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A study of equilibrium points with application to constitutive modelling

K. FRISCHMUTH (WARSZAWA)

The paper contains an analysis of the behavior of a general constitutive model during relaxation processes. Basic definitions and a minimal set of assumptions are introduced to prove some theorems on equilibrium states as well as on the set of all equilibrium points. The results of Sect. 5 are the starting point for a discussion of unstable processes. Some of the present results are generalizations of theorems, which were earlier obtained by other authors. Several restrictions for the modelling of viscoplastic media follow from the paper.

Praca zawiera jakościową analizę zachowania się ogólnego modelu konstytutywnego w trakcie procesów relaksacji. Wprowadzono podstawowy zbiór definicji oraz minimalny układ założeń by wykazać kilka istotnych twierdzeń o punktach i stanach równowagi jak i o zbiorze punktów równowagi. Wyniki uzyskane w punkcie 5 są punktem wyjścia do dyskusji procesów niestabilnych. Część wyników pracy uogólnia rezultaty uzyskane przez innych autorów. Praca nakłada pewne ograniczenia na modelowanie ośrodków lepkoplastycznych.

В работе содержится качественный анализ поведения общей определяющей модели в ходе процессов релаксации. Вводится основное множество определений, а также минимальная система предположений для выявления нескольких существенных теорем о точках и состояниях равновесия, как и о множестве точек равновесия. Результаты, полученные в 5 точке являются исходным пунктом к дискуссии о неустойчивых процессах. Часть результатов работы обобщает результаты, полученные другими авторами. Работа накладывает определенные ограничения на моделирование вязкопластических сред.

1. Introduction

In 1967 Coleman and Gurtin [1] formulated their thermodynamics with internal state variables. They also introduced the concept of the domain of attraction of an equilibrium state together with the notion of the (quasi-) asymptotic stability of equlibrium states and assumed the stability postulate, which for each temperature and deformation requires the existence of an asymptotically stable in the large equilibrium state.

Using the new language of Noll's mathematical theory of materials [9], we can think of equilibrium states as being "relaxed states" and treat the constitutive model proposed by Coleman and Gurtin as a semi-elastic material element with internal state variables. In general, the semi-elastic material element is characterized by a one-to-one correspondence between configurations and relaxed states, so that this element can describe a viscoelatic material and is incapable to model a viscoplastic one, which is possible only when there is a nontrivial dependence of the relaxed state on the past deformation-temperature history.

In the paper [8] W. Kosiński and K. C. Valanis discussed the asymptotic stability and constitutive continuity of a material with internal state variables. A uniqueness theorem for quasi-asymptotically stable equilibrium states was proved, that is a sufficient condition for the stability postulate was given. Let us note again that under this condition the constitutive model assumed here cannot describe viscoplasticity.

It results from the above considerations that to formulate a constitutive model of either a real material or a body, one should check all qualitative features of the behaviour of the admitted model not to be to restrictive. For example, these features ought to contain the existence, the uniqueness and the continuous dependence of the evolution equation solution on the initial data, the asymptotic stability and the uniqueness of equilibrium states and, last not least, the existence of both unstable processes and equilibrium states.

The properties listed here are criteria for the choice of material structures which are capable to describe the behaviour of a given class of real materials. The aim of the present paper is twofold. First we prove a theorem for asymptotically stable equilibrium states, which is a generalization of the uniqueness theorem contained in [8]. Then we consider the case when asymptotic stability does not occur and prove that in this case there are nontrivial relations between the mentioned qualitative features of the model. We use the concept of invariant sets, and point out that the domains of attraction and their boundaries as well as the set of starting points of unstable processes are invariant.

