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An unsteady Faxen's relation for the force including interaction effects

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THE PAPER deals with unsteady hydrodynamic interactions between N spherical particles, immersed in an incompressible, unbounded fluid. It is assumed that the flow is governed by the unsteady Stokes equations. The integral equation approach, involving the Green function, depending explicitly on the time variable is used. The unsteady hydrodynamic drag, exerted on the j-th spherical particle in the presence of N-1 other particles, is calculated under some simplifying assumptions.

Praca niniejsza dotyczy niestacjonarnych oddziaływań hydrodynamicznych między N sztywnymi kulami, umieszczonymi w cieczy nieściśliwej. Przepływ cieczy opisują niestacjonarne równania Stokesa. Celem opisu oddziaływań hydrodynamicznych wykorzystano funkcję Greena, zależącą w sposób jawny od czasu. Obliczono, przyjmując dodatkowe założenia upraszczające, niestacjonarny opór hydrodynamiczny j-tej kuli, w obecności N-1 innych kul.

В работе рассматириваются нестационарные гидродинамические взаимодействия между N жесткими шарами, помещенными в несжимаемой жидкости. Течение жидкости описывают нестационарные уравнения Стокса. Для описания гидродинамического взаимодействия была использована функция Грина, зависящая от времени. Произведен расчет нестационарного гидродинамического сопротивления j шара, при наличии N-1 других шаров.

1. Introduction

ONE of the many problems concerning the motion of particles in viscous flows at low Reynolds numbers, is to examine hydrodynamic interactions between immersed particles. In the case of steady-state flows considered in the framework of the steady Stokes equations, the subject has received a great deal of attention over the years. It was mainly because of its importance in the theory of suspensions. However, under many conditions the unsteady effects become important. Let us mention here the hydrodynamic interaction of particles, being close to each other [1], the settling of particles under the gravity force [2], the influence of the Brownian motion of particles [3]. The first step in studying the unsteady effects is to consider them in the framework of the unsteady Stokes equations.

In this paper we confine our attention to the unsteady Stokes drag exerted on the spherical rigid particle, knowing the velocity of the fluid in the absence of the particle, and the velocity of the immersed particle. This kind of relations is called Faxen's relations. A simple example of Faxen's relation is the Boussinesq' formula derived on the basis of unsteady Stokes equations of motion. It gives the force $\mathbf{F}(t)$ exerted on a single sphere of radius a, moving at the velocity $\mathbf{V}(t)$ in an incompressible fluid, being at rest at infinity:

$$\mathbf{F}(t) = \left[-6\pi a\mu - \frac{2}{3}\pi a^3\varrho\frac{d}{dt} - 6\sqrt{\pi}a^2\sqrt{\varrho\mu}\int_0^t \frac{d\tau}{\sqrt{t-\tau}}\frac{d}{d\tau}\right]\mathbf{V}, \tag{1.1}$$

where μ, ϱ denote, respectively, the viscosity, and the density of the fluid.

In this formula the first term gives the Stokes drag, the second term — the effect of the inertia of the liquid, the last one takes into account the history of the sphere motion. This formula has been recently extended in a number of papers which followed the paper by MAZUR and BEDEAUX [4]. Mazur and Bedeaux considered the drag force exerted on the sphere, immersed in an arbitrary, unsteady Stokes flow of the fluid. Further, this approach was modified to the case of compressible fluids [5], of slip boundary conditions on the sphere surface [6], of the impact of the initial distribution of fluid velocity [7], etc.

In this paper we consider the unsteady motion of N rigid spheres in an incompressible fluid under the assuption that the flow obeys the linearized Navier–Stokes equations. The presence of spheres is described, following [4, 10], by so-called induced forces distributed on the surfaces of spheres. In the first part of the paper we discuss a set of integral equations relating the induced forces and the velocities of spheres. In the derivation of these equations no simplification is involved. The Faxen's type relation for the force exerted on the j-th sphere, in the presence of $N-1$ other spheres, is deduced in the second part of this paper. This is done for the case when the spheres are moving translationally with the velocity $\mathbf{V}_k(t)$, $k = 1, \ldots, N$, in the fluid being at rest at infinity. In this step some simplifying assumptions are accepted.

2. Governing equations

The time-dependent positions of the N spheres are specified in the fixed coordinate system $\mathbf{r}(x, y, z)$. An arbitrary point on the surface of the j-th sphere is indicated by $\mathbf{R}_j$, $j = 1, \ldots, N$, whereas the centre of the sphere is given by $\mathbf{R}_j^0$. In addition, for each sphere of radius a_j, a local coordinate system is introduced, $\mathbf{r}_j(\Omega_j) = \mathbf{R}_j(\Omega_j, t) - \mathbf{R}_j^0(t)$. In these local coordinate systems Ω_j give the angle coordinates of the vector $\mathbf{r}_j$, with some other vector chosen as the polar axis.

The velocity of the j-th sphere is given by

$$\dot{\mathbf{R}}_j(\Omega_j, t) = \dot{\mathbf{R}}_j^0(t) + \boldsymbol{\omega}_j(t) x \mathbf{r}_j, \quad |\mathbf{r}_j| = a_j, \tag{2.1}$$

and consists of the translational velocity $\dot{\mathbf{R}}_j^0(t)$ and the angular velocity $\boldsymbol{\omega}_j(t)$.

