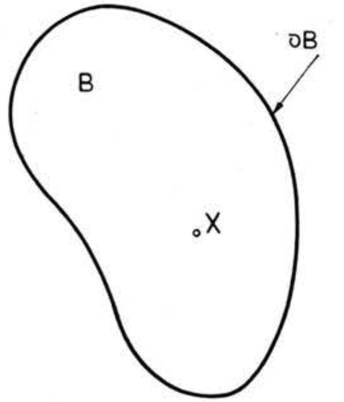
In Sect. 7 the criteria of stability of an equilibrium intrinsic state are investigated by means of the second Liapounov method. The particular form of the Liapounov function is assumed. The criteria of stability are stated for two cases. The first when a set of solution functions is endowed with the properties of the metric space, and the second for the metric generated by the norm in a Hilbert space.

Section 8 focusses on the discussion of the criteria of stability obtained for an equilibrium intrinsic state. It is pointed out that the criteria of asymptotic stability obtained are stronger than those required by the thermodynamic postulate.

2. Flow process for dissipative solids

In what follows we shall consider only pure mechanical processes. Therefore temperature is assumed to be constant in time and uniformly distributed in a body $\mathscr{B}$, cf. Fig. 1.

The isothermal flow process for dissipative solids with internal imperfections is determined in the material description by:



$\vartheta(X, t) = \vartheta^* = \text{const}$

$X \in B, \; t \in [0, d_p]$

FIG. 1.

(i) the constitutive equation for the Piola–Kirchhoff stress tensor

$$\mathbf{T} = \hat{\mathbf{T}}(\sigma), \tag{2.1}$$

where σ denotes the intrinsic state which is given by the pair — the strain tensor field $\mathbf{E}$ and the field of internal state vector $\boldsymbol{\alpha}$, i.e.

$$\sigma = (\mathbf{E}, \boldsymbol{\alpha}) \in \Sigma \tag{2.2}$$

and Σ denotes the intrinsic state space.

Basing on the previous results (cf. [17, 18]), we can write

$$\mathbf{T} = 2\varrho_0 \partial_{\mathbf{E}} \hat{\Psi}(\sigma), \tag{2.3}$$

where $\hat{\Psi}$ denotes the free energy constitutive function and ϱ_0 is the mass density in the reference configuration;

(ii) the evolution equation for the internal state variable in the form

$$\partial_t \boldsymbol{\alpha}(\mathbf{X}, t) = \mathscr{L} \boldsymbol{\alpha}(\mathbf{X}, t) + \hat{\mathbf{f}}(\sigma), \tag{2.4}$$

where $\mathscr{L}$ is a linear spatial differential operator, $\mathbf{f}$ is a nonlinear function of σ and ∂_t denotes differentiation with respect to time;

(iii) the equation for the strain tensor $\mathbf{E}$, i.e.

$$\mathbf{E} = \frac{1}{2}(\mathbf{H} + \mathbf{H}^T + \mathbf{H}^T\mathbf{H}), \tag{2.5}$$

where $\mathbf{H} = \nabla_{\mathbf{X}} \mathbf{u}$ denotes the displacement gradient;

(iv) the Cauchy equation of motion in the form

$$\text{Div}(\mathbf{FT}) + \varrho_0 \mathbf{b} = \varrho_0 \partial_t \mathbf{v}, \tag{2.6}$$

where $\mathbf{F}$ is the deformation gradient and $\mathbf{v} = \partial_t \mathbf{u}$ is the velocity vector field;

(v) the initial values

$$\mathbf{u}(\mathbf{X}, 0) = 0, \quad \mathbf{v}(\mathbf{X}, 0) = \mathbf{v}^0(\mathbf{X}), \quad \boldsymbol{\alpha}(\mathbf{X}, 0) = \boldsymbol{\alpha}^0(\mathbf{X}) \tag{2.7}$$

for $\mathbf{X} \in \mathscr{B}$;

(vi) the boundary conditions

$$\mathbf{u}(X, t) = \mathbf{u}^1(X, t),$$
$$\mathbf{a}\partial_{\mathbf{n}}\boldsymbol{\alpha}(X, t) + \mathbf{b}\boldsymbol{\alpha}(X, t) = 0, \tag{2.8}$$

for $(X, t) \in \partial\mathscr{B} \times [0, d_P]$, where $\mathbf{n}$ is the unit outward normal vector on $\partial\mathscr{B}$, $\mathbf{u}^1$, $\mathbf{a}$, $\mathbf{b}$ are bounded functions on $\partial\mathscr{B} \times [0, d_P]$.

By the solution $\varphi = \{\mathbf{u}, \mathbf{v}, \boldsymbol{\alpha}\}$ we understand such functions $\mathbf{u}$, $\mathbf{v}$ and $\boldsymbol{\alpha}$ which satisfy Eqs. (2.3)–(2.6) with the initial-boundary value conditions (2.7)–(2.8). The solution φ describes the isothermal inelastic flow process for a given body $\mathscr{B}$.

3. Discussion of a modified material structure with internal state variables

The purpose of introducing internal state variables to the material structure is as follows;

1. They are used to describe the internal dissipation of a material.

2. They represent a very suitable way of describing the internal changes of a material. The internal changes are caused by the following phenomena:

(i) internal arrangements of dislocations in solids(¹);

(ii) arrangements of point defects (migration, generation and annihilation of vacancies and interstitials) in the material of a body(²);

(iii) distribution of imperfections (migration, nucleation and growth of voids, cracks, etc.) in solids.

In every case the transport phenomena (kinetics of dislocations, point defects and imperfections) play an important role. To describe effects of transport phenomena we have to introduce partial differential evolution equations for the internal state variables assumed(³).

3. They play a very important role in the description of cooperative phenomena (synergic effects) in solids.

4. There exists the possibility of interpreting the internal state variables in the phenomenological theory basing on statistical methods used on the microscopic level.

It is useful to have the mathematical definition of the internal state variables as follows:

DEFINITION 1. *A vector valued function $\boldsymbol{\alpha}(\cdot, \cdot)$ is called the internal state variable for a body $\mathscr{B}$ if and only if*

(i) *it satisfies the evolution equation in the form*

$$\partial_t\boldsymbol{\alpha}(X, t) = A\boldsymbol{\alpha}(X, t), \tag{3.1}$$

where $A = \mathscr{L} + \hat{\mathbf{f}}$, $\mathscr{L}$ is a spatial differential operator and $\hat{\mathbf{f}}$ is a nonlinear function defined on Σ with values in $\mathscr{V}_n$, i.e. $\hat{\mathbf{f}}\colon \Sigma \to \mathscr{V}_n$;

(¹) This interpretation of the internal state variables was also suggested by J. KESTIN and J. R. RICE [11].

(²) For a thorough discussion of point defects and transport phenomena see C. P. FLYNN [2].

(³) The concept of the description of transport phenomena in solids within the framework of the internal state variable material structure was introduced in Ref. [17].

