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Exact solutions of boundary value problems for two-velocity models of the Boltzmann equation

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EXISTENCE and uniqueness of solutions of boundary value problems (BVPs) with general boundary data for two-velocity models of the Boltzmann equation are investigated. Explicit examples of the nonunique positive solutions are given and their bifuraction from the unique solutions is discussed. Stability properties of the solutions are studied numerically.

Zbadano istnienie i jednoznaczność rozwiązań zagadnień brzegowych z ogólnymi warunkami brzegowymi dla dwuprędkościowych modeli równania Boltzmanna. Znaleziono przykłady niejednoznacznych rozwiązań dodatnich i przedyskutowano bifurkację rozwiązania jednoznacznego. Zbadano numerycznie stabilność rozważanych rozwiązań.

Исследованы существование и единственность решений граничных задач с общими граничными условиями для двухскоростных моделей уравнения Больцмана. Найдены примеры неоднозначных положительных решений и обсуждена бифуркация однозначного решения. Исследована численно стабильность рассматриваемых решений.

1. Introduction

ONE OF THE CENTRAL problems in the mathematical kinetic theory of gases, important for applications and for conceptual reasons, is the problem of existence and uniqueness of stationary solutions of the Boltzmann equation in physical domains for general boundary data.

Due to difficulties in dealing with the true Boltzmann equation, various models of the equation became popular. In this paper we solve a general boundary value problem for a class of the semilinear hyperbolic systems of equations which describe the so-called two-velocity models of the Boltzmann equation. The most popular example is the Carleman model; our results are applicable to the general class of the two-velocity models in one space dimension with admissible linear terms corresponding to sources (sinks). The system of equations similar to those, considered in this paper, occurs also in mathematical models of various physical phenomena in biological systems, chemicals reactions, ecology, binary gas mixtures [6].

We demonstrate the existence of positive solutions of the related boundary value problems for a general class of the boundary conditions, and indicate explicitly the submanifolds in the phase space of the parameters, characterizing the boundaries (boundary parameters), for which there are exactly two, one or zero solutions. We also discuss examples of existence of exactly two positive solutions for a fixed choice of the boundary data. The number of solutions depends on the numerical values of the boundary parameters. In the models discussed in this paper one can evaluate the corresponding solutions explicitly, due to the particularly simple collision operators.

We also discuss influence of the linear terms (sources, sinks) on the number of admissible solutions, and consider some further related problems, such as solutions in semibounded domains, and solutions of the relevant linearized problems.

Finally we discuss some numerical results related to the time evolution of the underlying IBVP (initial BVP) for the two-velocity models. The solutions of the considered BVPs can be compared with the time asymptotic limits of the related IBVPs in the cases in which such solutions exist. In this respect we discuss some numerical results for various choices of the initial and boundary data. In the cases of existence of more than one stationary state, we give some numerical indications related to their stability.

2. Formulation of the problem

We consider the following IBVP for the density functions $N_i(t,x)$, $i = 1,2$,

$$(2.1)_a \qquad \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right) N_1 = \alpha N_1^2 + \beta N_1 N_2 + \gamma N_2^2 + \varepsilon_1 N_1 + \varepsilon_2 N_2 \equiv Q\,,$$

$$(2.1)_b \qquad \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right) N_2 = -Q\,,$$

$$t \in [0,\infty)\,, \qquad x \in [-L, L]\,, \qquad L > 0\,.$$

Initial data

$$(2.1)_c \qquad N_i(0,x) = N_{i0}(x)\,, \qquad i = 1,2\,.$$

Boundary data

$$(2.1)_d \qquad N_1^- = \alpha_1 N_2^- + \beta_1\,,$$

$$(2.1)_e \qquad N_2^+ = \alpha_2 N_1^+ + \beta_2\,,$$

where

$$\alpha_i \in R^1_+\,, \qquad \beta_i \in R^1\,, \qquad \varepsilon_i \in R^1\,, \qquad i = 1,2\,,$$

and

$$(2.1)_f \qquad N_1^- \equiv N_1(t,-L)\,, \qquad N_2^- \equiv N_2(t,-L)\,,$$

$$(2.1)_g \qquad N_1^+ \equiv N_1(t,+L)\,, \qquad N_2^+ \equiv N_2(t,+L)\,,$$

with the requirement, that at $t = 0$ the boundary data match the initial data.

The physical model underlying the problem formulated above can be described in the following way. The system of equations (2.1) describes a set of particles, moving with the velocities ± 1 along the x-axis inside the interval $[-L, L]$. The functions $N_1(t,x)$, $N_2(t,x)$ are the density functions of the particles travelling with the velocities respectively $+1$ and -1. The particles undergo collisions between themselves and with the boundaries. The intermolecular interaction is described by the collision operator Q, whereas the collisions with the boundaries are characterized by the boundary conditions $(2.1)_d - (2.1)_g$. At the boundaries we allow multiplication, absorption, and boundary sources (sinks), described by the parameters α_1, α_2, β_1, β_2 in Eqs. $(2.1)_d$ and $(2.1)_e$. In the operator Q the quadratic terms describe the binary collisions, while the linear terms correspond to bulk sources or sinks. The parameters α_1, β_1, α_2, β_2 will be called the boundary parameters of the IBVP $(2.1)_a - (2.1)_g$. The choice of the rhs in Eqs. $(2.1)_a$, $(2.1)_b$ gives the conservation of the mass flux inside the considered space domain: by adding Eqs. $(2.1)_a$ and $(2.1)_b$ we obtain the continuity equation: $\varrho_t + (\varrho U)_x = 0$, where $\varrho = N_1 + N_2$, and $\varrho U = N_1 - N_2$. The system as a whole is of course not conservative in general, due to the boundary sources (sinks) and multiplication or absorption at the boundaries (except some particular cases, discussed below). The choice