The presence of the spheres in the flow is accounted by time-dependent, point forces $\mathbf{f}_j(\Omega_j, t)$, distributed on the surfaces of the spheres. These surfaces are given by $\delta(\mathbf{r} - \mathbf{R}_j(\Omega_j, t))$. Thus the collection of all spheres acts as the source term in the equation of motion:

$$\sum_{j=1}^{N} \int d\Omega_j \delta[\mathbf{r} - \mathbf{R}_j(\Omega_j, t)]\mathbf{f}_j(\Omega_j, t). \tag{2.2}$$

The hydrodynamic interactions between the suspended spheres are considered in the framework of the linearized Navier–Stokes equations. For an incompressible fluid, being under the action of an external, time-dependent force $\mathbf{f}^{ext}(\mathbf{r}, t)$, they have the form

$$(2.3)\qquad \left(\varrho\frac{\partial}{\partial t}-\mu\nabla^2\right)\mathbf{v}(\mathbf{r}, t)+\nabla p(\mathbf{r}, t) = \mathbf{f}^{ext}(\mathbf{r}, t)+\sum_{j=1}^{N}\int d\Omega_j\,\delta[\mathbf{r}-\mathbf{R}_j(\Omega_j, t)]\mathbf{f}_j(\Omega_j, t), \quad \nabla\cdot\mathbf{v}(\mathbf{r}, t) = 0,$$

where $\mathbf{v}(\mathbf{r}, t), p(\mathbf{r}, t)$ denote the velocity and the pressure fields.

The initial condition reads

$$(2.4)\qquad \mathbf{v}(\mathbf{r}, 0) = \mathbf{0}.$$

The stress tensor in the fluid is denoted by $P(\mathbf{r}, t)$:

$$(2.5)\qquad P_{kl}(\mathbf{r}, t) = \delta_{kl}p(\mathbf{r}, t)-\mu\left(v_{k,l}(\mathbf{r}, t)+v_{l,k}(\mathbf{r}, t)\right).$$

Following [4, 10], the equations of motion (2.3) and (2.4) are supposed to be applied in the whole space. Thus it is necessary to specify the divergence of the stress tensor also inside the volumes occupied by the suspended spheres. It can be checked that Eqs. (2.3) and (2.4) are satisfied if this divergence is given by the following relations:

$$(2.6)\qquad \begin{aligned} \nabla\cdot\mathbf{P}(r_j, t) &= -\varrho\frac{\partial}{\partial t}[\dot{\mathbf{R}}_j^0(t)+\boldsymbol{\omega}_j(t)x\mathbf{r}_j], \quad |\mathbf{r}_j| < a_j,\\ \nabla\cdot\mathbf{P}(\mathbf{r}_j, t) &= -\varrho\frac{\partial}{\partial t}[\dot{\mathbf{R}}_j^0(t)+\boldsymbol{\omega}_j(t)x\mathbf{r}_j]+\mathbf{f}_j(\Omega_j, t), \quad |\mathbf{r}_j| = a_j.\end{aligned}$$

In the volumes of the spheres, the stress tensor is expressed in terms of the time-derivative of the velocities of the spheres. However, there appear additional terms — the induced forces $\mathbf{f}_j(\Omega_j, t)$ — on the surfaces of the spheres. In virtue of the above relations, the force $\mathbf{F}_j(t)$ exerted on the j-th sphere by the fluid assumes the form

$$(2.7)\qquad \mathbf{F}_j(t) = -\int\nabla\cdot\mathbf{P}(\mathbf{r}_j, t)d\mathbf{r}_j = \frac{4}{3}\pi a^3\varrho\frac{\partial}{\partial t}\dot{\mathbf{R}}_j^0(t)-\int\mathbf{f}_j(\Omega_j, t)d\mathbf{r}_j.$$

To obtain the relation of the force $\mathbf{f}_j(\Omega_j, t)$ to the velocities of the spheres, we write the velocity field as the function of the forces exerted on the fluid, using the integral equation approach. Hence the velocity field can be expressed in terms of the convolution integrals:

$$(2.8)\qquad \mathbf{v}(\mathbf{r}, t) = \mathbf{v}_0(\mathbf{r}, t)+\int_0^t dt'\int_{E_3}d\mathbf{r}'\mathbf{G}(\mathbf{r}-\mathbf{r}'; t-t')\sum_{j=1}^{N}\int d\Omega_j\,\delta[\mathbf{r}'-\mathbf{R}_j(\Omega_j', t')]\mathbf{f}_j(\Omega_j', t'),$$

$$(2.9)\qquad \mathbf{v}_0(\mathbf{r}, t) = \int_0^t dt'\int_{E_3}d\mathbf{r}'\mathbf{G}(\mathbf{r}-\mathbf{r}'; t-t')\mathbf{f}^{ext}(\mathbf{r}', t').$$

The integration with respect to $\mathbf{r}'$ is done over the whole space E_3. Here $\mathbf{v}_0(\mathbf{r}, t)$ denotes the unperturbed fluid velocity, i.e. the velocity of the fluid in the absence of the spheres. The dynamical Oseen tensor $\mathbf{G}(\mathbf{r}, t)$ is defined by

$$(2.10)\qquad \mathbf{G}(\mathbf{r}, t) = \int\frac{d\omega}{2\pi}\int\frac{d\mathbf{k}}{(2\pi)^3}\frac{\exp[i\mathbf{k}\cdot\mathbf{r}+i\omega t]}{i\omega\varrho+\mu|\mathbf{k}|^2}\left[\mathbf{1}-\frac{\mathbf{kk}}{|\mathbf{k}|^2}\right],$$

where the space-time Fourier representation is used.