(ii) *it satisfies the initial condition*

$$\boldsymbol{\alpha}(X, 0) = \boldsymbol{\alpha}^0(X), \quad X \in \mathscr{B}; \tag{3.2}$$

(iii) *the domain of the differential operator* $\mathscr{L}$ *is defined by* ([4])

$$\operatorname{Dom}\mathscr{L} = \{\boldsymbol{\alpha} \in \mathscr{H} : \mathbf{a}\partial_n \boldsymbol{\alpha}(X, t) + \mathbf{b}\boldsymbol{\alpha}(X, t) = 0 \quad \textit{for} \quad X \in \partial\mathscr{B}\}, \tag{3.3}$$

where $\mathscr{H}$ *denotes a real Hilbert space.*

This definition is very general and when the differential operator $\mathscr{L}$ vanishes, then it reduces to the classical formulation([5]).

It is noteworthy that the boundary conditions which restrict the domain of the spatial differential operator $\mathscr{L}$ are of the homogeneous type, that is with zero boundary values. This feature is very important and is connected with the physical interpretation of the internal state variables. Experiment also suggests this theoretical assumption. It is clear that the internal state variables can describe the dissipative mechanisms of such a nature that are independent of boundary values for those variables. In other words, by means of internal state variables we can describe only typical internal mechanisms (dissipative phenomena) which are not controlled directly through the boundary of a body. They can be controlled only by means of other state variables like deformation (or temperature for thermodynamic processes).

Let us focus our attention on the constitutive equations and the evolution equations postulated by the modified material structure with internal state variables, i.e. Eqs. (2.1) and (2.4) together with the initial values $(2.7)_3$ and the boundary conditions $(2.8)_2$ for the internal state variable $\boldsymbol{\alpha}$. So we have a set of equations as follows:

$$\begin{gathered}
\sigma = \big(\mathbf{E}(X, t), \boldsymbol{\alpha}(X, t)\big), \\
\mathbf{T} = \hat{\mathbf{T}}(\sigma), \\
\partial_t \boldsymbol{\alpha}(X, t) = \mathscr{L}\boldsymbol{\alpha}(X, t) + \hat{\mathbf{f}}(\sigma), \\
\boldsymbol{\alpha}(X, 0) = \boldsymbol{\alpha}^0(X), \quad X \in \mathscr{B}, \\
\mathbf{a}\partial_{\mathbf{n}} \boldsymbol{\alpha}(X, t) + \mathbf{b}\boldsymbol{\alpha}(X, t) = 0, \quad (X, t) \in \partial\mathscr{B} \times [0, d_P].
\end{gathered} \tag{3.4}$$

Let us assume that we start our analysis from a given intrinsic state σ_0 at time $t = 0$ and we suppose a deformation process $P = \mathbf{E}_{[0,t]}$, consistent with the boundary conditions $(2.8)_1$. At the end of this process we have a new intrinsic state σ which is determined by the evolution function as follows

$$\sigma = \hat{\mathrm{e}}(\sigma_0, \mathbf{E}_{[0,t]}) = \big(\mathbf{E}(X, t), \mathfrak{F}(\mathbf{E}_{[0,t]}, \boldsymbol{\alpha}^0(\cdot), \mathbf{a}, \mathbf{b}, \partial\mathscr{B})\big), \tag{3.5}$$

([4]) It is noteworthy that the spatial differential operator $\mathscr{L}$ in general has the form

$$\mathscr{L} = \begin{pmatrix} \mathscr{L}_1 \\ \mathscr{L}_2 \\ \vdots \\ \mathscr{L}_n \end{pmatrix}$$

and some of $\mathscr{L}_i$ vanish identically.

([5]) Cf. with the proposition given first by B. D. COLEMAN and M. E. GURTIN [1].

where $\mathfrak{F}$ denotes the solution functional of our evolution problem. The stress tensor **T** at particle X at the end of the supposed deformation process, i.e. at time t is determined by

$$\mathbf{T}(\mathrm{X}, t) = \hat{\mathbf{T}}\big(\mathbf{E}(\mathrm{X}, t), \mathfrak{F}(\mathbf{E}_{[0,t]}, \boldsymbol{\alpha}^0(\cdot), \mathbf{a}, \mathbf{b}, \partial\mathscr{B})\big). \tag{3.6}$$

The classical formulation can be obtained by the assumption $\mathscr{L} \equiv 0$, then

$$\begin{aligned} \sigma = \hat{\mathrm{e}}(\sigma_0, \mathbf{E}_{[0,t]}) &= \big(\mathbf{E}(\mathrm{X}, t), \mathfrak{F}(\mathbf{E}_{[0,t]}, \boldsymbol{\alpha}(\mathrm{X}))\big), \\ \mathbf{T}(\mathrm{X}, t) &= \hat{\mathbf{T}}\big(\mathbf{E}(\mathrm{X}, t), \mathfrak{F}(\mathbf{E}_{[0,t]}, \boldsymbol{\alpha}^0(\mathrm{X}))\big). \end{aligned} \tag{3.7}$$

So, for the modified material structure the constitution of a material at particle X does depend on the history of the deformation $\mathbf{E}_{[0,t]}$, and the initial value $\boldsymbol{\alpha}^0(\cdot)$ in a whole body $\mathscr{B}$ as well as on the boundary conditions given by **a**, **b** and the shape of the body $\partial\mathscr{B}$ itself.

4. Flow process as a dynamical system

Let $\mathscr{T}$ be a subset of R^+ (for our purposes we can assume $\mathscr{T}$ as an interval $[0, d_{\mathrm{P}}]$), and let $\mathscr{A}(\mathsf{R}^+, \phi)$ define a set of solution function space. By ϕ we denote the set of values

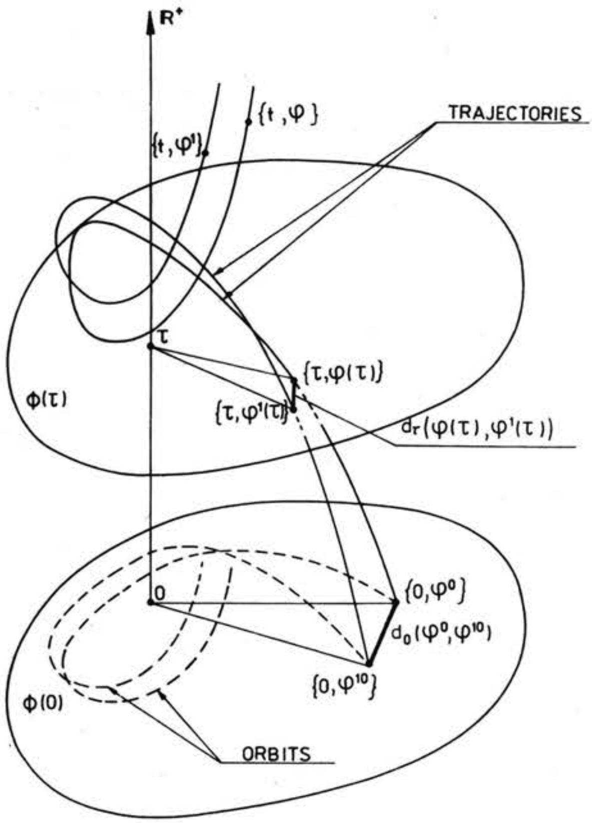


FIG. 2.

of the solution functions. So, $\varphi \in \mathscr{A}(\mathsf{R}^+, \phi)$ and $\varphi: \mathsf{R}^+ \to \phi$. For each $\tau \in \mathscr{T}$ and $\varphi \in \mathscr{A}(\mathsf{R}^+, \phi)$ we introduce a notation φ_τ for the translate of φ defined on R^+ by

$$\varphi_\tau(t) = \varphi(\tau+t), \quad t \in \mathsf{R}^+. \tag{4.1}$$

We may treat our flow process as a dynamical system corresponding to a triple $(\mathsf{R}^+, \mathscr{T}, \phi)$, cf. R. J. KNOPS and E. W. WILKES [12].