All the results are stated in the framework of a general constitutive model in which the response is function of a pair of arguments, consisting of the configuration and the method of preparation [7, 11] (or simply an internal parameter). The evolution of the parameter is controlled by the process in the configuration space. Specifying the parameter, the evolution functions and classes of admissible processes, we can describe materials with memory or with internal variables as well as rate- or differential -type materials. The parameter can be a past history, an array of tensors or, for nonlocal theories, — an element of a space of functions or distributions defined on a material body. In order to include the case, when $\mathscr{K}$ is not a metric space, but the topology of $\mathscr{K}$ is generated by a uniformity using the constitutive functional [9], we assume $\mathscr{K}$ to be a Hausdorff space. The special case when $\mathscr{K}$ is a Banach function space of past histories [3] is studied in [5]. In this paper another uniqueness theorem is proved and conclusions for viscoplastic modelling are given.

2. Constitutive assumptions

In most of all theories of dissipative materials we are concerned with a response function which is determined by a pair of arguments. One of them, the actual value of the process, governs the reaction in (thermo-) elasticity. The other one characterizes the history of the process. This argument can be an element of a space of past histories, but often it is more convenient to express the essential features of the past history by a finite number of parameters. Thus each process being a continuation of the history determines a change of the second argument. We assume that the evolution of the second argument is determined entirely by its actual value and by the process, so that the constitutive equation has the form

$$Z = F(P, K),$$

where Z is the reaction, P is the configuration, i.e. the actual value of the process, K is a parameter and F is the response function.

Then we admit that the configuration and the reaction are elements of Banach spaces and K belongs to a Hausdorff space $\mathscr{K}$. Hence

$$F: B \times \mathscr{K} \to B_r.$$

We denote the evolution functions by T_t

$$T_t: \mathscr{K} \times C([0, t], B) \to \mathscr{K}, \quad t \in R^+.$$

If at the beginning of a process Π, $\Pi: [0, t] \to B$, the value of the parameter is equal to K, then at the end it takes the value $T_t(K, \Pi) = K(t)$.

At last the classes of admissible processes $C([0, t], B) =: C_t$ are assumed to include all constant functions, and the family of evolution functions $T_t(\cdot, \cdot)$ to satisfy the following axiom:

$$\forall t_1, t_2 \in R^+ \forall \Pi_1 \in C_{t_1}, \Pi_2 \in C_{t_2} \forall K \in \mathscr{K}\ \Pi_1 * \Pi_2 \in C_{t_1+t_2}$$
$$\Rightarrow T_{t_1}\big(T_{t_2}(K, \Pi_2), \Pi_1\big) = T_{t_1+t_2}(K, \Pi_1 * \Pi_2),$$

where

$$(\Pi_1 * \Pi_2)(t) := \begin{cases} \Pi_2(t) & \text{for} \quad t \in [0, t_2) \\ \Pi_1(t-t_2) & \text{for} \quad t \in [t_2, t_1+t_2]. \end{cases}$$

Let us now assume the evolution functions to possess the continuity property given by

POSTULATE 1. $\forall\, t \in R^+\ \forall\, \Pi \in C_t\, T_t(\cdot, \Pi): \mathscr{K} \to \mathscr{K}$ is continuous.

REMARK. When we consider a process (configuration) in a subspace B_0 of B, for example in the subspace of all configurations with a vanishing temperature gradient, we will write $\pi(p)$ instead of $\Pi(P)$. Furthermore we will identify each constant process and its corresponding configuration.

DEFINITION. *If* $\forall\, t \in R^+\ T_t(K, p) = K$, *then we call the pair* (K, p) *an equilibrium state and* K — *an equilibrium point corresponding to* p.

POSTULATE 2 of asymptotic rest property.

$$\forall K \in \mathscr{K} \forall p \in B_0 \exists K_\infty \in \mathscr{K}\ K_\infty = \lim_{t \to \infty} T_t(K, p).$$

LEMMA 1. If K_∞ is the limit of a relaxation process $t \mapsto T_t(K, p)$, then (K_∞, p) is an equilibrium state.

P r o o f. $T_t(K_\infty, p) = T_t(\lim_{\tau \to \infty} T_\tau(K, p), p) = \lim_{\tau \to \infty} T_{t+\tau}(K, p) = K_\infty$.