The no-slip boundary conditions are assumed on the surfaces of the spheres:

$$\dot{\mathbf{R}}_j(\Omega_j, t) = \mathbf{v}(\mathbf{R}_j(\Omega_j, t), t) = \int\limits_{E_3} d\mathbf{r}\, \delta[\mathbf{r}-\mathbf{R}_j(\Omega_j, t)]\mathbf{v}(\mathbf{r}, t). \tag{2.11}$$

They give the coupling between the flow of the fluid and the velocities of the spheres. Using these boundary conditions as well as Eqs. (2.8) and (2.9) for the fluid velocity, one obtains N coupled integral equations for the unknown $\mathbf{f}_j(\Omega_j, t)$:

$$\begin{aligned}\dot{\mathbf{R}}_j(\Omega_j, t) &= \mathbf{v}_0\big(\mathbf{R}_j(\Omega_j, t), t\big) \\ &+ \int\limits_0^t dt' \int d\Omega_j' \mathbf{G}[\mathbf{R}_j(\Omega_j, t)-\mathbf{R}_j(\Omega_j', t'); t-t']\mathbf{f}_j(\Omega_j', t') \\ &+ \sum_{k\neq j}^{N} \int\limits_0^t dt' \int d\Omega_k' \mathbf{G}[\mathbf{R}_j(\Omega_j, t)-\mathbf{R}_k(\Omega_k', t'); t-t']\mathbf{f}_k(\Omega_k', t').\end{aligned} \tag{2.12}$$

In Eq. (2.12) the induced forces $\mathbf{f}_j(\Omega_j, t)$ depend on the relative instantaneous velocities of the spheres with respect to the fluid, $\mathbf{V}_j(\Omega_j, t)$:

$$\mathbf{V}_j(\Omega_j, t) \equiv \dot{\mathbf{R}}_j(\Omega_j, t)-\mathbf{v}_0\big(\mathbf{R}_j(\Omega_j, t), t\big). \tag{2.13}$$

The first integral on the r.h.s. contains the tensor $\mathbf{G}(\mathbf{R}_j(\Omega_j, t)-\mathbf{R}_j(\Omega_j', t'); t-t')$, which acts on the force induced on the j-th sphere in the absence of other spheres in the fluid. The last term involving the summation over the spheres $k \neq j$ is due to the hydrodynamic interactions between the suspended spheres. These interactions depend on the trajectories of the spheres, as seen from the arguments $\mathbf{R}_j(\Omega_j, t)-\mathbf{R}_k(\Omega_k', t')$ of the tensors considered. This dependence can be expressed in terms of the initial distances between the spheres ($\mathbf{R}_{jk} = \mathbf{R}_j^0(t)|_{t=0}-\mathbf{R}_k^0(t)|_{t=0}$), and the velocities of the spheres in the time interval considered. It should be noted that the integrals over the time variable depend on the time interval considered, and that the integration over space is confined to the integration over the surfaces of the spheres.

3. Hydrodynamic interaction tensors

To recast the basic set of equations (2.12), we expand $\mathbf{V}_j(\Omega_j, t)$ and $\mathbf{f}_j(\Omega_j, t)$ in terms of the normalized spherical harmonics Y_l^m [8]. The expansion is chosen in view of the fact that the space integration in Eq. (2.12) is to be done over the surfaces of the spheres

$$\mathbf{V}_j(\Omega_j, t) = \sqrt{4\pi} \sum_{l,m} \mathbf{V}_{j,lm}(t)\, Y_l^m(\theta_j, \phi_j), \qquad |\mathbf{r}_j| = a_j, \tag{3.1}$$

$$\mathbf{f}_j(\Omega_j, t) = \frac{1}{\sqrt{4\pi}a_j^2} \sum_{l,m} \mathbf{f}_{j,lm}(t) Y_l^m(\theta_j, \phi_j), \quad |\mathbf{r}_j| = a_j, \tag{3.2}$$

where $0 \leqslant l < \infty$, $|m| \leqslant l$,

$$\mathbf{r}_j = (|\mathbf{r}_j|, \Omega_j) = (|\mathbf{r}_j|, \theta_j, \phi_j).$$

Further, the dynamical Oseen tensor is presented in terms of the space-Fourier representation:

$$\mathbf{G}(\mathbf{R}, t) = \frac{1}{\varrho}\int \frac{d\mathbf{k}}{(2\pi)^3}\exp\{i\mathbf{k}\cdot\mathbf{R}-\nu|\mathbf{k}|^2t\}\left[\mathbf{1}-\frac{\mathbf{k}\mathbf{k}}{|\mathbf{k}|^2}\right], \quad \nu = \mu/\varrho, \tag{3.3}$$

where the time variable appears explicitly.

The last auxiliary step consists in the expansion

$$\exp(\pm i\mathbf{k}\cdot\mathbf{R}) = 4\pi\sum_{l,m}(\pm i)^l j_l(Rk)Y_l^m(\theta,\phi)Y_l^{-m}(\chi,\xi), \tag{3.4}$$

where, as previously, the spherical polar coordinates $\mathbf{R} = (R, \theta, \phi)$, and $\mathbf{k} = (k, \chi, \xi)$ are used; $j_l(Rk)$ denotes the spherical Bessel function of the first kind.

