Drag on a sphere oscillating in a dusty gas

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IN THE PRESENT paper rectilinear oscillation of a sphere in an infinite expanse of viscous, incompressible fluid having uniform distribution of solid spherical particles is studied. The problem is solved by the method of separation of variables and particular attention is focused on the drag acting on the sphere due to fluid stresses. An exact formula for the drag is obtained in terms of two parameters and graphs have been drawn to study the variation of these parameters.

Przeanalizowano zagadnienie kuli drgającej w nieskończonej objętości lepkiego płynu nieściśliwego zawierającego równomiernie rozłożone cząsteczki kuliste. Problem rozwiązano za pomocą rozdzielenia zmiennych, zwracając szczególną uwagę na siły oporu działające na kulę, a wynikające z naprężeń przenoszonych przez płyn. Ścisły wzór na siłę oporu zawiera dwa parametry, a przedstawione wykresy pokazują wpływ zmienności tych parametrów na wielkość siły.

Проанализирована задача колеблющегося шара в бесконечном объеме вязкой несжимаемой жидкости, содержавшей равномерно распределенные сферические частицы. Проблема решена при помощи разделения переменных, обращая особенное внимание на силы сопротивления, действующие на шар и вытекающие из напряжений переносимых через жидкость. Точная формула для силы сопротивления содержит два параметра, а представленные диаграммы показывают влияние переменности этих параметров на величину силы.

Nomenclature

u velocity of fluid particles,
v velocity of dust particles,
p pressure,
ϱ density,
ν kinematic viscosity,
F_1 body force vector,
g acceleration due to gravity,
U_0 amplitude of oscillation of the sphere,
σ frequency of oscillation,
f density ratio of particles to fluid (per unit volume),
τ particle relaxation time.

1. Introduction

THE STUDY of fluids having uniform distribution of solid spherical particles plays an important role in many technical areas such as fluidization, environmental pollution, combustion, blood flow through capillaries, pneumatic conveyance of small grain-like parti-

cles, flow in rocket tubes, etc. The basic theory of multi-phase flows is given in a recent book by Soo [1]. The development of the subject can be seen in a number of papers [2–12]. LIU [3] has solved the Stokes problem. Later, HEALY and YANG [10] have found the exact solution of the Rayleigh problem by the Laplace transform technique. MICHAEL [11] has investigated some spherical flows by the perturbation method. Recently, INDRASENA and ZARTY [12] have studied the rotary oscillation of a sphere in a dusty gas.

The present paper deals theoretically with an oscillating system which can be used when the fluid is large with uniform distribution of dust particles. In this system a sphere which is suspended in an infinite expanse of dusty, incompressible, viscous fluid, executes rectilinear oscillations about its mean position. The problem is solved by the method of separation of variables and analytical expressions for the components of fluid velocity are obtained. An exact formula for the drag experienced by the sphere due to fluid stresses is established in terms of the amptitude, frequency of oscillation and two parameters, known as drag parameters. Graphs have been drawn to study the variation of these parameters with frequency of oscillation. It is observed that the presence of dust particles increases the magnitude of the drag. The drag experienced by a sphere in clean viscous liquid has been obtained as a particular case of the present investigation.

2. Basic equations

The equations of motion of unsteady flow of a viscous, incompressible fluid with uniform distribution of dust particles are given below [2]:

$$(2.1) \qquad \nabla \cdot \bar{u} = 0,$$

$$(2.2) \qquad \frac{\partial \bar{u}}{\partial t} + (\bar{u} \cdot \nabla)u = -\frac{1}{\varrho}\nabla p + \frac{1}{\varrho}\bar{F}_1 + \nu\nabla^2\bar{u} + \frac{f}{\tau}(\bar{v} - \bar{u}),$$

$$(2.3) \qquad \tau\left[\frac{\partial \bar{v}}{\partial t} + (\bar{v} \cdot \nabla)\bar{v}\right] = \bar{u} - \bar{v},$$

$$(2.4) \qquad \nabla \cdot \bar{v} = 0.$$

It is assumed that 1) the interaction between the phases take place according to the Stokes drag law; 2) there is negligible particle interaction; 3) sedimentation is neglected; 4) there is no radial migration of the particles; 5) the volume occupied by the particulate phase is constant and 6) Brownian motion is neglected.

