Arch. Mech., 33, 4, pp. 483-500, Warszawa 1981

Finite endochronic theory for ratcheting and cyclic plasticity

W. KOSIŃSKI (WARSZAWA) and T. NASHIRO (CINCINNATI)

MATERIAL objective nonlinear constitutive relations of the endochronic theory are constructed. In the free energy function terms up to the third order in principal values of a tensorial state variable and the right Cauchy–Green strain tensor are retained. The operational definition of the internal variable is given. Systematic approximation to all constitutive relations is performed. The dual constitutive relations are derived as well. A number of problems are solved by means of the derived equations.

Skonstruowano nieliniowe, materialnie obiektywne, związki konstytutywne teorii endochronicznej. W funkcji energii swobodnej zachowano wyrazy do trzeciego rzędu w głównych wartościach tensorowej zmiennej stanu i prawego tensora Cauchy'ego–Greena. Podano operacyjną definicję zmiennej wewnętrznej. Przeprowadzono systematyczną aproksymację wszystkich związków konstytutywnych. Wyprowadzono także związki dualne. Rozwiązano kilka problemów, wykorzystując wyprowadzone równania.

Построены нелинейные, материально объективные, определяющие соотношения эндохронической теории. В функции свободной энергии сохранены члены до третьего порядка в главных значениях тензорной переменной состояния и правого тензора Коши–Грина. Дается операционное определение внутренней переменной. Проведена систематическая аппроксимация всех определяющих соотношений. Выведены также дуальные соотношения. Решено некоторое количество задач, используя выведенные уравнения.

1. Introduction

THE PAPERS BY COLEMAN and GURTIN [1] and VALANIS [2] published in the late 60's should be treated as the first serious applications of the internal state variable approach to thermodynamics of nonlinear continua.

The first attempt at describing nonelastic, mainly viscoplastic, materials within thermodynamics with internal state variables is found in the paper [3] by PERZYNA and WOJNO. After that paper appeared, a number of articles were published. There the concept of internal variables (internal parameters) was used in the constitutive modelling of nonelastic (plastic) material behaviour.

In 1971 VALANIS [4] proposed his theory of plasticity without a yield surface but with the intrinsic time and in the conceptual formalism of the internal variable theory. Several versions of Valanis' endochronic theory (ET) were published together with its applications to dynamic and quasi-static problems of plasticity and viscoplasticity, and fracture. Moreover, ET has been used in the description of cyclic phenomena in plasticity.

Although ET has been constructed in the framework of the general nonlinear continuum mechanics approach, most of its applications have been performed under the assumptions of infinitesimal strains.

The main aim of this paper is to construct material objective nonlinear constitutive relations of the endochronic theory of plasticity which can be successfully used in the description of complex ratcheting and cyclic hardening or softening phenomena.

The existing plasticity theories (among them the original Valanis endochronic theory) can describe ratcheting and cyclic phenomena only up to some level of account (i.e. strain and cycle number).

In the authors' opinion there is no theory of plasticity in the framework of which the following features could be described([1]):

1) nonlinear dependence of ratchet strain on the amplitude of cyclic strain;

2) nonlinear dependence of ratchet strain on steady stress;

3) cyclic hardening and cyclic softening, i.e. changes in the number od cycles of the peak stress and the shape of the hysteresis loop.

Applying the internal state variable approach constitutive and evolution equations of the endochronic theory of plasticity are derived; these equations are more general than those of ET. Moreover, the equations satisfy the principle of material objectivity.

For an isotropic material we retain in the free energy function terms up to the third order in principal values of a tensorial state variable and the right Cauchy–Green strain tensor. The operational definition requires the internal variable to coincide with the permanent strain at unloaded (i.e. stress-free) state. The assumption of plastic incompressibility is made.

Systematic approximation to all constitutive relations of the developed theory by polynomials in the tensorial state variable and the Cauchy–Green strain tensor is performed. The procedure leads to the nonlinear evolution equation with only one material constant.

The dual constitutive relations (i.e. in terms of the stress and the internal variable) are derived as well. For the problems of cyclic plasticity appropriate time scale and time measure are proposed.