$$\alpha = -1, \quad \beta = \varepsilon_1 = \varepsilon_2 = 0, \quad \gamma = 1$$

in Eqs $(2.1)_a$, $(2.1)_b$ corresponds to the Carleman model. Other interesting examples are the Ruijgrok-Wu model ($\alpha = 0, \beta = 1, \gamma = \varepsilon_1 = \varepsilon_2 = 0$) [3], and the McKean model ($\alpha = 0, \beta = -1, \gamma = 1, \varepsilon_1 = \varepsilon_2 = 0$) [4]. The parameters $\varepsilon_1, \varepsilon_2$ describe sinks and/or sources (in the Ruijgrok-Wu model—annihilation and creation of particles, cf. [3]).

The choice $\alpha_1 = \alpha_2 = 1$ and $\beta_1 = \beta_2 = 0$ in Eqs. $(2.1)_d - (2.1)_g$ corresponds to the specular reflection boundary conditions. For the Carleman model this case was treated by FITZGIBBON [1] by semigroup techniques, and by ILLNER and REED [2], who proved a global existence and uniqueness theorem for a large class of initial data, and investigated the time asymptotic limit of the solutions. We also note, that the IBVPs and the corresponding BVPs for general discrete velocity models were recently investigated by KAWASHIMA [7], who proved existence and stability theorem for various configurations of initial and/or boundary data (e.g. constant boundary data and initial data close to Maxwellian distribution functions).

In the first part of the paper we discuss the following BVP, related to Eqs. $(2.1)_a$ – $(2.1)_g$:

$$(2.2)_a \qquad \frac{d}{dx}N_1 = Q,$$

$$(2.2)_b \qquad -\frac{d}{dx}N_2 = -Q,$$

$$(2.2)_c \qquad N_1^- = \alpha_1 N_2^- + \beta_1,$$

$$(2.2)_d \qquad N_2^+ = \alpha_2 N_1^+ + \beta_2,$$

where now $N_i = N_i(x)$, $i = 1, 2$, $x \in [-L, L]$,

$$\alpha_i \in R^1_+, \qquad \beta_i \in R^1,$$

$$(2.2)_e \qquad N_1^- \equiv N_1(-L), \qquad N_2^- \equiv N_2(-L),$$

$$(2.2)_f \qquad N_1^+ \equiv N_1(+L), \qquad N_2^+ \equiv N_2(+L),$$

and the operator Q is defined in Eq. $(2.1)_a$.

In the next two sections we construct explicit solutions of Eqs. $(2.2)_a$ – $(2.2)_d$, then prove uniqueness of the solutions for some intervals of the boundary parameters and nonuniqueness for some other cases. We also discuss dependence of the solutions on the boundary parameters (bifurcations of the unique solutions).

3. Construction of the solution

The system $(2.2)_a$, $(2.2)_b$ of ODE can be solved explicitly. For simplicity we discuss in this section the case

$$\alpha + \beta + \gamma = 0, \quad \varepsilon_i = 0, \quad i = 1, 2,$$

corresponding e.g. to the Carleman or the McKean models. An example of the model for which the above assumptions is not satisfied, is the RUIJGROK-WU model [3], which will be discussed afterwards.

Under these assumptions, the general solution $(2.2)_a - (2.2)_b$ is

$$(3.1)_a \qquad N_1 = -\frac{b}{a} + \left(N_1^- + \frac{b}{a}\right) e^{a(L+x)},$$

$$(3.1)_b \qquad N_2 = N_1 - C,$$

where $C \neq 0$ and N_1^- are arbitrary constants (for physical applications N_1^- must be nonnegative), and

$$(3.1)_c \qquad a = -(\beta + 2\gamma)C, \quad b = \gamma C^2.$$

The case $C = 0$ gives $N_1 \equiv N_2$, which does not match the boundary conditions $(2.2)_c$, $(2.2)_d$, unless $\beta_i = 0$, $\alpha_i = 1$, $i = 1, 2$.

In general, the existence and the number of solutions of the BVP $(2.2)_a - (2.2)_d$ depends on the values of the boundary parameters of the problem, as will be seen below from the construction.

In the following we also assume

$$(3.1)_d \qquad \alpha_1 \neq 1 \wedge \alpha_2 \neq 1, \quad \alpha_1 > 0, \quad \alpha_2 > 0.$$

REMARK

The case $\alpha_1 = 1$, $\alpha_2 \neq 1$ and $\alpha_1 = \alpha_2 = 0$ lead to the unique solution for almost all pairs (β_1, β_2), while $\alpha_1 = \alpha_2 = 1$ gives in general an indeterminate problem, as can be checked out by a straightforward calculation. These, and other particular cases are discussed in Appendix.

