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A cylindrical tank response as an example of shakedown of non-Clapeyronian systems (*)

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GENERAL theorems of the shake-down theory may not hold for non-Clapeyronian systems. Consequences of this fact are studied using an example of elastic-plastic response of cylindrical storage tanks with flat bottoms resting on a rigid subgrade. Unlikely to the limit analysis problem, unilateral character of the bottom support can give either a rise or a decrease in the shakedown pressure, depending on the geometry and load configuration.

Ogólne twierdzenia teorii przystosowania konstrukcji mogą nie zachowywać ważności w przypadku układów nieclapeyronowskich. Skutki tego faktu badane są na przykładzie sprężysto-plastycznego zachowania się walcowych zbiorników o dnach spoczywających na sztywnym podłożu. Jednostronny charakter oparcia dna może wpływać w odróżnieniu od problemu nośności granicznej na obniżenie albo podwyższenie ciśnienia przystosowania.

Общие теоремы теории приспособления конструкций могут не сохранять важности в случае неклапейроновских систем. Следствия этого факта исследуются на примере упруго-пластического поведения цилиндрических резервуаров с днами, находящимися на жестком основании. Односторонний характер опирания дна может влиять, в отличие от задачи предельной несущей способности, на снижение или повышение давления приспособления.

1. Introduction

APPEARANCE of certain plastic deformations in steel structures has been generally accepted provided their growth is limited by the condition of shakedown. However, the fundamental theorems of the shakedown [1] and, therefore, its classical techniques (see, e.g., [2]) can be employed only in the case of Clapeyronian systems, i.e. systems with boundary conditions being prescribed on fixed parts of the boundary. Unfortunately, this is not always the case, e.g., for unilateral constraints occurring at the contact between bottom of a tank and the subgrade. It follows from the theorem on alternating plasticity [2] that for structures undergoing a single-parameter pulsating loading, shakedown values do not exceed the double of the elastic load. We want to show here that in presence of unilateral constraints the shakedown load can become higher than this limit in some practically important cases.

2. Formulation of the problem

We consider a vertical circular cylindrical liquid-storage tank of the radius R and height H (Fig. 1a) composed of a thin-walled shell of the thickness (in its lower part) h_s, and a flat bottom plate of the thickness h_b freely resting on the subgrade. We restrict our analysis to the most important element of the structure—the bottom-to-shell

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connection (Fig. 1b). To emphasize better the non-Clapeyronian character of the system response in the presence of unilateral supports, the subgrade is considered to be rigid. It corresponds also to some important practical cases of a continuous concrete slab under the bottom steel plate or of a concrete ring under the external part of the bottom.

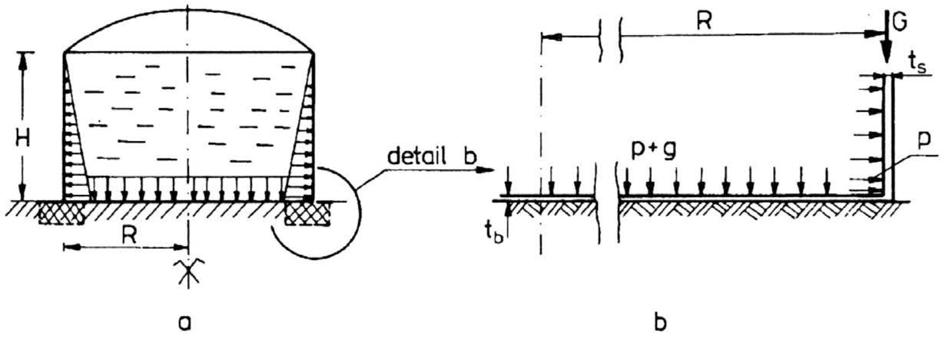

FIG. 1. a. Tank on a rigid subgrade or on a rigid ring. b. Detail of the bottom-to-shell connection.

The simplest loading–unloading process is considered, with the liquid pressure at the bottom level p varying from 0 to $\overline{p}$ (Fig. 2) and the vertical edge load G (the shell weight and loads transmitted from the roof) assumed to be constant. The bottom weight g, although negligible, may be taken into consideration to exclude lifting-up of the whole tank when vertical loads on the plate vanish. Membrane force in the bottom due to the shear force acting at the shell edge is disregarded; its contribution to the stress never exceeds, following the elementary elastic analysis (e.g., [3]), one per cent of the total value.

