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H-theorem and trend to equilibrium in the kinetic theory of gases

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THE PRESENT status of the H-theorem and its role in establishing the trend to equilibrium of a dilute gas are reviewed. In particular the alleged counter-examples are shown to contain some features which do not allow the use of the standard arguments.

Omawia się stan obecny twierdzenia H i jego rolę w ocenie dążności gazów rozrzedzonych do przyjmowania stanu równowagi. Wykazano w szczególności, że rzekome kontrargumenty zawierają pewne cechy, które nie pozwalają na stosowanie standardowych metod rozumowania.

Рассматривается существующее состояние теоремы H и ее роль при оценке стремления разреженных газов к состоянию равновесия. Особенно подчеркивается, что мнимые контраргументы содержат некоторые особенности не позволяющие применять стандартных методов.

1. Introduction

IN A RECENT survey paper on "The physics of transition flow" [1], written in collaboration with R. HERCZYŃSKI, Professor FISZDON examined the problem of the asymptotic behaviour of the derivative dH/dt of Boltzmann's H function when $t \to \infty$. In that paper the authors state that if S is the entropy defined as follows

$$S = -\int f_M \log f_M \, d\boldsymbol{\xi} \, d\mathbf{x}, \tag{1.1}$$

(f_M being the local Maxwellian) and H is given as usual by

$$H = \int f \log f \, d\boldsymbol{\xi} \, d\mathbf{x} \tag{1.2}$$

(f being the actual distribution function satisfying the Boltzmann equation), then "it is tempting to assume ... that

$$\frac{dH}{dt} = -\frac{S+H}{\tau}, \tag{1.3}$$

where τ is some relaxation time .... However, it is not clear how to substantiate this choice starting from the Boltzmann equation".

It is the aim of this paper to point out that, although there are no arguments to justify Eq. (1.3), it seems highly reasonable to expect that the following inequality holds (in the absence of heat sources on the boundary):

$$\frac{dH}{dt} \leqslant -\frac{S+H}{\tau}, \tag{1.4}$$

where τ is a suitable relaxation time.

An inequality of the form shown in Eq. (1.4), though less handy than the corresponding equality, is useful in the discussion of the present status of the H-theorem and its use for the purpose of inferring the decay of f to a Maxwellian distribution. In fact, the discussion of an inequality such as the one considered above offers the opportunity of providing a serene assessment of the present status of the H-theorem in connection with the study of the time asymptotic behaviour of the solutions of the Boltzmann equation. Doubts on the use of the traditional interpretation of the H-theorem [2] have been cast in a recently published book [3] which exhibits, as counterexamples, previously known solutions of both the Boltzmann equation [4] and Maxwell's transfer equations for the moments [5].

It is this author's opinion that an entirely rigorous discussion of the asymptotic trend of the solutions of the Boltzmann equation can find its place only in the framework of a mathematical approach starting with the proof of an existence theorem in the large. Though this basic theory has registered some improvements [6–18] in the last few years, we are still far from such a theorem in the important case of the space inhomogeneous Boltzmann equation.

In this paper no new theorems will be proven, but conditions will be given on the initial and boundary data which are conjectured to be sufficient, together with a judicious (but, unfortunately, not yet available) choice of the function space where solutions have to be located, for proving both existence in the large and the H-theorem. If the possibility of reaching the latter difficult goal is granted, then further conditions are required in order to draw the traditional conclusions concerning the asymptotic behaviour for long times. While some of these conditions concern the smoothness of the solution and are therefore part and parcel of a still missing existence theory, there are also restrictions on boundary data (for a bounded domain) or on the behaviour of the initial data at large distances (for an unbounded domain). These conditions are such as to exclude the counterexamples to existence and to decay to an equilibrium distribution given in the aforementioned book [3]. The present contribution is intended as a step toward the solution of the first main problem of kinetic theory, as defined by Truesdell and Muncaster, i.e. "to discover and specify the circumstances that give rise to solutions which persist forever".

In the next section the formal proof of the H-theorem as well as the cases in which the proof has been made rigorous are reviewed. Then the inequality (1.4) is discussed. Finally, sufficient conditions for drawing consequences from the formal H-theorem are stated and counterexamples discussed.

