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Spatial amplification and Squire's theorem

T. CEBECI (LONG BEACH) and K. STEWARTSON (LONDON)

SPATIAL and temporal model disturbances to a two-dimensional shear flow are considered. The „zarf" or absolute neutral curve for a Blasius flow is obtaied and the lower branch is shown to differ from the two-dimensional neutral curve. For a fixed frequency the most amplified spatial oscillation, as the Reynolds number varies, is not two-dimensional but very nearly so.

Rozważono przestrzenne i czasowe zaburzenia modalne dwuwymiarowego przepływu ścinania. Otrzymano absolutną krzywą obojętną dla przepływu Blasiusa i wykazano, że jej dolna gałąź różni się od dwuwymiarowej krzywej obojętnej. Przy ustalonej częstości i zmiennej liczbie Reynoldsa najsilniej wzmocnione drgania przestrzenne są prawie dwuwymiarowe.

Рассматриваются пространственные и временные модальные возмущения двумерного течения среза. Получена абсолютная кривая нейтральная для течения Блезиза и показано, что его нижняя ветвь отличается от двумерной нейтральной кривой. При установленной частоте и переменном числе Рейнольдса, наиболее мощное усиление пространственных колебаний почти двумерные.

1. Introduction

THE LINEAR theory of the stability of a plane parallel flow is important both for the determination of the highest Reynolds number, Re, at which all sufficiently weak disturbances decay and also as a predictor, or at least a correlator, of the onset of transition to turbulent flow. In its second role it clearly has shortcomings especially in its neglect of nonlinear effects and of external disturbances but it is the best available procedure and may be regarded as an early groping towards a proper theory of transition.

Consider a pseudo-plane parallel flow, in a semi-infinite fluid bounded on one side by the plane $y = 0$, with the velocity components $[u(y), 0, w(y)]$ relative to Cartesian coordinates (x, y, z). We make a small disturbance to this flow in which the velocity component normal to the plane $y = 0$ has the form

$$\psi(y)\exp[i\alpha x+i\beta z-i\omega t], \tag{1.1}$$

where α, β, ω are constants and ψ is a function of y only; the other components of the perturbation velocity take on equivalent forms. On substituting these forms into the Navier–Stokes equation and ignoring the fact that the basic flow does not quite satisfy them at the high values of the Reynolds number in which we are interested, we find (e.g. GREGORY, STUART and WALKER, 1955) that ψ is a solution of the differential equation

$$\psi''''-2(\alpha^2+\beta^2)\psi''+(\alpha^2+\beta^2)^2\psi = iR\{(\alpha u+\beta w-\omega)([\psi''-(\alpha^2+\beta^2)\psi] -(\alpha u''+\beta w'')\psi\} \tag{1.2}$$

together with the boundary conditions

$$\psi(0) = \psi'(0) = 0, \quad \psi \to 0 \quad \text{as} \quad y \to \infty, \tag{1.3}$$

where primes denote differentiation with respect to y. Here R is a representative Reynolds number of the flow and the coordinates have all been appropriately scaled. For example, if the basic flow is a Blasius boundary layer, of special interest to us in this paper, the Reynolds number is based on the displacement thickness δ^* of the boundary layer at the special point of interest 0 of the wall. All coordinates are measured from 0 as origin and scaled with δ^*, the time is scaled by δ^*/U_∞ where U_∞ is the mainstream velocity, u by U_∞ and $w = 0$. As defined R is proportional to the square root of the distance of 0 from the leading edge.