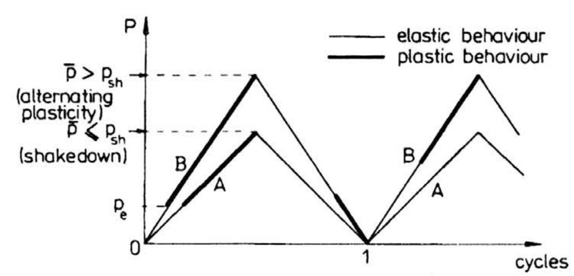

FIG. 2. Loading-unloading cycles: A—shakedown, B—inadaptation by alternating plasticity.

In the process of filling-in the tank the pressure of the liquid will cause the edge part of the shell to bend, what results in bending of the bottom by the edge couple M_A and in driving up the external annulus of the plate (Fig. 3a). At pressure p_e plastic deformations appear first at the plate edge, its thickness h_b not exceeding the thickness of the shell h_s. If the shell is sufficiently thick ($h_s \geq \sqrt{3/2}h_b$), its edge zone will never become plastic.

Plastic zones appearing at the outer faces of the plate spread along the faces and across the thickness (Fig. 5), and at the pressure p_0 the edge cross-section becomes fully plastic and a plastic hinge is formed, Fig. 3b. The process follows then with the edge moment $M_A = M_P$ constant and the angle between the plate and the shell $(\pi/2) + \varphi^h$ increasing.

As the pressure decreases from its maximum value $\overline{p}$, the unloading response is elastic with the relative plastic rotation $\overline{\varphi}^h$ remaining unchanged. The shell wall tries to restore the vertical position, the edge moment M_A decreases rapidly and eventually changes its sign. Following an approach by the sandwich model or by substitutive plastic rotations in virtual hinges [4], the case $M_A = 0$ would correspond to the contact of the bottom plate completely restored (Fig. 3c). In reality, however, it is prevented by the finite spread of the plastically deformed zone (dashed line in Fig. 3c). At a certain pressure p_1 depending upon the edge load G, the edge loses the contact with the subgrade (in the approximate approaches it appears at $M_A = 0$), (Fig. 3d). When the unloading continues, we have two possibilities, Fig. 2:

i. Unloading remains elastic down to $p = 0$ and, therefore, a response to the following loading cycles not exceeding the preceding peak value $\overline{p}$ will be elastic ($\overline{p} < \overline{p}_{\text{sh}}$, the structure shakes down).

ii. If the peak load $\overline{p}$ and, therefore, the plastic rotation accumulated during the loading process is sufficiently large, bending in the opposite direction will result in a new plastic yielding, with a plastic hinge eventually formed (Fig. 3e).

The latter situation would be inadmissible since it leads to alternating plastic strain increments in subsequent loading–unloading cycles and, in consequence, to the low-cycle fatigue of the connection ($\overline{p} > \overline{p}_{\text{sh}}$). Safety of the tank requires the unloading process to proceed elastically.

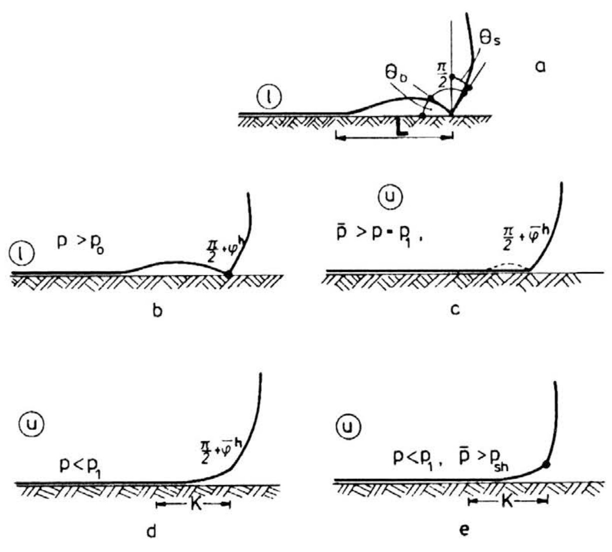

FIG. 3. Consecutive stages of deformation of the bottom-to-shell connection; l—loading, u—unloading; b and e—plastic hinges formed.