DEFINITION 2. *A dynamical system is a set $\mathscr{A}(\mathsf{R}^+, \phi)$ of functions defined on R^+ taking values in ϕ such that*

(i) $\varphi_\tau \in \mathscr{A}(\mathsf{R}^+, \phi)$ *whenever* $\varphi \in \mathscr{A}(\mathsf{R}^+, \phi)$ *and* $\tau \in \mathscr{T}$;

(ii) $\lim\limits_{t\to 0} \varphi_\tau(t) = \varphi(\tau)$, $\varphi \in \mathscr{A}(\mathsf{R}^+, \phi)$, $\tau \in \mathscr{T}$.

We can say that the function φ is the solution and the motion is the translate φ_τ.

Similarly to [12] we define the trajectory as the set of all pairs $(t, \varphi_\tau(t))$ for $t \in \mathsf{R}^+$, that is as a graph of the motion, cf. Fig. 2, and the orbit as the projection of the trajectory onto ϕ, that is the set of values $pr_\phi(\varphi_\tau(t))$.

Let $\phi(\tau)$ be the subset of ϕ defined as follows

$$\phi(\tau) = \{\varphi(\tau): \varphi \in \mathscr{A}(\mathsf{R}^+, \phi), \tau \in \mathscr{T} \subseteq \mathsf{R}^+\}. \tag{4.2}$$

We shall call $\phi(0)$ the set of initial values of the motion φ_τ for our process.

To define the neighbourhood of a solution $\varphi \in \mathscr{A}(\mathsf{R}^+, \phi)$ we shall use the metric d. In some circumstances it is necessary to indicate both the set of the solution functions and the metric. In this case we shall write $\mathscr{A}_\mathsf{d}(\mathsf{R}^+, \phi)$.

The inelastic flow process defined by Eqs. (2.3)–(2.8) is dissipative, so we can define for our dynamical system the function

$$\hat{i}: \Sigma \times \mathscr{B} \to \mathsf{R}^+ \tag{4.3}$$

which is called the internal dissipation function. Basing on the previous results (cf. Ref. [17, 18]) we have

$$\hat{i}(\sigma, \mathrm{X}) = -\frac{1}{\vartheta^*} \partial_\alpha \hat{\Psi}(\sigma) \cdot [\mathscr{L}\boldsymbol{\alpha} + \hat{\mathbf{f}}(\sigma)](\mathrm{X}) \geqslant 0. \tag{4.4}$$

The internal dissipation function $\hat{i}$ can be used to define the storage function

$$\mathrm{S}_{t_1}(\sigma_1) = \mathrm{S}_{t_0}(\sigma_0) + \int_{t_0}^{t_1} \hat{i}(\sigma)\, dt, \tag{4.5}$$

where $[t_0, t_1] \subset [0, d_\mathrm{P}]$, cf. J. C. WILLEMS [19] and M. E. GURTIN [5].

It will be convenient to introduce a nonlinear operator $\mathsf{T}_{(\cdot)}$ such that

$$\varphi_\tau(t) = \mathsf{T}_t \varphi(\tau), \tag{4.6}$$

i.e. φ_τ defined by Eq. (4.6) is an element of $\mathscr{A}_\mathsf{d}(\mathsf{R}^+, \phi)$ for any $\varphi(\tau) \in \phi(\tau)$, and $\mathsf{T}_{(\cdot)}$ is a mapping $\mathsf{T}_{(\cdot)}: \phi(\tau) \to \mathscr{A}_\mathsf{d}(\mathsf{R}^+, \phi)$.

The nonlinear operator $\mathsf{T}_{(\cdot)}$ is defined by Eqs. (2.3)–(2.8).

5. Criteria of instability of inelastic flow process

5.1. Criteria of discontinuous solution. Liapounov function

DEFINITION 3. *A solution $\varphi \in \mathscr{A}_d(R^+, \phi)$ is said to be Liapounov unstable if and only if for some $\tau \in \mathscr{T}$, the mapping $T_{(\cdot)}$ from $\phi(\tau)$ to $\mathscr{A}_d(R^+, \phi)$ is discontinuous at φ with respect to the neighbourhoods of φ induced on $\phi(\tau)$ and $\mathscr{A}_d(R^+, \phi)$ by d_τ and d, respectively.*

This means that a process described by the solution φ is unstable if and only if for at least one instant $\tau \in \mathscr{T}$, there exists a positive real number ε such that for each positive number δ there exists a $\varphi^1(\tau) \in \phi(\tau)$ such that

$$(5.1)\qquad d_\tau\big(\varphi(\tau), \varphi^1(\tau)\big) < \delta \quad \text{and} \quad d(\varphi_\tau, \varphi^1_\tau) \geqq \varepsilon,$$

where $\varphi^1_\tau(t) = T_t\varphi^1(\tau)$ and

$$(5.2)\qquad d(\varphi, \varphi^1_\tau) = \sup_{t\in R^+} d\big(\varphi_\tau(t), \varphi^1_\tau(t)\big)$$

and $\varphi, \varphi^1 \in \mathscr{A}_d(R^+, \phi)$.

We have now the fundamental theorem of instability (cf. J. E. GILBERT and R. J. KNOPS [4] and R. J. KONPS and E. W. WILKES [12]):

THEOREM 1. *A solution $\varphi \in \mathscr{A}_d(R^+, \phi)$ is unstable if and only if there exist positive definite functions (Liapounov functions)*

$$(5.3)\qquad V_{\tau, t}, \quad \tau \in \mathscr{T}, \quad t \in R^+ \quad \textit{defined on} \quad \phi\times\phi$$

such that

(i) *the mapping $T_{(\cdot)}$ from $\phi(\tau)$ to $\mathscr{A}_{V_\tau}(R^+, \phi)$ is discontinuous at φ for some $\tau \in \mathscr{T}$;*

(ii) *the identity mapping I is continuous from $\mathscr{A}_d(R^+, \phi)$ to $\mathscr{A}_{V_\tau}(R^+, \phi)$; where*

$$(5.4)\qquad V_\tau(\varphi_\tau, \varphi^1_\tau) = \sup_{t\in R^+} V_{\tau, t}\big(\varphi_\tau(t), \varphi^1_\tau(t)\big).$$

