Arch. Mech., 43, 6, pp. 795–798, Warszawa 1991

BRIEF NOTES

A generalized variational principle for piezoelectromagnetism in an elastic medium

J. S. YANG (NEW JERSEY)

A VARIATIONAL principle is developed for piezoelectromagnetism in an elastic medium with displacement, strain, stress, electric displacement, magnetic field, scalar and vector potentials of the electromagnetic fields as independent variables. All the governing equations and boundary conditions can be obtained as the stationary conditions of the functional.

1. Introduction

VARIATIONAL PRINCIPLES for piezoelectromagnetism in a solid medium can be found in a series of publications [1–11]. All these variational principles have a common feature that they all have constraints. Using the Lagrange multiplier, the constraints can be removed and a variational principle without constraints can be obtained. The principle in nature is like the Hu-Washizu Principle in elasticity [12].

2. Constitutive relations of piezoelectromagnetism

For a piezoelectromagnetic elastic body, the internal energy density U is a function of strain ε_{ij}, electric displacement D_i and magnetic induction B_i

$$U = U(\varepsilon_{ij}, D_i, B_i),$$

and

$$\delta U = \sigma_{ij}\delta\varepsilon_{ij} + E_i\delta D_i + H_i\delta B_i,$$

where σ_{ij} is stress, E_i electric field and H_i magnetic field.

The electric enthalpy H can be introduced by partial Legendre transform as follows:

$$\begin{aligned} H &= H(\varepsilon_{ij}, E_i, B_i) \\ &= U - E_iD_i, \end{aligned}$$

and

$$\delta H = \sigma_{ij}\delta\varepsilon_{ij} - D_i\delta E_i + H_i\delta B_i.$$

With H, the constitutive relations assume the following form:

$$\sigma_{ij} = \frac{\partial H}{\partial \varepsilon_{ij}}, \quad D_i = -\frac{\partial H}{\partial E_i}, \quad H_i = \frac{\partial H}{\partial B_i}.$$

3. Governing equations and boundary conditions

Let the space region occupied by the piezoelectromagnetic elastic body be V, the boundary surface of V be S, the unit outward normal of S be n_i, and S can be partioned in the following way:

$$S_u \cup S_\sigma = S_\phi \cup S_D = S_A \cup S_H = S\,,$$
$$S_u \cap S_\sigma = S_\phi \cap S_D = S_A \cap S_H = \emptyset\,.$$

The governing equations and boundary conditions are
dynamics

$$\sigma_{ji,j} + \rho f_i = \rho \ddot{u}_i \quad \text{in } V\,;$$

geometry

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \quad \text{in } V\,;$$

potential representation

$$E_i = -\phi_{,i} - \dot{A}_i\,,$$
$$B_i = \varepsilon_{ijk} A_{k,j} \quad \text{in } V\,;$$

Maxwell's equation

$$D_{i,i} = \rho_e\,,$$
$$\varepsilon_{ijk} H_{k,j} - \dot{D}_i = J_i \quad \text{in } V\,;$$

constitutive relations

$$\sigma_{ij} = \frac{\partial H}{\partial \varepsilon_{ij}}\,, \qquad D_i = -\frac{\partial H}{\partial E_i}\,, \qquad H_i = \frac{\partial H}{\partial B_i} \quad \text{in } V\,;$$

boundary conditions

$$\begin{aligned} u_i - \overline{u}_i &= 0 && \text{on } S_u\,,\\ \sigma_{ji} n_j - \overline{t}_i &= 0 && \text{on } S_\sigma\,,\\ \phi - \overline{\phi} &= 0 && \text{on } S_\phi\,,\\ D_i n_i - \overline{d} &= 0 && \text{on } S_D\,,\\ \varepsilon_{ijk} n_j A_k - \overline{a}_i &= 0 && \text{on } S_A\,,\\ \varepsilon_{ijk} n_j H_k - \overline{h}_i &= 0 && \text{on } S_H\,, \end{aligned}$$

where ρ is mass density, f_i force per unit mass, ρ_e density of electric charge, J_i density of electric current, and they are all considered to be known quantities. $\overline{u}_i, \overline{t}_i, \overline{\phi}, \overline{d}, \overline{a}_i$ and $\overline{h}_i$ are quantities prescribed on the boundary. ϕ and A_i are scalar and vector potentials of the electromagnetic fields. Because of the potential representation two homogeneous ones of the four Maxwell's equations are identically satisfied and the other two are left.