2. Squire's theorem

The existence of a nontrivial solution of Eq. (1.2) subject to Eq. (1.3) is only possible if there is a relation connecting α, β, ω, R. This relation, if it exists, is not necessarily unique but it is generally believed possible to identify the one of importance to the stability theory and we shall assume that this can be done. For temporal stability α, β are assumed to be real and Eq. (1.2) provides a means of computing ω as a function of R. We are then considering the evolution of a given local disturbance as a function of time and the relation may be written in the form

$$\omega = \alpha f(\alpha^2+\beta^2, \alpha R) \tag{2.1}$$

since $w = 0$. Hence if the growth rate ω_i of a disturbance is equal to ω_{3i} when $R = R_3$, $\alpha = \alpha_3$, $\beta = \beta_3 \neq 0$, it is equal to $\omega_{3i}\alpha_2/\alpha_3$ when $R = R_2$, $\alpha = \alpha_2$, $\beta = 0$,

$$\alpha_2 = (\alpha_3^2+\beta_3^2)^{1/2} > \alpha_3, \quad R_2 = \frac{\alpha_3 R_3}{\sqrt{\alpha_3^2+\beta_3^2}} < R_3. \tag{2.2}$$

This is SQUIRE's theorem (1933) and it tells us that as the Reynolds number is increased the first mode of disturbance to become unstable at $R = R_c$ is two-dimensional with $\beta = 0$. Further for every unstable three-dimensional mode there is a more unstable two-dimensional mode at a lower Reynolds number. It does not necessarily follow, however, that at larger values of R the most unstable disturbances are two-dimensional although this does appear to be the case in plane Poiseuille flow (WATSON 1960, MICHAEL 1961) and in Blasius flow (CRIMINALE and KOVASZNAY 1962).

3. Spatial amplification

The most successful of the methods of predicting transition is the "e^n" method originated by SMITH and GAMBERONI (1956) and VAN INGEN (1956). In this method it is often assumed that ω is real, corresponding to spatial modes of instability and in two-dimensional studies Eq. (2.1) is used to compute α as a function of R and ω. Thus we envisage a finite disturbance at the origin 0 for all time, analyze it into Fourier components and determine their amplitudes proportional to

$$\exp(-\alpha_i x). \tag{3.1}$$

We now use the principle of sectional analysis to extend this formula not only to large x, on the scale of the boundary-layer thickness, but also to finite distances down the flat plate. This is done first by noting that if the imposed frequency remains constant as the wave advances down the plate, ω must also change taking the form $\omega_1 R/R_1$ where R_1 is the Reynolds number at which the disturbance originates. Next we replace the relation (3.1) by

$$\exp - \int \alpha_i dx \tag{3.2}$$

and change the variable x to the physical distance x^* along the plate using the formulas

$$\delta^* = \delta_0 x^* \sqrt{\frac{\nu}{U_\infty x^*}}, \quad R = \frac{U_\infty \delta^*}{\nu}, \quad x^* - x_0^* = x\delta^*, \tag{3.3}$$

where $\delta_0 = 1.721$, and x_0^* is the distance from the leading edge at the origin of x. It then follows that Eq. (3.2) may be replaced by

$$\exp\left[-\frac{2}{\delta_0^2}\int_{R_1}^{R} \alpha_i\left(\frac{\omega_1 \hat{R}}{R_1}, \hat{R}\right) d\hat{R}\right]. \tag{3.4}$$

Strictly Eq. (3.4) may only be identified with Eq. (3.1) if the variation of R is small but we now assume that it represents a reasonable approximation to the growth of the initial disturbance as R changes through finite values. The lower limit of integration is taken to be a point on the neutral curve (where $\alpha_i = 0$) since at smaller values of R modal disturbances determined by the solution of Eq. (1.2) are strictly negligible when compared with those associated with the continuous spectrum. Transition is estimated by computing the maximum value of the exponent as ω_1 varies. In two-dimensional or axisymmetrical flows with moderate pressure gradients, transition occurs at that Reynolds number R_T at which the exponent first reaches 8.98 for some ω_1 (CEBECI and BRADSHAW 1977).