These conditions have the following mathematical from

$$(\mathrm{i}')\qquad \bigvee_{\tau\in\mathscr{T}}\ \bigvee_{\varepsilon>0}\ \bigwedge_{\delta>0}\ \bigvee_{\varphi^1(\tau)\in\phi(\tau)} d_\tau\big(\varphi(\tau), \varphi^1(\tau)\big) < \delta \wedge V_\tau(\varphi_\tau, \varphi^1_\tau) \geqslant \varepsilon;$$

$$(\mathrm{ii}')\qquad \bigwedge_{\eta>0}\ \bigvee_{\zeta(\eta,\tau)>0} d(\varphi_\tau, \varphi^1_\tau) < \zeta \Rightarrow V_\tau(\varphi_\tau, \varphi^1_\tau) < \eta.$$

Consistently with the inelastic flow process formulated, one may assume ([6])

$$(5.5)\qquad V_{\tau,t}\big(\varphi_\tau(t), \varphi^1_\tau(t)\big) = \int_{\mathscr{B}} \left\{\left|\hat{\Psi}\big(\varphi_\tau(t)\big) - \hat{\Psi}\big(\varphi^1_\tau(t)\big)\right| + \frac{1}{2}\varrho_0|\mathbf{v}^2 - \mathbf{v}^2_1|\right\} dV.$$

DEFINITION 4. *An element $\varphi^* = \{\mathbf{u}^*, 0, \boldsymbol{\alpha}^*\}$ of $\mathscr{A}_d(R^+, \phi)$ is an equilibrium solution if and only if*

$$(5.6)\qquad \mathbf{v}^* = \partial_t \mathbf{u}^* = 0, \quad \partial_t \boldsymbol{\alpha}^* = 0$$

and $\boldsymbol{\alpha}^*$ is determined in a body $\mathscr{B}$ by the solution of the boundary value problem

$$(5.7)\qquad \begin{gathered} \mathscr{L}\boldsymbol{\alpha}^*(X) + \hat{\mathbf{f}}\big(\mathbf{E}^*(X), \boldsymbol{\alpha}^*(X)\big) = 0, \\ \mathbf{a}\partial_n \boldsymbol{\alpha}^*(X) + \mathbf{b}\boldsymbol{\alpha}^*(X) = 0 \quad \textit{for} \quad X \in \partial\mathscr{B}, \end{gathered}$$

and $\mathbf{E}^*(X)$ is given by Eq. (2.5) with $\mathbf{H} = \nabla_X \mathbf{u}^*$.

([6]) Cf. with the proposition given first by M. E. GURTIN [5].

DEFINITION 5. *The intrinsic state* $\sigma^* = (\mathbf{E}^*(\mathbf{X}), \boldsymbol{\alpha}^*(\mathbf{X}))$ *is called an equilibrium state provided* $\boldsymbol{\alpha}^*(\mathbf{X})$ *is given by the solution of the boundary value problem* (5.7).

In many practical cases we are interested in the investigation of the flow process near the equilibrium solution $\varphi^* = \{\mathbf{u}^*, 0, \boldsymbol{\alpha}^*\}$ (or near the equilibrium intrinsic state $\sigma^* = (\mathbf{E}^*, \boldsymbol{\alpha}^*)$).

It is convenient to introduce the Liapounov function as follows

$$(5.8) \qquad \mathsf{V}_{\tau,t}(\varphi_\tau(t), \varphi^*) = \int_{\mathscr{B}} \left\{ [\hat{\Psi}(\varphi_\tau(t)) - \hat{\Psi}(\varphi^*)] + \frac{1}{2}\varrho_0 \mathbf{v}^2 \right\} dV.$$

This function has very important properties. It vanishes at the equilibrium solution φ^* and it has a very simple interpretation by using the storage function $\mathrm{S}(\sigma)$.

5.2. Criteria of bifurcation. Branching of solution

The solution $\varphi \in \mathscr{A}(\mathsf{R}^+, \phi)$ of the inelastic flow process in said to be unique if for each $\tau \in \mathscr{T}$

$$(5.9) \qquad \psi(\tau) = \varphi(\tau)$$

implies

$$(5.10) \qquad \psi_\tau(t) = \mathsf{T}_t \psi(\tau) = \varphi_\tau(t) = \mathsf{T}_t \varphi(\tau), \quad \tau \in \mathscr{T}, \quad t \in \mathsf{R}^+.$$

There exists a connection between Liapounov stability and uniqueness.

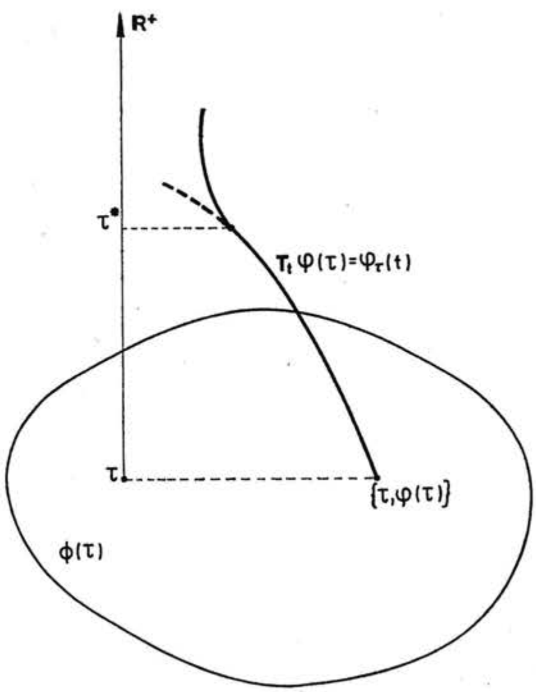


FIG. 3.

A. A. MOVCHAN [15] proved the theorem (cf. also J. E. GILBERT and R. J. KNOPS [4]) as follows.

THEOREM 2. *If* $\varphi \in \mathscr{A}(\mathsf{R}^+, \phi)$ *is Liapounov stable, then it is unique.*

An immediate consequence of the uniqueness theorem is that nonuniqueness of a solution automatically implies its instability.

Let us consider the solution

$$\varphi_\tau(t) = \mathsf{T}_t\varphi(\tau) \in \mathscr{A}(\mathsf{R}^+, \phi) \tag{5.11}$$

of the inelastic flow process. For some particular value $t = \tau^*$ the solution φ is not unique. Starting from the point $\{\tau^*, \varphi_\tau(\tau^*)\}$ on the trajectory we can expect more than one solution, cf. Fig. 3. So we have at $\{\tau^*, \varphi_\tau(\tau^*)\}$ branching of the solution. In other words the mapping $t \to \mathsf{T}_t\varphi(\tau)$ is a nonsingle-valued (multi-) function.

The criterion of bifurcation (branching of solution) is connected with the uniqueness theorems for the process considered, cf. W. KOSIŃSKI [13].

The criteria of bifurcation have been broadly investigated by R. HILL [7–9]. See the review papers by R. HILL [10] and by J. P. MILES [14].