2. Formal and rigorous H-theorems

We consider the Boltzmann equation for a simple monatomic gas (for a recent treatment of the case of polyatomic molecules see Ref. [19]):

$$\frac{\partial f}{\partial t}+\boldsymbol{\xi}\cdot\frac{\partial f}{\partial \mathbf{x}}+\mathbf{X}\cdot\frac{\partial f}{\partial \boldsymbol{\xi}} = Q(f,f), \quad \mathbf{x}\in\Omega\subset R^3, \quad \boldsymbol{\xi}\in R^3, \tag{2.1}$$

where t is time, $\mathbf{x}$ the position vector and $\boldsymbol{\xi}$ the velocity vector of a molecule, $\mathbf{X}$ the force per unit mass acting on a molecule (a function of $\mathbf{x}$ and t; dependence on $\boldsymbol{\xi}$ is allowed provided $(\partial/\partial\boldsymbol{\xi})\cdot\mathbf{X}=0)$, $f=f(\mathbf{x},\boldsymbol{\xi},t)$ the distribution function (a mass density in $\Omega\times R^3$, if Ω is the space domain where the equation has to be solved). $\partial/\partial\mathbf{x}$ and $\partial/\partial\boldsymbol{\xi}$ are the gradient operators in ordinary and velocity space, $Q(f,f)$ the collision operator [20, 21]

$$Q(f,f)=\int[f(\boldsymbol{\xi}')f(\boldsymbol{\xi}'_*)-f(\boldsymbol{\xi})f(\boldsymbol{\xi}_*)]\,B(\theta,|\mathbf{V}|)\,d\boldsymbol{\xi}_*\,d\theta\,d\varepsilon, \tag{2.2}$$

where $\boldsymbol{\xi}_*$ is the velocity of the target molecule in a collision, $\mathbf{V}=\boldsymbol{\xi}-\boldsymbol{\xi}_*$ the relative velocity and

$$\begin{aligned}\boldsymbol{\xi}'&=\boldsymbol{\xi}-\boldsymbol{\alpha}(\boldsymbol{\alpha}\cdot\mathbf{V}),\\ \boldsymbol{\xi}'_*&=\boldsymbol{\xi}_*+\boldsymbol{\alpha}(\boldsymbol{\alpha}\cdot\mathbf{V}).\end{aligned} \tag{2.3}$$

Here $\boldsymbol{\alpha}$ is a unit vector forming an angle θ with $\mathbf{V}$ and ε the angle of $\boldsymbol{\alpha}$ about $\mathbf{V}$.

If Ω is not the entire space, there will be boundary conditions on $\partial\Omega$ which will be assumed to be local in time and space and linear in f:

$$|\boldsymbol{\xi}\cdot\mathbf{n}|\,f(\mathbf{x},\boldsymbol{\xi},t)=\int_{\boldsymbol{\xi}'\cdot\mathbf{n}<0}R(\boldsymbol{\xi}'\to\boldsymbol{\xi};\mathbf{x},t)\,f(\mathbf{x},\boldsymbol{\xi}',t)\,|\boldsymbol{\xi}'\cdot\mathbf{n}|\,d\boldsymbol{\xi}' \qquad (\mathbf{x}\in\partial\Omega,\ \boldsymbol{\xi}\cdot\mathbf{n}>0), \tag{2.4}$$

where, if f_w is a Maxwellian describing a gas in local equilibrium with the wall, $R(\boldsymbol{\xi}'\to\boldsymbol{\xi};\mathbf{x},t)$ satisfies the two relations

$$\int_{\boldsymbol{\xi}\cdot\mathbf{n}>0}R(\boldsymbol{\xi}'\to\boldsymbol{\xi};\mathbf{x},t)\,d\boldsymbol{\xi}=1, \tag{2.5}$$

$$\int_{\boldsymbol{\xi}\cdot\mathbf{n}>0}f_w(\boldsymbol{\xi}';\mathbf{x},t)\,R(\boldsymbol{\xi}';\mathbf{x},t)\,|\boldsymbol{\xi}'\cdot\mathbf{n}|\,d\boldsymbol{\xi}'=|\boldsymbol{\xi}\cdot\mathbf{n}|\,f_w(\boldsymbol{\xi};\mathbf{x},t). \tag{2.6}$$