We denote $T(K, p) := \lim_{t \to \infty} T_t(K, p)$.

REMARK. The lemma yields the following equality:

$$\forall t \in R^+ p \in B_0 \forall K \in \mathscr{K}\ T_t\big(T(K, p), p\big) = T(K, p).$$

On the other hand from the definition of $T(K, p)$ we have

$$\forall t \in R^+ \forall p \in B_0 \forall K \in \mathscr{K}\ T\big(T_t(K, p), p\big) = T(K, p).$$

POSTULATE 3 of asymptotic contuinuity.

$$\forall p \in B_0\ T(\cdot, p): \mathscr{K} \to \mathscr{K} \text{ is continuous.}$$

3. Quasi-asymptotic stability

DEFINITION. *For a given equilibrium state (K, p) we define its domain of attraction* $D(K, p)$ by

$$D(K, p) := \{\tilde{K} \in \mathscr{K}:\ T(\tilde{K}, p) = K\}.$$

REMARKS. $D(K, p)$ is a) non-empty because $K \in D(K, p)$ and b) the inverse image of the point K under the function $T(\cdot, p)$

$$D(K, p) = T(\cdot, p)^{-1}(\{K\}).$$

Thus $K_1 \neq K_2 \Rightarrow D(K_1, p) \cap D(K_2, p) = \emptyset$ and if $P3$ holds, then all the domains $D(K, p)$ are closed.

DEFINITION. *If there exists a neighbourhood $U_{K,p} \ni K$, such that $U_{K,p} \subset D(K, p)$, an equilibrium state (K, p) is called quasi-asymptotically stable.*

An equilibrium state (K, p) is called Liapunov stable, if for each neighbourhood V of K there exists a neighbourhood $U \ni K$ such that $\forall\, t \in R^+\ T_t(U, p) \subset V$.

An equilibrium state is asymptotically stable if it is both quasi-asymptotically and Liapunov stable.

Now we can formulate

THEOREM 1. *If $P2$ and $P3$ hold, then for any $p \in B_0$ the existence of a quasi-asymptotically stable equilibrium state (K, p) excludes the existence of another equilibrium point K_1 different from K, which corresponds to the configuration p and belongs together with K to the same connected component of the parameter space $\mathscr{K}$.*

Proof. Let (K, p) be a quasi-asymptotically stable equilibrium state. Then in view of $P3$ $D(K, p)$ is closed. Let us now show that it is also open. By the assumption there exists an open set $U_{K,p} \subset D(K, p)$ with $K \in U_{K,p}$. Since the domains of attraction are disjoint, in the set $U_{K,p}$ there are no equilibrium points different from K, which correspond to the configuration p. Thus

$$T(\cdot, p)^{-1}(U_{K,p}) = \{\tilde{K} \in \mathscr{K}:\ T(\tilde{K}, p) \in U_{K,p}\} = \{\tilde{K} \in \mathscr{K}:\ T(\tilde{K}, p) = K\} = D(K, p).$$

Because of the assumed continuity of $T(\cdot, p)$ the first set being the inverse image of the open set $U_{K,p}$ is open. Hence we proved that $D(K, p)$ is a non empty open-closed subset of $\mathscr{K}$, which implies that $D(K, p)$ is a whole connected component of the space $\mathscr{K}$.

COROLLARY 1. If K_1 and K_2 are two different equilibrium points corresponding to the same configuration p, and the pair (K_1, p) forms a quasi-asymptotically stable state, then there is not any continuous process in the space $\mathscr{K}$, $K(\cdot): [0, t] \to \mathscr{K}$, with $K(0) = K_1$ and $K(t) = K_2$.

COROLLARY 2. If all the equilibrium points of the assumed model corresponding to all the configurations $p \in B_0$ are quasi-asymptotically stable and $\mathscr{K}$ is connected, then there exists a one-to-one correspondence between the configurations and the relaxed states, which means that the model is semi-elastic.