3. Formulation of the boundary-value problem

We consider a sphere of radius a oscillating rectilinearly along the vertical diameter $\theta = 0$ with velocity $U_0\cos\sigma t$. If $\dfrac{U_0}{a\sigma}$ is small, i.e. if the spatial amplitude of oscillation is small compared with the radius, then the inertial terms in Eqs. (2.2) and (2.3) can be neglected. Elimination of $\bar{v}$ between the resulting linearised equations yields

$$(3.1) \qquad \left(1 + f + \tau\frac{\partial}{\partial t}\right)\frac{\partial \bar{u}}{\partial t} = -\left(1 + \tau\frac{\partial}{\partial t}\right)\left(\frac{1}{\varrho}\nabla p + \frac{1}{\varrho}\bar{F}_1\right) + \nu\left(1 + \tau\frac{\partial}{\partial t}\right)\nabla^2\bar{u}.$$

Choosing a spherical polar coordinate system (r, θ, ϕ), taking $\bar{e}_r, \bar{e}_\theta, \bar{e}_\phi$ as the base vectors of the system and assuming the motion to be symmetric about the vertical diameter, the vector $\bar{u}$ can be written as

(3.2) $$\bar{u} = u_r(r, \theta, t)\bar{e}_r + u_\theta(r, \theta, t)\bar{e}_\theta.$$

Using Eq. (3.2) Eq. (3.1) can be resolved into the following equations:

(3.3)

$$\left(1+f+\tau\frac{\partial}{\partial t}\right)\frac{\partial u_r}{\partial t} = -\frac{1}{\varrho}\left(1+\tau\frac{\partial}{\partial t}\right)\frac{\partial p'}{\partial r} + \nu\left(1+\tau\frac{\partial}{\partial t}\right)\left(\nabla^2 u_r - \frac{2u_r}{r^2} - \frac{2\cot\theta}{r^2}u_\theta - \frac{2\partial u_\theta}{r^2\partial\theta}\right),$$

(3.4)

$$\left(1+f+\tau\frac{\partial}{\partial t}\right)\frac{\partial u_\theta}{\partial t} = -\frac{1}{\varrho}\left(1+\tau\frac{\partial}{\partial t}\right)\left(\frac{1}{r}\frac{\partial p}{\partial\theta}\right) + \nu\left(1+\tau\frac{\partial}{\partial t}\right)\left(\nabla^2 u_\theta - \frac{u_\theta}{r^2\sin^2\theta} + \frac{2}{r^2}\frac{\partial u_r}{\partial\theta}\right).$$

In view of Eq. (2.1), the components u_r, u_θ can be expressed in terms of a function $\psi(r, \theta, t)$ as

(3.5) $$u_r = -\frac{1}{r^2\sin^2\theta}\frac{\partial\psi}{\partial\theta}, \quad u_\theta = \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r},$$

which simplify Eqs. (3.3) and (3.4) respectively to

(3.6)

$$\left(1+f+\tau\frac{\partial}{\partial t}\right)\frac{\partial}{\partial t}\left(\frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta}\right) = \left(1+\tau\frac{\partial}{\partial t}\right)\left(\frac{1}{\varrho}\frac{\partial p}{\partial r}\right) + \nu\left(1+\tau\frac{\partial}{\partial t}\right)\left[\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\nabla_1^2\psi)\right],$$

(3.7)

$$\left(1+f+\tau\frac{\partial}{\partial t}\right)\frac{\partial}{\partial t}\left(\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}\right) = -\frac{1}{\varrho}\left(1+\tau\frac{\partial}{\partial t}\right)\left(\frac{1}{r}\frac{\partial p'}{\partial\theta}\right) + \nu\left(1+\tau\frac{\partial}{\partial t}\right)\left[\frac{1}{r\sin\theta}\frac{\partial}{\partial r}(\nabla_1^2\psi)\right],$$

where

$$\nabla_1^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} - \frac{\cot\theta}{r^2}\frac{\partial}{\partial\theta}$$

and

$$p' = p - gr\cos\theta + \text{const.}$$

Eliminating p' from Eq. (3.6) and (3.7), one finds that

(3.8) $$\left(1+f+\tau\frac{\partial}{\partial t}\right)\frac{\partial}{\partial t}(\nabla_1^2\psi) = \nu\left(1+\tau\frac{\partial}{\partial t}\right)(\nabla_1^4\psi).$$

Since the velocity of fluid at the surface of the sphere must be $U_0\cos\sigma t$ parallel to the direction of the oscillation of the sphere, the boundary conditions are

(3.9)
$$u_r(a, \theta, t) = U_0 e^{i\sigma t}\cdot\cos\theta,$$
$$u_\theta(a, \theta, t) = -U_0 e^{i\sigma t}\sin\theta.$$

4. Solution of the problem

The form of the boundary conditions suggests that the function ψ can be assumed as

(4.1) $$\psi(r, \theta, t) = F(r) \cdot e^{i\sigma t} \cdot \sin^2\theta.$$

In Eqs. (3.2), (3.9), (4.1) and in all subsequent equations only the real parts are to be taken whenever physical quantities are represented by complex quantities.