We continue with the assumption $(3.1)_d$. To determine the constants N_1^-, C in Eqs. $(3.1)_a$, $(3.1)_b$ we proceed as follows. In our problem there exist five unknown constants: C, and the four constants $N_1^\pm$, $N_2^\pm$, related by $(2.2)_c$, $(2.2)_d$. Equation $(3.1)_b$ written at $x = \mp L$ give two additional equations

$$(3.2) \qquad N_2^- = N_1^- - C, \quad N_2^+ = N_1^+ - C.$$

The fifth equation is obtained from Eq. $(3.1)_a$, written at $x = +L$. After some algebra, this set of five equations can be reduced to the following two equations for C and N_1^-,

$$(3.3) \qquad e^{-2\delta LC} = \frac{\alpha_1 - 1}{\alpha_2 - 1} \cdot \frac{-\beta_2\delta + (\gamma - \alpha_2\gamma - \delta)C}{-\beta_1\delta + (\gamma - \alpha_1\gamma + \delta\alpha_1)C},$$

$$(3.3)_a \qquad N_1^- = \frac{\alpha_1 C - \beta_1}{\alpha_1 - 1},$$

where

$$(3.3)_b \qquad \delta = \beta + 2\gamma.$$

From this construction we see that the problem of the existence and uniqueness of solutions of the BVP $(2.2)_a - (2.2)_d$ is equivalent to the problem of the existence and uniqueness of the algebraic system (3.3), $(3.3)_a$.

Equation (3.3) can be rewritten in the following form

$$(3.4) \qquad f(x) \equiv e^{-nx} - t - \frac{m}{x+1} = 0,$$

where n, m, t are given functions of the boundary and collisional parameters

$$(3.4)_a \qquad n = -\frac{2L\delta^2\beta_1}{(\beta + \gamma)\alpha_1 + \gamma},$$

$$m = \frac{\alpha_1 - 1}{\alpha_2 - 1} \cdot \left[\frac{\beta_2}{\beta_1} - \frac{\gamma(1-\alpha_2) - \delta}{(\beta+\gamma)\alpha_1 + \gamma} \right], \tag{3.4$_b$}$$

$$t = \frac{\alpha_1 - 1}{\alpha_2 - 1} \cdot \left[\frac{(1-\alpha_2)\gamma - \delta}{(\beta+\gamma)\alpha_1 + \gamma} \right], \tag{3.4$_c$}$$

and

$$x = \frac{-C \cdot (\alpha_1 + 1)}{2\beta_1}, \quad \beta_1 \neq 0. \tag{3.4$_d$}$$

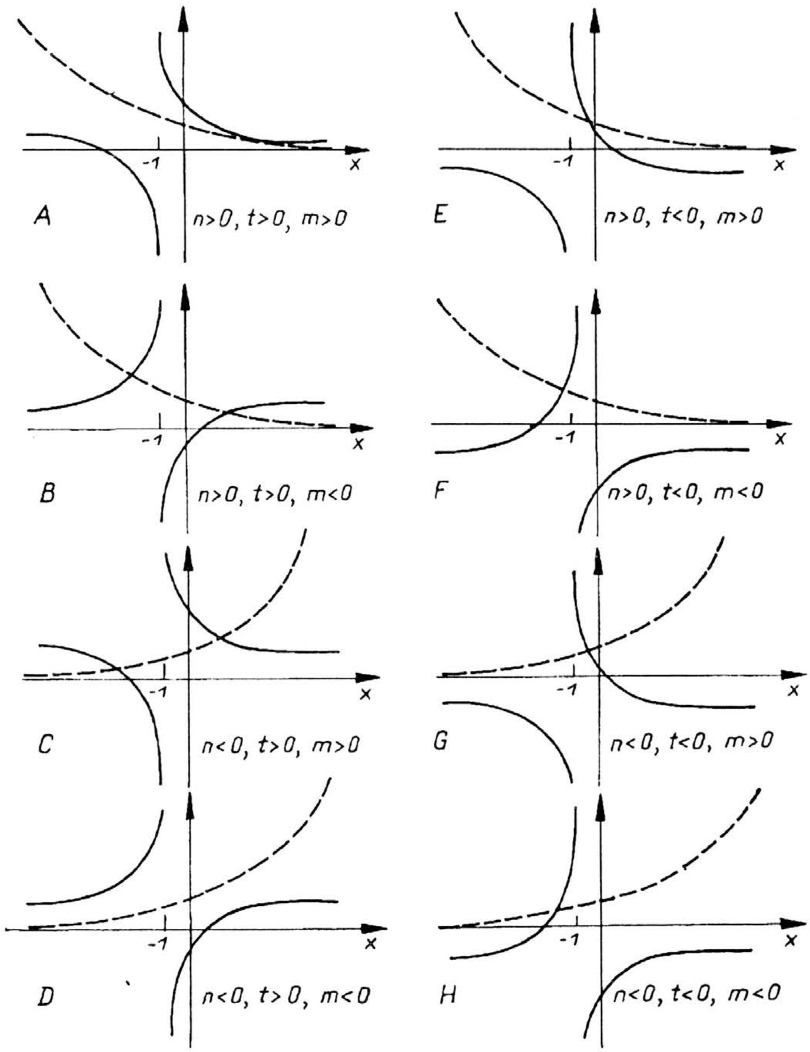

FIG. 1. Examples of graphics of e^{-nx} – – –, and $t + \dfrac{m}{1+x}$ ————.