REMARK. Usually $\mathscr{K}$ is a Banach space and consequently connected. Moreover, if we assume that

a) $\Pi \in C_t$, $t_1 \leqslant t$ implies that $\Pi|_{[0,t_1]} \in C_{t_1}$ and

b) $\tau \mapsto T_\tau(K, \Pi|_{[0,\tau]})$ is continuous for each $\Pi \in \bigcup_{t \in R^+} C_t$ and $K \in \mathscr{K}$

and at last restrict $\mathscr{K}$ to contain an element, from which by an admissible process in the configuration space each other element can be reached, then $\mathscr{K}$ will be arcwise connected.

In [8] the authors considered the material structure with internal variables. They assumed $\mathscr{K}$ to be a connected subset of R^n and the solution of the evolution equation to exist for all $t \in R^+$ and to depend continuously uniformly with respect to time on the initial value and on the process. Thus our postulates $P1$ and $P3$ together are weaker than the assumption of constitutive continuity made by KOSIŃSKI and VALANIS. At last our postulate $P2$ is equivalent to the so-called postulate of asymptot icstability introduced in [8]. Hence the main result of [8], i.e. the uniqueness of the quasi-asymptotically stable equilibrium state under the assumptions mentioned above, is a special case of our Theorem 1.

It should be pointed out that the proof in [8] is based on the assumption that there are two distinct quasi-asymptotically stable equilibrium states corresponding to one configuration, which yields a contradiction. Thus the result that a quasi-asymptotically stable equilibrium state (K, p) excludes the existence of even non-quasi-asymptotically stable equilibrium points corresponding to p, is essentially new.

4. Invariant sets

In what follows we assume that the postulates $P1$, $P2$ and $P3$ hold, that $\mathscr{K}$ is connected, and consider the equilibrium point set E_p without the assumption of quasi-asymptotic stability. By Theorem 1 an equilibrium state is not quasi-asymptotically stable if and only if there exists another equilibrium state for the same configuration.

THEOREM 2. *If the postulates $P1$, $P2$, $P3$ hold and the parameter space $\mathscr{K}$ is connected, then for each given p the set of equilibrium points $E_p \subset \mathscr{K}$ corresponding to p is connected.*

P r o o f. For equilibrium points K we have $T(K, p) = K$, and on the other hand $P1$ implies that for each $p \in B_0$ and $K \in \mathscr{K}$ $T(K, p)$ is an equilibrium point. Thus we can represent E_p in the form

$$E_p := \{K \in \mathscr{K}: \exists \tilde{K} \in \mathscr{K}\ T(\tilde{K}, p) = K\} = T(\mathscr{K}, p)$$

from which we conclude that E_p being the image of a connected set under a continuous function is connected.(1)

COROLLARY 3. If $\mathscr{K}$ is a connected metric space and the family $T_t(\cdot, \cdot)$ satisfies the postulates $P1$, $P2$ and $P3$, then for a given configuration p the set E_p is either a singleton or uncountable.

THEOREM 3. *If $P1$ and $P2$ hold, then E_p is closed.*

P r o o f. Since $\mathscr{K}$ is a Hausdorff space and each $T_t(\cdot, p)$ is continuous, each set

$$F_t := \{K \in \mathscr{K}: T_t(K, p) = id(K) = K\}$$

is closed as the set on which two continuous functions $T_t(\cdot, p)$ and $id(\cdot)$ are equal. But by the definition there is

$$E_p = \bigcap_{t \in R^+} F_t = \{K \in \mathscr{K}: \forall t \in R^+\ T_t(K, p) = K\}.$$

Hence E_p as an intersection of closed sets is closed.

(1) For definitions and theorems from topology see [4].

REMARK. In the special case, when $\mathscr{K}$ satisfies the first countability axiom, we can prove this result in terms of sequences, namely

$$T_t(e, p) = T_t(\lim_{n\to\infty} e_n, p) = \lim_{n\to\infty} T_t(e_n, p) = \lim_{n\to\infty} e_n = e$$

for any $t \in R^+$ when $\{e_n\}_0^\infty \subset E_p$ and $e = \lim e_n$.