6. Stability of equilibrium intrinsic states

Let the set $\Omega(\mathsf{R}^+, \Sigma)$ define an intrinsic state function space with Σ as the set of values of its functions, i.e. $\sigma \in \Omega(\mathsf{R}^+, \Sigma)$ and $\sigma: \mathsf{R}^+ \to \Sigma$. We also introduce a notation σ_τ for the translate of σ defined on R^+ by

$$\sigma_\tau(t) = \sigma(\tau + t), \quad t \in \mathsf{R}^+, \quad \tau \in \mathscr{T}. \tag{6.1}$$

It is straightforward to use the dynamical system methods to investigate the evolution of an intrinsic state σ during the inelastic flow process of a body $\mathscr{B}$.

Let us focus our attention on the motion (evolution) generated by small perturbations of the internal state variable $\boldsymbol{\alpha}$ around the equilibrium value $\boldsymbol{\alpha}^*$. Hence we consider the evolution of the intrinsic state $\bar{\sigma}$ defined by (7)

$$\bar{\sigma}(\mathrm{X}, t) = \big(\mathbf{E}^*(\mathrm{X}), \boldsymbol{\alpha}(\mathrm{X}, t)\big), \quad \bar{\sigma} \in \Omega(\mathsf{R}^+, \bar{\Sigma}), \tag{6.2}$$

where $\mathbf{E}^*(\cdot)$ denotes the equilibrium distribution of the strain tensor field in a body $\mathscr{B}$.

DEFINITION 6. *A dynamical system corresponding to a triple* $(\mathsf{R}^+, \mathscr{T}, \bar{\Sigma})$ *is a set* $\Omega(\mathsf{R}^+, \bar{\Sigma})$ *of functions defined on* R^+ *taking values in* $\bar{\Sigma}$ *such that*

(i) $\bar{\sigma}_\tau \in \Omega(\mathsf{R}^+, \bar{\Sigma})$ *whenever* $\bar{\sigma} \in \Omega(\mathsf{R}^+, \bar{\Sigma})$, $\tau \in \mathscr{T}$;

(ii) $\lim_{t \to 0} \bar{\sigma}_\tau(t) = \bar{\sigma}(\tau)$, $\bar{\sigma} \in \Omega(\mathsf{R}^+, \bar{\Sigma})$, $\tau \in \mathscr{T}$;

(iii) *the function* $\bar{\sigma}$ *is subjected to the restriction as follows*

$$\partial_t \boldsymbol{\alpha}(\mathrm{X}, t) = \mathscr{L}\boldsymbol{\alpha}(\mathrm{X}, t) + \hat{\mathbf{f}}\big(\mathbf{E}^*(\mathrm{X}), \boldsymbol{\alpha}(\mathrm{X}, t)\big), \tag{6.3}$$

with

$$\boldsymbol{\alpha}(\mathrm{X}, 0) = \boldsymbol{\alpha}^0(\mathrm{X}), \quad \mathrm{X} \in \mathscr{B}; \tag{6.4}$$

(7) It is noteworthy that $\mathbf{E}^*$ is fixed in what follows, so $\bar{\Sigma}$ corresponds to the so-called $\mathbf{E}^*$-section of Σ denoted usually by $\Sigma_{\mathbf{E}^*} \subset \Sigma$ (cf. W. NOLL [16]).

(iv) *the domain $\mathscr{D}$ of the differential operator $\mathscr{L}$ is specified by the assumed properties for solution* $\boldsymbol{\alpha}$ *and restricted by the boundary condition*

$$\mathbf{a}\partial_n \boldsymbol{\alpha}(X, t) + \mathbf{b}\boldsymbol{\alpha}(X, t) = 0, \quad (X, t) \in \partial\mathscr{B} \times [0, d_P]. \tag{6.5}$$

Let $\bar{\Sigma}(\tau)$ be the subset of $\bar{\Sigma}$ defined as follows

$$\bar{\Sigma}(\tau) = \{\bar{\sigma}(\tau): \bar{\sigma} \in \Omega(R^+, \bar{\Sigma}), \tau \in \mathscr{T} \subseteq R^+\}. \tag{6.6}$$

We shall call $\bar{\Sigma}(\tau)$ the set of initial values of the intrinsic state $\bar{\sigma}_\tau$, cf. Fig. 4.

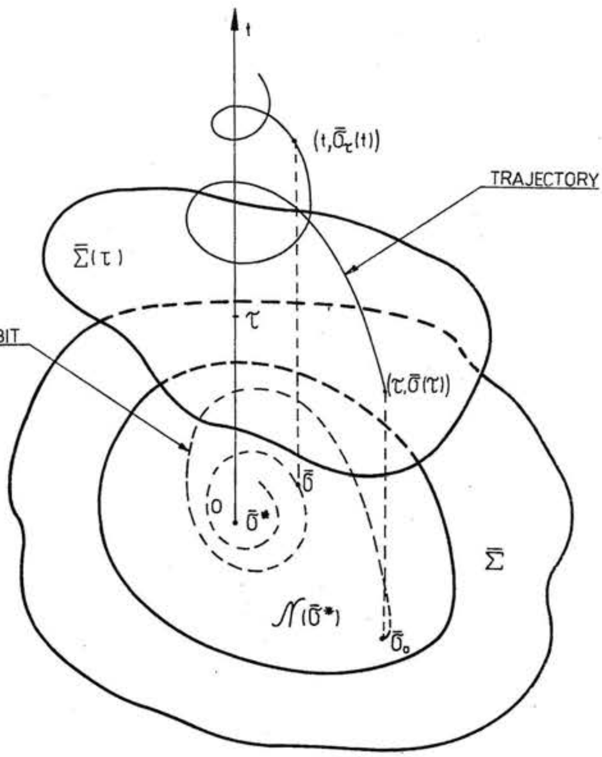


FIG. 4.

The mapping $T^*_{(\cdot)}$ defined by

$$T^*_{(\cdot)}: \bar{\Sigma}(\tau) \rightarrow \Omega(R^+, \bar{\Sigma}) \tag{6.7}$$

is the fundamental evolution operator, i.e.

$$\bar{\sigma}_\tau(t) = T^*_{(\cdot)}\bar{\sigma}(\tau). \tag{6.8}$$

For the internal state variable $\boldsymbol{\alpha}$ we can write

$$\boldsymbol{\alpha}(X, t) = T^*_{(\cdot)}\boldsymbol{\alpha}_0(\cdot). \tag{6.9}$$

The evolution operator $T^*_{(\cdot)}$ is determined by the evolution equation (6.3) and the boundary condition (6.5).

By $\Omega_{\mathsf{d}}(\mathsf{R}^+, \Sigma)$ we shall denote a set of functions of intrinsic state (solutions) provided d is a metric. If this metric is generated by the norm in a Hilbert space $\mathscr{H}$, then $\Omega_{||\cdot||}(\mathsf{R}^+, \bar{\Sigma})$ denotes a set of solutions.

