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Elastic-plastic states in solidifying castings(*)

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IN THIS PAPER the thermo-elastic-plastic model for determination of stresses in a solidifying casting is proposed. The temperature fields, solidification rates, stresses and strains have been determined for a solidifying axially-symmetrical casting. The effect of plastic strains on the instant of generation and size of shrinkage gap has been discussed. The problems have been solved numerically by means of the finite element method.

W pracy zaproponowano model termo-sprężysto-plastyczny do wyznaczania naprężeń w krzepnącym odlewie (ciele narastającym). Wyznaczono pola temperatury, prędkości krzepnięcia, naprężenia i odkształcenia w krzepnącym i stygnącym odlewie osiowo-symetrycznym. Przedyskutowano wpływ odkształceń plastycznych na moment powstawania i wielkość szczeliny skurczowej. Zadania rozwiązano metodą elementów skończonych.

В работе предложена термо-упруго-пластическая модель определения напряжений в затвердевающей отливке (наращиваемом теле). Определены температурные поля, скорости затвердевания, напряжения и деформации в затвердевающей и остывающей осесимметричной отливке. Обсуждено внияние пластических деформаций на момент возникновения и величину усадочного (газового) зазора. Задачи разрешены численным методом конелных элементов.

1. Introduction

IN THE CASE of casting solidification one deals with the problems pertaining to high temperatures and their high gradients, which produce considerable thermal loads resulting in stresses generated both in the casting and chill mould. The value of stresses produced in the casting/mould system influences the instant of generation and development of shrinkage gap (i.e., region in which the separation of casting from the mould occurs) that, in turn, affects the rate of thermal load changes. Furthermore, the knowledge of the level of stresses generated in the solidifying casting supplies some information about the control of the process of founding and enables one to avoid the development of cracks in castings being crystalized during and after their solidification. The information about the level of stress produced by thermal loads enables one to estimate whether the chill mould will be damaged in the process of casting or adapt to the repeated thermal loads during its multiple

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use. The knowledge of stresses in solidifying castings is then very important in the foundry practice. The cognitive aspects of the solidification mechanics play also a meaningful role for the growth of solid phase occurring in a solidifying casting differentiates this problem from the classical formulations of thermomechnics. Moreover, the problem of formulation of adequate constitutive relationships for solidifying material appears also in this case. This problem is, however, neglected in the present work by assuming the casting or the chill mould material to be elastic-plastic. In the process of solidification three fundamental groups of phenomena are distinguished, and namely the thermal, diffusive and mechanical ones. These phenomena are coupled with each other [1, 2, 4] but assuming, according to the majority of researchers dealing with this problem, that the thermal phenomena dominate in this process, the thermo-mechanical coupling is disregarded in this study. Therefore, one considers the mechanical phenomena produced by thermal loads but not the opposite effects. The continuity conditions at the solidification front (cf. [1, 8, 18]) are not fully considered in this work. It is assumed that the stresses transferred from liquid to the solid phase are equal to zero, if the liquid is not subject to extermal pressure (die casting pressure). The problem of proper determination of stress on the solidification surface, as seen from the side of solid, is not yet, as it seems, finally solved.

In considering the stresses in solidifying castings the paper by B. A. BOLEY and J. H. WEINER [19] should be mentioned, where the elastic-plastic model was applied to a solidifying casting. The reviews of other works in this field are presented in Refs. [9, 14, 17].

The present paper is a continuation of previous attempts presented in earlier works [13, 14] but extended by the application of the elastic-plastic model which is mainly based on papers [7, 11, 16] by B. RANIECKI.

The consideration concerning the numerical solutions is founded on the work by M. KLEIBER [5]. The concept applied does not considerably differ from that described by M. KLEIBER and T. NIEZGODA [6] concerning the thermo-elastic-plastic states in the axially-symmetrical problem.

In that paper the solidification of axi-symmetrical casting was considered. The temperature field, the solidification rates, the stress and strain fields developing during solidification and cooling of the casting were determined. The influence of plastic strains on initial development of the shrinkage gap was also studied.

Consider a casting (solidifying material) $\mathscr{V}$. The liquid phase $\mathscr{V}_L$ is separated from the solid phase $\mathscr{V}_S$ by solidification surface $s(t)$. The solid phase/mould or solid phase air (or shrinkage gap $\mathscr{V}_G$) interface is denoted by Γ_S, the liquid phase/surrounding (or mould) interface by Γ_L, and the mould/surrounding interface by Γ_0 (Fig. 1).

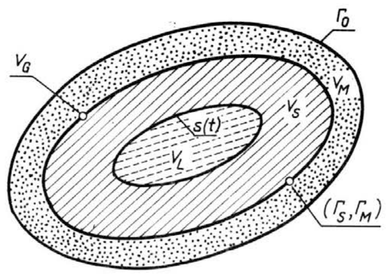

FIG. 1. Casting in a mould. Identification of subregions.

The particles of casting are identified by spatial coordinates x^α referred to the instantaneous configuration β_t [1, 18]. The time of solidification, t, of individual particles of casting x^α is denoted by

(1.1) $$t = \psi(x^\alpha).$$

For a fixed time, t, the points x^α determined from Eq. (1.1) describe the solidification surface $s = s(t)$. The projections of velocity of points of surface s on the normal n_α are independent of the method of parametrization,

(1.2) $$n_\alpha \dot{x}^\alpha = \omega,$$

and are called the solidification velocity.

2. Mathematical model of solidification

The differential equations describing the thermal field in the process of solidification of a casting and its mould (region $\mathcal{V}_M \cup \mathcal{V}_S \cup \mathcal{V}_L$) are assumed in the following form [1, 2, 15]:

(2.1) $$\nabla_\alpha(\underset{r}{\lambda}\partial_\alpha \underset{r}{\Theta}) - (\rho \underset{r}{c_{ef}})\dot{\underset{r}{\Theta}} = 0,$$

where $c_{ef}(\Theta)$ is the so-called effective thermal capacity, ∇_α is the operator of covariant differentiation, $\alpha = 1, 2, 3$ — spatial coordinates, r denotes, respectively: M — mould, L— liquid phase, S — solid phase of casting, λ — thermal conductivity, ρ — density.

The substantial derivative ($\dot{\Theta}$) is identified with the partial derivative with respect to time ($\dot{\Theta} = \partial\Theta/\partial t$), thus neglecting the effect of convective motions on the solution. In order to simplify the notation, subscript (r) is also neglected, bearing in mind that the equation of heat condution concerns the region of mould, liquid and solid phase of casting. The majority of industrial steel grades are the alloys that crystallize within the range of temperature between the liquidus and solidus. For non-uniform temperature distribution in the casting cross-section the presence of crystallization zone produces a blurred crystallization front containing a mixture of dendritic crystals and liquid, called the biphase zone. If the heat of crystallization is produced within the range of temperature ($\Theta_l - \Theta_s$), then the effective thermal capacity (c_{ef}) for individual phase [4, 15] will be:

(2.2)
$$c_{ef}(T) = \begin{cases} C_l(\Theta), & \text{for } \Theta > \Theta_{liq}, \\ C_{sol} - L\dfrac{d\phi}{d\Theta}, & \text{for } \Theta_{sol} < \Theta < \Theta_{liq}, \\ C_s(\Theta), & \text{for } \Theta < \Theta_{sol}, \end{cases}$$

$\phi = (V_s/V_0)$ is the volume fraction of solid phase in the biphase region is the Lafent heat of fusion.