DEFINITION. *For a given $p \in B_0$ a subset $I \subset \mathscr{K}$ is called an invariant set if*

$$\forall t \in R^+ \, T_t(I, p) \subset I.$$

REMARKS. a) $\emptyset$ and $\mathscr{K}$ are invariant for each p.

b) Since for each $t \in R^+$, $p \in B_0$ and $K \in \mathscr{K}$ $T(T_t(K, p), p) = T(K, p)$ each domain of attraction is an invariant set.

c) For a fixed p the sum and the intersection of each family of invariant sets are invariant.

d) If $P1$ holds, then the closure of an invariant set is invariant.

P r o o f. d) Because $T_t(\cdot, p)$ is continuous, for each subset $I \subset \mathscr{K}$ we have

$$T_t(\mathrm{cl}\, I, p) \subset \mathrm{cl}\, T_t(I, p).$$

For an invariant I we have $T_t(I, p) \subset I$, and consequently

$$\mathrm{cl}\, T_t(I, p) \subset \mathrm{cl}\, I.$$

These inclusions yield $T_t(\mathrm{cl}\, I, p) \subset \mathrm{cl}\, I$, which ends the proof.

LEMMA 2. If the postulates $P1$ and $P2$ are fulfilled, then the boundary of each domain of attraction is an invariant set.

P r o o f. In view of remark b) $D(K, p)$ is invariant for each equilibrium state (K, p). Thus by d) $\mathrm{cl}\, D(K, p)$ is invariant. So it remains to show that a point from the boundary $\mathrm{fr}\, D(K, p)$ cannot enter the interior $\mathrm{int}\, D(K, p)$. Indeed, since

$$T\big(T_t(\tilde{K}, p), p\big) = T(\tilde{K}, p)$$

we have

$$T_t(\cdot, p)^{-1}\big(D(K, p)\big) = D(K, p).$$

But it follows from the continuity of $T_t(\cdot, p)$ that

$$T_t(\cdot, p)^{-1}\big(\mathrm{int}\, D(K, p)\big) \subset \mathrm{int}\, T_t(\cdot, p)^{-1}\big(D(K, p)\big),$$

which in turn shows that

$$T_t(\cdot, p)^{-1}\big(\mathrm{int}\, D(K, p)\big) \subset \mathrm{int}\, D(K, p),$$

$$\text{or} \quad T_t\big(\mathscr{K} \setminus \mathrm{int}\, D(K, p), p\big) \subset \mathscr{K} \setminus \mathrm{int}\, D(K, p).$$

So both $\mathrm{cl}\, D(K, p)$ and $\mathscr{K} \setminus \mathrm{int}\, D(K, p)$ are invariant and, consequently, by c) the set

$$\mathrm{cl}\, D(K, p) \cap \big(\mathscr{K} \setminus \mathrm{int}\, D(K, p)\big) = \mathrm{fr}\, D(K, p)$$

is invariant.

REMARK. It should be pointed out that the condition $K_0 \in \mathrm{fr}\, D(K, p)$ is not sufficient for another equilibrium point K_1 to exist, so that $K_0 \in \mathrm{fr}\, D(K_1, p)$. However, in some cases, for instance if E_p is finite, such an implication can occur. In such circumstances the lemma becomes a straightforward consequence of the remarks and of the equality

$$\operatorname{fr} D(K, p) = \bigcup_{\substack{\tilde{K} \in E_p \\ \tilde{K} \neq K}} \left(\operatorname{cl} D(K, p) \cap \operatorname{cl} D(\tilde{K}, p)\right).$$

COROLLARY 4. The intersection of an attraction domain closure family is invariant.

COROLLARY 5. If $P1$, $P2$ and $P3$ hold, $\mathscr{K}$ is connected, and at least two equilibrium points corresponding to p exist, then each of them lies on the boundary of its domain of attraction.