The formal proof of the H-theorem rests on two inequalities: the first is due to Boltzmann and reads

$$\mathscr{V}\equiv\int\log f\,Q(f,f)\,d\boldsymbol{\xi}\leqslant 0 \tag{2.7}$$

equality holding if and only if f is (almost everywhere in velocity space) a Maxwellian; the second one is much more recent ([22–25]; see discussion in Refs. [3 and 21]) and has the following form:

$$\mathscr{B}\equiv\int f\log f\,\boldsymbol{\xi}\cdot\mathbf{n}\,d\boldsymbol{\xi}\leqslant-\frac{(\mathbf{q}\cdot\mathbf{n})_{\rm solid}}{RT_w}, \tag{2.8}$$

where T_w is the wall temperature, R the gas constant, $\mathbf{q}$ the heat flow vector, $\mathbf{n}$ the normal unit vector pointing into the gas. The subscript "solid" means that $\mathbf{q}\cdot\mathbf{n}$, which is, in general, discontinuous at the surface, has to be evaluated on the solid side. Equation (2.8) holds if the values of f for $\boldsymbol{\xi}\cdot\mathbf{n}>0$ are related to those for $\boldsymbol{\xi}\cdot\mathbf{n}<0$ through Eq. (2.4). Equality applies if and only if $f=f_w$ (the "only if" part does not apply if $R(\boldsymbol{\xi}'\to\boldsymbol{\xi};\mathbf{x},t)$ is a delta function).

Some comments are in order here concerning the classical inequality (2.7) which is a trivial consequence of the properties of the logarithm and of the identity [2, 3, 20, 21]

$$\int\varphi Q(f,f)\,d\boldsymbol{\xi}=-\frac{1}{4}\int(\varphi'+\varphi'_*-\varphi-\varphi_*)(f'f'_*-ff_*)\,B(\theta,|\mathbf{V}|)\,d\varepsilon\,d\theta\,d\boldsymbol{\xi}\,d\boldsymbol{\xi}_*.$$

In order to prove this identity, one uses the fact that the absolute value of the Jacobian of $\boldsymbol{\xi}'$ and $\boldsymbol{\xi}'_*$ with respect to $\boldsymbol{\xi}$ and $\boldsymbol{\xi}_*$ for *fixed* $\boldsymbol{\alpha}$ is unity; a trivial consequence of the *linearity* of Eqs. (2.3) and of the fact that the transformation expressed by these equations is its own inverse [20]. In addition, one exploits the fact that if θ' is the angle between $\mathbf{V}' = \boldsymbol{\xi}' - \boldsymbol{\xi}'_*$ and $-\boldsymbol{\alpha}$, then $\theta = \theta'$ as well as the trivial observation that changing $\boldsymbol{\alpha}$ into $-\boldsymbol{\alpha}$ does not matter. In order to conclude that

$$\left| \frac{\partial(\boldsymbol{\xi}', \boldsymbol{\xi}'_w, \theta', \varepsilon')}{\partial(\boldsymbol{\xi}, \boldsymbol{\xi}_*, \theta, \varepsilon)} \right| = 1, \tag{2.10}$$

it only suffices to remark that we are ***not obliged*** to use θ and ε as angles to identify $\boldsymbol{\alpha}$ (though we may find it useful) and hence we can keep $\boldsymbol{\alpha}$ fixed when we change variables from $\boldsymbol{\xi}$, $\boldsymbol{\xi}_*$ to $\boldsymbol{\xi}'$, $\boldsymbol{\xi}'_*$. To be more explicit:

$$\sin\theta\, d\theta\, d\varepsilon\, d\boldsymbol{\xi}\, d\boldsymbol{\xi}_* = \sin\eta\, d\eta\, d\varphi\, d\boldsymbol{\xi}\, d\boldsymbol{\xi}_* = \sin\eta\, d\eta\, d\varphi\, d\boldsymbol{\xi}'\, d\boldsymbol{\xi}'_* = \sin\theta'\, d\theta'\, d\varepsilon'\, d\boldsymbol{\xi}'\, d\boldsymbol{\xi}'_*, \tag{2.11}$$

where η and φ are polar angles independent of $\mathbf{V}$ and use has been made of the well-known rotation invariance of the area element of the unit sphere. Equation (2.10) is a consequence of Eq. (2.11) and of $\theta' = \theta$.