In the general case parameter Φ is determined from the kinetics of crystal growth at the solidification front. For sufficiently large castings one may assume that overcooling of the alloy at the crystallization front is small and the fraction of solid phase may be approximately determined from the equilibrium system of the alloy considered. For binary systems the fraction of solid phase is determined by the lever arm principle [4] or may be calculated from the equations of diffusion [2, 15].

Equations (2.1) are completed with:

(*i*) *initial conditions*

(2.3)
$$\underset{r}{\Theta}(x^\beta, 0) = \underset{r}{\Theta}_0(x^\beta),$$

(*ii*) *boundary conditions*

at the mould/surroundings boundary

(2.4)
$$\lambda_M \frac{\partial \Theta_M}{\partial_n} = -\alpha_\theta(\Theta_M|_{\Gamma_0} - \Theta_{sur}),$$

at the casting/surroundings boundary

(2.5)
$$\underset{r}{\lambda}\frac{\partial \Theta_r}{\partial_n} = -\alpha_\theta(\underset{r}{\Theta}|_{\Gamma_0} - \Theta_{sur}),$$

at the casting/shrinkage gap protective coating/mould boundaries

(2.6)
$$\lambda_S \frac{\partial \Theta_S}{\partial n}\Big|_{\Gamma_0} = \frac{\lambda_T}{\delta_T}\,(\Theta_S|_{\Gamma_S} - \Theta_M|_{\Gamma_M}),$$
$$\lambda_M \frac{\partial \Theta_M}{\partial n}\Big|_{\Gamma_F} = \frac{\lambda_T}{\delta_T}\,(\Theta_S|_{\Gamma_S} - \Theta_M|_{\Gamma_M}),$$

where α_Θ is the surface film conductance, λ_T — equivalent thermal conductivity of protective coating, δ_T ($\delta_T = \delta_C + \delta_G$) is the total thickness of protective coating (δ_C) and the width of shrinkage gap (δ_G).

(iii) Continuity conditions on the solidification surface for the model with a sharp solidification front [1, 2, 3]

(2.7)
$$\lambda_S \frac{\partial \Theta_S}{\partial n}\bigg|_{s(t)} - \lambda_L \frac{\partial \Theta_L}{\partial n}\bigg|_{s(t)} = \rho_s L \omega,$$
$$\Theta_S\bigg|_{s(t)} = \Theta_L\bigg|_{s(t)} = \Theta_{kr}(C),$$

where the freezing point Θ_{kr} is dependent on concentration of the substitute of alloy (C) at the solidification front.

3. Mathematical model of stress generation in castings

Let us derive the Cauchy relations for the growing solids (solidifying casting), with the linear part [1, 13, 14]:

(3.1)
$$\varepsilon_{\alpha\beta} = \frac{1}{2}(\nabla_\beta u_\alpha + \nabla_\alpha u_\beta + \dot{u}_\alpha\bigg|_{\psi(\chi\alpha)} \psi_{,\beta} + \dot{u}_\beta\bigg|_{\psi(\chi\alpha)} \psi_{,\alpha}),$$

where $\dot{u}^\alpha = du^\alpha/dt$ is the displacement rate.

By differentiating Eq. (3.1) with respect to time one gets

(3.2)
$$\dot{\varepsilon}_{\alpha\beta} = \frac{1}{2}(\nabla_\beta \dot{u}_\alpha + \nabla_\alpha \dot{u}_\beta).$$

Therefore, for time $t = \psi(x^\alpha)$. one obtains

(3.3)
$$\varepsilon_{\alpha\beta}\bigg|_{\psi(x\alpha)} = 0,$$

that is the initial condition, expressed in strains, for the already solidified part of casting (solid phase).

The constitutive relations are assumed in the classical form

(3.4)
$$\mathbf{T} = \mathbf{E} \circ \mathbf{e}'',$$

where

(3.5) $$\mathbf{e}^e = \mathbf{e} - \mathbf{e}^p - \mathbf{e}^\theta.$$

$\mathbf{T} = \mathbf{T}(\sigma^{\alpha\beta})$	tensor of stresses,
$\mathbf{E} = \mathbf{E}(E^{\alpha\beta\gamma\delta}(\theta))$	tensor of material constants,
$\mathbf{e}^e = \mathbf{e}(\varepsilon^e_{\alpha\beta})$	tensor of elastic strains,
$\mathbf{e} = \mathbf{e}(\varepsilon_{\alpha\beta})$	tensor of total strains,
$\mathbf{e}^\theta = \mathbf{e}(\varepsilon^\theta_{\alpha\beta})$	tensor of dilatational strains,
$\mathbf{e}^p = \mathbf{e}(\varepsilon^p_{\alpha\beta})$	tensor of plastic strains.

In the theory of thermoplasticity the constitutuve relations are expressed in terms of rates [6, 12],

(3.6) $$\dot{\mathbf{T}} = \mathbf{E} \circ \dot{\mathbf{e}}^e + \dot{\mathbf{E}} \circ \mathbf{e}^e.$$

The equilibrium equations are expressed in terms of stress rates (disregarding the mass forces),

(3.7) $$\nabla_\beta \dot{\sigma}^{\alpha\beta} = 0.$$

Similar equations are also valid for the mould.

The boundary conditions on the free surfaces of casting and mould are

(3.8) $$\dot{\sigma}^{\alpha\beta} n_\beta = \begin{cases} 0 & \text{for free surface,} \\ \dot{t} & \text{for loaded surface.} \end{cases}$$

At the casting/mould interface

(3.9) $$\left.\underset{S}{\dot{\sigma}^{\alpha\beta}}\right|_{\Gamma_S} n_\beta = \left.\underset{M}{\dot{\sigma}^{\alpha\beta}}\right|_{\Gamma_M} n_\beta,$$

for

(3.10) $$\left.\underset{S}{\sigma^{\alpha\beta}}\right|_{\Gamma_S} n_\beta n_\alpha < 0 \quad \text{and then} \quad \delta_G = 0,$$

and for

(3.11) $$\left.\underset{S}{\sigma^{\alpha\beta}}\right|_{\Gamma_S} n_\beta n_\alpha \geq 0,$$

and then a shrinkage gap exists and is equal to:

(3.12) $$\delta_G = \left(u^\alpha\Big|_{\Gamma_M} - u^\alpha\Big|_{\Gamma_S}\right) n_\alpha > 0.$$

The boundary conditions at the solidification front are

(3.13) $$\left.\underset{S}{\sigma^{\alpha\beta}}\right|_{s(t)} = 0,$$

(3.14) $$\left.\underset{S}{\dot{\sigma}^{\alpha\beta}}\right|_{s(t)} n_\beta = 0, \text{ for } \chi^\alpha \subset s(t),$$

where **n** is unit normal in the direction of fluid. The condition is relevant only for the time period of coexistence of two phases.

As it has been mentioned before, one applies the model of non-isothermal plastic flow in which the plastic strain $\dot{\mathbf{e}}^p$ is expressed by relationship to be determined from consistency relation

(3.15) $$\dot{\mathbf{e}}^p = \lambda \frac{\partial f}{\partial \mathbf{T}},$$

and

(3.16) $$\dot{f} = 0,$$

where λ is a scalar and f is called the yield function.