A number of problems solved by means of the derived equations shows the effectiveness of ET in a broad domain of plasticity.

It should be pointed out that there exists a number of concepts of generalization of the classical ET to problems of cyclic plasticity, ratcheting and strain-rate effect, for example H. C. WU, K. C. VALANIS and A. R.-F. YAO [5], A. R.-F. YAO [6], H. C. WU and M. C. YIP [7], K. C. VALANIS and H. C. WU [8].

2. Framework of the theory

In the internal state variable approach of irreversible thermodynamics a state of a material particle is given by a set of the following quantities defined for the particle: the right Cauchy–Green strain tensor $\mathbf{C}$, internal variables $q_1, \ldots, q_n$ and the system of thermal variables, namely the temperature and the temperature gradient. Restricting our attention

([1]) When a material element (a sample) is under a uniaxial steady stress and superposed cyclic stress or strain in a perpendicular direction, a permanent deformation (called ratchet strain) accumulates in the direction of the steady stress implying large strain. This phenomenon is called ratcheting.

to isothermal processes, the temperature is assumed to be constant with the vanishing temperature gradient.

The internal variables may be tensors, scalars or vectors. If they are tensors or vectors, they remain invariant with the rotation of the spatial system of reference unless the spatial (Euler) description is adopted instead of the material (Lagrange) one.

For the purpose of the present paper we assume that one tensorial (symmetric) internal state variable $\mathbf{q}$ behaves as a scalar in the spatial system.

This and the previous assumptions mean that two tensors $\mathbf{C}$ and $\mathbf{q}$ have the same geometric properties and both define a state.

In the internal state variable approach (in the case considered), constitutive equations for the free energy ψ and the second Piola-Kirchhoff stress $\mathbf{T}$ given as functions $\Psi(\mathbf{C}, \mathbf{q})$ and $\mathcal{T}(\mathbf{C}, \mathbf{q})$, respectively, are accompanied with the evolution equation for the internal state variable $\mathbf{q}$. In the classical approach the form of the equation is as follows:

$$\dot{\mathbf{q}} = \mathbf{a}(\mathbf{C}, \mathbf{q}), \quad \mathbf{q}(0) = \mathbf{q}_0, \tag{2.1}$$

where the superposed dot denotes the time (material) derivative. Assuming every process to be compatible with the second law of thermodynamics, the following potential relation

$$\mathbf{T} = 2\varrho \frac{\partial}{\partial \mathbf{C}} \Psi(\mathbf{C}, \mathbf{q}) \tag{2.2}$$

and the dissipation inequality

$$\frac{\partial}{\partial \mathbf{q}} \Psi(\mathbf{C}, \mathbf{q}) \cdot \mathbf{a}(\mathbf{C}, \mathbf{q}) \leqslant 0 \tag{2.3}$$

can be derived as the necessary and sufficient conditions.

It is obvious that the evolution equation (2.1) leads to the rate-dependent description, i.e. the theory with viscosity. The original idea of VALANIS [4] was to formulate a rate-independent internal variable theory. To do this the (natural) time derivative in the left-hand side of Eq. (2.1) should be replaced with a "new" time derivative. This new time measure, called by Valanis intrinsic, is in the simplest case an arc-length ξ in the deformation space, i.e.

$$\dot{\xi}^2 = \dot{\mathbf{C}} \cdot \mathbf{P}[\dot{\mathbf{C}}], \tag{2.4}$$

where $\mathbf{P}$ is a fourth order tensor. In order to describe more complex strain-history dependent phenomena, the differentiation with respect to some new time scale z may be incorporated in Eq. (2.1) provided the time scale z is a positive and monotonically increasing function of a time measure ξ and the state variables, i.e.

$$\frac{dz}{d\xi} = h(\xi, \mathbf{C}, \mathbf{q}) > 0, \quad z|_{\xi=0} = 0. \tag{2.5}$$