In the above theory β is taken to be zero and the basic shear flow also has $w = 0$. Recently, however, (CEBECI and STEWARTSON 1979, NAYFEH 1979), the two-dimensional theory of spatial amplification has been extended to three-dimensional disturbances of a three-dimensional basic shear so that $w \neq 0$. Now emphasis is laid on the original disturbances being a spatial group of waves localized in the neighborhood of 0 and we are interested in the amplification properties of those waves from 0 which reach large values of x, z. These waves form a group centered on particular values of α and β in Eq. (1.1), given by a solution of Eq. (1.2) which also satisfies

$$\partial\alpha/\partial\beta = -\tan\theta = -z/x. \tag{3.5}$$

The partial differentiation is carried out holding ω and R constant, these two quantities being prescribed by the choice of disturbance and its source, 0. The condition (3.5) enables α, β to be found as functions of ω, R, θ which again may not be unique but are such that we can expect to identify the most unstable members of a discrete family of solutions. The generalization of Eq. (3.4) to a three-dimensional shear flow requires us to integrate

$$\alpha_i + \beta_i \tan\theta = -B \tag{3.6}$$

with respect to R for appropriate values of θ, ω. CEBECI and STEWARTSON (1979) applied these ideas to the prediction of transition on a rotating disk using a numerical scheme devised by CEBECI and KELLER (1977). They were able to identify an absolute neutral curve for spatial amplification which they named a zarf [lit. envelope (Turk.)] which is the locus of the minimum value of R on the neutral curve for each ω. On the zarf

$$\alpha_i = \beta_i = 0, \quad \partial\alpha/\partial\beta = -\tan\theta_A(\omega), \tag{3.7}$$

where $\theta_A(\omega)$ is a real function of ω. This curve is identical with the zarf for temporal disturbances, its projection onto the $\omega - R$ plane being the envelope of all neutral curves as α varies for fixed β. Further, its projection onto the $\alpha - R$ plane separates out the regions where $\omega_i < 0$ for all neighboring real β from that where $\omega_i > 0$ for at least one neighboring real value of β. A similar remark applies to the $\beta - R$ plane.

The calculations of CEBECI and STEWARTSON (1979), for the rotating disk problem, originated at the zarf and, as R increased, θ was held fixed at the value θ_A it took on this curve. Some measure of success was achieved in that the static wave with $\omega = 0$ was found to have the maximum growth rate and the corresponding value of θ_A to be about $-8°$; experimentally the well-known spiral vortices (GREGORY, STUART and WALKER 1955) are also associated with $\omega = 0$ and their direction corresponds to $\theta = -13°$. The value of n is $= 10$ at the beginning of transition, when the spiral arms appear, and is ≈ 20 at the onset of turbulence.

Clearly, however, this mode of procedure for determining n is not the most general nor can it lead to the greatest possible value of n at transition. For the disturbance amplifies at different rates in different directions and these rates also vary with R, so that we cannot assume the most unstable direction is always θ_A. Our aim in this paper is to investigate whether a more general approach, in which the local direction θ of the group is allowed to vary with R so that B is always a maximum, is likely to make a significant difference to the choice of n. We shall also examine whether the basic result of Squire's theorem can be generalized to the working rule that in spatial amplification two-dimensional disturbances are the most unstable when $w = 0$ and, if not, to estimate the likely errors involved in making such an assumption.

4. Results

Let us first examine the properties of B when β is small. With the assumption that $w = 0$, it follows from Eq. (1.2) that α may be expressed as a power series in β^2 whose coefficients are functions of ω and R; both real quantities in spatial coefficients are functions of ω and R, both real quantities in spatial amplification theory, and taking the form

$$\alpha = \alpha_0(\omega, R) + \beta^2\alpha_2(\omega, R) + \beta^4 a_4(\omega, R) + \ldots. \tag{4.1}$$