We have spelled out the details of this argument because Eq. (2.10) is said to be the consequence "of direct if lengthy calculations" in the book by TRUESDELL and MUNCASTER [3]. The above proof seems to have been misunderstood if one can judge from a few letters of criticism from the readers of a book [20] where the proof itself is sketched. An indirect criticism without a precise reference seems to be contained in a paper by SCHNUTE [26] who remarks that it is not uncommon to infer that Eq. (2.10) is a consequence of the fact that the transformation $(\boldsymbol{\xi}, \boldsymbol{\xi}_*, \theta, \varepsilon) \to (\boldsymbol{\xi}', \boldsymbol{\xi}'_*, \theta', \varepsilon')$ is its own inverse. This argument would indeed be false when applied to this (nonlinear) transformation, but it is correct when applied to the linear transformation $(\boldsymbol{\xi}, \boldsymbol{\xi}_*) \to (\boldsymbol{\xi}', \boldsymbol{\xi}'_*)$ for a fixed $\boldsymbol{\alpha}$. Schnute's criticism has been echoed by TRUESDELL and MUNCASTER [3].

Concerning the inequality (2.8), we remark that if the scattering kernel is a delta function, i.e.

$$R(\boldsymbol{\xi}' \to \boldsymbol{\xi}; \mathbf{x}, t) = \delta(\boldsymbol{\xi} - \boldsymbol{\xi}(\boldsymbol{\xi}')), \tag{2.12}$$

where $\boldsymbol{\xi}(\boldsymbol{\xi}')$ is a (differentiable) function which we assume to be uniquely invertible, then Eq. (2.6) implies (adopting a reference frame where the wall is locally at rest):

$$\left| \frac{\partial \boldsymbol{\xi}}{\partial \boldsymbol{\xi}'} \right| |\boldsymbol{\xi}(\boldsymbol{\xi}') \cdot \mathbf{n}| = |\boldsymbol{\xi}' \cdot \mathbf{n}|, \tag{2.13}$$

$$|\boldsymbol{\xi}(\boldsymbol{\xi}')| = |\boldsymbol{\xi}'|. \tag{2.14}$$

The restriction of Eqs. (2.13) and (2.14) to $\boldsymbol{\xi}' \cdot \mathbf{n} < 0$ can be dropped by defining $\boldsymbol{\xi}(\boldsymbol{\xi}')$ to coincide with its own inverse when $\boldsymbol{\xi}' \cdot \mathbf{n} > 0$. All the transformations defined in the entire R^3 and satisfying Eq. (2.14) are known to be linear and to have a Jacobian with unit absolute value. If we restrict ourselves to the subset of these transformatons for which Eq. (2.13) is also satisfied (the Jacobian factor may now be suppressed), we find that

$$\boldsymbol{\xi}(\boldsymbol{\xi}') \cdot \mathbf{n} = -\boldsymbol{\xi}' \cdot \mathbf{n} \tag{2.15}$$

i.e. the linear transformation $\boldsymbol{\xi}' \to \boldsymbol{\xi}(\boldsymbol{\xi}')$ transforms $\boldsymbol{\xi}' \cdot \mathbf{n}$ into $-\boldsymbol{\xi}' \cdot \mathbf{n}$. It is otherwise a rotation in a plane tangential to the wall. If we add the somewhat natural restriction that $\boldsymbol{\xi}(\boldsymbol{\xi}')$ lies in the plane of $\mathbf{n}$ and $\boldsymbol{\xi}'$, we are left only with specular reflection and the parity transformation $\boldsymbol{\xi}' = -\boldsymbol{\xi}$, a result first discovered by SCHNUTE [26]. In either case Eq. (2.8) holds trivially because both sides vanish.

Once Eqs. (2.7) and (2.8) are established we have only to assume that f is sufficiently well behaved to conclude that in a bounded domain

(2.16) $$\dot{H} \leqslant \int_{\partial\Omega} \frac{(\mathbf{q}\cdot\mathbf{n})_{\text{solid}}}{RT_w}\, dS,$$

where dS is the surface element on $\partial\Omega$ and

(2.17) $$H = \iint f \log f d\boldsymbol{\xi}\, d\mathbf{x}$$

is the celebrated H-function (actually a functional of f) introduced by BOLTZMANN [27]. Equation (2.16) is an extension of Boltzmann's H-theorem and appears to have been proved formally in full detail for the first time in Ref. [24] (see, however, also Refs. [22] and [23] and discussion in Refs. [3] and [21]). Originally, the theorem was established with no boundary term at all or in the presence of specularly reflecting boundaries [28, 2]. In unbounded regions, the condition $|\mathbf{x}|^2 f \log f |\boldsymbol{\xi}\cdot\mathbf{n}| d\boldsymbol{\xi} \to 0$ for $|\mathbf{x}| \to \infty$ has to be added for Eq. (2.16) to hold.