Proof. By the continuity of $T(\cdot, p)$ the boundary $\operatorname{fr} D(K, p)$ is contained in $D(K, p)$, and $\operatorname{fr} D(K, p)$ is not empty because $D(K, p)$ is a proper subset of $\mathscr{K}$. Thus we can take a $K_0 \in \operatorname{fr} D(K, p)$. Now, by the lemma 2 we have

$$\forall t \in R^+ \; T_t(K_0, p) \in \operatorname{fr} D(K, p),$$

and hence the limit $\lim_{t \to \infty} T_t(K_0, p) = T(K_0, p) = K$ belongs to $\operatorname{fr} D(K, p)$.

REMARK. This fact can be also proved making use of the Theorem 2.

REMARK. In many practical cases $T_t(K, p)$ is the solution of an initial-value problem for an ordinary differential equation, starting at K. Under well-known conditions this solution depends continuously on K and t, and at $t = 0$ we have $T_0(K, p) = K$ for each K. If we assume an equilibrium state $(\tilde{K}, p)$ to be Liapunov stable, then we have the following properties:

a) $\forall K \in \mathscr{K} \forall p \in B_0 \; T_0(K, p) = K$,

b) the function T_p: $\overline{R^+} \times D(\tilde{K}, p) \to D(\tilde{K}, p)$,

$$T_p(t, K) := \begin{cases} T_t(K, p) & \text{for} \quad t < \infty, \\ T(K, p) = \tilde{K} & \text{for} \quad t = \infty \end{cases}$$

is continuous.

Taking a closed invariant subset $I \subset D(K, p)$, we infer that $K \in I$ and the function $H: I \times [0, 1]| \to I$

$$H(i, x) = T_p\left(\frac{x}{1-x}, i\right)$$

is a homotopy.

Since a sphere in R^n cannot be contracted to a point and is no retract of the closed ball, the following facts are worth-while mentioning:

a) neither a domain of attraction nor its boundary can be homeomorphic with a sphere in R^n.

b) The equilibrium point set E_p cannot be the boundary of a set which is homeomorphic with the closed ball in R^n.

Examples. a) $\mathscr{K} = R$, $E_p = [a, b]$

$$D(x, p) = \begin{cases} (-\infty, x] & \text{for} \quad x = a, \\ \{x\} & \text{for} \quad x \in (a, b), \\ [x, +\infty) & \text{for} \quad x = b. \end{cases}$$

b) By our last remark the following situation cannot occur

$$\mathscr{K} = R^2, \quad E_p = \{(x, 0): x \geqslant 0\}, \quad D\big((x, 0), p\big) = \{(u, v): u^2 + v^2 = x^2\}.$$

5. Existence theorems for unstable equilibrium states

Let us now assume that $P1$ and $P2$ hold, but $T(\cdot, p)$ is not a continuous function. In this case several equilibrium states corresponding to p must exist, and more than one of them can be asymptotically stable. The aim of this section is to prove the existence of Liapunov unstable equilibrium states for p. For the sake of simplicity we start from the special case in which one of the domains of attraction is not closed, and make use of the Lemma 2. Then, in order to give a general proof of the existence of instability points, we consider the set SING(p) of all points at which the function $T(\cdot, p)$ is not continuous.

THEOREM 4. *When $P1$ and $P2$ hold and one of the domains of attraction $D(K, p)$ is not closed, then there exists an equilibrium state (K_1, p), $K_1 \in \mathrm{fr}\, D(K, p)$, which is not Liapunov stable.*

Proof. By assumption, there exists a point $K_0 \in \mathrm{fr}\, D(K, p) \setminus D(K, p)$. Again, as in Corollary 5, we conclude that $K_1 := T(K_0, p)$, different from K by choice, lies on the boundary of $D(K, p)$, and is also an equilibrium point corresponding to p. We can take disjoint neighbourhoods U and U_1 of correspondingly K and K_1. Since each neighbourhood V of K_1 contains a point K_V from $D(K, p)$, there exists a time t_V such that $T_{t_V}(K_V, p) \in U$, and, consequently, $T_{t_V}(K_V, p) \notin U_1$. Now, because V was arbitrary, K_1 is Liapunov unstable.