On the zarf α, β, ω, R are all real and it is clear from Eq. (4.1) that one possible form for it is $\beta = 0$, i.e. the two-dimensional neutral curve. If we now suppose that ω, R are chosen to be on this neutral curve, so that $\omega = \omega_2(R)$, α_0 is real but nothing can be said of course about α_2, α_4. An admissible solution (i.e. one with $\partial\alpha/\partial\beta$ real) must satisfy

$$\alpha_{2i}\beta_r + \alpha_{2r}\beta_i = 0, \tag{4.2}$$

when β is small, where the suffixes r, i denote, respectively, the real and imaginary parts of the complex number, and the growth rate is

(4.3) $$B = -(\alpha_i+\beta_i \tan\theta) = -\alpha_{2i}(\beta_r^2+\beta_i^2).$$

Hence if the two-dimensional neutral curve also corresponds to the maximum growth rate $\alpha_{2i} > 0$, and we may then reasonably expect it to form part of the curve in the (ω, R) plane dividing regions with negative amplification of all admissible solutions from those in which there is at least one direction θ for which positive amplification occurs. On the other hand, if $\alpha_{2i} < 0$, there are values of θ along which amplification takes place even though there is none at $\theta = 0$. This condition on α_{2i} is actually the same as that for the formation of the kidney-shaped curve (CRIMINALE and KOVASZNAY 1962, GASTER and DAVEY 1968) in the plane of real α and real β on which ω is also real for a constant values of R.

The values of α_0, α_2 have been computed at a number of points on the lower branch of the two-dimensional neutral curve and the results are set out in Table 1. It is deduced that $\alpha_{2i} = 0$ when $R = 710$ and we shall refer to this point as R_b,. So far as we can tell $\alpha_{2i} > 0$ on the upper branch of this neutral curve.

Table 1. Values of α_0, α_2 on the lower branch of the two-dimensional neutral curve as functions of R.

R	α_0	α_2
690	0.2165	$-0.66+0.005i$
772	0.2003	$-0.72-0.015i$
943	0.1741	$-0.85-0.058i$
1089	0.1609	$-0.93-0.083i$
1333	0.1450	$-1.05-0.123i$
1540	0.1353	$-1.14-0.153i$
1722	0.1285	$-1.21-0.175i$

Let us now suppose, as indeed appears to be the case for Blasius flow, that $\alpha_{4i} > 0$. Then in the neighborhood of R_b, α_{2i} is small and, taking β_r to be small as well,

(4.4) $$\theta = 2\alpha_{2r}\beta_r+O(\beta_r^3), \qquad B = -\alpha_{2i}\beta_r^2-\alpha_{4i}\beta_r^4$$

and hence at the maximum growth rate

(4.5) $$\beta_r^2 = -\alpha_{2i}/(2\alpha_{4i}).$$

It thus appears that the correct curve in the $\omega-R$ plane from which the integral in Eq. (3.4), as amended by Eq. (3.6), should be started is the two-dimensional neutral curve only if $R < R_b$. In other words, the zarf coincides with this curve if $R < R_b$ but bifurcates from it if $R > R_b$. In order to find the shape of the zarf near $R = R_b$, we expand α in Eq. (4.1) in powers of $(\omega-\omega_b)$, $(R-R_b)$ retaining the leading terms only and obtaining

(4.6) $$\alpha = \alpha_b+(\omega-\omega_b)L+(R-R_b)M+\beta^2\alpha_{2r}+i\beta^2[N(R-R_b)+P(\omega-\omega_b)]+\beta^4\alpha_4+\ \dots,$$

where L, M, P, N are constants of which N, P are real. Then from the conditions defining the zarf we have

$$\begin{aligned}
\beta^2 &= -\frac{M_i P + L_i N}{2L_i \alpha_{4i}}(R-R_b)+\ \dots,\\
\alpha-\alpha_b &= \left[\frac{M_r L_i - L_r M_i}{L_i} - \frac{M_i P + L_i N}{2\alpha_{4i} L_i}\,\alpha_{2r}\right](R-R_b)+\ \dots, \qquad (4.7)\\
\omega-\omega_b &= -\frac{M_i}{L_i}(R-R_b)+O(R-R_b)^2.
\end{aligned}$$