In order to make the proof of Eq. (2.16) rigorous, one might specify sufficient conditions for the formal steps to be correct. This would be easy to do but there is little advantage in it because the theorem might be true under more general conditions (perhaps in a weak sense).

It is much better to examine the particular cases when the H-theorem has been actually proved. The starting point is, generally speaking, the proof of an existence and uniqueness theorem under suitable restrictions on initial and, if necessary, boundary data. Then under the same or more restrictive conditions, the existence of H for all times follows, together with Eq. (2.16).

In the space homogeneous case ($\partial f/\partial \mathbf{x} = 0$), the right hand side in Eq. (2.16) as well as the integration with respect to $\mathbf{x}$ in Eq. (2.16) have to be omitted, of course. This is the only case in which satisfactory results are available.

It is true that the problem of a trend toward equilibrium of a spatially homogeneous gas is trivial from the fluid dynamic point of view because there is no change at all in density, velocity, temperature and hence no fluid-dynamics in the usual sense of the word, but interesting facts do occur in velocity space and the mathematics is by no means trivial.

The first existence and uniqueness theorem for this problem was obtained by CARLEMAN [6] in the case of hard sphere molecules. He obtained a strong nonlinear result in the Banach space with the norm

(2.18) $$||f||_\alpha = \operatorname*{Ess\,sup}_{\boldsymbol{\xi}} [|f(\boldsymbol{\xi})|\,(1+|\boldsymbol{\xi}|^2)^\alpha],$$

where $\alpha > 3$. Uniqueness and uniqueness in the large are shown in the cone of positive functions together with the existence of the H-functional as a non-increasing function of t and the tendency to a Maxwellian distribution for $t \to \infty$.

WILD [29] and MORGENSTERN [9] proved similar results for the simpler case of Maxwell's potential with angular cutoff, but their results are less complete than Carleman's as far as the tendency toward equilibrium is concerned. The best available results for the space homogeneous case are those of ARKERYD [10] who was able to prove that, for a gas of rigid spheres or with angular cutoff, there is global existence of a positive solution if $f_0 \geqslant 0$, $(1+|\boldsymbol{\xi}|^2)f_0$ and $f_0 \log f_0$ belong to L^1.

If α can be taken to be not less than 2, then the solution is shown to be unique and the assumption on $f_0 \log f_0$ is not needed for existence. If the latter hypothesis is satisfied, however, it follows that $f \log f$ is L^1 for any $t > 0$ and the H-theorem is valid; in addition if $\alpha > 2$, there is a trend to equilibrium (i.e. f tends to a Maxwellian for $t \to \infty$) in the sense of weak convergence in L^1.

Recently, L. ARKERYD has shown [30] how to handle noncutoff potentials. In particular, he showed that in the case of inverse power molecules with exponents $s > 3$ there exists a weak solution of the Boltzmann equation provided $f_0(1+|\xi|^2) \in L^1_+$ and $f_0 \log f_0 \in L^1$. The solution conserves mass and momentum; energy is only shown not to increase.

The treatment of space-dependent solutions is much more difficult and decidedly incomplete at this moment. Since we are here mainly interested in the asymptotic trend for large times, we leave aside any proof of existence lacking a global character, apart from the first proof, due to GRAD [7], and the only proof allowing the presence of a body force, due to GLICKSON [31]. MORGENSTERN [8] and POVZNER [11] solved the Cauchy problem for a certain altered version of the Boltzmann equation, containing a "mollifying kernel" in the collision term. This mollifier removes the main difficulty for an existence proof in L^1 for the inhomogeneous problem, i.e. the impossibility of assigning a norm to a product such as $f(\mathbf{x}, \boldsymbol{\xi}, t)f(\mathbf{x}, \boldsymbol{\xi}_*, t)$. It is to be noted that for the space homogeneous case the mollifier disappears; accordingly, Morgenstern's and Povzner's results apply to the unmodified Boltzmann equation if $\partial f/\partial \mathbf{x} = 0$.