Equation (3.8) on using Eq. (4.1) simplifies to

(4.2) $$D_1^2 F(r) - n^2 D_1 F(r) = 0,$$

where

$$D_1 = \frac{d^2}{dr^2} - \frac{2}{r^2}$$

and

$$n^2 = \frac{i\sigma}{\nu} \frac{(1+f+\tau i\sigma)}{(1+\tau i\sigma)}.$$

The boundary conditions (4.1) in terms of function F become

(4.3) $$F(a) = -\frac{U_0 a^2}{2}, \quad F'(a) = -U_0 a.$$

The general solution of Eq. (4.2) subject to the condition that u_r and u_θ vanish as $r \to \infty$, is given by

(4.4) $$F(r) = \frac{A}{r} + B\left(\frac{1}{r} + n\right)e^{-nr}.$$

Using the boundary conditions (4.3), the constants A and B can be evaluated as

(4.5) $$A = -\frac{1}{2} U_0 a^3 - \frac{3}{2} \frac{U_0 a}{n^2}(1 + na),$$

$$B = \frac{3}{2} \frac{U_0 a}{n^2} e^{na}.$$

Integration of Eqs. (3.6) and (3.7) yields

(4.6) $$p' = -\frac{A\mu n^2}{r^2}\cos\theta \cdot e^{i\sigma t} + \frac{Ce^{-t/\tau}}{r^2} + \text{const},$$

where C is an arbitrary constant.

From Eqs. (3.2), (4.1), (4.4) and (4.5) it follows that

(4.7) $$u_r(r, \theta, t) = U_0\left(\frac{a}{r}\right)^3\left[1 + \frac{3(1+na)}{n^2 a^2}\left\{1 - \frac{(1+nr)}{(1+na)}e^{-n(r-a)}\right\}\right]\cos\theta \cdot e^{i\sigma t},$$

(4.8) $$u_\theta(r, \theta, t) = \frac{U_0}{2}\left(\frac{a}{r}\right)^3\left[1 + \frac{3(1+na)}{n^2 a^2}\left\{1 - \frac{(1+nr+n^2r^2)}{(1+na)}e^{-n(r-a)}\right\}\right]\sin\theta \cdot e^{i\sigma t}.$$

We can calculate the force which must be applied to the moving sphere to maintain its oscillations. This is equal in magnitude to the drag $D(t)$ due to fluid stresses and is given by

(4.9)
$$D(t) = \int_0^\pi (\tau_{rr}\cos\theta - \tau_{r\theta}\sin\theta)_{r=a} 2\pi a^2 \sin\theta - d\theta,$$

where

(4.10)
$$(\tau_{rr})_{r=a} = -\frac{U_0\mu}{2a}(3 + 3na + n^2a^2)\cos\theta \cdot e^{i\sigma t},$$

$$(\tau_{r\theta})_{r=a} = \frac{3U_0\mu}{2a}(1 + na)\cdot \sin\theta \cdot e^{i\sigma t}.$$

Equation (4.9), on using (4.10) leads to

$$D(t) = \frac{2\pi\mu}{3} U_0 a(9 + 9na + a^2n^2)e^{i\sigma t},$$

which on separation of the real part gives

(4.11)
$$D(t) = M'\sigma U_0(K\sin\sigma t - K'\cos\sigma t),$$

where

(4.12)
$$K = \frac{9}{2}\frac{|n|\sin\phi}{a\alpha} + \frac{1}{2}\frac{|n|^2}{\alpha}\sin 2\phi,$$

(4.13)
$$K' = \frac{9}{2a^2\alpha} + \frac{9|n|\cos\phi}{2a\alpha} + \frac{|n|^2}{2\alpha}\cos 2\phi.$$

Where $M' = \dfrac{4\pi a^3\varrho}{3}$ is the mass of fluid displaced by the sphere and $|n|$, ϕ and α are defined by

(4.14)
$$|n|^2 = \frac{\sigma}{\nu}\left[\frac{(1+f)^2 + \tau^2\sigma^2}{1 + \tau^2\sigma^2}\right]^{\frac{1}{2}},$$

$$2\phi = \frac{\pi}{2} + \tan^{-1}\frac{\tau\sigma}{1+f} - \tan^{-1}\tau\sigma,$$

$$\alpha = \frac{\sigma}{\nu}.$$

For convenience, the quantities K and K' introduced in the expression (4.11) may be designated as drag parameters.