For ξ we postulate a general equation

$$\dot{\xi} = k(\mathbf{D}, \dot{\mathbf{C}}), \quad \xi|_{t=0} = 0, \tag{2.6}$$

where $\mathbf{D}$ is the stretching and k is a non-negative function, homogeneous of order one, i.e. satisfying $k(c\mathbf{D}, c\dot{\mathbf{C}}) = |c|k(\mathbf{D}, \dot{\mathbf{C}})$ for any real c. In the applications the following form will be used:

$$\dot{\xi}^2 = k_1^2|\mathrm{I}_{\dot{\mathbf{C}}}|^2 + k_2^2|\mathrm{II}_{\dot{\mathbf{C}}}| + k_3|\mathrm{III}_{\dot{\mathbf{C}}}|^{2/3},$$

k_i — material constants.

It is not difficult to show (compare VALANIS [4]) that under these assumptions the relations (2.2) and (2.3) still hold even if we have replaced the evolution equation (2.1) with the equation

$$\frac{d\mathbf{q}}{dz} = A(\mathbf{C}, \mathbf{q}), \quad \mathbf{q}|_{z=0} = \mathbf{q}_0. \tag{2.7}$$

Now we are going to give a more precise definition of $\mathbf{q}$. We assume the tensor $\mathbf{q}$ to be a measure of inelastic (permanent) deformation in the material coordinate system. From the experimental point of view the measurement of permanent deformations can be only performed in unloaded (stress-free) states in which total strains are permanent strains. Hence the tensor $\mathbf{q}$ should coincide in those states with the total strain.

Mathematically the following relation

$$\mathbf{T} = \mathscr{T}(\mathbf{C}, \mathbf{q})|_{\mathbf{C}=\mathbf{q}} = \mathbf{0} \tag{2.8}$$

expresses this fact.

Since a rate-independent (inviscid) plastic material cannot begin to change neither the stress nor the plastic strain at any unloaded state, the next relation

$$\frac{d\mathbf{q}}{dz} = A(\mathbf{C}, \mathbf{q})|_{\mathbf{C}=\mathbf{q}} = \mathbf{0} \tag{2.9}$$

is physically obvious.

We make at this stage an important assumption regarding materials to which the proposed equations are to be applied, namely in all deformation processes an inelastic deformation is isochoric. This incompressibility assumption in the inelastic (plastic) region is natural when metallic media are considered. As a consequence of this requirement we have the following constraint:

$$\det\mathbf{q} = 1. \tag{2.10}$$

Note that in a small strain theory Eq. (2.10) is replaced by $\mathrm{tr}\mathbf{q} = 0$.

Essential for any endochronic theory is the concept of intrinsic time. It consists of the time measure (2.6) and the time scale (2.5).

Valanis has used the time scale of the form

$$\frac{dz}{d\xi} = \frac{1}{1+\beta\xi}, \quad \beta = \text{const}. \tag{2.11}$$

In problems of cyclic plasticity the time scale reflects the deformation history and must cover the limit case $\xi \to \infty$. In this limit case the Valanis time scale does not make any sense, for $dz/d\xi$ tends to zero and, consequently, (cf. Eq. (2.7)) $\mathbf{q}$ vanishes, i.e. the material behaves as a perfectly elastic one. This is why Valanis' original time scale cannot be used for the analysis of cyclic loading.

In experiments with cyclic loading, periodic inputs either in strain (strain-control tests) or in stress (stress-control tests) are used. In steady-state outputs, i.e. stress or strain, are periodic as well. Hence the constitutive relations proposed should have some periodic properties. Two possibilities are conceivable. The first one is to introduce a periodic function (e.g. $\sin\xi$) into the time scale, but this form restricts considerations to special loading conditions and is not applicable to nonperiodic loadings. Another possibility is to introduce state variables (cf. Eq. (2.5)) in the definition of time scale imitating BAŽANT and his co-workers (cf. [9]). The form (2.5) is different from that of Bažant's for it includes the dependence on the internal variable. The reason for this is the following: when the cycle number is small, the material is in general in transient stage, where the hysteresis curve is not constant. In order to describe this, one has to incorporate the dependence on $\mathbf{q}$; moreover, in this stage the accumulation rate of ratchet strain is not constant, either.