The number of zeros of $f(x)$ in Eq. (3.4) gives the number of different solutions of the BVP $(2.2)_a - (2.2)_f$. In Fig. 1 we show various possibilities of existence of the solutions to Eq. (3.4). Each case corresponds to a submanifold in the phase space of the boundary parameters of the problem: α_1, α_2, β_1, β_2. By inspection of Fig. 1 we see that there are four possibilities. The number of solutions varies between zero and three; the latter case occurs for rather "narrow" intervals of the boundary data, corresponding e.g. to Fig. 1E, and originates from different convexity of the relevant functions (cf. formula $(4.1)_a$). Several statements related to the considered BVPs can now be easily proved. We prove

PROPOSITION 1. If

$$(\alpha_1 - 1)(\alpha_2 - 1)[\beta + \gamma(\alpha_2 + 1)][\beta\alpha_1 + \gamma(\alpha_1 + 1)] > 0 , \tag{3.5$_a$}$$

and

$$(\alpha_1 - 1)(\alpha_2 - 1)\{\beta_2[\beta\alpha_1 + \gamma(\alpha_1 + 1)] + \beta_1[\beta + \gamma(1 + \alpha_2)]\} > 0 , \tag{3.5$_b$}$$

then the BVP $(2.2)_a - (2.2)_d$ has a unique solution.

Proof

The conditions $(3.5)_a$, $(3.5)_b$ are equivalent, respectively, to $t < 0$, $m \cdot n < 0$, as can be easily seen from Eqs. $(3.4)_a - (3.4)_c$. The proof follows by inspection of the graphics in Fig. 1F, G. □

REMARK

For the Carleman and McKean models the sufficient conditions $(3.5)_a$, $(3.5)_b$ for the existence of the unique solution are equivalent to the conditions

$$(\alpha_1 - 1)(\alpha_2 - 1) > 0 , \qquad \beta_1(\alpha_2 + 1) + \beta_2(\alpha_1 + 1) > 0 ,$$

where the first inequality means that both walls simultaneously multiply or reduce the number of outcoming particles in the process of the collisions with the walls. Note that the second inequality is satisfied automatically if the boundary sources are positive, (cf. Eq. $(3.1)_d$). In the latter case we have checked numerically that if, in addition, the reflection coefficients $\alpha_i < 1$, $i = 1, 2$, then, in all the considered cases of boundary data, the solutions are positive, whereas if $\alpha_i > 1$, $i = 1, 2$, then there are no positive solutions. Physically, in the former case, absorption on the boundaries is compensated by the boundary sources, whereas in the latter case, the boundary sources and multiplication of particles due to collisions with the boundaries, prevent existence of a positive stationary state.

PROPOSITION 2. If

$$(\alpha_1 - 1)(\alpha_2 - 1)[\beta + \gamma(\alpha_2 + 1)] \cdot [\beta\alpha_1 + \gamma(\alpha_1 + 1)] < 0 , \tag{3.5$_c$}$$

and

$$(\alpha_1 - 1)(\alpha_2 - 1)\{\beta_1[\beta + \gamma(1 + \alpha_2)] + \beta_2[\beta\alpha_1 + \gamma(1 + \alpha_1)]\} > 0 , \tag{3.5$_d$}$$

then the BVP $(2.2)_a - (2.2)_d$ has exactly two solutions.

Proof

As can be easily seen, the inequalities $(3.5)_c$ and $(3.5)_d$ are equivalent to respectively $t > 0$ and $m \cdot n < 0$. The proof follows immediately by inspection of the graphics in Fig. 1B, C. □

Below we specify these general sufficient conditions for the cases of the most popular models, i.e. the Carleman and the McKean models.

PROPOSITION 2a. If

$$(\alpha_1 - 1)(\alpha_2 - 1) < 0\,, \qquad \beta_1(\alpha_2 + 1) + \beta_2(\alpha_1 + 1) < 0\,, \tag{3.6$_a$}$$

then there exist exactly two solutions of the BVP $(2.2)_a - (2.2)_d$ with the Carleman collision operator.

PROPOSITION 2b. If

$$(\alpha_1 - 1)(\alpha_2 - 1) < 0\,, \qquad \beta_1\alpha_2 + \beta_2 < 0\,, \tag{3.6$_b$}$$

then there exist exactly two solutions of the BVP $(2.2)_a - (2.2)_d$ with the McKean collision operator.

REMARK

The first inequalities in $(3.6)_a$, $(3.6)_b$ correspond to the situations in which one of the walls multiplies the number of particles in the collision with the wall, whereas on the other wall prevails absorption—the collisions with the wall reduce the number of outcoming particles. The second inequalities in $(3.6)_a$, $(3.6)_b$ require at least one of β_i to be negative, i.e. a boundary sink on one of the boundaries.