In what follows we shall use the Huber-Mises yield condition and the model of isotropic hardening. Thus,

(3.17) $$f = \left(\frac{3}{2}\mathbf{D}\cdot\mathbf{D}\right)^{1/2} - Y(\Theta,\ \varepsilon^p_{\text{ef}}),$$

where **D** is the deviator of stress tensor, $\varepsilon^p_{\text{ef}}$ is the effective plastic strain defined by

(3.18) $$\dot{\varepsilon}^p_{\text{ef}} = \left(\frac{2}{3}\,\dot{\mathbf{e}}^p\cdot\dot{\mathbf{e}}^p\right)^{1/2}.$$

Y is the actual level of effective stress $D_{\text{ef}} = \left(\frac{3}{2}\mathbf{D}\cdot\mathbf{D}\right)^{1/2}$ corresponding to point x^α lying on the simple (uniaxial) tension (or compression) curve.

We shall also use the notion of hardening modulus $\varkappa$ defined by (cf. Fig. 2)

(3.19) $$\varkappa \equiv \frac{\partial Y}{\partial \varepsilon^p_{\text{ef}}},$$

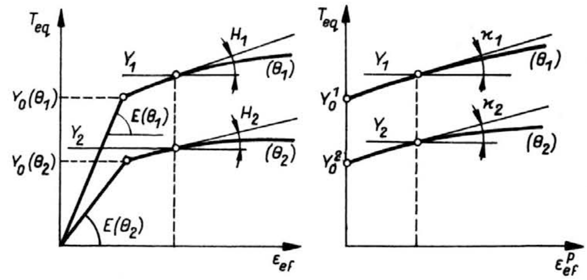

FIG. 2. Curves of uniaxial tension of material tested.

and the notion of thermal softening modulus

$$Y_\theta \equiv \frac{\partial Y}{\partial \Theta}. \tag{3.20}$$

By differentiating Eq. (3.17) and using conditions (3.16), one gets

$$\dot{f} = \frac{3}{2Y}\mathbf{D}\cdot\dot{\mathbf{D}} - Y_\theta\dot{\Theta} - \varkappa\dot{\varepsilon}^p_{ef} = 0. \tag{3.21}$$

Hence

$$3\mathbf{D}\cdot\dot{\mathbf{T}} - 2YY_\theta\dot{\Theta} - 2Y\varkappa\dot{\varepsilon}^p_{ef} = 0. \tag{3.22}$$

Taking into account Eq. (3.15) and using Eqs. (3.4)—(3.6), relations (3.6) being expressed in the form

$$\dot{\mathbf{T}} = \mathbf{E}\circ(\dot{\mathbf{e}} - \dot{\mathbf{e}}^p - \dot{\mathbf{e}}^\Theta) + \dot{\mathbf{E}}\circ\mathbf{e}^e, \tag{3.23}$$

one obtains the unknown scalar ($\dot{\lambda}$) expressed in terms of the total strains [12],

$$\dot{\lambda} = Y\frac{\mathbf{D}\circ\mathbf{E}\circ(\dot{\mathbf{e}} - \dot{\mathbf{e}}^\Theta) + \mathbf{D}\circ\dot{\mathbf{E}}\circ\mathbf{e}^e - \frac{2}{3}YY_\theta\dot{\Theta}}{\frac{3}{2}\mathbf{D}\circ\mathbf{E}\circ\mathbf{D} + \frac{2}{3}Y^2\varkappa}. \tag{3.24}$$

Relation (3.24) expressed in terms of tensor components assumes the form

$$\dot{\lambda} = Y\frac{D_{\alpha\beta}E^{\alpha\beta\gamma\delta}(\dot{\varepsilon}_{\gamma\delta} - \dot{\varepsilon}^\Theta_{\gamma\delta}) + D_{\alpha\beta}\dot{E}^{\alpha\beta\gamma\delta}\varepsilon^e_{\gamma\delta} - \frac{2}{3}YY_\theta\dot{\Theta}}{\frac{3}{2}D_{\alpha\beta}E^{\alpha\beta\gamma\delta}D_{\gamma\delta} + \frac{2}{3}Y^2\varkappa}. \tag{3.25}$$

The final form of plastic flow rules for isotropic hardening material (with temperature-dependent properties at the yield point $f = 0$ is [7, 11]

$$\text{(3.26)} \qquad \dot{\mathbf{e}}^p = \begin{cases} \dfrac{3}{2}\dfrac{\mathbf{D}}{\mathrm{Y}}\dot{\lambda} & \text{if } \dot{\lambda} > 0, \\ 0 & \text{otherwise.} \end{cases}$$

It can easily be seen that $\dot{\lambda}$ is equal to the effective plastic strain rate $\dot{\lambda} = \dot{\varepsilon}^p_{\text{ef}}$.

Equations (3.25)–(3.26) were used in the finite element formulation of the problem considered.

4. Formulation of problems expressed in terms of FEM

Application of the Bubnov-Galerkin method to the equations of heat conductivity (2.1) yields

$$\text{(4.1)} \qquad \int\limits_{\mathcal{V}} \left(\nabla_\alpha (\lambda \partial_\alpha \Theta) - \rho c_{\text{ef}} \frac{\partial \Theta}{\partial t} \right) \phi \, d\mathcal{V} = \int\limits_{\partial \mathcal{V}} (q - \overset{*}{q}) \phi \, d\Gamma,$$

where (ϕ) is a selected basic function [5, 10], $(\phi \in H^1)$, $q = \lambda \partial_\alpha \Theta n_\alpha$ — the flux on $\partial \mathcal{V}$, $\overset{*}{q}$ — the flux imposed on $\partial \mathcal{V}$, n_α — the vector normal to $\partial \mathcal{V}$.

Equation (4.1) may be written in the form

$$\text{(4.2)} \qquad \int\limits_{\mathcal{V}} \left(\lambda \partial_\alpha \Theta \partial_\alpha \phi + \rho c_{\text{ef}} \frac{\partial \Theta}{\partial t} \phi \right) d\mathcal{V} = \int\limits_{\partial \mathcal{V}} \lambda \partial_\alpha \Theta \phi n_\alpha \, d\Gamma + \int\limits_{\partial \mathcal{V}} (q - \overset{*}{q}) \phi \, d\Gamma.$$

By substituting boundary conditions (2.4), one obtains

$$\text{(4.3)} \qquad \int\limits_{\mathcal{V}} \left(\lambda \partial_\alpha \Theta \partial_\alpha \phi + \rho c_{\text{ef}} \frac{\partial \Theta}{\partial t} \phi \right) d\mathcal{V} + \int\limits_{\Gamma_0} \alpha_\theta \Theta \phi \, d\Gamma = \int\limits_{\Gamma_0} \alpha_\theta \Theta^{\text{sur}} \phi \, d\Gamma.$$