To describe the behaviour of a material in long deformation histories and in a wide range of cycle number, let the time scale be an explicit function of the time measure. Therefore the function $h(\xi, \mathbf{C}, \mathbf{q})$ in Eq. (2.5) is split as follows:

$$\frac{dz}{d\xi} = h_1(\mathbf{C}, \mathbf{q}) h_2(\xi), \tag{2.12}$$

where $h_1(\mathbf{C}, \mathbf{q})$ has to express the periodic property of material behaviour and $h_2(\xi)$ — the influence of long deformation histories on $dz/d\xi$. We have found that the following forms of h_1 and h_2 are suitable:

$$h_1(\mathbf{C}, \mathbf{q}) = (\mathrm{II}_{(\mathbf{C}-\mathbf{q})})^{c_2}, \quad h_2(\xi) = c_3 + c_4 e^{-c_5}, \tag{2.13}$$

c_i, $i = 2, \ldots, 5$ material constants.

For problems of material behaviour and some boundary-value problems it is often more suitable to formulate constitutive relations in terms of stress and internal variable (compare [5, 6]). In such a dual formulation one treats the stress tensor $\mathbf{T}$ and the internal state variable $\mathbf{q}$ as a state and introduce the complementary energy function

$$\varphi = \mathrm{tr}(\mathbf{TC}) - 2\varrho\Psi \quad \text{or} \quad \varphi = \mathrm{tr}(\mathbf{TE}) - \varrho\hat{\Psi} \tag{2.14}$$

using the Lagrange strain tensor

$$\mathbf{E} = \frac{1}{2}(\mathbf{C}-\mathbf{1}) \tag{2.15}$$

and the reduced internal variable

$$\mathbf{q} = \frac{1}{2}(\mathbf{q}-\mathbf{1}) \tag{2.16}$$

and the free energy function

$$\psi = \hat{\Psi}(\mathbf{E}, \hat{\mathbf{q}}). \tag{2.17}$$

We can get the Lagrange strain tensor $\mathbf{E}$ by the relation

$$\mathbf{E} = \frac{\partial\varphi}{\partial\mathbf{T}} \tag{2.18}$$

Similarly, we formulate the evolution equation, the intrinsic time measure and time scale in terms of the state variables $\mathbf{T}, \hat{\mathbf{q}}$:

$$\frac{d\hat{\mathbf{q}}}{dz} = \hat{B}(\mathbf{T}, \hat{\mathbf{q}}), \quad \dot{\xi}^2 = l(\dot{\mathbf{T}}),$$

$$\frac{dz}{d\xi} = g(\mathbf{T}, \hat{\mathbf{q}}, \xi).$$

Note that the scalar-valued function l as well as k in Eq. (2.6) will depend on the invariants of $\dot{\mathbf{T}}$, and $\mathbf{D}$ and $\dot{\mathbf{C}}$, respectively. In the next section we derive explicit expressions for the functions introduced.

3. Second-order constitutive equation

Assuming that the material under consideration is isotropic, the representation theorem for an isotropic scalar-valued function of two symmetric tensors $\mathbf{C}$ and $\mathbf{q}$ (or $\mathbf{E}$ and $\hat{\mathbf{q}}$) states that there exist only 10 independent invariants through which the free energy function can be expressed. In the case of $\mathbf{E}$ and $\hat{\mathbf{q}}$ as the state variables the following set of invariants can be chosen:

$$\begin{aligned} &J_1 = 2\mathrm{I}_{\mathbf{E}} = \mathrm{I}_{\mathbf{C}} - 3, \quad J_2 = 4\mathrm{II}_{\mathbf{E}} = \mathrm{II}_{\mathbf{C}} - 2(\mathrm{I}_{\mathbf{C}} - 3) - 3, \\ &J_3 = 8\mathrm{III}_{\mathbf{E}} = \mathrm{III}_{\mathbf{C}} - 1 - (\mathrm{II}_{\mathbf{C}} - 3) + \mathrm{I}_{\mathbf{C}} - 3, \\ &J_4 = 2\mathrm{I}_{\hat{\mathbf{q}}}, \quad J_5 = 4\mathrm{II}_{\hat{\mathbf{q}}}, \quad J_6 = 8\mathrm{III}_{\hat{\mathbf{q}}}, \quad J_7 = 4\mathrm{tr}(\mathbf{E}\hat{\mathbf{q}}), \\ &J_8 = 8\mathrm{tr}(\mathbf{E}^2\hat{\mathbf{q}}), \quad J_9 = 8\mathrm{tr}(\mathbf{E}\hat{\mathbf{q}}^2), \quad J_{10} = 16\mathrm{tr}(\mathbf{E}^2\hat{\mathbf{q}}^2). \end{aligned} \tag{3.1}$$

We have decided to introduce the invariants (3.1) since they vanish in the reference configuration, where

$$\mathbf{C} = \mathbf{q} = \mathbf{1} \quad \text{and} \quad \mathbf{E} = \hat{\mathbf{q}} = \mathbf{0}.$$

The aim of this section is to derive the explicit equations of the second-order theory for isotropic plastically incompressible materials. To fulfill this we need the precise definition of the order as well as the list of all invariants with their orders.

DEFINITION. *When a function f is a polynomial function in a small quantity a, then the order of f is defined as the exponent of the lowest degree term.*

Our main assumption is:

The principal extensions δ_i, $i = 1, 2, 3$ are of the first order in some small parameter a.

According to the classical result, the strain tensor $\mathbf{C}$ in the basis of its principal direction has the form

$$[\mathbf{C}] = \begin{bmatrix} (1+\delta_1)^2 & 0 & 0 \\ 0 & (1+\delta_2)^2 & 0 \\ 0 & 0 & (1+\delta_3)^2 \end{bmatrix},$$

while the Lagrange strain tensor

$$[\mathbf{E}] = \begin{bmatrix} \delta_1 & 0 & 0 \\ 0 & \delta_2 & 0 \\ 0 & 0 & \delta_3 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} \delta_1^2 & 0 & 0 \\ 0 & \delta_2^2 & 0 \\ 0 & 0 & \delta_3^2 \end{bmatrix}.$$

In view of our definition of the internal variable $\mathbf{q}$, the following representation is obvious on the basis of the eigenvectors of $\mathbf{q}$, namely

$$[\mathbf{q}] = \begin{bmatrix} (1+\beta_1)^2 & 0 & 0 \\ 0 & (1+\beta_2)^2 & 0 \\ 0 & 0 & (1+\beta_3)^2 \end{bmatrix}$$

and, consequently,

$$[\hat{\mathbf{q}}] = \begin{bmatrix} \beta_1 & 0 & 0 \\ 0 & \beta_2 & 0 \\ 0 & 0 & \beta_3 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} \beta_1^2 & 0 & 0 \\ 0 & \beta_2^2 & 0 \\ 0 & 0 & \beta_3^2 \end{bmatrix}.$$

Note that at any unloaded state (i.e. $\mathbf{T} = \mathbf{0}$) the "permanent" extensions β_i, $i = 1, 2, 3$ coincide with the (total) principal extensions δ_i, $i = 1, 2, 3$. So we assume that β_i, $i = 1, 2, 3$, are of the first order.

It is a standard procedure to show that

$$\text{ord}(J_1) = 1, \quad \text{ord}(J_2) = 2, \quad \text{ord}(J_3) = 3. \tag{3.2}$$

It should be noted that the incompressibility condition in the plastic region (2.10) changes the orders of the invariants of $\mathbf{q}$. Both the invariants J_4 and J_5 are of the second order in the extensions β_i. In fact

$$J_4 = \text{tr}\mathbf{q} - 3 = 2(\beta_1+\beta_2+\beta_3) + (\beta_1^2+\beta_2^2+\beta_3^2) = -4(\beta_1\beta_2+\beta_2\beta_3+\beta_3\beta_1) + o(a^3),$$

where we have used the equalities

$$\begin{aligned} \det\mathbf{q} &= (1+\beta_1)^2(1+\beta_2)^2(1+\beta_3)^2 = 1, \\ 2(\beta_1+\beta_2+\beta_3) &= -(\beta_1^2+\beta_2^2+\beta_3^2) - 4(\beta_1\beta_2+\beta_2\beta_3+\beta_3\beta_1) + o(a^3). \end{aligned} \tag{3.3}$$