An examination of the graphics in Fig. 1A, D, corresponding to

$$t > 0\,, \qquad m \cdot n > 0\,, \tag{3.6$_c$}$$

indicates that there are other possibilities for the existence of more than one solution. Of course, in these cases there are also possibilities of zero or one solution, cf. bifurcation of the solutions, discussed in the next section.

PROPOSITION 3a. If

$$n > 0\,, \qquad t > 0\,, \qquad m > 0\,, \qquad t + m < 1\,, \tag{3.6$_d$}$$

then there exist exactly two solutions of the BVP(2).

PROPOSITION 3b. If

$$n < 0\,, \qquad t > 0\,, \qquad m < 0\,, \qquad t + m > 1\,, \tag{3.6$_e$}$$

then there exist exactly two solutions of the BVP(2).

As can be easily seen, for the Carleman and for the McKean models these conditions again require at least one β_i to be negative, i.e. a boundary sink on one of the wall. In the next section we discuss examples, in which there exist exactly two positive solutions for all the boundary parameters positive.

4. Numerical results

First we consider the Carleman and the McKean model with all the boundary parameters positive. This implies $n < 0$, and $m \cdot t < 0$, as can be seen from Eqs. $(3.4)_a - (3.4)_c$. These inequalities define submanifolds in the 4-dimensional space of boundary parameters, in which we should look for nonunique positive solutions with all the boundary data positive. Inspection of Fig. 1 indicates, that the only possibility to obtain nonunique solutions with all the boundary parameters positive, is the configuration shown in Fig. 1D.

Positivity of the resulting distribution functions in these and all other cases must then be checked from the analytic expressions in Eqs. $(3.1)_a - (3.1)_c$. Below we give results for some choices of the boundary data, and normalization $L = 1$.

A. The Carleman model,

$$(4.1)_a \qquad \alpha_1 = 1.05\,, \quad \alpha_2 = 0.7\,, \quad \beta_1 = 0.07\,, \quad \beta_2 = 0.2\,.$$

The corresponding positive solutions of Eqs. $(2.2)_a$, $(2.2)_b$ are shown in Fig. 2A. Note that in this case all the boundary parameters are positive.

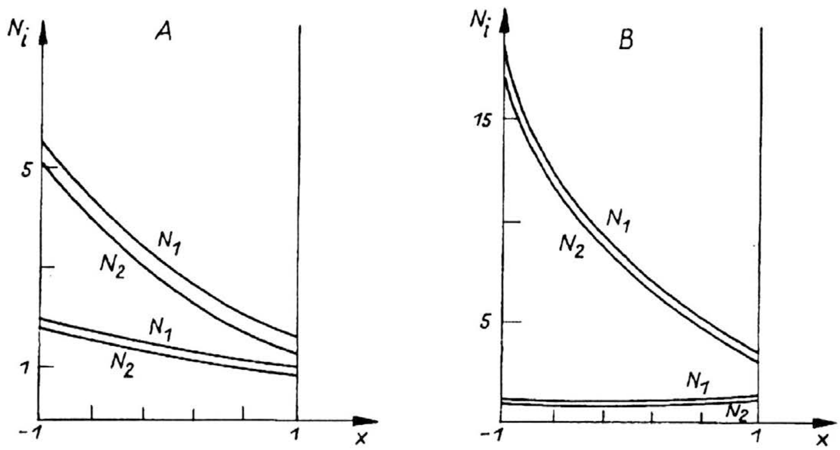

FIG. 2A. Two positive solutions, Carleman model. B. Two positive solutions, McKean model.

B. The McKean model,

$$(4.1)_b \qquad \alpha_1 = 1.05\,, \quad \alpha_2 = 0.7\,, \quad \beta_1 = 0.05\,, \quad \beta_2 = 0.2\,.$$

The corresponding positive solutions $(2.2)_a$, $(2.2)_b$ are shown in Fig. 2B. The boundary parameters are also positive. Finally we give an example, in which there exist exactly two positive solutions of the BVP $(2.2)_a$, $(2.2)_b$ in spite of a boundary sink on the left wall.

C. The Carleman model,

$$(4.1)_c \qquad \alpha_1 = 1.1\,, \quad \alpha_2 = 0.2\,, \quad \beta_1 = -0.1\,, \quad \beta_2 = 0.2\,.$$

In this case there are two solutions of Eq. (3.3)

$$(4.1)_d \qquad C_1 \cong -0.8\,, \quad C_2 \cong 0.58\,.$$

The resulting distribution functions, i.e. the solutions of Eqs. $(2.2)_a$, $(2.2)_b$ are positive (cf. [5a]), and correspond to the configuration of Fig. 1C, as can be seen by a straightforward calculation.

4.1. Bifurcation of the solutions

The space B of the boundary parameters is 4-dimensional. Analysis of Fig. 1A, D, E, H indicates possibilities of bifurcation of solutions in some submanifolds of B. Below we discuss 1-dimensional submanifolds, i.e. we fix three boundary parameters, and discuss behaviour of the solutions for the varying fourth parameter. We choose the Carleman model.