Some studies [13, 15, 16] have mainly dealt with situations close to equilibrium by exploiting results previously obtained by GRAD [32]. These papers employ norms in spaces whose physical relevance is difficult to envision and place restrictions on the deviation of the initial data from equilibrium. Minimal restrictions ($f_0 \in L^1$) are required by the global existence and uniqueness theorem given in Refs. [17] and [34], where, however, the Boltzmann equation is set on a (periodic) lattice, i.e. the term $\boldsymbol{\xi} \cdot (\partial f/\partial \mathbf{x})$ is replaced by a finite difference approximation in order to avoid the introduction of a mollifier. While the latter paper is restricted to cutoff Maxwell molecules, this assumption is relaxed in a subsequent study [18] where the collision term with $B(\theta, |\mathbf{V}|) < M(1+|\mathbf{V}|^\beta)(0 \leqslant \beta < 2)$ are treated under the additional assumption that $(1+|\boldsymbol{\xi}|^2)f_0 \in L^1_+$. In the same paper the existence of the limit when the lattice spacing goes to zero is investigated under the assumptions $(1+|\boldsymbol{\xi}|^2)f_0 \in L^1_+$ and $f_0 \log f_0 \in L^1$. If $f_{(n)}$ is the solution corresponding to a lattice with step 2^{-n} and associated, in a suitable sense, to an initial datum on a (continuous) cube with unit side, then there is a subsequence $\{f_{(n_k)}\}$ $(k = 1, 2, \ldots)$ converging weakly in L^1 to a limit f. It is to be stressed, however, that it is not known in what sense, if any, this weak limit is a solution of the original Boltzmann equation.

The H-theorem for the space inhomogeneous case is investigated by GUIRAUD [14] for a gas enclosed in a bounded convex domain with general boundary conditions of the form (2.4), under the usual restriction of an initial datum close to the wall Maxwellian f_w (assumed to be uniform). As a matter of fact, he proves the existence and uniqueness of a solution f, which is also shown to be nonnegative and bounded from below by a constant times f_w at least for long times ($t > \bar{t}$). This result shows that at least for $t > \bar{t}$ the H-function is defined and tends to a definite limit. In general, the gas exchanges energy with the wall and the H-theorem holds in the form (2.16); this does not contradict the existence of $\lim_{t \to \infty} H(t)$ because the integral

$$\int_{\infty}^{t} dt \int_{\partial\Omega} \frac{(\mathbf{q}\cdot\mathbf{n})_{\text{solid}}}{RT_w} dS \geqslant H(\infty) - H(\bar{t}) \tag{2.19}$$

is shown to be bounded.

It is to be remarked that all the global proofs reviewed here require one of the following tricks:

a) introduction of a mollifier [8, 11],

b) restriction to initial data suitably close to an equilibrium state [13, 15, 16],

c) possibility of using an *a priori* bound on some moments (such as density and energy per unit mass) [6, 7, 9, 10, 17, 18].

The proofs based on tricks a) and c) are automatically valid for the full Boltzmann equation in the space homogeneous case and this explains the progress reached in this particular case. For the space dependent solutions, tricks a) and c) require an actual modification of the Boltzmann equation (mollified collision term, finite difference approximation); accordingly these tricks can be viewed as efforts toward a better understanding of the difficulties and could become tools for handling the unmodified Boltzmann equation if passage to the limit of a deltalike mollifier or a zero stepsize were shown to be correct. For the moment, if one does not want to accept modifications of the adamantine structure the Boltzmann equation, he has no choice but to accept the restriction described under item b) above.

One can guess that sooner or later it will be possible to prove the existence (and uniqueness) of a solution f, belonging to a function set $\mathcal{S}$ of the initial value problem of Eq. (2.1) in the large (for a compact set Ω or an unbonded region) provided the initial datum f_0 is chosen in $\mathcal{S}$ and boundary conditions are such as to ensure that the momentum, energy and H influxes (through the boundary $\partial\Omega$ in the bounded case or an arbitrarily large sphere for an unbounded domain) remain bounded. The counterexample to global existence, quoted by TRUESDELL and MUNCASTER [3] and going back to BOLTZMANN [4], i.e. a Maxwellian with mass velocity $\mathbf{v} = -\mathbf{x}/(\tau - t)(\tau > 0$, a constant) has a total mass influx which is already unbounded at $t = 0$. This solution actually describes a tremendous implosion with terrific mass supply from infinity. No wonder that it ceases to exist for $t = \tau$.