Function $\Theta = \Theta(x^\alpha, t)$ is written in the form

$$\Theta(x^\alpha, t) = \sum_i \phi_i(x^\alpha) \Theta_i(t).$$

The region considered ($\mathscr{V}$) is then divided into N_e finite elements what implies N_b boundary elements. By substitung (4.4) into (4.3) one gets

$$(4.5)\quad \sum_{i}^{N_e} \int_{\mathscr{V}_e} (\lambda\,\partial_\alpha \phi_j \partial_\alpha \phi_i)\,\Theta_j\, d\mathscr{V} + \int_{\mathscr{V}_e} \rho c_{ef}\, \phi_j \phi_i \frac{\partial \Theta_j}{\partial t}\, d\mathscr{V} + \sum_{i}^{N_b} \int_{\Gamma_{0e}} \alpha_{i\theta}\, \phi_j \phi_i\, \Theta_j\, d\Gamma = \sum_{i}^{N_b} \int_{\Gamma_{0e}} \alpha_{i\theta}\, \phi_j \phi_i \Theta_j^{sur}\, d\Gamma.$$

A certain approximation with respect to time in the time space I, is also assumed because the temperature in the nodes of mesh is time-dependent (cf. Eq. (4.4)). This guarantees a higher accuracy of numerical calculation and, therefore, a more accurate estimation of the position of the solidification front in the casting [3, 10]. Therefore, we write

$$(4.6)\qquad \Theta(t) = \omega^s(t)\,\Theta^s,$$

where the functions $\omega_i^s(t)$ are the basic functions of space I. By introducing the notations

$$(4.7)\qquad \begin{aligned} A_{ij} &= \int_{\mathscr{V}} (\lambda\,\partial_\alpha \phi_i \partial_\alpha \phi_j)\, d\mathscr{V}, \\ B_{ij} &= \frac{1}{\Delta t} \int_{\mathscr{V}} \rho c_{ef}\, \phi_i \phi_j\, d\mathscr{V}, \\ F_{ij} &= \int_{\Gamma_0} \alpha_{i\theta}\, \phi_i \phi_j\, d\Gamma, \end{aligned}$$

Eqs. (4.5) will take the form

$$(4.8)\quad \beta A_{ij}\Theta_j^s + B_{ij}\Theta_j^s + \beta F_{ij}\Theta_j^s = \beta F_{ij}\Theta_j^{sur} + B_{ij}\Theta_j^{s-1} - (1-\beta)A_{ij}\Theta_j^{s-1} - (1-\beta)F_{ij}\Theta_j^{s-1},$$

where $\beta \in (0, 1\rangle$, $\Delta t = t^s - t^{s-1}$ — finite increment of time t, $\Theta_j^s = \Theta_j(s)$, s — the successive level of time t.

For $\beta = 1$ one obtains a classical implicit differential scheme and for $\beta = 0.5$ the Crank-Nicholson scheme [3, 10]. In the first step of solution

$\Theta_j^{s-1} = \Theta_j^0$, the initial condition (2.3) is used. Integrals (4.7) are evaluated by the Gauss-Legendre method of numerical integration. By integrating along the boundary and integral elements of the region considered, the coefficients of the system of equations are determined,

$$\mathbf{H}\mathbf{\Theta}^s = \mathbf{D}, \tag{4.9}$$

where

$$\begin{aligned} \mathbf{H} &= \beta\mathbf{A} + \mathbf{B} + \beta\mathbf{F}, \\ \mathbf{D} &= (\mathbf{B} - (1-\beta)\mathbf{A} - (1-\beta)\mathbf{F})\mathbf{\Theta}^{s-1} + \beta\mathbf{F}\mathbf{\Theta}^{sur}. \end{aligned} \tag{4.10}$$

Solution of this system of equations gives the temperatures in the nodes of mesh ($\Theta_j = \Theta_j(t)$) for time $t = \sum_1^s \Delta_i t$.

Now, let us return to integral $(4.7)_2$. The integral is split into two integrals

$$B_{ij} = B^*_{ij} + Q^*_{ij}. \tag{4.11}$$

where

$$B^*_{ij} = \frac{1}{\Delta t}\int_{\mathcal{V}} \rho c_v \phi_i \phi_j d\mathcal{V}, \tag{4.12}$$

$$Q^*_{ij} = \frac{1}{\Delta t}\int_{\mathcal{V}} \rho_s L \phi_i \phi_j d\mathcal{V}, \neq 0 \quad \text{for} \quad i = j. \tag{4.13}$$

Here c_v is the specific heat at constant volume, whereas L is the latent heat of fusion. Relations (4.10) are now modified to the form

$$\begin{aligned} \mathbf{H} &= \beta\mathbf{A} + \mathbf{B}^* + \beta\mathbf{F}, \\ \mathbf{D} &= \mathbf{Q}^* + (\mathbf{B}^* - (1-\beta)\mathbf{A} - (1-\beta)\mathbf{F})\mathbf{\Theta}^{s-1} + \beta\mathbf{F}\mathbf{\Theta}^{sur}. \end{aligned} \tag{4.14}$$

Thus it is possible to obtain the solution with a „sharp" solidification front by searching for one surface and not for the biphase region. By distinguishing the solidification surface $s = s(t)$, integral (4.13) may be expressed in the form

$$Q^*_{ij} = \frac{1}{\Delta t}\int_s \rho_s L \phi_i \psi_j d\xi ds. \tag{4.15}$$

By assuming that $d\xi = \omega dt$ (increment of the solidified layer in the direction normal to s), continuity condition (2.7) may be written in the form

$$(4.16) \qquad \lambda_s \partial_\alpha \Theta^s \Big|_s n_\alpha - \lambda_l \partial_\alpha \Theta^l \Big|_s n_\alpha = \frac{d\xi}{dt} \rho L.$$

For finite time increments $dt \to \Delta t$ one may assume

$$(4.17) \qquad \underset{i}{\Delta\xi} = \Delta t \underset{i}{\chi} \Big|_s \Theta_i^s \Big|_s ,$$

with

$$(4.18) \qquad \underset{i}{\chi} \Big|_s = (\lambda_s \psi_{i,\alpha} \Big|_s n_\alpha - \lambda_l \psi_{i,\alpha} \Big|_s n_\alpha) / (\rho L).$$

Successive position of solidification front is determined by the relation

$$(4.19) \qquad \xi^s = \xi^{s-1} + \Delta t \chi^i \Big|_s \Theta_i^s \Big|_s ,$$

where for time $t = 0$, $(s = 0)$, $\xi^0 = 0$.

Condition (4.16) is satisfied by the iteration procedure determining, in r-th iteration,

$$(4.20) \qquad \xi^{s,r} = \xi^{s-1} + \Delta t \chi^i \Big|_s \Theta_i^{s,r} \Big|_s .$$

The process of iteration is completed, if the required accuracy of solution is satisfied, that is if

$$(4.21) \qquad |\Theta_i^{s,r} \Big|_s - \Theta_{kr}(C_l \Big|_s)| \le \varepsilon ,$$

where $\Theta_{kr}(C_l|_s)$ is the solidification temperature expressed in terms of concentration in the liquid phase at the solidification front. By using the algorithm proposed, the temperature fields in the axially-symmetrical casting and mould and solidification rates for casting manufactured in sand and chill moulds have been determined.