Thus the bifurcated zarf intersects the two-dimensional neutral curve at right angles in the $\beta - R$ plane, at a finite angle in the $\alpha - R$ plane and touches it in the $\omega - R$ plane. These

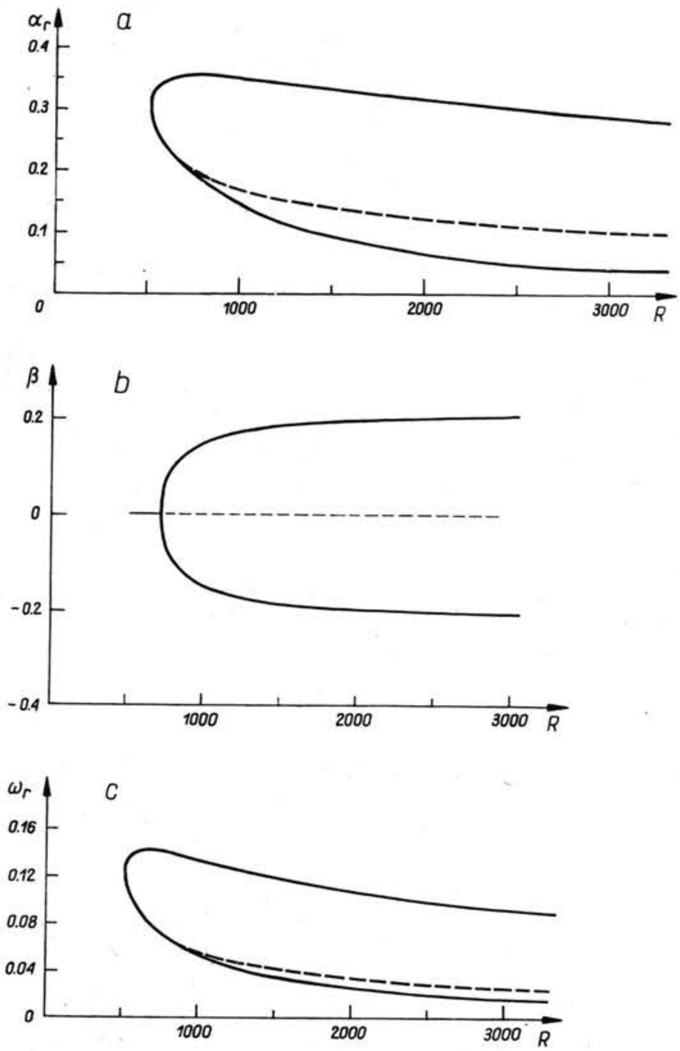

FIG. 1.a. Projection of the zarf on the α_r, R plane. Dashed line is the two-dimensional neutral curve. b. Projection of the zarf on the β_r, R plane. c. Projection of the zarf on the ω_r, R plane.

properties are displayed in Fig. 1 in which the shape of the zarf has been calculated up to values of R in excess of 3000. Considerable differences between the two curves may be seen especially in those for α and β. It may be shown (CEBECI and STEWARTSON 1981) that in the limit $R \to \infty$, $|\beta| \to 0.212$ and $\alpha \sim 149/R$ on the zarf.

A general point is worth making. The computation of admissible solutions of Eq. (1.2) depends on the assumption of a Taylor series expansion of α as a function of β, ω, R all of whose coefficients are finite. Should this not prove to be the case, the computational procedure breaks down. Contrariwise, if the numerical procedure breaks down, which we find to be the case at some values of ω and R, either it is not delicate enough to converge, with the constraints applied, or the Taylor-series principle has broken down. We shall assume that the numerical failure is due to the failure of this principle and that groups of waves with such phase velocities (i.e. α_r, β_r) cannot penetrate to large values of (x, z) except in a weaker form than has been envisaged. This conclusion is tentative and the phenomenon merits further study.