This procedure allows to perform the integrations over the surfaces of the spheres, and leads to the following set of equations for the expansion coefficients $\mathbf{f}_{j,lm}(t)$, entering Eq. (3.2):

$$\mathbf{V}_{j,l_1m_1}(t) = \int\limits_0^t dt' \sum_{l_2,m_2}\mathbf{T}_{l_1m_1}^{l_2m_2}(\mathbf{R}_j^0(t)-\mathbf{R}_j^0(t'); t-t')\mathbf{f}_{j,l_2m_2}(t') \tag{3.5}$$
$$+\sum_{k\neq j}\int\limits_0^t dt' \sum_{l_2,m_2}\mathbf{T}_{l_1m_1}^{l_2m_2}(\mathbf{R}_j^0(t)-\mathbf{R}_k^0(t'); t-t')\mathbf{f}_{k,l_2m_2}(t'); \quad j = 1, \ldots, N.$$

The tensors $\mathbf{T}_{l_1m_1}^{l_2m_2}$ acting on the expansion coefficients of the induced forces are called hydrodynamic interaction tensors. They were introduced by YOSHIZAKI and YAMAKAWA [9], for the steady-state flows. The idea of Yoshizaki and Yamakawa of using the hydrodynamic interaction tensors is extended here to take into account the unsteady effects. The tensor $\mathbf{T}_{l_1m_1}^{l_2m_2}(\mathbf{R}_j^0(t)-\mathbf{R}_k^0(t'); t-t')$ gives the (l_1, m_1) component of the disturbed flow, at time t, generated on the surface of the j-th sphere by the (l_2, m_2) component of the force $\mathbf{f}_k$ $(k \neq j)$, distributed on the k-th sphere, in the time interval considered. Thus, the memory effects intervening the interactions between the spheres can, by means of Eq. (3.5), be analyzed. The form and properties of the tensors $\mathbf{T}_{l_1m_1}^{l_2m_2}$ are discussed in the Appendix.

4. The force $\mathbf{F}_j(t)$ exerted on the j-th sphere

To illustrate how the hydrodynamic interaction tensors work, let us consider the translation of N spheres in a fluid being at rest at infinity. Let the instantaneous relative velocities of the spheres with respect to the fluid be given by

$$\mathbf{V}_{j,lm}(t) = \left\{\begin{matrix}\mathbf{V}_j(t), & l = 0 \\ 0, & l \geqslant 1\end{matrix}, \quad j = 1, \ldots, N\right\}. \tag{4.1}$$

Due to the symmetry of the spheres

$$\int \mathbf{f}_j(\mathbf{r}_j, t)d\mathbf{r}_j = \mathbf{f}_{j,00}(t). \tag{4.2}$$

Hence, to calculate the force exerted on the j-th sphere, only the expansion coefficient having indexes equal to zero, $\mathbf{f}_{j,00}(t)$, is needed. This coefficient will be obtained under the additional assumptions:

(i) The hydrodynamic interactions are regarded within the linear approximation with respect to the sphere velocities. Hence the tensors $\mathbf{T}^{l_2 m_2}_{l_1 m_1}$ are required in the lowest order approximation with respect to $\mathbf{V}_j(t)$. Under such conditions, the integral equations entering Eq. (3.5) become the convolution integrals with respect to time. The Laplace transform of Eq. (3.5) gives

$$(4.3) \qquad \mathbf{V}_{j,l_1 m_1}(p) = \sum_{l_2, m_2} \mathbf{T}^{l_2 m_2}_{j.l_1 m_1}(p)\mathbf{f}_{j,l_2 m_2}(p) + \sum_{k \neq j} \sum_{l_2, m_2} \mathbf{T}^{l_2 m_2}_{l_1 m_1}(\mathbf{R}_{jk}, p)\mathbf{f}_{k,l_2 m_2}(p),$$

where p denotes the variable conjugate to t,

$$R_{jk} = |\mathbf{R}^0_j(t=0) - \mathbf{R}^0_k(t=0)|, \qquad \mathbf{T}^{l_2 m_2}_{j.l_1 m_1}(p) = \mathbf{T}^{l_2 m_2}_{l_1 m_1}(\mathbf{R}_{jk} = 0, p).$$

The solution for $\mathbf{f}_{j,00}(p)$ found using the iteration procedure reads

$$(4.4) \qquad \mathbf{f}_{j,00}(p) = \tilde{\mathbf{T}}^{00}_{j.00}(p)\mathbf{V}_{j,00}(p) - \tilde{\mathbf{T}}^{00}_{j.00}(p) \sum_{k \neq j}^{N} \mathbf{T}^{00}_{00}(\mathbf{R}_{jk}, p)\tilde{\mathbf{T}}^{00}_{k.00}(p)\mathbf{V}_{k,00}(p)$$

$$+ \tilde{\mathbf{T}}^{00}_{j.00}(p) \sum_{k \neq j}^{N} \sum_{l \neq k}^{N} \sum_{l_1, m_1, m_2} \mathbf{T}^{l_1 m_1}_{00}(\mathbf{R}_{jk}, p)\tilde{\mathbf{T}}^{l_1 m_2}_{k.l_1 m_1}(p)\mathbf{T}^{00}_{l_1 m_2}(\mathbf{R}_{kl}, p)\tilde{\mathbf{T}}^{00}_{l.00}(p)\mathbf{V}_{l,00}(p) + \ldots,$$

$$\tilde{\mathbf{T}} \cdot \mathbf{T} = \mathbf{1}.$$

The first term gives the forces due to the interaction of a single, j-th sphere with the fluid, whereas the second term describes the direct hydrodynamic interaction between the j-th and k-th sphere. These two terms contain only the interaction tensors with indexes equal to zero. The last term and all higher order terms take into account the indirect interactions among the spheres, and the appropriate interaction tensors have the indexes different from zero as well. However, due to the properties of the tensors considered (see Appendix), dropping the terms with $l_j \neq 0$ implies that the force is calculated up to terms of order $0\left(\frac{A}{R}\right)^3$. Here A denotes the maximum value of a_j, and R — the minimum value od the distances R_{jk}. As the time of the start of the motion $t = 0$ is chosen arbitrarily, the condition $A < |\mathbf{R}_j(t) - \mathbf{R}_k(t)|$ is assumed to be fulfilled in the whole time interval considered. This fact leads to the second assumption:

(ii) $\mathbf{f}_{j,00}(t)$ correct to terms of $0\left(\frac{A}{R}\right)^3$ will be considered. Within this approximation, only the tensors $\tilde{\mathbf{T}}_j(p) \equiv \tilde{\mathbf{T}}^{00}_{j.00}(p)$, and $\mathbf{T}(\mathbf{R}_{jk}, p) \equiv \mathbf{T}^{00}_{00}(\mathbf{R}_{jk}, p)$ are to be calculated. It follows from the estimation (see Appendix)

$$(4.5) \qquad AF_{l_1 l_2, t}(R, t-t') \leqslant \left(\frac{A}{R}\right)^{l_1 + l_2 + 1} \mathscr{H}\left(\frac{R}{\sqrt{\nu(t-t')}}, t-t'\right), \qquad t > t'.$$

(iii) Expanding the interaction tensors with respect to p allows to compare the drag force obtained here with the Boussinesq' formula (1.1).

Under these conditions, the interaction tensor $\tilde{\mathbf{T}}_j(p)$ which gives the translational friction coefficient of the sphere of radius a_j reads

$$(4.6) \qquad \tilde{\mathbf{T}}_j(p) = 6\pi\mu a_j \left(1 + \frac{a_j}{\sqrt{\nu}}\sqrt{p} + \frac{1}{3}\frac{a_j^2}{\nu}p\right)\mathbf{1}, \quad \text{for} \quad \frac{a_j}{\sqrt{\nu}}\sqrt{p} < 1.$$

As it should be, (compare Eq. (2.7)), Eq. (4.6) differs from Eq. (1.1) only by the term due to the inertia of the liquid displaced by the rigid sphere.

From the above formula the physical meaning of "small" p is clear; the condition is: $p < \nu/a_j^2$.

The tensor $\mathbf{T}(\mathbf{R}_{jk}, p)$, describing the direct interaction between two spheres, reads

$$(4.7)\qquad \mathbf{T}(\mathbf{R}_{jk}, p) = \frac{1}{8\pi\mu R_{jk}}\left[\mathbf{1}+\mathbf{e}_{jk}\mathbf{e}_{jk}+\frac{a_j^2+a_k^2}{R_{jk}^2}\left(\mathbf{1}-\frac{1}{3}\mathbf{e}_{jk}\mathbf{e}_{jk}\right)\right]-\frac{\mathbf{1}}{6\pi\mu R_{jk}}\left(\frac{R_{jk}}{\sqrt{\nu}}\sqrt{p}\right)$$
$$+\frac{3}{32\pi\mu R_{jk}}\left[\mathbf{1}-\frac{1}{3}\mathbf{e}_{jk}\mathbf{e}_{jk}+\frac{2}{9}\frac{a_j^2+a_k^2}{R_{jk}^2}(\mathbf{1}+\mathbf{e}_{jk}\mathbf{e}_{jk})\right]\left(\frac{R_{jk}^2}{\nu}p\right);$$
$$\mathbf{e}_{jk}\mathbf{e}_{jk} = \frac{\mathbf{R}_{jk}\mathbf{R}_{jk}}{|\mathbf{R}_{jk}|^2},\quad \text{for}\quad \frac{R_{jk}}{\sqrt{\nu}}\sqrt{p} < 1.$$

This form resembles that of the dynamical Oseen tensor for the point forces:

$$(4.8)\qquad \mathbf{G}(\mathbf{R}_{jk}, p) = \frac{1}{8\pi\mu R_{jk}}[\mathbf{1}+\mathbf{e}_{jk}\mathbf{e}_{jk}]-\frac{1}{6\pi\mu R_{jk}}\left(\frac{R_{jk}}{\sqrt{\nu}}\sqrt{p}\right)\mathbf{1}$$
$$+\frac{3}{32\pi\mu R_{jk}}\left[\mathbf{1}-\frac{1}{3}\mathbf{e}_{jk}\mathbf{e}_{jk}\right]\left(\frac{R_{jk}^2}{\nu}p\right).$$

The differences are due to the impact of the finite radii of the spheres. The first terms in the expressions (4.7), and (4.8) coincide with the steady-state Oseen tensors.

Using the interaction tensors, the drag on the j-th sphere, in the presence of $N-1$ other spheres, can be presented as

$$(4.9)\qquad \mathbf{F}_j(p) = \frac{4}{3}\pi a_j^3 \varrho p \mathbf{V}_j(p)-\tilde{\mathbf{T}}_j(p)\mathbf{V}_j(p)+\tilde{\mathbf{T}}_j(p)\sum_{k\neq j}\mathbf{T}(\mathbf{R}_{jk}, p)\tilde{\mathbf{T}}_k(p)\mathbf{V}_k(p)$$
$$-\tilde{\mathbf{T}}_j(p)\sum_{k\neq j}^{N}\sum_{l\neq k}^{N}\mathbf{T}(\mathbf{R}_{jk}, p)\tilde{\mathbf{T}}_k(p)\mathbf{T}(\mathbf{R}_{kl}, p)\tilde{\mathbf{T}}_l(p)\mathbf{V}_l(p)+\ \ldots$$

The structure of this expression is the same as that for the steady-state interactions [9]. Qualitatively speaking, this similarity is due to the simplifying assumptions. Unsteady effects are incorporated into the hydrodynamic interaction tensors. The first two terms give the hydrodynamic drag in the case $N = 1$.