3. Analysis of subsequent stages of the tank response

Since the shell thickness is definitely higher than that of the bottom plate, the shell is assumed to remain elastic and the analysis is oriented, first of all, at determining the variation of the bending moment in the plate. The depth of the lifted up bottom zone subject to bending is small as compared with the plate radius and, therefore, the zone can be treated, as in the standard elastic analysis (e.g., [3]), as a plate strip undergoing cylindrical bending. We have thus a beam of the span L or K (Fig. 4) to be determined from the contact condition and with the plate stiffness D_b. Elastic analysis taking into account the two-dimensional bending [5] shows that for $R > 300h_b$ the influence of the latter is negligible.

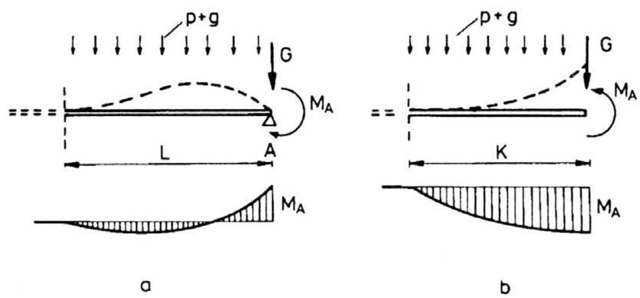

FIG. 4. Uplifted zone of the bottom plate: a—at loading, b—at unloading.

At the first stage of the loading process the strip that loses contact is loaded by the edge couple M_A and by the transversal load $p + g$, and is supported at its edge point A (Fig. 4a). Continuity conditions (rotation and the bending moment disappearing at the inner edge and equality of the shell and plate rotation along the joint) give

$$M_A = (p+g)L^2/4, \tag{3.1}$$

$$\frac{L^3(p+g)}{6D_b} + \frac{L^2 l(p+g)}{2D_s} = pl^3\left(1 - \frac{l}{H}\right), \tag{3.2}$$

where the rigidities and the shell characteristic length are, respectively

$$D_s = \frac{Eh_s^3}{12(1-\nu^2)}, \qquad D_b = \frac{Eh_b^3}{12(1-\nu^2)}, \qquad l = \left(\frac{R^2h_s^2}{3(1-\nu^2)}\right)^{1/4}.$$

If the bottom weight g is disregarded a single, the length of the uplifted zone L will be constant and the moment M_A will be proportional to p. This stage ends at $p = p_e$ with the first plastic deformation appearing at the plate edge, i.e. when

$$M_A = M_e = \sigma_Y h_b^2/6,$$

where σ_Y is the yield point in the bi-axial plane strain tension.

To take into account the increased deformability of the partially plastic zone (Fig. 5), plastic curvature should be integrated across the zone. The continuity conditions described already in the elastic stage give now:

$$M_A = \frac{(p+g)L^2}{4} - \frac{6\Delta D_b}{L^2}, \tag{3.3}$$

$$\frac{(p+g)L^3}{24D_b} + \frac{(p+g)L^2 l}{8D_s} - \frac{pl^3}{4D_s}\left(1 - \frac{l}{H}\right) + \theta^p - \frac{3\Delta}{L}\left(1 + \frac{lD_b}{LD_s}\right) = 0, \tag{3.4}$$

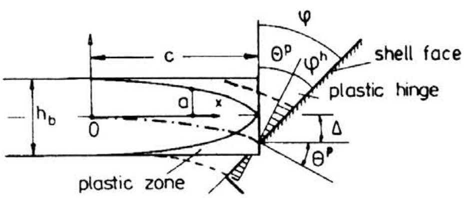

FIG. 5. Plastic zones in the partially yielding bottom plate.

where θ^p and Δ are, respectively, the edge rotation and displacement (Fig. 5) due to plastic deformation,

$$(3.5) \qquad \theta^p = \frac{1}{D_b}\int_0^c \left(\frac{M_e}{\eta} - M\right) dx\,, \qquad \Delta = \frac{1}{D_b}\int_0^c \left(\frac{M_e}{\eta} - M\right)(c-x)\, dx\,,$$

c is the length of the plastic zone and $\eta = 2a/h_b$ its dimensionless depth (Fig. 5). Since θ^p and Δ depend upon M_A and L, an iterative procedure is needed. However, in the loading process L is nearly constant and the influence of the term Δ appears to be negligible. Hence, the length of the plastic zone is

$$(3.6) \qquad c \cong \frac{L}{4}\left(3 - \sqrt{1 + \frac{8}{m}}\right), \qquad m = \frac{M_A}{M_e}\,.$$