The symbol $o(a^3)$ denotes a term of the third order in a. The order of J_5 is two, for

$$\frac{1}{4}J_5 = \text{II}_{\hat{\mathbf{q}}} = \beta_1\beta_2+\beta_2\beta_3+\beta_3\beta_1 + o(a^3).$$

Note that in the present case there is no invariant of $\mathbf{q}$ of the first order. The order of J_6 is three. For the consistent notation we introduce the new invariant J_5' of the third order, using the standard method (see RIVLIN [10]),

$$-J_5' = J_6 = \text{III}_{\mathbf{q}} - 1 - (\text{II}_{\mathbf{q}} - 3) + \text{I}_{\mathbf{q}} - 3 = -\text{II}_{\mathbf{q}} + \text{I}_{\mathbf{q}}. \tag{3.4}$$

For further derivation we choose J_4 and J_5' as the complete set of the invariants of $\mathbf{q}$. Basing on the last results it is not difficult to show that

$$\text{ord}(J_7) = 2, \quad \text{ord}(J_8) = 3, \quad \text{ord}(J_9) = 3, \quad \text{ord}(J_{10}) = 4. \tag{3.5}$$

Now we make the fundamental assumption on the second order approximation.

The free energy function is a polynomial function of the J's invariants and contains terms up to and including the third order.

The mathematical formula of this assumption is the following expression:

$$\psi = \Psi(J_1, J_2, J_3, J_4, J_5', J_7, J_8, J_9) = a_0 + a_1 J_1 + a_2 J_2 + a_3 J_3 + a_4 J_4 + a_5 J_5' + a_7 J_7 + a_8 J_8 + a_9 J_9 + a_{10} J_1^2 + a_{11} J_1^3 + a_{12} J_1 J_2 + a_{14} J_1 J_4 + a_{17} J_1 J_7, \tag{3.6}$$

where a's are the material constants.

We assume that in the natural state (the reference configuration) $\psi = 0$ and $\mathbf{T} = \mathbf{0}$, then $a_0 = a_1 = 0$. We can see that the free energy has 12 constants in this case.

The second Piola–Kirchhoff stress relation in the case considered will be

$$\mathbf{T}\frac{1}{2\varrho} = [2(a_2 + 2a_{10})\mathrm{I}_{\mathbf{E}} + 4(a_{12} + 3a_{11})\mathrm{I}_{\mathbf{E}}^2 + 4(a_3 + a_{12})\mathrm{II}_{\mathbf{E}}]\mathbf{1} + [2a_{14}\mathrm{I}_{\hat{\mathbf{q}}} + 4a_{17}\mathbf{E}\cdot\hat{\mathbf{q}}]\mathbf{1} - 2[a_2 + 2(a_3 + a_{12})\mathrm{I}_{\mathbf{E}}]\mathbf{E} + 4a_3\mathbf{E}^2 + 2a_7\hat{\mathbf{q}} + 4a_{17}\mathrm{I}_{\mathbf{E}}\hat{\mathbf{q}} + 4a_8(\mathbf{E}\hat{\mathbf{q}} + \hat{\mathbf{q}}\mathbf{E}) + 4a_9\hat{\mathbf{q}}^2, \tag{3.7}$$

where the potential relation (2.2) has been used.

In order to determine the material constants we make the following assumption:

The linear part of the initial response of the material from its natural state (i.e. $\mathbf{E} = \hat{\mathbf{q}} = \mathbf{0}$) is given by the classical Lamé constants λ and μ.

This gives us the following relationships:

$$-2\varrho a_2 = \mu, \quad 4\varrho(a_2 + 2a_{10}) = \lambda.$$

Hence

$$a_2 = -\frac{\mu}{2\varrho}, \quad a_{10} = \frac{1}{8\varrho}(\lambda + 2\mu). \tag{3.8}$$

The condition (2.8) gives us the further constraints on the material constants.