Application of the Bubnov-Galerkin method to the equilibrium equations (3.7) with a certain basic function (φ), $(\varphi \in H^1)$ selected in the region $\mathscr{V}$, yields

(4.22)
$$\int_{\mathscr{V}} \nabla_\beta \dot{\sigma}^{\alpha\beta} \varphi d\mathscr{V} = \int_{\partial\mathscr{V}} (\dot{t}^\alpha - \dot{t}^\alpha_*) \varphi d\Gamma,$$

where $\dot{t}^\alpha = \dot{\sigma}^{\alpha\beta} n_\beta$ is the stress vector (flux) on the boundary of the region considered $\mathscr{V}$, $\dot{t}^\alpha_*$ — the known stress vector on $\partial\mathscr{V}$. By applying the Gauss-Ostrogradski theorem one obtains, from Eq. (4.22), the weak form as follows:

(4.23)
$$\int_{\mathscr{V}} \dot{\sigma}^{\alpha\beta} \partial_\beta \varphi d\mathscr{V} = \int_{\partial\mathscr{V}} \dot{t}^\alpha_* \varphi d\Gamma.$$

After substituting constitutive relations (3.6) (suitable for the axially-symmetrical problem) into (4.23), using the Cauchy relations (3.2) and discretizing the region considered by means of finite elements where the unknown functions ($\mathbf{u} = \mathbf{u}(u^\alpha)$) are approximated by the so-called nodal functions assigned to elements [5, 10],

(4.24)
$$\mathbf{u}(x^\alpha) = u_i \varphi_i(x^\alpha), \quad \mathbf{u}_{,\alpha}(x^\alpha) = u_i \varphi_{i,\alpha}(x^\alpha),$$

one gets the system of equations in the form

(4.25)
$$\mathbf{K}\dot{\mathbf{U}} = \dot{\mathbf{R}} + \sum_{e=1}^{N_e} \int_{\mathscr{V}_e} \Phi \circ \mathbf{E} \circ \dot{\mathbf{e}}^e d\mathscr{V} + \sum_{e=1}^{N_e} \int_{\mathscr{V}_e} \Phi \circ \dot{\mathbf{E}} \circ \mathbf{e}^e d\mathscr{V},$$

where $\mathbf{U}$ is the vector of displacements

(4.26)
$$\dot{\mathbf{U}} = \left\{ \left(\underset{1}{\dot{u}^\alpha}, \ldots, \underset{q}{\dot{u}^\alpha} \right) \Big|_{e=1}, \ldots, \left(\underset{1}{\dot{u}^\alpha_\alpha}, \ldots, \underset{q}{\dot{u}^\alpha} \right) \Big|_{e=N_e} \right\}^T,$$

and q is the number of nodes in a mesh element, $\mathbf{R}$ — vector of external loads

(4.27)
$$\dot{\mathbf{R}} = \sum_{e=1}^{N_b} \int_{\partial\mathscr{V}_e} \left\{ \left(\underset{1}{\dot{t}^\alpha_*}, \ldots, \underset{b}{\dot{t}^\alpha_*} \right) \Big|_e \right\}^T \varphi d\Gamma,$$

b is the number of nodes in the loaded boundary elements, $\mathbf{K}$ — the matrix of stiffness

(4.28)
$$\mathbf{K} = \sum_{e=1}^{N} \int_{\mathscr{V}_e} \Phi^T \circ \mathbf{E} \circ \Phi d\mathscr{V},$$

Φ — matrix of the corresponding derivatives of the approximation functions: ϕ, ($\phi = \phi(\varphi_{,\alpha})$, φ — the vector of approximating functions (cf. Eq. (4.24)).

The thermal and mechanical loads are assumed to be independent of plastic strains; the stress tensor **T** is split into two parts. The first one corresponds to the thermal and mechanical loads ($\mathbf{T}^{\theta}$), and the second one is related to the plastic strain ($\mathbf{T}^{p}$)

$$\mathbf{T} = \mathbf{T}^{\theta} + \mathbf{T}^{p}. \tag{4.29}$$

The solution of the system of Eqs. (4.25) yields the displacement rates resulting from the increment of thermal and mechanical loads, namely the stress tensor $\mathbf{T}^{\theta}$. The problem has been solved by means of the implicit scheme, and the integration of displacement function has been performed as follows:

$$\mathbf{U}(t+\Delta t) = \mathbf{U}(t) + ((1\text{-}\vartheta)\dot{\mathbf{U}}(t) + \vartheta\dot{\mathbf{U}}(t+\Delta t))\Delta t, \tag{4.30}$$

where $\vartheta \in (0, 1\rangle$. The parameter ϑ has been assumed to be equal to 0.5 except the procedure of searching for the instant of generation of the shrinkage gap. However, its value has always been within the range $(0, 1\rangle$. If condition (3.26) is not satisfied at the successive stage of solution (successive increment of load), the increment of plastic strain rate $\delta\dot{\mathbf{e}}^{p}$ will be determined from the relation (3.25)—(3.26), and then the process of iteration at each iteration stage, has been performed in order to solve, the following system of equations:

$$\mathbf{K}\delta^{i}\dot{\mathbf{U}} = \sum_{e=1}^{N_e} \int_{\mathscr{V}_e} \Phi^{T} \circ \mathbf{E} \circ \delta^{i}\dot{\mathbf{e}}^{p}\, d\mathscr{V}, \tag{4.31}$$

and

$$\dot{\mathbf{T}}_{i}^{\theta p}(t+\Delta t) = (1-\vartheta)\dot{\mathbf{T}}^{\theta p}(t) + \vartheta\left(\dot{\mathbf{T}}^{\theta}(t+\Delta t) + \sum_{k=1}^{i-1} \dot{\mathbf{T}}_{k}^{p}\right). \tag{4.32}$$

The total displacements at this stage of solution are equal to

$$\mathbf{U}(t+\Delta t) = \mathbf{U}(t) + ((1-\vartheta)\dot{\mathbf{U}}(t) + \vartheta\left(\dot{\mathbf{U}}^{\theta}(t+\Delta t) + \sum_{k=1}^{i} \delta^{k}\dot{\mathbf{U}})\right)\Delta t. \tag{4.33}$$

The process of iteration is completed when the required accuracy of solution is reached. In this case the following condition has to be satisfied:

$$\left|\delta^{i}\Delta\varepsilon_{\mathrm{ef}}^{p} - \delta^{i-1}\Delta\varepsilon_{\mathrm{ef}}^{p}\right| \le \varepsilon. \tag{4.34}$$

This method of approaching the solution of the problem considered seems to be the most advantageous if it is aimed at the determination of stresses in solidifiying castings, because then it is possible to use the iterative procedures with variable stiffness matrix **K**. In this solution the modified Newton-Raphson algorithm [5] is used for determining the stresses in an axially-symmetrical casting and mould, generated during solidification and cooling of the casting. Special attention has been paid to establishing the effect of instantaneous plastic strains on the instant of generation and size of the shrinkage gap.

5. Numerical examples

The results of calculation are presented in successive diagrams for a separated part of the casting (Fig. 3).

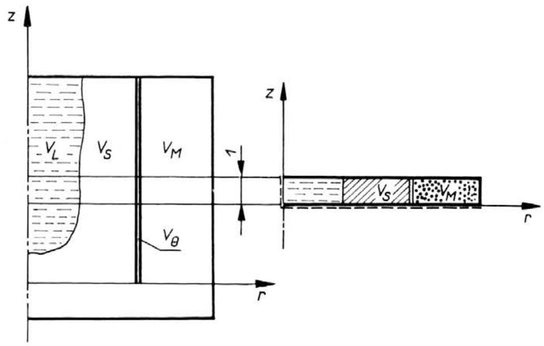

FIG. 3. A segment of axially-symmetrical casting and the method of their fixing.