In the studies to determine n we selected a point on the lower branch of the two-dimensional neutral curve for Blasius flow with $\beta = 0$, to start from and followed the behaviour of B as β, R varied defining ω in terms of R as in Eq. (3.4). For example, we took $R_1 = 1018$, $\omega_1 = 0.05345$, $\alpha_1 = 0.16678$, $\beta = 0$ and then computed both B and $\tan\theta$ for a set of values of β holding ω, R constant. It was found that B duly vanished at $\beta = 0$ but increased with β_r ($\alpha_{2i} \simeq -0.07$). At $\beta \simeq 0.150-0.005i$ it seemed that a maximum value of B of approximately 0.07×10^{-3} was very nearly attained. Simultaneously, the value of θ decreased and nearly reached a minimum of $\simeq -9.4°$, the corresponding phase velocity being in a direction making an angle $\simeq -45°$ to the mainstream. The two extrema are inferred to occur at slightly different values of β_r. However, further calculations at larger values of β_r failed to converge since it proved impossible to find a β_i such that $|\partial\alpha_i/\partial\beta_r|$ is small enough to satisfy our tolerance requirements (< 0.001). Another calculation was carried out at $R_1 = 1540$, $\omega_1 = 0.03909$ the breakdown occurring at $\beta_r > 0.120$ with B and $\tan\theta$ still definitely increasing. The principal results of these two calculations are displayed in Table 2.

A related calculation was also carried out at $R_1 = 1540$, $\omega_1 = 0.03909$ in which $\beta_i = 0$ and β_r were allowed to vary. The reults are of interest in connection with the experiments of KLEBANOFF, TIDSTROM and SARGENT (1962) on the evolution of waves from a vibrating ribbon in a boundary layer with surface spacers of cellophane tape. This calculation showed good agreement in estimates of B with the admissible solution of Table 2 except for $\beta_r = 0.105$ and the growth rate achieved a maximum at $\beta_r \simeq 0.16$ when the spanwise wave-length is about 2/3 of the chordwise wave length. The same ratio occurs in the experiments carried out at $R_1 = 1635$. The present calculations were stopped at $\beta_r = 0.240$ but it seems clear that if they were continued, the growth rate would be found to vanish again at $\beta_r \simeq 0.3$. In the β, R-plane the corresponding point is well outside the zarf but there is no contradiction since the zarf also satisfies the criterion that on it $\omega_i = \alpha_i = \beta_i = 0$ and ω_r is an extremum regarded as a function of α_r, β_r with R fixed. Here $\omega = 0.03971$ while $\omega = 0.0345$ on the lower branch of the zarf.

Next the variation of B with R was computed by choosing $R_1 = 1333$, $\omega_1 = 0.04342$

Table 2. Variation of B, $\tan\theta$ as functions of β_r, for $R = 1018$ and $R = 1540$.

$R = 1018$			$R = 1540$		
β_r	$B \times 10^3$	$\tan\theta$	β_r	$B \times 10^3$	$\tan\theta$
0	0	0	0	0	0
0.015	0.016	−0.027	0.015	0.034	−0.034
0.030	0.063	−0.053	0.030	0.135	−0.067
0.045	0.138	−0.077	0.045	0.298	−0.097
0.060	0.234	−0.099	0.060	0.511	−0.123
0.075	0.342	−0.118	0.075	0.762	−0.144
0.090	0.453	−0.134	0.090	1.037	−0.160(¹)
0.105	0.556	−0.147	0.105	1.826	−0.184(¹)
0.120	0.638	−0.156			
0.135	0.690	−0.162			
0.150	0.703	−0.165			

(¹) convergence marginal

and defining ω for other values of R by $\omega = \omega_1 R/R_1$. In order to obtain as large a value of n as possible, we define

$$n = \frac{2}{\delta_0^2} \int_{R_z}^{R_T} B_{\max}\, dR, \tag{4.8}$$

where $B_{\max}$ is defined to be the largest attainable value for B at a particular value of R as β varies, with $\omega = \omega_1 R/R_1$, and the requirement that the solution of Eq. (1.2) be admissible while R_z, $\omega_z (= \omega_1 R_z/R_1)$ is a point on the zarf. In Table 3 we display the variation of $B_{\max}$ with R and compare it with that of B_2 which is equal to $-\alpha_i$ when $\beta = 0$.