1. We fix

$$\alpha_2 = 0.7\,, \quad \beta_1 = 0.07\,, \quad \beta_2 = 0.2\,.$$

Then there exists $1.05 < \alpha_0 < 1.051$ so that for $\alpha_1 = \alpha_0$ the BVP $(2.2)_a - (2.2)_d$ has exactly one solution, the solution is positive, and for $\alpha_1 > \alpha_0$ there are no solutions. In physical terms, multiplication on the left boundary becomes too large to maintain a steady configuration. For $1 < \alpha_1 < \alpha_0$ there are two (positive) solutions. In Fig. 3A we plot the total densities ϱ of the solutions, where

$$\varrho = \int_{-L}^{L} (N_1(x) + N_2(x))\, dx\,.$$

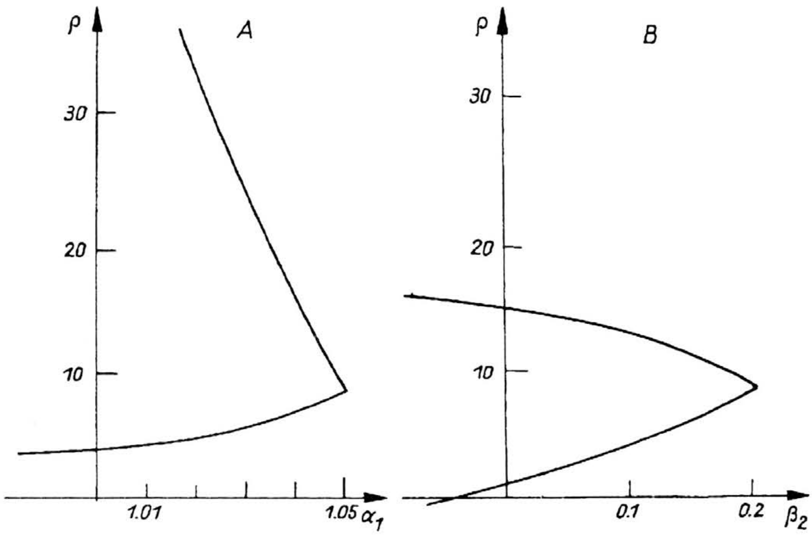

FIG. 3. Bifurcation of the unique solution.

2. We fix

$$\alpha_1 = 1.05\,, \qquad \alpha_2 = 0.7\,, \qquad \beta_1 = 0.07\,.$$

Then there exists $0.2 < \beta_0 < 0.201$ so that for $\beta_2 = \beta_0$ the considered BVP has exactly one solution, the solution is positive, and for $\beta_2 > \beta_0$ there are no solutions. In physical terms, the source on the right boundary is too strong to maintain a steady configuration. For $\beta_2 < \beta_0$ there are two solutions. Their total densities are shown in Fig. 3B. Note that one of the solutions becomes negative for a certain value of the boundary sink on the right boundary.

A careful examination of Fig. 1E indicates possibilities of three solutions of (3.4) for rather specific configurations of the boundary data. We found some intervals of the boundary data, which lead to three positive solutions of the BVP $(2.2)_a - (2.2)_d$. This comes from different convexity of the graphics of the relevant functions in Fig. 1E, and happens e.g. for the Carleman model, if

$$(4.1)_c \qquad \alpha_1 = 1.01\,, \quad \alpha_2 = 1.01\,, \quad \beta_1 = -0.8\,, \quad \beta_2 = -0.9\,.$$

Several interesting problems related to stability of these and, other nonunique solutions, and to the underlying time-dependent problems, are under investigation.

5. Influence of linear source and sink-type terms

To investigate the influence of the linear terms of the source and sink type on the number of admissible solutions we discuss below the following generalization of the Carleman model

$$(5.1)_a \qquad \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right) N_1 = -N_1^2 + N_2^2 + \varepsilon_1 N_1 + \varepsilon_2 N_2 \equiv Q\,,$$

$$(5.1)_b \qquad \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right) N_2 = -Q\,,$$

$$x \in [-L, L]\,, \qquad L > 0\,,$$

with the boundary data:

$$(5.1)_c \qquad N_1^- = \alpha_1 N_2^- + \beta_1\,,$$

$$(5.1)_d \qquad N_2^+ = \alpha_2 N_1^+ + \beta_2\,,$$

where

$$(5.1)_e \qquad \alpha_i \in R^1_+\,, \qquad \beta_i \in R^1\,, \qquad i = 1, 2\,,$$

and

$$(5.1)_f \qquad N_1^- \equiv N_1(-L)\,, \qquad N_2^- \equiv N_2(-L)\,,$$

$$(5.1)_g \qquad N_1^+ \equiv N_1(+L)\,, \qquad N_2^+ \equiv N_2(+L)\,.$$

This model is a particular case of the general two-velocity model $(2.1)_a$, $(2.1)_b$.

The procedure similar to that discussed above leads to the result, that the problem of the existence and uniqueness of solutions to Eqs. $(5.1)_a - (5.1)_d$ can be reduced to the same problem for the equation

$$(5.2) \qquad se^{-nC} = t + \frac{C - m}{C^2 + pC + q}\,,$$

where n, m, s, t, p, q are given algebraic functions of the boundary parameters. Analysis of Eq. (5.2) indicates that for some intervals of the boundary parameters there exist more than one (at most four) solutions. We omit details.

The linear terms of the type introduced above appear e.g. in the WU–RUIJGROK model [3] of interactions of elementary particles. Below we briefly discuss stationary solutions of this model.