5. Conclusions

The unsteady hydrodynamic interactions between N spheres suspended in an incompressible fluid are considered on the basis of the linearized Navier–Stokes equations. The integral equation approach, involving the Green function (the dynamical Oseen tensor) depending explicitly on the time variable, is used. The complexity of the time-dependent interactions results from the memory effects, due to the dependence of the hydrodynamic interaction tensors on the trajectories of the spheres. However, when the spheres suspended in the flow move slowly enough to neglect the dependence of the interaction tensors on the velocities of these spheres, the description of the interactions becomes substantially simpler.

In the particular case of translationally moving N spheres, and with the use of some additional assumptions, the hydrodynamic drag exerted on the j-th sphere is calculated. This particular result can be treated as an extension of the Boussinesq' formula. The form of this drag force resembles that for the steady-state flows, whereas the unsteady effects have the impact on the hydrodynamic interaction tensors.

Appendix. The properties of hydrodynamic interaction tensors

A. General formulae for the hydrodynamic interaction tensors

Substituting the expansions of $\mathbf{V}_j$, $\mathbf{f}_j$, $\exp(i\mathbf{k}\mathbf{r}_j)$, listed in Eqs. (3.1), (3.2) and (3.4), to the integral equations (2.12), and taking into account the orthogonality of Y_l^m harmonics, one obtains the hydrodynamic interaction tensors in the form

$$\text{(A.1)}\qquad \mathbf{T}_{l_1 m_1}^{l_2 m_2}(\mathbf{R}_j^0(t)-\mathbf{R}_k^0(t'), t-t') = \frac{1}{2\pi^2\varrho} i^{l_1-l_2}\int\frac{d\mathbf{k}}{(2\pi)^3}\exp\{i\mathbf{k}[\mathbf{R}_j^0(t)-\mathbf{R}_k^0(t')]\}$$
$$\cdot\exp[-\nu|\mathbf{k}|^2(t-t')]\left(\mathbf{1}-\frac{\mathbf{k}\mathbf{k}}{|\mathbf{k}|^2}\right)j_{l_1}(a_j k)j_{l_2}(a_k k)\,Y_{l_1}^{-m_1}(\chi,\xi)\,Y_{l_2}^{m_2}(\chi,\xi),\qquad t>t'.$$

The time dependence of the tensor $\mathbf{T}_{l_1 m_1}^{l_2 m_2}$ can be more conveniently presented as

$$\text{(A.2)}\qquad \mathbf{T}_{l_1 m_1}^{l_2 m_2}(\mathbf{R}_j^0(t)-\mathbf{R}_k^0(t'), t-t') = \frac{1}{2\pi^2\varrho} i^{l_1-l_2}\int d\mathbf{x}\,d\mathbf{x}'\delta(\mathbf{x}-\mathbf{R}_j^0(t))\,\delta(\mathbf{x}'-\mathbf{R}_k^0(t'))$$
$$\cdot\int\frac{d\mathbf{k}}{(2\pi)^3}\exp\{i\mathbf{k}(\mathbf{x}-\mathbf{x}')-\nu|\mathbf{k}|^2(t-t')\}\left(\mathbf{1}-\frac{\mathbf{k}\mathbf{k}}{|\mathbf{k}|^2}\right)j_{l_1}(a_j k)j_{l_2}(a_k k)$$
$$\cdot\,Y_{l_1}^{-m_1}(\chi,\xi)\,Y_{l_2}^{m_2}(\chi,\xi),\qquad t>t'.$$

Remembering that (see Eq. (3.4))

$$e^{i\mathbf{k}(\mathbf{x}-\mathbf{x}')}\equiv e^{i\mathbf{k}\mathbf{D}} = 4\pi\sum_{l,m} i^l j_l(|\mathbf{D}|k)\,Y_l^{-m}(\chi,\xi)\,Y_l^m(\varkappa,\gamma),$$

where in spherical polar coordinates $\mathbf{D}=(D,\varkappa,\gamma)$, $\mathbf{k}=(k,\chi,\xi)$, the interaction tensors become

$$\text{(A.3)}\qquad \mathbf{T}_{l_1 m_1}^{l_2 m_2}(\mathbf{R}_j^0(t)-\mathbf{R}_k^0(t'), t-t') = \int d\mathbf{x}d\mathbf{x}'\delta(\mathbf{x}-\mathbf{R}_j^0(t))\,\delta(\mathbf{x}'-\mathbf{R}_k^0(t'))$$
$$\cdot\sum_{l,m}F_{l_1 l_2, l}(|\mathbf{D}|, t-t')\,\mathbf{K}_{l_1 m_1, lm}^{l_2 m_2}Y_l^m(\varkappa,\gamma),\qquad t>t'.$$