REMARK. The assumptions of Theorem 4 are satisfied if

a) $P1$, $P2$ hold, $\mathscr{K}$ is connected, (K, p) is quasi-asymptotically stable and $T(\cdot, p)$ is not continuous.

Proof. By the Lemma 2 $D(K, p)$ and its boundary $\mathrm{fr}\, D(K, p)$ are then disjoint, so that $D(K, p)$ is open and cannot be closed because otherwise $T(\cdot, p)$ would be a constant and, consequently, a continuous function.

b) $P1$, $P2$ hold, E_p is finite and $T(\cdot, p)$ is not a continuous function.

Proof. If E_p is finite, then $T(\cdot, p)$ is continuous if and only if all domains of attraction are closed, so that there must exist one domain of attraction which is not closed.

c) $P1$, $P2$ hold and the boundaries of two domains of attraction have common points, i.e. $\mathrm{cl}\, D(K_1, p) \cap \mathrm{cl}\, D(K_2, p) \neq \emptyset$.

Proof. Because $D(K_1, p) \cap D(K_2, p) = \emptyset$ either $D(K_1, p)$ or $D(K_2, p)$ cannot be closed.

COROLLARY 6. If $\mathscr{K}$ is connected, $P1$ and $P2$ hold and (K, p) is a quasi-asymptotically stable equilibrium state, then either K is the only equilibrium point corresponding to p or there exists at least one Liapunov unstable equilibrium state (K_1, p) with K_1 at the boundary of $D(K, p)$.

Let us now consider the following example:

$$\mathscr{K} = R^3 \ni (x, y, z),\ T_t\big((x, y, z), p\big) = (x, ye^{-axt}, 0),$$

where a is a positive function of p.

It is easy to see that

a) $T_0 \neq id$ is the projection on the x-y-plane,

b) $P1$ and $P2$ hold,

c) $T((x, y, z), p)$ is discontinuous in all points of the form $(0, y, z)$ with $y \neq 0$,

d) $E_p = \{(x, y, 0) : x = 0 \vee y = 0\}$ is connected.

For $x = 0$ we have $D((0, y, 0), p) = \{(0, y, z): z \in R\}$ and for $y = 0$, $x \neq 0$ there is $D((x, 0, 0), p) = \{(x, y, z): y \in R, z \in R\}$. So all the domains of attraction are closed. But all the points $(0, y, 0)$ with $y \neq 0$ are Liapunov unstable because all the points $(x, y, 0)$ with $x \neq 0$ are attracted to points which lie outside the $y/2$-ball with the center in $(0, y, 0)$.

This example suggests that the thesis of Theorem 4 holds even in more general cases. To prove this hypothesis, we introduce the following:

DEFINITION. *We say, that $K \in \mathscr{K}$ satisfies sing (p) with V, if V is an open neighbourhood of T (K, p) and for each open neighbourhood U of K there exists a point $\tilde{K} \in U$ such that $T(\tilde{K}, p) \notin V$.*

DEFINITION. SING (p) *is the set of all $K \in \mathscr{K}$ which satisfy sing (p) with some $V \subset \mathscr{K}$.*

REMARK. $T(\cdot, p)$ is discontinuous if and only if SING (p) is non-empty.

LEMMA 3. If $P1$ and $P2$ hold, then for each p SING(p) is an invariant set. Moreover, if K satisfies sing(p) with V, then for each $t \in R^+$ $T_t(K, p)$ satisfies sing(p) with the same V.

Proof. Let us take an arbitrary open neighbourhood $U_t \ni T_t(K, p)$. Because $T_t(\cdot, p)$ is continuous, we can find an open $U_0 \ni K$, such that $T_t(U_0, p) \subset U_t$. Because K satisfies sing(p) with V, there exists a point $\tilde{K} \in U_0$ such that $T(\tilde{K}, p) \notin V$. Hence we have

$$\tilde{K}(t) := T_t(\tilde{K}, p) \in U_t$$

and

$$T(\tilde{K}_t, p) = T(T_t(\tilde{K}, p), p) = T(\tilde{K}, p) \notin V.$$

On the other hand there is the relation

$$T(T_t(K, p), p) = T(K, p) \in V.$$

This proves that $T_t(K, p)$ satisfies sing(p) with V.