The thermophysical constants used in determining the temperature fields and solidification rate were taken from the literature and are equal:

for the casting made of steel constaining 0.2% C:

$$\lambda_s = 35\ [\mathrm{W/(mK)}],\ c_s = 644\,[\mathrm{J/(kgK)}],\ \rho_s = 7760\ [\mathrm{kg/(m^3)}],$$

$$\lambda_l = 23,\ c_l = 837,\ \rho_l = 7300,$$

for the chill mould $\lambda_m = 54$, $c_m = 510$, $\rho_m = 7500$,

for the sand mould $\lambda_m = 1.28$, $c_m = 1080$, $\rho_m = 1650$,

for the protective coating : $\lambda_c = 04$—1.0, $c_c = 1670$, $\rho_c = 1600$.

The thickness of protective coating was assumed to be $\delta_c = 0.2$ mm. The temperatures of liquid metal and crystallization were equal to $\Theta = 1820$ K and $\Theta_{cr_e} = 1750$ K, respectively. The initial temperature of mould was $\Theta_{im} = 403$ K, and the ambient temperature $\Theta_{sur} = 303$ K.

The casting solidification rates for different protective coating are presented in Fig. 4.

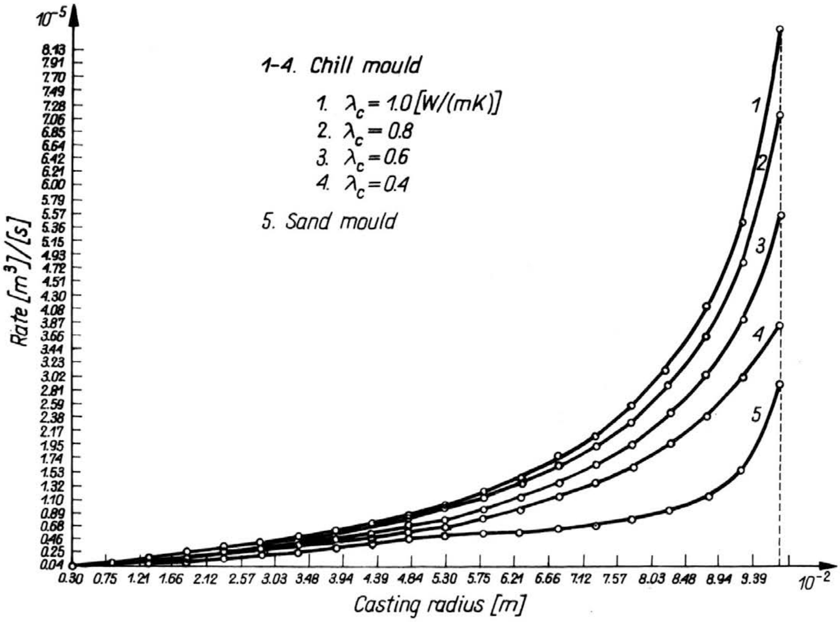

FIG. 4. Casting solidification rates for different protective coatings.

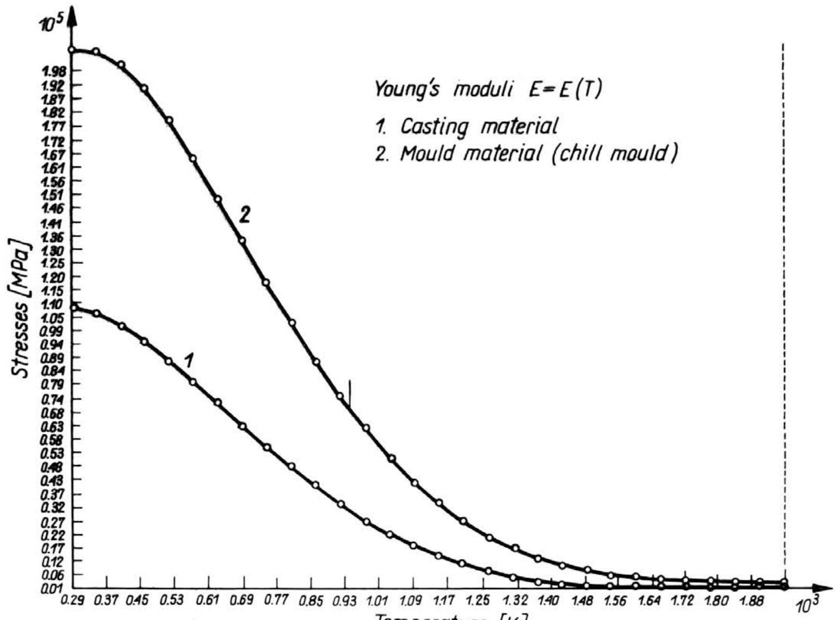

FIG. 5. Young's moduli $E = E(\Theta)$.

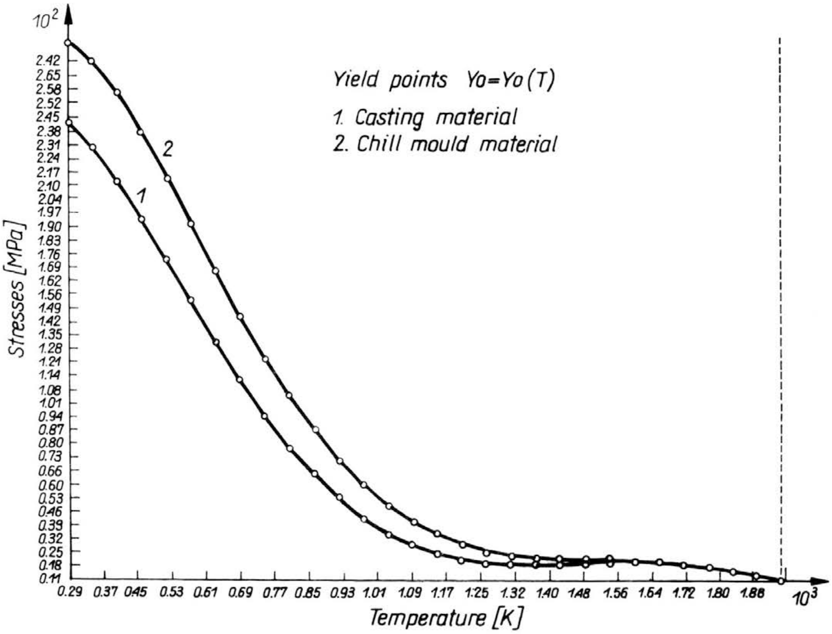

FIG. 6. Yield points $Y_0 = Y_0(\Theta)$.