In the above formula the integration over $\mathbf{k}$ space is written down in the spherical polar coordinates (k,χ,ξ), and the integrals which should be calculated are

$$\text{(A.4)}\qquad F_{l_1 l_2, l}(|\mathbf{D}|, t-t') = \frac{2}{\pi\varrho}\int_0^\infty j_{l_1}(a_j k)j_{l_2}(a_k k)j_l(Dk)e^{-\nu k^2(t-t')}k^2dk,\qquad t>t',$$

and

$$\text{(A.5)} \qquad \mathbf{K}^{l_2 m_2}_{l_1 m_1, lm} = i^{l_1-l_2+l} \int \sin\chi d\chi d\xi \left(\mathbf{1} - \frac{\mathbf{kk}}{|\mathbf{k}|^2}\right) Y^{-m_1}_{l_1}(\chi, \xi) Y^{m_2}_{l_2}(\chi, \xi) Y^{-m}_{l}(\chi, \xi).$$

The integral (A.4) in the limit of the steady-state flows was calculated by YOSHIZAKI and YAMAKAWA [9]. It should be noted that the time variable does not appear in the tensor $\mathbf{K}^{l_2 m_2}_{l_1 m_1, lm}$.

Hence the explicit form of this tensor found in [9] can be applied under unsteady conditions as well.

B. The dependence of the tensor $\mathbf{T}^{l_2 m_2}_{l_1 m_1}(\mathbf{R}^0_j(t) - \mathbf{R}^0_j(t'), t-t')$ on the j-th sphere velocity $\dot{\mathbf{R}}^0_j(t)$

In general, the interaction tensors exhibit a nonlinear dependence on the velocities of the spheres. To discuss this effect, let us write down the tensor $\mathbf{T}^{l_2 m_2}_{l_1 m_1}(\mathbf{R}^0_j(t) - \mathbf{R}^0_j(t'), t-t')$ in the form similar to (A.1):

$$\text{(B.1)} \qquad \mathbf{T}^{l_2 m_2}_{l_1 m_1}(\mathbf{R}^0_j(t) - \mathbf{R}^0_j(t'), t-t') = \frac{1}{2\pi^2 \varrho} i^{l_1-l_2} \int \frac{d\mathbf{k}}{(2\pi)^3} \exp\{i\mathbf{k}[\mathbf{R}^0_j(t) - \mathbf{R}^0_j(t')]\}$$
$$\cdot \exp[-\nu k^2(t-t')] \left(\mathbf{1} - \frac{\mathbf{kk}}{|\mathbf{k}|^2}\right) j_{l_1}(a_j k) j_{l_2}(a_j k) Y^{-m_1}_{l_1}(\chi, \xi) Y^{m_2}_{l_2}(\chi, \xi).$$

The displacement of the centre of the j-th sphere during the time interval $t-t'$ can be approximated by

$$\text{(B.2)} \qquad \mathbf{R}^0_j(t) - \mathbf{R}^0_j(t') = \int_{t'}^{t} \dot{\mathbf{R}}^0_j(\tau) d\tau \approx \dot{\mathbf{R}}^0_j(\tau^*)(t-t'), \quad t' < \tau^* < t.$$

For small displacements the $\mathbf{T}^{l_2 m_2}_{l_1 m_1}$ becomes

$$\text{(B.3)} \qquad \mathbf{T}^{l_2 m_2}_{l_1 m_1}(\mathbf{R}^0_j(t) - \mathbf{R}^0_j(t'), t-t') \approx \frac{1}{2\pi^2 \varrho} i^{l_1-l_2} \int \frac{d\mathbf{k}}{(2\pi)^3} \{1 + i\mathbf{k}\dot{\mathbf{R}}^0_j(\tau^*)(t-t') + \ldots\}$$
$$\cdot \exp[-\nu k^2(t-t')] \left(\mathbf{1} - \frac{\mathbf{kk}}{|\mathbf{k}|^2}\right) j_{l_1}(a_j k) j_{l_2}(a_j k) Y^{-m_1}_{l_1}(\chi, \xi) Y^{m_2}_{l_2}(\chi, \xi).$$

Hence, in the lowest order approximation with respect to $\dot{\mathbf{R}}^0_j(t)$, the tensors $\mathbf{T}^{l_2 m_2}_{l_1 m_1}$ do not depend on the velocities of the spheres. A similar line of reasoning holds for the tensors $\mathbf{T}^{l_2 m_2}_{l_1 m_1}(\mathbf{R}^0_j(t) - \mathbf{R}^0_k(t'), t-t')$. The lowest order approximation will be used to calculate the drag on the j-th sphere in the presence of $N-1$ other spheres (compare the specification (i)).

C. The estimation of the $\mathbf{T}^{l_2 m_2}_{l_1 m_1}$ tensors with respect to $\left(\frac{A}{R}\right)$

The absolute values of distances between two spheres in the flow enter the $F_{l_1 l_2, l}$ $(D, t-t')$ functions. However, under the specification (i), the $F_{l_1 l_2, l}$ functions depend only on the distances between the spheres at time $t = 0$. This property enables to estimate the tensors $\mathbf{T}^{l_2 m_2}_{l_1 m_1}$ with respect to A/R, where A denotes the maximum value of

a_j, $j = 1, \ldots, N$, and R is the minimum value of R_{jk}, $j = 1, \ldots, N$, $k \neq j$. It follows from the formula (A.4) that the $F_{l_1 l_2, l}$ can be presented in the form

$$\text{(C.1)} \qquad AF_{l_1 l_2, l} = \frac{\sqrt{\pi}}{4\varrho} \frac{\Gamma\left(\frac{3}{2} + \frac{1}{2}(l_1 + l_2 + l)\right)}{\Gamma\left(l_1 + \frac{3}{2}\right)\Gamma\left(l_2 + \frac{3}{2}\right)\Gamma\left(l + \frac{2}{3}\right)} \left(\frac{A}{R}\right)^{l_1+l_2+1} \left(\frac{R}{2\sqrt{\nu(t-t')}}\right)^{l_1+l_2+l+1}$$