LEMMA 4. Provided that $P1$ and $P2$ hold, and if K satisfies sing(p) with V, then $T(K, p)$ satisfies sing(p) with the same V.

Proof. For each open neighbourhood $U \ni T(K, p)$ there exists a time $t \in R^+$ such that $T_t(K, p) \in U$. By Lemma 3, $T_t(K, p)$ satisfies sing(p) with V. So in the set U there are points which are attracted to equilibrium points outside V. To complete the proof let us recall that $T(\cdot, p)$ is idempotent, so that V is an open neighbourhood of $T(T(K, p), p) = T(K, p)$.

REMARK. In the last example SING(p) is not closed, which illustrates the fact that it is essential for $T_t(K, p)$ to satisfy sing(p) for all $t \in R^+$ with the same V. Now we can formulate.

THEOREM 5. If $\mathscr{K}$ is a regular space and $P1$, $P2$ hold, but $T(\cdot, p)$ is not continuous, then there exists an equilibrium point K, corresponding to p, which is Liapunov unstable.

Proof. SING(p) is non-empty, so by Lemma 4 there exists in SING(p) an equilibrium point K, corresponding to p. Let us assume that K satisfies sing(p) with V. Then, since $\mathscr{K}$ is regular, an open neighbourhood $W \ni K$, such that cl $W \subset V$, can be found. Now, for an arbitrary open neighbourhood U of K, there exists a point K_U in U such that

$T(K_U, p) \notin V$, and, consequently $T(K_U, p) \notin \mathrm{cl}\, W$. Thus there must exist a time $t_U \in R^+$ with $T_{t_U}(K_U, p) \notin \mathrm{cl}\, W$, which proves that there is the relation

$$\forall U \ni K \exists K_U \in U \exists t_U \in R^+ \, T_{t_U}(K_U, p) \notin W.$$

This is exactly the negation of the definition of Liapunov stability.

REMARK. The assertion that $T(\cdot, p)$ is discontinuous in K means that the process $t \mapsto T_t(K, p)$ in $\mathscr{K}$ is unstable in the following sense. Small perturbations of the initial value may cause large perturbations for large times. By Lemma 3 we see that we can perturbate the process $t \mapsto T_t(K, p)$ at an arbitrary time $t \in R^+$ to get the same large perturbations for times $\tau > t$. So an unstable process cannot become stable for large times and, moreover, even the limit of an unstable process is unstable. The existence of such unstable processes implies the existence of unstable equilibrium points. According to this we can consider the unstable case without introducing a concept of stability of processes. This is the most interesting conclusion from this section.

Concluding remarks

Thermodynamic conditions are well known not to ensure the stability of the evolution equation solution [1, 2]. However, GURTIN [6], by constructing a Liapunov function for the motion of a material body subjected to some special kind of loading and interface conditions, proved a generalization of the energy criterion for the stability of nonelastic bodies. He made use of a thermodynamic inequality and assumed the asymptotic stability in the large for the evolution in the parameter space. When the evolution of the parameter is a contraction in a complete metric space, this assumption together with the postulates $P1$, $P2$ and $P3$ are fulfilled. This suggests that the introduction of parameters lying in a Hilbert space, which are controlled by dissipative evolution equations [10], cannot change qualitatively the stability behaviour of the body.

If one intends to create in the sense of our postulates a continuous model of a visco-plastic material and to endow this model with the asymptotic stability, which is physically desirable, then a new concept of an asymptotic stability has to be introduced. Namely, the domains of attraction should be neighbourhoods of the equilibrium points with respect to a constitutive topology, introduced in accordance with the real stability behaviour of the material.

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