Table 3. Table of the maximum value of B as a function of R deduced from the solution of the Orr-Sommerfeld equation with $\partial\alpha/\partial\beta = -\tan\alpha$(real) and $\omega = \omega_1\, R/R_1$ where $R_1 = 1333$ and ω_1 is the value of ω at $R_1 = 1333$ for which $\alpha_i = 0$ when $\beta = 0$. Also shown is a table of $B_2 = -\alpha_i$ computed for the same values of R_ω, but with $\beta = 0$. If $R \geqslant 1806$, $B_{\max} = B_2$.

R	β_r	$\tan\theta$	$B_{\max} \times 10^2$	$B_2 \times 10^2$
1333	0.12	−0.175	0.14(¹)	0
1440	0.135	−0.164	0.36	0.27
1540	0.12	−0.155	0.55	0.50
1633	0.09	−0.127	0.72	0.70
1722	0.06	−0.088	0.87	0.87
1806	0	0	1.00	1.00

(¹) No admissible solutions found for $\beta_r > 0.12$

Also included are the corresponding values of β_r, $\tan\theta$. We see that the differences between the two values of B are negligible or zero for $R > 1540$ and that the increase in the value of n due to the three-dimensional effects amounts to 0.17 only. Another calculation was carried out for $R_1 = 943$. In this intstance the difference between the two values of B is negligible or zero for $R \geqslant 1089$ and the increase in the value of n is 0.04.

A similar phenomenon occurs when β is kept real and R allowed to increase, with $\omega = \omega_1 R/R_1$. For example at $R_1 = 1540$, $\omega_1 = 0.0391$, B vanishes at $\beta_r = 0$ and has a maximum of 0.0018 at $\beta_r = 0.165$. The difference between the corresponding values of B diminishes as R increases, the relevant numbers being 0.0038 and 0.0047 at $R = 1721$, 0.0069 and 0.0072 at $R = 1886$; thereafter the differences are negligible.

5. Discussion

The principal result of this paper is that even if the basic shear flow is two-dimensional it does not follow that the zarf satisfies $\beta = 0$. So far as α, β are concerned, the zarf shows considerable departures from the two-dimensional neutral curve on the lower branch although the two are coincident on the upper branch. This phenomenon is important when small amounts of cross flow are present, for it follows that the corresponding zarfs are not necessarily close to the neutral curve with $\beta = 0$. An example is the yawed stagnation flow (Cebeci and Stewartson 1979).

Secondly, we have established that while the most amplified waves at a given Reynolds number R and frequency $\omega/2\pi$ of oscillation do not necessarily have a phase and group velocity in the direction of the mainstream outside the boundary layer, in general little is lost by adopting such a working rule. It is least successful near the lower branch of the two-dimensional neutral curve for the shear flow studied, namely Blasius flow, and in order to achieve the largest possible value of n in the e^n method for predicting transition, the integration of Eq. (3.4) or Eq. (4.6) should strictly start at the zarf rather than at this neutral curve. The contribution to n from this neighborhood of R, ω is, however, quite small and, as the Reynolds number is advanced, the growth rates do increase but the differences between the two-dimensional and the maximum values diminish rapidly and soon the two coincide. The ultimate effect on n is to raise it by 0.17 for $R_1 = 1333$ when admissible three-dimensional disturbances are incorporated. Since 1333 is close to the crucial value of R_1 for the largest possible value of n at transition where $R \leqslant 2900$ and $n = 8.94$ when two-dimensional disturbances only are taken into account (Cebeci and Bradshaw 1977, p. 307), it is unlikely that the fuller study which permits $\beta \neq 0$ would increase n beyond 9.2. In view of the complicated nature of the method of predicting transition and its imprecise character for Blasius flow, such a change may be ignored for practical purposes.