An open ball with the centre in $\bar{\sigma}_0$ and the radius r is the set in $\Omega_{\mathsf{d}}(\mathsf{R}^+, \bar{\Sigma})$ and is defined as follows:

$$\mathrm{K}(\bar{\sigma}_0, r) \equiv \{\bar{\sigma} \in \Omega_{\mathsf{d}}(\mathsf{R}^+, \bar{\Sigma}) : \mathsf{d}(\bar{\sigma}_0, \bar{\sigma}) < r\}. \tag{6.10}$$

Similarly in $\Omega_{||\cdot||}(\mathsf{R}^+, \bar{\Sigma})$ we have

$$\mathrm{K}(\bar{\sigma}_0, r) \equiv \{\bar{\sigma} \in \Omega_{||\cdot||}(\mathsf{R}^+, \bar{\Sigma}) : ||\boldsymbol{\alpha} - \boldsymbol{\alpha}^0|| < r\}. \tag{6.11}$$

DEFINITION 7. *A subset $\mathscr{N}$ of $\Omega_{\mathsf{d}}(\mathsf{R}^+, \bar{\Sigma})$ is said to be a neighbourhood of $\bar{\sigma}_0 \in \Omega_{\mathsf{d}}^{*}(\mathsf{R}^+, \bar{\Sigma})$ if and only if there exists a positive number r such that* $\mathrm{K}(\bar{\sigma}_0, r) \subseteq \mathscr{N}$.

We can now introduce the definition of Liapounov stability (basing on the first Liapounov method).

DEFINITION 8. *An equilibrium intrinsic state $\bar{\sigma}^* \in \Omega_{\mathsf{d}}(\mathsf{R}^+, \bar{\Sigma})$ is said to be Liapounov stable if and only if, for each $t \in \mathsf{R}^+$, the mapping $\mathsf{T}^*_{(\cdot)}$ from $\bar{\Sigma}(\tau)$ to $\Omega_{\mathsf{d}}(\mathsf{R}^+, \bar{\Sigma})$ is continuous*

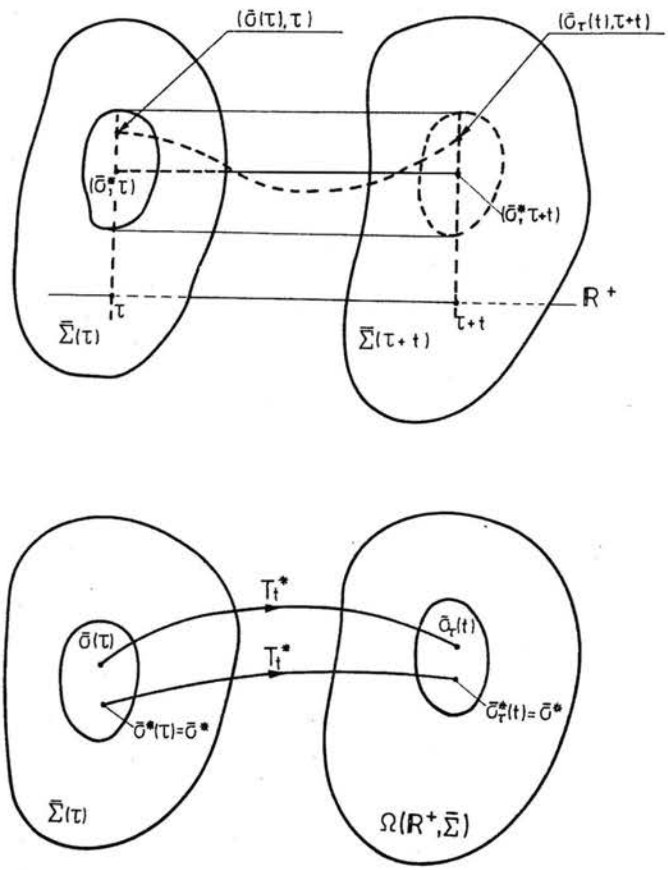


FIG. 5.

at σ^*. That is, for each $\tau \in \mathcal{T}$ and $t \in \mathsf{R}^+$ and for each positive real number ε, there exists a positive real number δ which depends on τ and ε such that

$$\mathsf{d}_\tau\big(\bar{\sigma}^*, \bar{\sigma}(\tau)\big) < \delta \Rightarrow \mathsf{d}(\bar{\sigma}^*, \bar{\sigma}_\tau) < \varepsilon \tag{6.12}$$

for $\bar{\sigma}^*, \bar{\sigma} \in \Omega_{\mathsf{d}}(\mathsf{R}^+, \bar{\Sigma})$ and

$$\mathsf{d}(\bar{\sigma}^*, \bar{\sigma}_\tau) = \sup_{t \in \mathsf{R}^+} \mathsf{d}\big(\bar{\sigma}^*, \bar{\sigma}_\tau(t)\big). \tag{6.13}$$

It is clear that for a given dynamical system an equilibrium intrinsic state may be stable for one choice of metric but not for another (cf. R. J. KNOPS and E. W. WILKES [12]).

Equivalent formulation of the definition of stability may be given in terms of neighbourhoods.

DEFINITION 9. *An equilibrium intrinsic state $\bar{\sigma}^* \in \Omega_{\mathsf{d}}(\mathsf{R}^+, \bar{\Sigma})$ is said to be stable relative to the set $(\bar{\Sigma}, \mathsf{d}_\tau, \mathsf{d})$ if and only if the map $\mathsf{T}^*_{(\cdot)}$ from $\bar{\Sigma}(\tau)$ to $\Omega(\mathsf{R}^+, \bar{\Sigma})$ is continuous at $\bar{\sigma}^*$ with the neighbourhoods of $\bar{\sigma}^*$ in $\bar{\Sigma}(\tau)$ and $\Omega(\mathsf{R}^+, \bar{\Sigma})$ defined respectively by d_τ and d* (cf. Fig. 5).

To have the most practical means of establishing stability conditions we shall use the theorem as follows:

THEOREM 3. *A sufficient condition for the equilibrium intrinsic state $\bar{\sigma}^* \in \Omega(\mathsf{R}^+, \bar{\Sigma})$ to be Liapounov stable is that d_τ and d satisfy the inequality*

$$\mathsf{d}\big(\bar{\sigma}^*, \bar{\sigma}_\tau(t)\big) \leqslant M_\tau(t)\,\mathsf{d}_\tau\big(\bar{\sigma}^*, \bar{\sigma}(\tau)\big), \qquad t \in \mathsf{R}^+ \tag{6.14}$$

for each $\tau \in \mathcal{T}$ and $\bar{\sigma} \in \Omega(\mathsf{R}^+, \bar{\Sigma})$, where $M_\tau(t)$ is a bounded real function on R^+.

COROLLARY 1. If $M_\tau(t)$ is independent of $\tau \in \mathcal{T}$, then $\bar{\sigma}^*$ is uniformly stable.

COROLLARY 2. If $M_\tau(t)$ tends to 0 as $t \to \infty$, then $\bar{\sigma}^*$ is asymptotically stable.

7. Criteria of stability of equilibrium intrinsic state

In order to give precise formulation of the second Liapounov method in a manner appropriate to investigate the evolution of the intrinsic states during an inelastic flow process, we introduce the Liapounov functions.