$$\cdot \frac{1}{\nu(t-t')} \Psi_2\left(\frac{3}{2} + \frac{1}{2}(l_1+l_2+l);\quad l_1 + \frac{3}{2},\ l_2 + \frac{3}{2},\ l + \frac{3}{2};\ \frac{A^2}{4\nu(t-t')},\right.$$

$$\left.\frac{A^2}{4\nu(t-t')},\ \frac{R^2}{4\nu(t-t')}\right), \qquad t > t',$$

where Γ is the Gamma function and Ψ_2 is the confluent hypergeometric series of three variables. Further, the above formula allows to write

$$\text{(C.2)} \qquad AF_{l_1 l_2, l} \leqslant \left(\frac{A}{R}\right)^{l_1+l_2+1} \mathscr{H}\left(\frac{R}{\sqrt{\nu(t-t')}},\quad t-t'\right), \qquad t > t',$$

i.e. the dependence on A/R can be extracted, H being the function of other variables.

D. The interaction tensors $\tilde{\mathbf{T}}^{00}_{j.00}(p)$, and $\mathbf{T}^{00}_{00}(\mathbf{R}_{jk}, p)$

The interaction tensors $\tilde{\mathbf{T}}^{00}_{j.00}(p)$, and $\mathbf{T}^{00}_{00}(\mathbf{R}_{jk}, p)$ calculated under the specification (i) read

$$\text{(D.1)} \qquad \tilde{\mathbf{T}}_j(p) \equiv \tilde{\mathbf{T}}^{00}_{j.00}(p) = 12\pi\varrho a_j^2 \sqrt{\nu p}\, e^{\frac{2a_j}{\sqrt{\nu}}\sqrt{p}} \cdot \left(e^{\frac{2a_j}{\sqrt{\nu}}\sqrt{p}} - 1\right)^{-1} \mathbf{1},$$

$$\text{(D.2)} \qquad \mathbf{T}(\mathbf{R}_{jk}, p) \equiv \mathbf{T}^{00}_{00}(\mathbf{R}_{jk}, p) = \frac{1}{16\pi\varrho a_j a_k R_{jk}^3} \frac{1}{p} \left\{\left[\beta^2 - \alpha^2\right.\right.$$

$$\left. + \exp\left[-\frac{R_{jk}}{\sqrt{\nu}}\sqrt{p}\right]\left(\mathrm{ch}\frac{\alpha}{\sqrt{\nu}}\sqrt{p} - \mathrm{ch}\frac{\beta}{\sqrt{\nu}}\sqrt{p}\right)\left(R_{jk}^2 + \sqrt{\frac{\nu}{p}}R_{jk} + \frac{\nu}{p}\right)\right]\mathbf{1}$$

$$\left. -3\mathbf{e}_{jk}\mathbf{e}_{jk}\left[\beta^2 - \alpha^2 + 2\exp\left[-\frac{R_{jk}}{\sqrt{\nu}}\sqrt{p}\right]\left(\mathrm{ch}\frac{\alpha}{\sqrt{\nu}}\sqrt{p} - \mathrm{ch}\frac{\beta}{\sqrt{\nu}}\sqrt{p}\right)\left(\frac{1}{3}R_{jk}^2 + \sqrt{\frac{\nu}{p}}R_{jk} + \frac{\nu}{p}\right)\right]\right\},$$

where

$$\alpha = a_j + a_k, \qquad \beta = a_j - a_k.$$

The formula (D.1) for arbitrary p differs from that obtained by MAZUR and BEDEAUX [4]. This difference is due to the approximation which is used here to calculate the tensor $\tilde{\mathbf{T}}_j(p)$. However, the expression (4.6), obtained here for $\frac{a_j}{\sqrt{\nu}}\sqrt{p} < 1$, coincides with Mazur's and Bedeaux's results.

References

1. J. HAPPEL, H. BRENNER, *Low Reynolds number hydrodynamics*, § 6.2, Noordhoff International Publishing, 1973.
2. S. LEICHTBERG, S. WEINBAUM, R. PFEFFER, M. J. GLUCKMAN, Phil. Trans. Roy. Soc. London, **282A**, 585, 1976.
3. T. S. CHOW, J. J. HERMANS, Kolloid-Z. u. Z. Polymere, **250**, 404, 1972.
4. P. MAZUR, D. BEDEAUX, Physica, **76**, 235, 1974.
5. D. BEDEAUX, P. MAZUR, Physica, **78**, 505, 1974; R. B. JONES, Physica, **95A**, 104, 1979.
6. A. M. ALBANO, D. BEDEAUX, P. MAZUR, Physica, **80A**, 89, 1975; B. U. FELDERHOF, Physica, **84A**, 569, 1976.
7. I. PIEŃKOWSKA, Arch. Mech., **33**, 469, 1981.
8. H. A. BETHE, E. E. SALPETER, *Quantum mechanics of one-and-two electron systems*, Handbuch der Physik, vol. XXXV, ed. S. FLÜGGE, Springer—Verlag, 1957.
9. T. YOSHIZAKI, H. YAMAKAWA, J. Chem. Phys., **73**, 578, 1980.
10. K. F. FREED, M. MUTHUKUMAR, J. Chem. Phys., **69**, 2657, 1978

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