3. Consequences of the H-theorem and counterexamples

If one accepts the H-theorem, he can draw some consequences. These consequences amount to saying that, under certain conditions, the distribution function will tend to a Maxwellian. The difficult point is to make these conditions explicit.

In the space homogeneous case and in the absence of body forces, the argument goes back to BOLTZMANN himself [27]. TRUESDELL and MUNCASTER [3] criticize the traditional "proof", especially in the form given by CHAPMAN and COWLING [2]. The argument under discussion starts from the fact that $\dot{H} \leqslant 0$ implies that H monotonously decreases in time unless f is a Maxwellian. It is then noted that H is bounded from below and H tends to a limit H_∞. This limit is asserted to correspond to a state of the gas in which $\dot{H} = 0$; this in turn is known to imply that f is a Maxwellian.

There are three difficult points in the argument: the boundedness of H from below, the fact that H tends to some limit (which then must be zero) and the existence of a limit of f when $t \to \infty$.

In the space homogeneous case the first difficulty is easily disposed of. This aspect is discussed in some detail by Chapman and Cowling who remark that, *roughly speaking*, the existence of the integral $\int f\xi^2 d\boldsymbol{\xi}$ implies that of H. In order to make their argument rigorous, let us consider the elementary inequality:

$$f \log f \geqslant f \log f_M + f - f_M, \tag{3.1}$$

where f_M is a Maxwellian having the same density, bulk velocity and temperature as f. In the space homogeneous case these moments of f are constant (because of the conservation equations) and f_M is time independent. Integrating the above inequality delivers the following result:

$$H \geqslant H_M, \tag{3.2}$$

where H_M is the (constant) value of H corresponding to $f = f_M$. Equation (3.2) says that H is bounded from below (assuming temperature to be finite).

We are dealing with the second difficulty ($\dot{H} \to 0$ for $t \to \infty$) that a deeper study of the properties of the Boltzmann equation is needed. In fact, the existence of a limit of H for $t \to \infty$ does not imply that $\dot{H} \to 0$. However, H is not just an arbitrary function of time but a functional of f, whose time derivative equals another functional of f. Accordingly although the traditional argument is rather cavalier, objections based on general remarks about the possible nonexistence of the limits of f and $\dot{H}$ for $t \to \infty$ are not decisive. In fact the quantity

$$h = \int (f \log f - f \log f_M + f_M - f)\, d\boldsymbol{\xi} = H - H_M \geqslant 0 \tag{3.3}$$

is a convex functional of f, vanishing when $f = f_M$ and satisfying $\dot{h} = \mathscr{V}$ where $\mathscr{V}$ is defined in Eq. (2.7).

It is tempting to assume that

$$\mathscr{V} \equiv \int \log f Q(f,f)\, d\boldsymbol{\xi} \leqslant -\lambda \varrho h, \tag{3.4}$$

where ϱ is, as usual, the density and λ a suitable constant. If Eq. (13.4) is assumed to be true, then

$$\dot{h} \leqslant -\lambda\varrho h \tag{3.5}$$

and, ϱ being constant,

$$h \leqslant h_0 e^{-\lambda\varrho t}. \tag{3.6}$$

Hence $h \to 0$ when $t \to \infty$ and $H \to H_M$. In addition, *if* f goes to a limit $\bar{f}$ when $t \to \infty$ in a function set where h is a continuous functional, then $\bar{f}$ must be f_M in agreement with the traditional argument.

We examine now the conjecture that Eq. (3.4) holds. The ratio

$$\frac{\mathscr{V}}{\varrho h} = \frac{\int \log f Q(f,f)\, d\boldsymbol{\xi}}{\left[\int f d\boldsymbol{\xi}\right]\left\{\int \left[f \log(f/f_M) + f_M - f\right] d\boldsymbol{\xi}\right\}} \tag{3.7}$$

is negative. The ratio is not well defined when $f = f_M$: it seems reasonable to argue that if $\mathscr{V}/\varrho h \to 0$ for some f, this will happen in a neighbourhood of f_M (in a suitable topology).