Let $\mathsf{V}_{\tau,t}$, where $\tau \in \mathcal{T}$, $t \in \mathsf{R}^+$, be positive-definite functions defined on $\bar{\Sigma} \times \bar{\Sigma}$. Additionally we denote by $\Omega_{\mathsf{V}_\tau}(\mathsf{R}^+, \bar{\Sigma})$ the set of functions $\Omega(\mathsf{R}^+, \bar{\Sigma})$ provided with the distance measure defined by

$$\mathsf{V}_\tau(\bar{\sigma}_1, \bar{\sigma}_2) = \sup_{t \in \mathsf{R}^+} \mathsf{V}_{\tau,t}\big(\bar{\sigma}_{1\tau}(t), \bar{\sigma}_{2\tau}(t)\big), \qquad \bar{\sigma}_1, \bar{\sigma}_2 \in \Omega(\mathsf{R}^+, \bar{\Sigma}). \tag{7.1}$$

The neighbourhoods in $\bar{\Sigma}(\tau)$ and $\Omega_{\mathsf{d}}(\mathsf{R}^+, \bar{\Sigma})$ are still defined by d_τ and d, respectively.

We now regard $\Omega_{\mathsf{V}_\tau}(\mathsf{R}^+, \bar{\Sigma})$ as being a set intermediate between $\bar{\Sigma}(\tau)$ and $\Omega_{\mathsf{d}}(\mathsf{R}^+, \bar{\Sigma})$ and we consider the mappings

(i) $\mathsf{T}^*_{(\cdot)}$ from $\bar{\Sigma}(\tau)$ to $\Omega_{\mathsf{V}_\tau}(\mathsf{R}^+, \bar{\Sigma})$ and defined by Eq. (6.7);

(ii) the identity mapping I from $\Omega_{\mathsf{V}_\tau}(\mathsf{R}^+, \bar{\Sigma})$ to $\Omega_{\mathsf{d}}(\mathsf{R}^+, \bar{\Sigma})$.

It is clear if $\mathsf{T}^*_{(\cdot)}$ is continuous at $\bar{\sigma}_1$ from $\bar{\Sigma}(\tau)$ to $\Omega_{\mathsf{V}_\tau}(\mathsf{R}^+, \bar{\Sigma})$ and I is continuous at $\bar{\sigma}_1$ from $\Omega_{\mathsf{V}_\tau}(\mathsf{R}^+, \bar{\Sigma})$ to $\Omega_{\mathsf{d}}(\mathsf{R}^+, \bar{\Sigma})$ then $\mathsf{T}^*_{(\cdot)}$ is continuous at $\bar{\sigma}_1$ from $\bar{\Sigma}(\tau)$ to $\Omega_{\mathsf{d}}(\mathsf{R}^+, \bar{\Sigma})$, cf. Fig. 6.

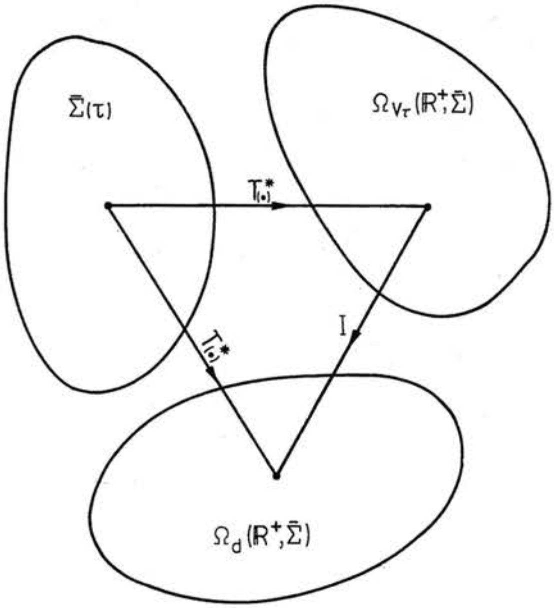


FIG. 6.

When this holds for each $\tau \in \mathscr{T}$, then $\bar{\sigma}_1$ is obviously stable. Conversely, if $\bar{\sigma}_1$ is stable, one such function $V_{\tau,t}$ always exists as can be seen by setting $V_{\tau,t}$ equal to d on $\bar{\bar{\Sigma}}\times\bar{\bar{\Sigma}}$.

Application of direct Liapounov conception to an equilibrium intrinsic state $\bar{\sigma}^*$ leads to the following theorem.

THEOREM 4. *The equilibrium intrinsic state* $\bar{\sigma}^* \in \Omega(R^+, \bar{\bar{\Sigma}})$ *is stable if and only if there exist positive-definite functions* $V_{\tau,t}$, *where* $t \in R^+$, $\tau \in \mathscr{T}$, *defined on* $\bar{\bar{\Sigma}}\times\bar{\bar{\Sigma}}$ *for which*

(i) *given a real positive number* ε, *there exists a real positive number* $\delta(\varepsilon, \tau)$ *such that*

$$d_\tau(\bar{\sigma}^*, \bar{\sigma}(\tau)) < \delta \Rightarrow V_\tau(\bar{\sigma}^*, \bar{\sigma}_\tau) < \varepsilon, \quad \bar{\sigma} \in \Omega(R^+, \bar{\bar{\Sigma}}); \tag{7.2}$$

(ii) *given a real positive number* η, *there exists a real positive number* $\varkappa(\eta, \tau)$ *such that*

$$V_\tau^{!}(\bar{\sigma}^*, \bar{\sigma}_\tau) < \varkappa \Rightarrow d(\bar{\sigma}^*, \bar{\sigma}_\tau) < \eta, \quad \bar{\sigma} \in \Omega(R^+, \bar{\bar{\Sigma}}). \tag{7.3}$$

V. I. ZUBOV [20] and A. A. MOVCHAN [15] prescribed the exact manner in which condition (i) of Theorem 4 is to be satisfied. After R. J. KNOPS and E. W. WILKES [12] we name these conditions (iii) and (iv):

(iii) *given a real positive number* ε *there exists a real number* $\delta(\varepsilon, \tau)$ *such that*

$$d_\tau(\bar{\sigma}^*, \bar{\sigma}(\tau)) < \delta \Rightarrow V_{\tau,0}(\bar{\sigma}^*, \bar{\sigma}(\tau)) < \varepsilon; \tag{7.4}$$

(iv) $V_{\tau,t}(\bar{\sigma}^*, \bar{\sigma}_\tau(t))$ *are non-increasing with respect to* t.

From Theorem 4 we have corollaries as follows:

COROLLARY 3. The equilibrium intrinsic state $\bar{\sigma}^*$ is uniformly stable if and only if the conditions (i) and (ii) of Theorem 4 hold uniformly in τ.

COROLLARY 4. The equilibrium intrinsic state $\bar{\sigma}^*$ is asymptotically stable if and only if the condition (i) holds together with the condition

$$\lim_{t\to\infty} V_{\tau,t}(\bar{\sigma}^*, \bar{\sigma}_\tau(t)) = 0. \tag{7.5}$$

To investigate the problem of evolution of intrinsic state $\bar{\sigma}$ generated by small disturbances of an internal state variable $\boldsymbol{\alpha}$ around its equilibrium value $\boldsymbol{\alpha}^*$ corresponding to the state $\bar{\sigma}^* = (\mathbf{E}^*, \boldsymbol{\alpha}^*)$, we postulate

$$\mathsf{V}_{\tau,t}(\bar{\sigma}^*, \bar{\sigma}_\tau(t)) = \int_{\mathscr{B}} [\hat{\Psi}(\bar{\sigma}_\tau(t)) - \hat{\Psi}(\bar{\sigma}^*)]\, dV, \tag{7.6}$$

where $\hat{\Psi}(\bar{\sigma}_\tau(t))$ denotes the value of the free energy function $\hat{\Psi}$ at $\bar{\sigma}_\tau(t)$.