For θ^p, a ready-to-use formula (see [4]) can be applied:

$$(3.7) \qquad \theta^p = \frac{M_e c}{4D_b}\left(\frac{1 - \sqrt{3 - 2m}}{m - 1} - \frac{m+1}{2}\right),$$

and Eqs. (3.3), (3.4) and (3.7) may be reduced to a single $p - M_A$ relationship. This stage of the loading process ends at $p = p_0$, when the edge cross-section becomes fully plastic ($M_A = M_p = 3M_e/2$) and a plastic hinge appears. The total inelastic rotation φ accumulated within the zone of partial yielding as well as in the hinge is

$$(3.8) \qquad \varphi = \theta^p + \varphi^h = \frac{pl^3}{4D_s}\left(1 - \frac{l}{H}\right) - \frac{M_p l}{2D_s} - \frac{M_p L}{6D_b} - \frac{2\Delta}{L}\,.$$

The depth of the zone without contact L is to be determined, as before, from Eq. (3.3). When displacement Δ is disregarded, simple formulae are available.

If the tank is unloaded starting from the maximum pressure $\overline{p}$, the maximum plastic rotation attained $\overline{\varphi}$ remains unchanged, whereas the bending moment along the joint rapidly decreases, as well as the length of the driven up zone. The continuity equation (3.4) becomes now

$$(3.9) \qquad \frac{p+g}{24D_b}L^3 + \frac{p+g}{8D_s}L^2 l - \frac{pl^3}{4D_s}\left(1 - \frac{l}{H}\right) + \overline{\varphi} = 0\,.$$

The process runs in an analogous way also if the unloading begins at $\overline{p} < p_0$. In this case Eq. (3.4) holds with the constant peak values θ^p and Δ.

If the unloading response is described with the use of the equivalent rotations concentrated in the hinge, the result will be similar to the response of sandwich structures, with the plate restoring its original flat form (Fig. 3c, solid line), and with $M_A = 0$ at the pressure

$$(3.10) \qquad p = p_1 = \frac{4D_s\overline{\varphi}}{L^3}\,.$$

Due to the finite spread of the plastic zone, the plate remains always deformed (Fig. 3c, dashed line). When the depth L decreases significantly, disregarding of the term Δ becomes unjustified and, finally, the previous solution ceases to be valid at $L < \overline{c}$, where $\overline{c}$ is the maximum spread of the plastic zone at $p = p_0$. The process of restoring the contact may continue until the edge reaction V_A

(3.11)
$$V_A = \frac{p+g}{2}L + \frac{M_A}{L}, \quad M_A < 0$$

(already negative now) reaches the value of the vertical load G, then the edge is driven up (Fig. 3d, 4b).

Let us consider the case when the depth of the driven-up zone exceeds the depth of the plastic zone $K > \overline{c}$. Bending moment at the point of the loss of contact must vanish and, therefore,

(3.12)
$$M_A = -GK - \frac{p+g}{2}K^2,$$

whereas the continuity condition for the edge rotations of the plate and the shell gives the equation for K

(3.13)
$$K^3\frac{p+g}{3D_b} + K^2\left(\frac{G}{D_b} + \frac{p+g}{4D_s}l\right) + \frac{KGl}{2D_s} + \frac{pl^3}{4D_s}\left(1 - \frac{l}{H}\right) = \overline{\varphi}.$$

The length K increases with p decreasing down to zero, but the validity of the solution is limited to the elastic response, it means until

(3.14)
$$-M_A > 2M_e - \overline{M}_A,$$

M_A being the edge moment at the maximum pressure $\overline{p}$. If this condition is not satisfied, plastic strains of opposite sign appear (Fig. 3e). It means the inadaptation at alternating plasticity under the load cycling in the range from 0 to $\overline{p}$.

4. Numerical examples

Calculations according to the solution given in Sect. 3 were performed for two standard oil tanks of capacity 100000 m^3 and 32000 m^3, their dimensions and mechanical characteristics are as follows:

For both tanks: $E = 21 \cdot 10^7$ MPa, $\sigma_Y = 250$ MPa, $\nu = 0.3$.

Tank 100000 m^3	Tank 32000 m^3
R = 4300 cm,	R = 2595 cm,
h_s = 3.2 cm, h_b = 1.7 cm,	h_s = 1.9 cm, h_b = 1.3 cm,
D_b = $0.945 \cdot 10^7$ Ncm,	D_b = $0.422 \cdot 10^7$ Ncm,
D_s = $6.30 \cdot 10^7$ Ncm,	D_s = $1.319 \cdot 10^7$ Ncm,
l = 91.26 cm,	l = 54.63 cm,
g = 0.0013 MPa,	g = 0.0010 MPa,
M_p = 18062 N,	M_p = 10562 N,
M_e = 12042 N,	M_e = 7043 N.