If this is granted, then one is led to examine the functional

$$J(p) = \frac{2(p, L_M p)}{\varrho_M (p,p)_M}, \tag{3.8}$$

where p is the perturbation of f_M (i.e. $f = f_M(1+p)$), L_M the collision operator linearized about f_M and the scalar product notation

$$(p,q)_M = \int f_M(\boldsymbol{\xi})\, p(\boldsymbol{\xi})\, q(\boldsymbol{\xi})\, d\boldsymbol{\xi} \tag{3.9}$$

is used. The functional in Eq. (3.8) is obtained by neglecting terms of order higher than second in both the numerator and the denominator of $\mathscr{V}/\varrho h$.

If the molecules are rigid spheres or interact with an inverse power law with the exponent $s \geqslant 5$, then a constant λ_M is known to exist such that $J(p) \leqslant -\lambda_M$ [20, 21]. This argument is, of course, not a proof of Eq. (3.4), but gives a strong support, in the author's opinion, to the conjecture that Eq. (3.4) holds.

We have to stress, however, that the validity of Eq. (3.4) is not enough to prove the trend to equilibrium since we have to assume that f has a limit $\bar{f}$ when $t \to \infty$.

This is really the basic point of the use of the H theorem because, if $f \to \bar{f}$ in a function set, where h and $\mathscr{V}$ are continuous functionals, then $\mathscr{V}$ and $\dot{h}$ must have a limit; if $\dot{h}$ has a limit, it must be zero. Hence $\mathscr{V}$ also tends to zero; then $\bar{f}$ equals f_M.

The proof of the existence of $\bar{f}$, however, requires an existence proof. This explains why any rigorous use of the H-theorem must follow an existence theorem. This kind of result, as explained in the previous section, has been obtained in the space homogeneous case, while its achievement in general appears to be of supreme difficulty.

We maintain that there is no objection to the extension of the consequences of the H-theorem in the space inhomogeneous case from a physical point of view, i.e. if we assume that a solution sufficiently smooth exists and we deduce its properties in a formal way. This of course can produce the objections of a critical mathematician who must first prove existence, smoothness, existence of an asymptotic behaviour, etc.; it is felt, however, that in the absence of such a refined theory it is important to argue in a formally correct way, leaving to more capable mathematicians the proofs of existence which are required on a stricter level of rigor.

In the space inhomogeneous case, in addition to smoothness, one has to impose an additional condition which is required in order to conclude that H is a limit: the integral in the right hand side of Eq. (2.16) must be nonpositive (no H is supplied to the gas). A sufficient condition is

$$(\mathbf{q}\cdot\mathbf{n})_{\text{solid}} \leqslant 0, \tag{3.10}$$

i.e. there is no local influx of energy into the gas. In addition, of course, a constant Maxwellian f_M to which Eq. (3.1) applies must be available.

In the space inhomogeneous case one has also to deal with the circumstance that the density goes to zero; then the H-theorem remains true if the function identical with zero in velocity space is considered to be a form of a degenerate Maxwellian. This circumstance appears in a gas expanding in a infinite space with Maxwellian distribution. As noted by PITTERI [33] and quoted by TRUESDELL and MUNCASTER [3] the ratio H/ϱ is larger than H_M/ϱ even in the limit $t\to\infty$ and f/ϱ does not tend to a Maxwellian; H tends to $H_M = 0$, however, and $f\to f_M\equiv 0$ for $t\to\infty$. A zero density is a problem, anyway, as Eq. (3.5) shows.

We also remark that the tendency to equilibrium is not applicable to the solution corresponding to the homo-energetic simple shearing [5,3] because there is no constant Maxwellian f_M which can play a role similar to that played by the local Maxwellian in the previous argument. In fact, the temperature is unbounded in time because work is exerted on the gas at a constant rate.

4. Concluding remarks

In view of the renewed interest which has arisen about the basic aspects of the kinetic theory of gases, it has seemed worthwhile to discuss the present *status* of the H-theorem with particular concern for the doubts which have been recently cast on its traditional interpretation. While an inequality of the form (1.4), or, equivalently, Eq. (3.5) may be used to remove one of the difficulties, the present discussion confirms that complete rigor can be obtained only if the problem is examined in a strictly mathematical way; it seems fair to say, however, that we have no indication that the traditional interpretation is wrong. In particular, sufficient conditions for a trend to equilibrium have been conjectured.

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