6. BVP for the Ruijgrok–Wu model

The RUIJGROK–WU model [3] has different mathematical properties and physical interpretation from the models considered in the previous sections. The mathematical differences come from the form of the nonlinearity in the collision term in Eq. $(2.1)_a$. Therefore it seems of some interest to study, how these differences pronounce on the level of exact solutions of the boundary value problems. In this section we comment on the following BVP for the Wu–Ruijgrok model

$$(6.1)_a \qquad \frac{d}{dx} N_1 = N_1 N_2 + \varepsilon_1 N_1 + \varepsilon_2 N_2 \equiv Q_1\,,$$

$$(6.1)_b \qquad -\frac{d}{dx} N_2 = -Q_1\,,$$

$$(6.1)_c \qquad N_1^- = \alpha_1 N_2^- + \beta_1 ,$$

$$(6.1)_d \qquad N_2^+ = \alpha_2 N_1^+ + \beta_2 ,$$

where

$$N_i = N_i(x), \quad i = 1,2, \quad x \in [-L, L], \quad \varepsilon < 0, \quad \varepsilon_2 > 0, \quad \alpha_i \in R^1_+, \quad \beta_i \in R^1 .$$

From Eq. $(6.1)_a$, $(6.1)_b$ we obtain

$$(6.2) \qquad \frac{d}{dx} N_1 = N_1^2 + \lambda N_1 - \varepsilon_2 C ,$$

where $\lambda = \varepsilon_1 + \varepsilon_2 - C$, C arbitrary.

Analytical form of the solution of Eq. (6.2) depends on the sign of the parameter $\Delta = \lambda^2 + 4\varepsilon_2 C$. For $\Delta > 0$ we obtain

$$(6.3) \qquad N_1 = \frac{N_a + N_b A e^{(N_a - N_b)x}}{1 + A e^{(N_a - N_b)x}} ,$$

where

$$(6.4) \qquad N_{a,b} = \frac{-\lambda \mp \sqrt{\Delta}}{2} ,$$

A is integration constant. The parameters C, A can be determined from Eqs. $(6.1)_c$, $(6.1)_d$. We omit details.

For $\Delta < 0$ the solution reads

$$(6.5) \qquad N_1 = \frac{S \cdot \tan[S(x + A)/2] - b}{2a} , \qquad S = \sqrt{-\Delta} ,$$

where A can be obtained from the relevant boundary conditions. As in the models considered previously, the number of solutions depends on the boundary parameters and on ε_1 and ε_2, and nonuniqueness can be seen from the analytical form of the solution. More detailed study of the relevant solutions is left as an exercise.

REMARK

Recently the Ruijgrok–Wu model has been studied in [8] in the frame of the so-called extended kinetic theory, in which the test particles are allowed to interact with a fixed background medium, and to undergo absorption or fission-like reactions, along with the usual elastic scattering.

7. BVP on a semiline

In the previous examples the particles were confined to the bounded domain. As one could expect, in the case of unbounded domains the results are qualitatively different. As an example we consider the following BVP on a semiline $[-L, \infty]$ for the general two-velocity model:

$$(7.1)_a \qquad \frac{d}{dx} N_1 = Q ,$$

$$(7.1)_b \qquad -\frac{d}{dx} N_2 = -Q ,$$

$$(7.1)_c \qquad N_1^- = \alpha_1 N_2^- + \beta_1 ,$$

where $N_i = N_i(x)$, $i = 1, 2$, $X \in [-L, \infty]$, Q is defined as in Eq. $(2.1)_a$,

$$(7.1)_d \qquad Q = \alpha N_1^2 + \beta N_1 N_2 + \gamma N_2^2 + \varepsilon_1 N_1 + \varepsilon_2 N_2 .$$

We easily prove, by solving the system of two equations: $(3.1)_b$ written at $x = -L$, and $(7.1)_c$, the following

PROPOSITION 4. If

$$\alpha_1 \neq 1\,, \quad \lim_{x\to\infty}(N_1 - N_2) = C_\infty \neq 0\,, \tag{7.2}$$

where C_∞ is a prescribed constant (flux at infinity), then the BVP $(7.1)_a - (7.1)_d$ has a unique solution. The solution is given by Eqs. $(3.1)_a$, $(3.1)_b$ with $C = C_\infty$, and

$$N_1^- = \frac{\beta_1 - \alpha_1 C_\infty}{1-\alpha_1}\,. \tag{7.2$_a$}$$

Note that in the case of $\alpha_1 = 1$ and arbitrary β, the method does not work, and in general we obtain a one-parameter family of solutions.

REMARK

Nonuniqueness of the BVPs considered in this paper is generated by the boundary conditions rather than by the nonlinear structure of the collision operator, as can be seen from the following example

$$\frac{d}{dx}N_1 = aN_1 + bN_2 \equiv Q_2\,, \tag{7.3$_a$}$$

$$-\frac{d}{dx}N_2 = -Q_2\,, \tag{7.3$_b$}$$

$$N_1^- = \alpha_1 N_2^- + \beta_1\,, \tag{7.3$_c$}$$

$$N_1^+ = \alpha_2 N_2^+ + \beta_2\,, \tag{7.3$_d$}$$

where $N_i = N_i(x)$, $i = 1, 2$, $x \in [-L, L]$, $a, b \in R^1$.