4. A variational principle

Consider the following functional:

$$
\begin{aligned}
\Pi &= \Pi(u_i, \varepsilon_{ij}, \sigma_{ij}, E_i, D_i, B_i, H_i, \phi, A_i) \\
&= \int_{t_0}^{t} dt \int_V \Big[\frac{1}{2}\rho \dot{u}_i \dot{u}_i + \rho f_i u_i - H - \rho_e \phi + J_i A_i \\
&\quad + \sigma_{ij}\left(\varepsilon_{ij} - \frac{1}{2}u_{i,j} - \frac{1}{2}u_{j,i}\right) - D_i(E_i + \phi_{,i} + \dot{A}_i) + H_i(B_i - \varepsilon_{ijk}A_{k,j}\Big]\, dV \\
&\quad + \int_{t_0}^{t} dt \int_{S_u} \sigma_{ji} n_j (u_i - \overline{u}_i)\, dS + \int_{t_0}^{t} dt \int_{S_\sigma} \overline{t}_i u_i \, dS \\
&\quad + \int_{t_0}^{t} dt \int_{S_\phi} D_i n_i (\phi - \overline{\phi})\, dS + \int_{t_0}^{t} dt \int_{S_D} \overline{d}\phi \, dS \\
&\quad + \int_{t_0}^{t} dt \int_{S_A} H_i(\varepsilon_{ijk} n_j A_k - \overline{a}_i)\, dS - \int_{t_0}^{t} dt \int_{S_H} \overline{h}_i A_i \, dS .
\end{aligned}
$$

With integration by parts, the following expression for the variation of Π can be obtained

$$
\begin{aligned}
\delta\Pi &= \int_V [\rho \dot{u}_i \delta u_i - D_i \delta A_i]_{t_0}^{t}\, dV \\
&\quad + \int_{t_0}^{t} dt \int_V \Big[(\sigma_{ji,j} + \rho f_i - \rho \ddot{u}_i)\delta u_i + (\varepsilon_{ij} - \frac{1}{2}u_{i,j} - \frac{1}{2}u_{j,i})\delta\sigma_{ij}\Big]\, dV \\
&\quad + \int_{t_0}^{t} dt \int_V [-(E_i + \phi_{,i} + \dot{A}_i)\delta D_i + (B_i - \varepsilon_{ijk}A_{k,j})\delta H_i]\, dV \\
&\quad + \int_{t_0}^{t} dt \int_V [(D_{i,i} - \rho_e)\delta\phi - (\varepsilon_{ijk}H_{k,j} - \dot{D} - J_i)\delta A_i]\, dV \\
&\quad + \int_{t_0}^{t} dt \int_V \left[\left(\sigma_{ij} - \frac{\partial H}{\partial \varepsilon_{ij}}\right)\delta\varepsilon_{ij} - \left(D_i + \frac{\partial H}{\partial E_i}\right)\delta E_i + \left(H_i - \frac{\partial H}{\partial B_i}\right)\delta B_i\right] dV \\
&\quad + \int_{t_0}^{t} dt \int_{S_u} (u_i - \overline{u}_i)\delta\sigma_{ji} n_j \, dS + \int_{t_0}^{t} dt \int_{S_\sigma} (\overline{t}_i - \sigma_{ji} n_j)\delta u_i \, dS \\
&\quad + \int_{t_0}^{t} dt \int_{S_\phi} (\phi - \overline{\phi})\delta D_i n_i \, dS + \int_{t_0}^{t} dt \int_{S_D} (\overline{d} - D_i n_i)\delta\phi \, dS \\
&\quad + \int_{t_0}^{t} dt \int_{S_A} (\varepsilon_{ijk} n_j A_k - \overline{a}_i)\delta H_i \, dS + \int_{t_0}^{t} dt \int_{S_H} (\varepsilon_{ijk} n_j H_k - \overline{h}_i)\delta A_i \, dS ,
\end{aligned}
$$

where δu_i, $\delta\varepsilon_{ij}$, $\delta\sigma_{ij}$, δE_i, δD_i, δB_i, δH_i, $\delta\phi$ and δA_i are independent variations. Hence we have the following variational principle:

Let

$$\delta u_i|_{t_0} = \delta u_i|_t = 0\,, \qquad \delta A_i|_{t_0} = \delta A_i|_t = 0\,,$$

then $\delta\Pi = 0$ implies

$$\sigma_{ji,j} + \rho f_i = \rho\ddot{u}_i \quad \text{in } V\,,$$
$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \quad \text{in } V\,,$$
$$E_i = -\,\phi_{,i} - \dot{A}_i\,,$$
$$B_i = \varepsilon_{ijk}A_{k,j} \quad \text{in } V\,,$$
$$D_{i,i} = \rho_e\,,$$
$$\varepsilon_{ijk}H_{k,j} - \dot{D}_i = J_i \quad \text{in } V\,,$$
$$\sigma_{ij} = \frac{\partial H}{\partial \varepsilon_{ij}}\,, \quad D_i = -\frac{\partial H}{\partial E_i}\,, \quad H_i = \frac{\partial H}{\partial B_i} \quad \text{in } V\,,$$
$$u_i - \overline{u}_i = 0 \quad \text{on } S_u\,,$$
$$\sigma_{ji}n_j - \overline{t}_i = 0 \quad \text{on } S_\sigma\,,$$
$$\phi - \overline{\phi} = 0 \quad \text{on } S_\phi\,,$$
$$D_i n_i - \overline{d} = 0 \quad \text{on } S_D\,,$$
$$\varepsilon_{ijk}n_jA_k - \overline{a}_i = 0 \quad \text{on } S_A\,,$$
$$\varepsilon_{ijk}n_jH_k - \overline{h}_i = 0 \quad \text{on } S_H\,.$$

References

1. E.P. EerNisse, *Variational method for electrostatic vibration analysis*, IEEE Trans., Sonics and Ultrason., SU-14, pp.59–67, 1967.
2. R. Holland and E.P. EerNisse, *Variational evaluation of admittances of multielectroded three-dimensional piezoelectric structures*, IEEE Trans. Sonics and Ultrason., SU-15, pp. 119–132, 1968.
3. H.F. Tiersten, *Linear piezoelectric plate vibrations*, Plenum Press, 137–139, 1969.
4. B.A. Auld, *Acoustic fields and waves in solids*, J. Wiley and Sons, 1, 276, New York 1973.
5. A.Z. Lazerson, B.F. Ryzhenko, S.G. Suchkov and Yu. Kh. Finkelstein, *Variational method for electroacoustic systems*, Radiotekh. Elektron., 20, pp. 441–443, 1975.
6. Y. Kagawa and T. Yamabuchi, *Finite element approach for a piezoelectric circular rod*, IEEE Trans. Sonics and Ultrason., SU–23, pp. 379–385, 1976.
7. R.D. Mindlin, *A variational principle for the equations of piezoelectromagnetism in a compound medium*, in: Complex Variable Analysis and its Applications (I.N. Vekua 70th Birthday Volume), pp. 397–400, Academy of Sciences USSR, Nauka, Moscow 1978.
8. A.N. Gubenkov, V.F. Kiritchenko, E.L. Kulikov and S.P. Pavlov, *Variational principle for analysis of piezoelectric devices with acoustoelectric interaction*, Akust. Zhur., 24, pp. 195–202, 1978.
9. D.F. Nelson, *Electric, optic and acoustic interactions in dielectrics*, J. Wiley and Sons, 1979.
10. T. Yamabuchi, Y. Kagawa and S. Takahashi, *Finite element analysis of ultrasonic circular rod vibrator unconstrained electric field model*, J. Acoust. Soc. Japan, 37, pp. 307–315, 1981.
11. H.-L. Zhang, *On variational principles of a piezoelectric body*, Acta Acust., 10, 4, July 1985.
12. K. Washizu, *Variational methods in elasticity and plasticity*, Second Ed., Pergamon Press, 1975.

DEPARTMENT OF CIVIL ENGINEERING AND OPERATIONS RESEARCH
PRINCETON UNIVERSITY, PRINCETON, NJ, USA.

Received August 14, 1991