The proposition (7.6) is a particular case of the previously postulated Eq. (5.8).

The assumed Liapounov function $\mathsf{V}_{\tau,t}$ in the form (7.6) is continuous at $\bar{\sigma}^*$ if and only if

$$\sup_{t \in \mathrm{R}^+} \left| \int_{\mathscr{B}} [\hat{\Psi}(\bar{\sigma}_\tau(t)) - \hat{\Psi}(\bar{\sigma}^*)]\, dV \right| \leqq c_1\, \mathsf{d}^2(\bar{\sigma}_\tau(t), \bar{\sigma}^*) \tag{7.7}$$

for all $\bar{\sigma}_\tau(t) \in \mathrm{K}(\bar{\sigma}^*, r)$, where c_1 is a positive number.

The function $\mathsf{V}_{\tau,t}$ is positive-definite if and only if

$$\inf_{t \in \mathrm{R}^+} \int_{\mathscr{B}} [\hat{\Psi}(\bar{\sigma}_\tau(t)) - \hat{\Psi}(\bar{\sigma}^*)]\, dV \geqq c_2\, \mathsf{d}^2(\bar{\sigma}_\tau(t), \bar{\sigma}^*) \tag{7.8}$$

for all $\bar{\sigma}_\tau(t) \in \mathrm{K}(\bar{\sigma}^*, r)$, where c_2 is a positive number.

We also assume the condition

$$\sup_{t \in \mathrm{R}^+} \dot{\mathsf{V}}_{\tau,t} = \sup_{t \in \mathrm{R}^+} \int_{\mathscr{B}} [\partial_\alpha \hat{\Psi}(\cdot) \cdot \partial_t \boldsymbol{\alpha}(\cdot, t)]\, dV \leqq -\beta \mathsf{d}^2(\bar{\sigma}_\tau(t), \bar{\sigma}^*) \tag{7.9}$$

for all $\bar{\sigma}_\tau(t) \in \mathrm{K}(\bar{\sigma}^*, r)$, where β is a positive number.

If the conditions (7.7)–(7.9) hold, then the conditions (i)–(iv) of Theorem 4 are valid together with the condition

$$\lim_{t \to \infty} \mathsf{V}_{\tau,t}(\bar{\sigma}_\tau(t), \bar{\sigma}^*) = 0. \tag{7.10}$$

Thus, the conditions (7.7)–(7.9) are sufficient to assure the asymptotic stability of the equilibrium intrinsic state $\bar{\sigma}^*$.

If a metric d is generated by the norm in a Hilbert space $\mathscr{H}$, i.e.

$$\mathsf{d}(\boldsymbol{\alpha}_\tau(t), \boldsymbol{\alpha}^*) = \sup_{t \in \mathrm{R}^+} \|\boldsymbol{\alpha}_\tau(t) - \boldsymbol{\alpha}^*\| = \sup_{t \in \mathrm{R}^+} (\boldsymbol{\alpha}_\tau(t) - \boldsymbol{\alpha}^*, \boldsymbol{\alpha}_\tau(t) - \boldsymbol{\alpha}^*)^{1/2}$$
$$= \sup_{t \in \mathrm{R}^+} \left[\int_{\mathscr{B}} (\alpha_{i_\tau}(t) - \alpha_i^*)(\alpha_{i_\tau}(t) - \alpha_i^*)\, dV \right]^{1/2}, \tag{7.11}$$

then the conditions (7.7)–(7.9) take the following form:

$$\sup_{t \in \mathrm{R}^+} \left| \int_{\mathscr{B}} [\hat{\Psi}(\mathbf{E}^*, \boldsymbol{\alpha}_\tau(t)) - \hat{\Psi}(\mathbf{E}^*, \boldsymbol{\alpha}^*)]\, dV \right| \leqq \bar{c}_1 \sup_{t \in \mathrm{R}^+} \|\boldsymbol{\alpha}_\tau(t) - \boldsymbol{\alpha}^*\|^2, \tag{7.12}$$

$$\inf_{t \in \mathrm{R}^+} \int_{\mathscr{B}} [\hat{\Psi}(\mathbf{E}^*, \boldsymbol{\alpha}_\tau(t)) - \hat{\Psi}(\mathbf{E}^*, \boldsymbol{\alpha}^*)]\, dV \geqq \bar{c}_2 \sup_{t \in \mathrm{R}^+} \|\boldsymbol{\alpha}_\tau(t) - \boldsymbol{\alpha}^*\|^2, \tag{7.13}$$

$$\sup_{t \in \mathrm{R}^+} \dot{\mathsf{V}}_{\tau,t} = \sup_{t \in \mathrm{R}^+} \int_{\mathscr{B}} [\partial_\alpha \hat{\Psi}(\mathbf{E}^*, \boldsymbol{\alpha}_\tau(t)) \cdot \partial_t \boldsymbol{\alpha}_\tau(t)]\, dV \leqq -\bar{\beta} \sup_{t \in \mathrm{R}^+} \|\boldsymbol{\alpha}_\tau(t) - \boldsymbol{\alpha}^*\|^2 \tag{7.14}$$

for all $\boldsymbol{\alpha}_\tau(t) \in \mathrm{K}(\boldsymbol{\alpha}^*, r)$, where $\bar{c}_1$, $\bar{c}_2$ and $\bar{\beta}$ are positive numbers.

The criteria of asymptotic stability of an equilibrium intrinsic state (7.12)–(7.14) can be compared with those obtained in the previous paper of the author [18] by means of the analytical theory of semi-groups ([8]). The criteria (7.12)–(7.14) have direct physical interpretation and it seems they have a more applicable character.

8. Discussion and conclusions

In the presentation of the first Liapounov method applied to the investigation of the evolution of the intrinsic states emphasis was laid on the fact that stability is a topological concept associated with continuity. This method was based on examining the evolution process directly.

On the other hand the essential features of the second Liapounov method lie in the fact that the criteria of instability of a flow process are determined by means of the Liapounov function whose properties are established directly without recourse to the solutions themselves. The second Liapounov method was applied to the investigation of the discontinuous solution of an isothermal flow process for dissipative solids with internal imperfections as well as to the analysis of the criteria of stability of an equilibrium intrinsic state.

The advantage of the second Liapounov method lies in the fact that we can obtain important informations on the behaviour of a flow process without solving the very complicated initial-boundary-value problem.

It is noteworthy that the criteria obtained for the asymptotic stability of an equilibrium intrinsic state (7.7)–(7.9) or (7.12)–(7.14) are stronger than those required by the second law of thermodynamics in the form of the Clausius–Duhem inequality (cf. the inequality (4.4)).

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([8]) Cf. also with the results obtained by K. Frischmuth [3] based on different mathematical postulates concerning the state space.

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