Proceeding as in the main example we construct (for $a + b \neq 0$, $\beta_1^2 + \beta_2^2 \neq 0$) two solutions of Eqs. $(7.3)_a-(7.3)_d$ for certain intervals of the boundary data. The case $a+b = 0$ corresponds e.g. to the Carleman model, linearized around a constant Maxwellian. In this case the solution is unique, as can be checked by elementary calculations. We omit details.

8. Initial boundary value problem

In this section we present results of a numerical study of time evolution for the IBVP $(2.1)_a - (2.1)_g$. Having in mind stability problems, it seems that the most interesting configurations of the boundary parameters are those, for which there are at least two positive solutions of the related BVP $(2.2)_a - (2.2)_d$. As an example, we report results for the Carleman model with the boundary data

$$\alpha_1 = 1.1\,, \quad \alpha_2 = 0.2\,, \quad \beta_1 = -0.1\,, \quad \beta_2 = 0.2\,, \tag{8.1}$$

for which there are exactly two positive solutions of the BVP (2) as discussed in the section on numerical results. We started with the constant initial data $N_{i0}(x) = 3$, $i = 1,2$, $x \in [-1, 1]$. Using an implicit numerical scheme, the time evolution of the data has been investigated. In the initial stages of the evolution, the distribution functions $N_i(t,x)$ approach the "upper" stationary state with larger total density. Then the solution reaches asymptotically the "lower", stable stationary state. Similar phenomena has been noticed for other choices of the boundary data. Some mathematical problems related to the stability of the solutions are under investigation.

We also found, that the time evolution from negative (partly or fully) initial data may lead to a stationary positive state. On the other hand, positive initial data do not guarantee in general, that the corresponding final, stationary state (if it exists) is also positive, as we have checked for several configurations, e.g. for the Carleman model with the boundary data $\alpha_1 = 0.5$, $\alpha_2 = 0.7$, $\beta_1 = -0.05$, $\beta_2 = -0.02$, and the initial data $N_1 = N_2 = 0.5$.

From the physical point of view, existence of boundary sinks (i.e. negative values of the parameters β_i) imposes some constraints on the initial distribution functions. In order to guarantee their positivity we can not start from "too small" initial data, otherwise e.g. N_1^- or N_2^- in Eqs. $(2.1)_d$ or $(2.1)_e$ may take negative values.

Of interest are also the boundary data, for which the considered BVPs have no solutions. Preliminary numerical calculations of the underlying IBVPs indicate, that the sup norm of the solutions tends to infinity with time (the particles accumulate in the domain).

Finally we note, that in the considered models one can calculate explicitly the local entropy profiles inside the domain, $E(t,x) = \sum_{i=1}^{2} N_i \ln(N_i)$ and the total entropy of the system, $E(t) = \int_{-L}^{L} E(t,x)\,dx$. Various problems related to the entropy production in the corresponding IBVPs are under investigation.

Appendix

For $\alpha_1 = \alpha_2 = 1$, $\beta_1 = \beta_2 = 0$ any pair (C, C) solves the IVP $(2.2)_a - (2.2)_d$ with the Carleman collision operator. Any solution is in this case of the form (C, C). For $\alpha_1 = \alpha_2 = 0$ there is always a unique solution, which of course can be written explicitly.

For $\beta_1 = \beta_2 = 0$, and $\alpha_1 \neq 1$, $\alpha_2 \neq 1$ there is either one or zero solutions, depending on the solutions of the equation

$$\text{(A.1)} \qquad e^{-2(\beta+2\gamma)LC} = \frac{-(1-\alpha_1)[\beta + \gamma(1+\alpha_2)]}{(1-\alpha_2)[\beta\alpha_1 + \gamma(1+\alpha_1)]}.$$

This equation has one or zero solutions, depending on the values of α_1, α_2 and the considered collision operator. In particular, for the Carleman and the McKean models, one can easily see that the (unique) solution exists if $(1-\alpha_1)(1-\alpha_2) < 0$, i.e. if one wall multiplies the number of particles in the collisions, whereas on the other one the absorption prevails. The solution is given by Eqs. $(3.1)_a$, $(3.1)_b$ with $N_1^- = (C\alpha_1)/(\alpha_1 - 1)$, where C solves Eqs. (A.1). In this case without boundary sources, if $(1-\alpha_1)(1-\alpha_2) > 0$ then there are no stationary states. It means, that if both walls simultaneously produce or absorb particles, then, in order to reach a nonzero stationary state, one should introduce boundary sinks or sources—the result, which is clear from the physical point of view. The existence of boundary sources increases the size of the intervals of values of α_i, for which stationary states can exist. We also note that in the considered BVPs the conditions of mass conservation on the boundaries, $\sum_{i=1}^{2} N_i^- u_i = 0 = \sum_{i=1}^{2} N_i^+ u_i$, $u_1 = -u_2 = 1$ are equivalent to the specular reflection boundary conditions. For the periodic boundary conditions, the solutions of the BVP $(2.2)_a - (2.2)_d$ depend on one free parameter.

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