The study also lends support to the hopes expressed by Cebeci and Stewartson (1979) that in calculations of n it is sufficient to start on the zarf. Indeed for Blasius flow one might go further and take $\beta = \theta = 0$ without significant loss of accuracy.

Finally, attention should be drawn to the abrupt and curious termination of the calculations with increasing β due to an inability to obtain admissible solutions. This phenomenon also occurred during the calculation of the absolute neutral curve for a rotating disk. In that problem the curve was eventually reached through a series of "false" solutions of which the final one simultaneously satisfied $\alpha_i = \beta_i = 0$ and $\partial\alpha_i/\partial\beta_r = 0$. It is an open question whether a similar device would enable us to increase the value of β_r in the Blasius studies, but it was felt that even if successful, there would be little change in our conclusions.

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References

1. T. CEBECI and P. BRADSHAW, *Momentum transfer in boundary layers*, Hemisphere/McGraw-Hill, Washington 1977.
2. T. CEBECI and H. B. KELLER, *Stability calculations for a rotating disc*, AGARD CPP-224, p. 7–1, 1977.
3. T. CEBECI and K. STEWARTSON, *On stability and transition in three-dimensional flows*, AIAA J., **18**, p. 398, 1980.
4. T. CEBECI and K. STEWARTSON, *On the prediction of transition in three-dimensional flows*, Presented at the IUTAM Symposium on Laminar-Turbulent Transition, Stuttgart 1980.
5. W. O. CRIMINALE and L. S. G. KOVASZNAY, *The growth of localized disturbances in a laminar boundary layer*, J. Fluid Mech., **14**, 59, 1962.
6. M. GASTER and A. DAVEY, *The development of 3-D wave packets in unbounded parallel flows*, J. Fluid Mech., **32**, 801, 1968.
7. N. GREGORY, J. T. STUART and W. S. WALKER, *On the stability of three-dimensional boundary layers with applications to the flow due to a rotating disk*, Phil. Trans. Roy. Soc., **A248**, pp. 155–199, London 1955.
8. P. S. KLEBANOFF, K. D. TIDSTROM and L. M. SARGENT, *The three-dimensional nature of transition*, J. F. M., **12**, 1–34, 1962.
9. A. H. NAYFEH, *Stability of three-dimensional boundary layers*, AIAA J., **18**, p. 406, 1980.
10. A. M. O. SMITH and N. GAMBERONI, *Transition pressure gradient and stability theory*, Proc. Int. Cong. Appl. Mech., **9**, 4, p. 234, Brussels 1956.
11. H. B. SQUIRE, *On the stability for three-dimensional disturbances of viscous fluid flow between parallel walls*, Proc. Roy. Soc., **A142**, 621–628, London 1933.
12. J. L. VAN INGEN, *A suggested semi-empirical method for the calculation of the boundary layer transition region*, Repts No VTH 71, VTH 74, Delft, Holland 1956.
13. A. N. NAYFEH, *Stability of three-dimensional boundary layers*, AIAA J., **18**, p. 406, 1980.
14. D. H. MICHAEL, *A note on the stability of plane parallel flow*, J. Fluid Mech., **10**, 525–528, 1961.
15. J. WATSON, *Three dimensional disturbances in flow between parallel planes*, Proc. Roy. Soc. Lond., **A254**, 562–569, 1958.

MECHANICAL ENGINEERING DEPARTMENT
CALIFORNIA STATE UNIVERSITY, LONG BEACH, CALIFORNIA, USA
and
DEPARTMENT OF MATHEMATICS
UNIVERSITY COLLEGE, LONDON, ENGLAND.

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