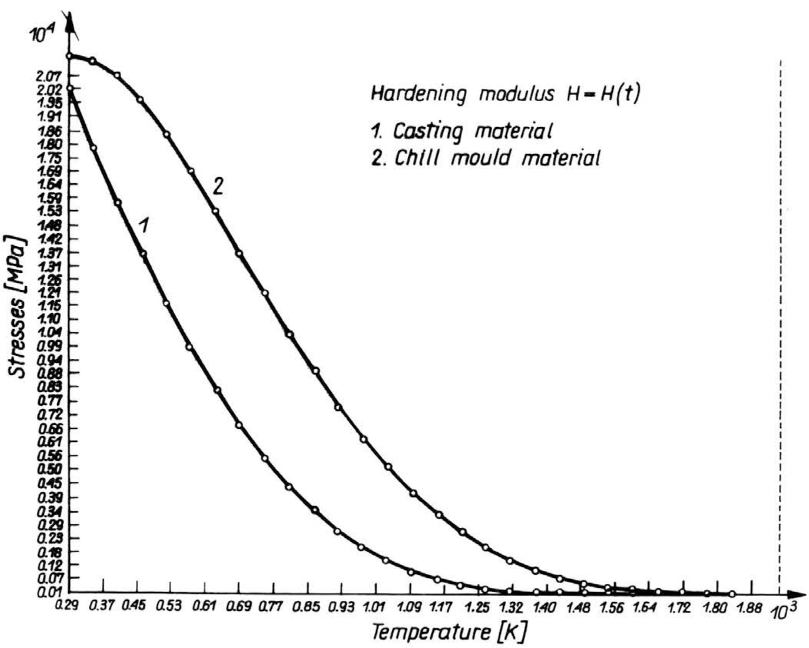

FIG. 7. Hardening coefficient $H_0 = H_0(\Theta)$.

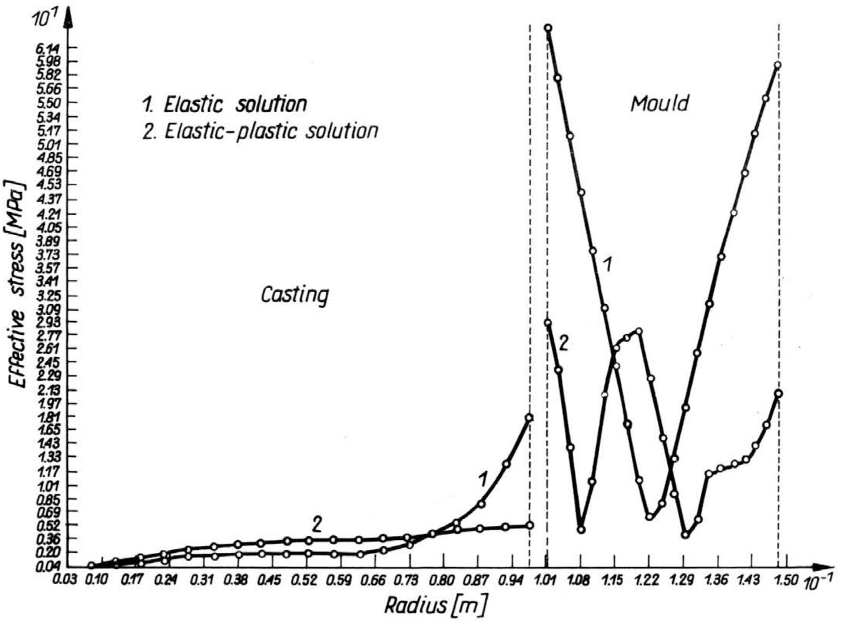

FIG. 8. Equivalent stresses at the instant of complete solidification.

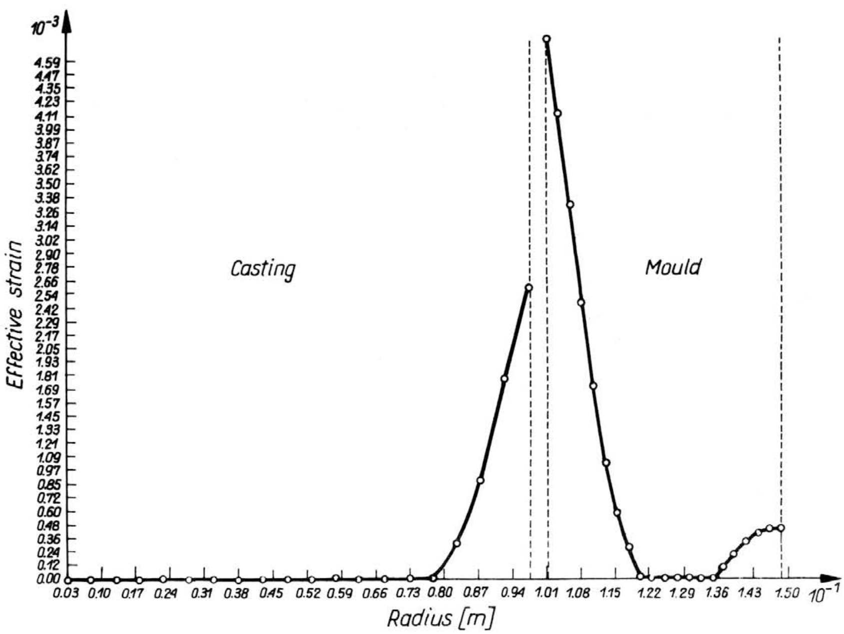

FIG. 9. Equivalent strains (plastic zones) at the instant of complete solidification.

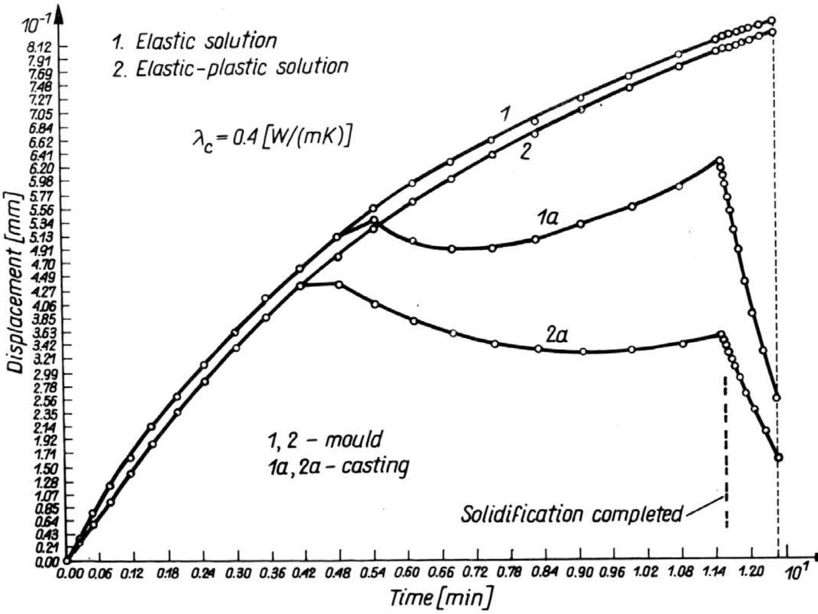

FIG. 10. Gap generation for elastic and elastic-plastic models.