5. Discussion

The two terms appearing in the expression (4.11) for $D(t)$ can be interpreted as follows. In the absence of fluid the force necessary to move the sphere of mass M is $-MU_0\sigma\sin\sigma t$. Expression (4.11) shows that, in addition, a further force $-\dot{M}U_0\sigma K\sin\sigma t$ in phase with the acceleration is required, because in the process of moving the sphere, fluid is neces-

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sarily moved as well. The second term in the expression (4.11) is the force that always opposes the movement of the sphere, and is thus a damping force out of phase with the acceleration. This force causes the decay of the oscillations of the sphere if left free. It is seen from Eq. (4.14) that for large values of $\tau\sigma$, the influence of the dust particles on the fluid motion is reduced and D, K, K' ultimately approach their corresponding values for clean viscous fluid. Figures 1 and 2 represent the variation of K and K' for
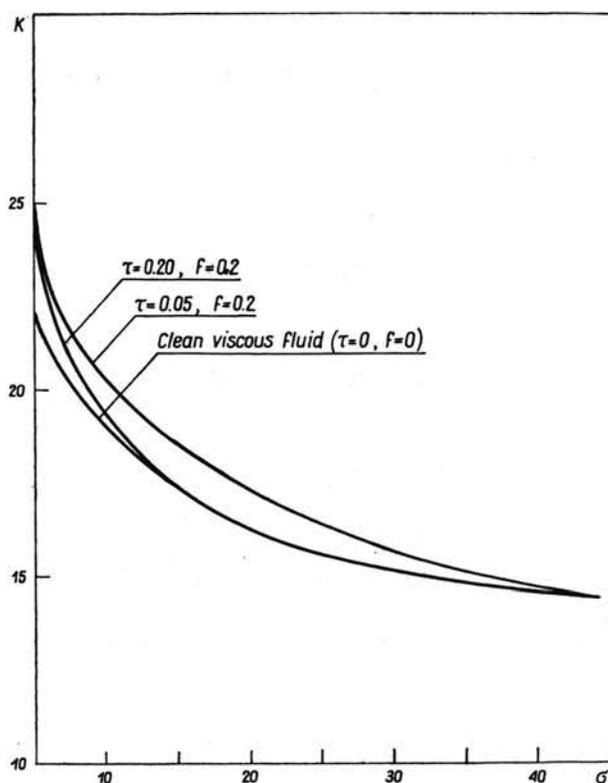

FIG. 1.

different frequencies of oscillation. These graphs show the decrease in the values of the drag parameters with the increase of frequency. There is an increase in the values of these parameters when compared to clean viscous fluid over the entire range of frequency considered. From Eqs. (4.11)–(4.14) it follows that for any given frequency the drag experienced by the sphere due to dusty fluid is more than the drag due to clean viscous fluid. This increase in drag is due to the presence of dust particles. If the masses of the dust particles are small, their influence on the fluid motion is reduced and ultimately as $m \to 0$ the fluid becomes ordinary viscous and the drag parameters simplify to

$$K = \frac{1}{2}(9\beta+1),$$

$$K' = \frac{9}{2}(\beta+2),$$

where

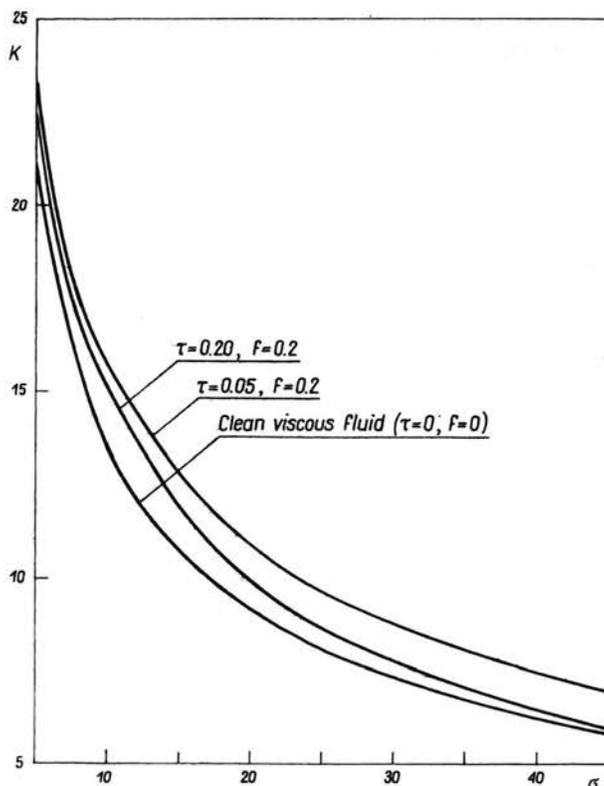$$\beta = \sqrt{\frac{\nu}{2a^2\sigma}}.$$

FIG. 2.

The expression for drag, (4.11) with the above values of K, K' is the same as the result given by LAMB [13]. The solution of the problem of steady motion of a sphere in ordinary viscous fluid can be deduced as a particular case of the present investigation.

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