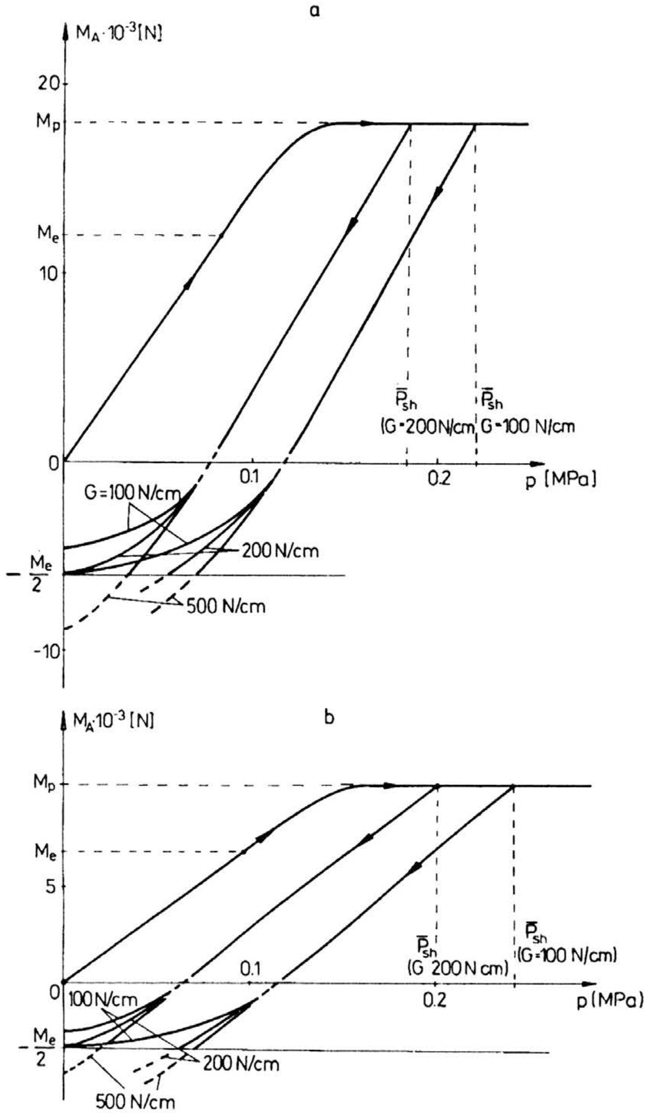

FIG. 6. Pressure-edge moment relationship $M_A = M_A(p)$ at loading and unloading: a—tank $V = 100000 \text{ m}^3$, b—tank $V = 32000 \text{ m}^3$.

Because the unloading process depends strongly upon the magnitude of the vertical load G, calculations have been performed for several values of G and the results are summarized in Fig. 6. Dashed segments of the unloading diagrams represent the stages, in which the obtained solutions are not strictly valid. The limit of applicability of the elastic analysis at unloading is $M_A = \overline{M}_A - 2M_e$. From this point at the axis $p = 0$ (normally it is $M_A = -M_e/2$) originates the unloading curve for the maximum pressure value $\overline{p} = \overline{p}_{\text{sh}}$ allowed by the requirement of shakedown after the first service cycle. The obtained values $\overline{p}_{\text{sh}}$ are compared with the elastic pressure p_e in the Fig. 7. It can be seen that, depending upon the value of the vertical load G, the shakedown limit can be either higher or lower than the classical value, i.e. the double of the elastic limit.

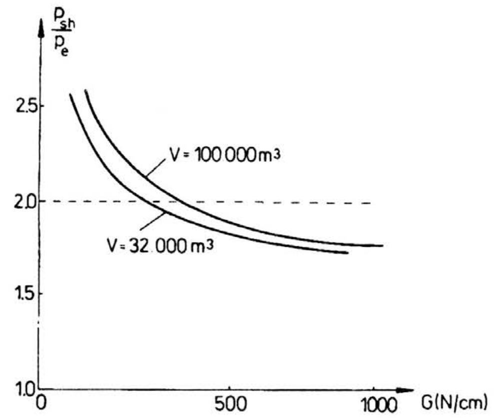

FIG. 7. Shakedown pressures depending upon the edge load G.

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