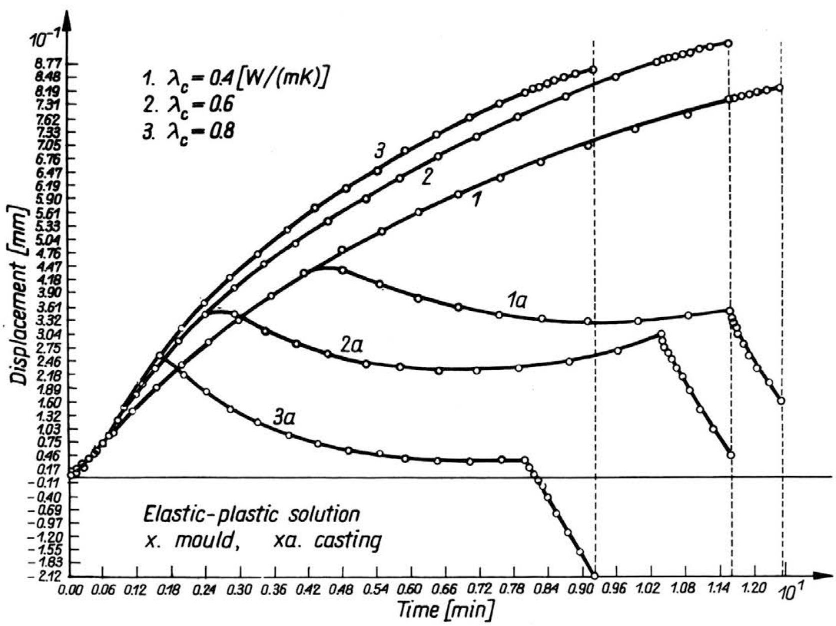

FIG. 11. Gap generation at three solidification rates.

The material constants, ie. Young's moduli, yield points, hardening coefficients have been found experimentially and their variations in terms of temperature are shown in Figs. 5, 6, and 7. Figures 8 and 9 are plotted in order to confirm the validity of the elastic-plastic model used for determining the stresses in the solidifying casting. One notices a considerable difference in the levels of stress calculated on the basis of elastic and elastic-plastic solutions.

On the basis of the results presented in Figs. 10 and 11 one may observe that the application of the elastic-plastic model enables us to determine accurately the instant of generation of the shirkage gap and its size. It is obvious that this phenomenon is also affected by the solidification rate, as confirmed by Fig. 11.

The differences (although insignificant) in the results obtained from the elastic and the elastic-plastic solution is manifested by comparing the size of shrinkage gap in the entire process of cooling; the results are presented in Fig. 12.

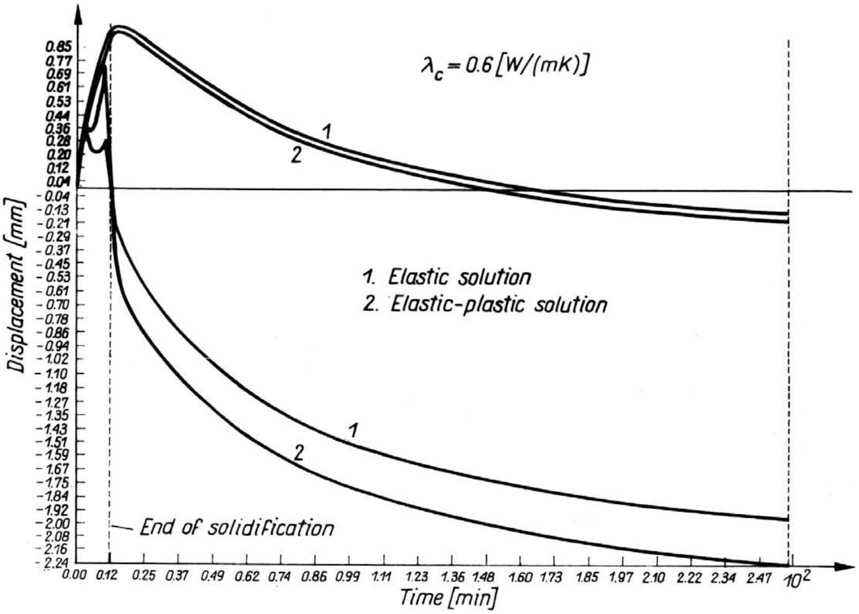

FIG. 12. Shrinkage gap in the process of solidification and cooling of casting.

References

1. A. BOKOTA and R. PARKITNY, *Couplings of thermal, diffusive and mechanical phenomena in the process of solidification.* Solidification of Metals and Alloys [in Polish], vol. 6, Ossolineum, Wrocław 1983.
2. A. BOKOTA and R. PARKITNY, *Coupled equations of casting solidification and examples of solutions* [in Polish], Reports of the Institute of Machine Building Technology of the Technical University of Wrocław, No 35, Ser. Conference No 12, Conference on Foundry Practice, 1987, pp. 14—18, Wrocław 1987.

3. P. BONNEROT and P. JAMET, *A second order finite element method for the one-dimensional Stefan problem*, Int. Jour. Num. Meth. Eng., 8, 811—820, 1974.
4. M. C. FLEMINGS, *Solidification processing*, New York 1974.
5. M. KLEIBER, *Finite element method in the nonlinear continuum mechnics* [in Polish], Warszawa-Poznań 1985.
6. M. KLEIBER and T. NIEZGODA, *Numerical analysis of thermoelasto-plastic problems*, Enging. Trans., **36**, 4, pp. 645—660, 1988.
7. M. KLEIBER and B. RANIECKI, *Elastic-plastic materials at finite strains*, in: Plasticity Today; Modelling, Methods and Applications, pp. 3—46, [Ed.] A. SAWCZUK and G. BIANCHI, Elsevier Applied Sci. Publ., 1985.
8. W. KOSIŃSKI, *Field singularities and wave analysis in continuum mechanics*, PWN, Warszawa 1986.
9. J. O. KRISTIANSSON, *Thermal stresses in the early stage of solidification of steel*, J. Thermal Stresses, 5, pp. 315—330, 1982.
10. G. MARCHOUK and V. AGOCHKOV, *Introduction aux méthodes des éléments finis* [transl. from the Russian ed.], Mir, Moskva 1985.
11. Z. MRÓZ and B. RANIECKI, *Variational principles in uncoupled thermoplasticity*, Intern. J. Engng. Sci., **11**, pp. 1133—1141, 1973
12. M. D. SNYDER and K. J. BATHE, *A solution procedure for thmermoelastic-plastic and creep problems*, Nuclear Eng. and Design **64**, 49—80, 1981.
13. R. PARKITNY, *Erstarrung von elastischen Material*, ZAMM, **62** T, 149—152, 1982.
14. R. PARKITNY, *Stress in growing solids*, Bull. Acad. Polon. Sci., **11**, 453—461, 1976.
15. R. PARKITNY, A. BOKOTA and N. SCZYGIOL, *The equations of solidification thermomechanics of casting being the binary alloys*, 1-st Conference on Mechanics, Proc. vol. 5, Praha 1987.
16. B. RANIECKI, *On the mechanics of solidification*, in: International Conference on Residual Stresses, ICRS2, [Ed.] G. BECK, S. DENIS, A. SIMON, Elsevier Appl. Sci., pp. 448—453, 1988.
17. G. RIEDER, *Zur Entstehung und Berechnung der Guss-spannungen*, ZAMM, **45**, 4, 153—170, 1965.
18. W. SAKWA, R. PARKITNY and A. BOKOTA, *Compatibility conditions on the solidification surface*, 2, pp. 17—21, 1981.
19. J. H. WEINER and B. A. BOLEY, *Elasto-plastic thermal stresses in a solidifying body*, J. Mech. Phys. Solids, **11**, 145—154, 1963.

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