At any unloaded state $J_4 = J_1$, $J_5 = J_2$, $J_7 = J_4^2 - 2J_5$ and hence

$$\mathbf{0} = \left(\frac{\lambda}{4\varrho} + a_{14}\right)J_4\mathbf{1} + \left(\frac{\mu}{\varrho_0} + 2a_7\right)\hat{\mathbf{q}} + (a_{12} + 3a_{11} + a_{17})J_4^2\mathbf{1} + 4(a_3 + 2a_8 + a_9)\hat{\mathbf{q}}^2 + (a_3 + a_{12} - 2a_{17})J_5\mathbf{1} + 2(a_{17} - a_3 - a_{12})J_4\hat{\mathbf{q}}.$$

This relation should be true for any symmetric tensor $\hat{\mathbf{q}}$ and its invariants J_4 and J_5 with only constraint $\det(2\hat{\mathbf{q}} + \mathbf{1}) = 1$. From this we get

$$\begin{aligned} -4\varrho a_{14} &= \lambda, \quad a_3 + a_{12} = a_{12} + 3a_{11} = a_{17} = 0, \\ -2\varrho a_7 &= \mu, \quad a_3 + 2a_8 + a_9 = 0. \end{aligned} \tag{3.9}$$

Substituting these relations into Eq. (3.7) gives the stress relation

$$\mathbf{T} = \lambda\mathrm{tr}(\mathbf{E} - \hat{\mathbf{q}})\mathbf{1} + 2\mu(\mathbf{E} - \hat{\mathbf{q}}) + 8\varrho\left[a_3\mathbf{E}^2 - \frac{1}{2}(a_3 + a_9)(\mathbf{E}\hat{\mathbf{q}} + \hat{\mathbf{q}}\mathbf{E}) + a_9\hat{\mathbf{q}}^2\right]. \tag{3.10}$$

Now the number of unknown constants reduces to two. The constant a_3 can be interpreted as the third-order material constant. In order to determine it we make the following assumption:

The free energy function Ψ can be decomposed into the purely elastic strain energy Ψ_e and the energy Ψ_i of the elasto-plastic coupling effects, i.e.

$$\Psi(\mathbf{E}, \hat{\mathbf{q}}) = \Psi_e(\mathbf{E}) + \Psi_i(\mathbf{E}, \hat{\mathbf{q}}),$$

where

$$\Psi_e = a_{10}J_1^2 + a_2 J_2 + a_{11}J_1^3 + a_{12}J_1 J_2 + a_3 J_3 = 4a_{10}\mathrm{I}_\mathbf{E}^2 + 4a_2 \mathrm{II}_\mathbf{E} + 8a_{11}\mathrm{I}_\mathbf{E}^3 + 8a_{12}\mathrm{I}_\mathbf{E}\mathrm{II}_\mathbf{E} + 8a_3 \mathrm{III}_\mathbf{E},$$

$$\Psi_i = a_4 J_4 + a_5 J_5' + a_7 J_7 + a_8 J_8 + a_9 J_9 + a_{14} J_1 J_4 + a_{17} J_1 J_7.$$

Now we identify the coefficient of $\mathrm{III}_\mathbf{E}$ with the third-order modulus in the strain energy function of acousto-elasticity. In that function defined per unit mass, the third-order modulus is denoted by $4\nu_3$. Hence we have the relation

$$2\varrho a_3 = \nu_3. \tag{3.11}$$

It should be noted that the material constant ν_3, in hyperelasticity known as the coefficient of second-order elasticity, can be measured using acoustic methods.

Under the condition (3.11) the final form of the stress relation will be

$$\mathbf{T} = 2\mathrm{tr}(\mathbf{E}-\hat{\mathbf{q}})\mathbf{1} + 2\mu(\mathbf{E}-\hat{\mathbf{q}}) + 4\nu_3\mathbf{E}^2 + 8\varrho a_9 \hat{\mathbf{q}}^2 - 2(\nu_3 + 2\varrho a_9)(\mathbf{E}\hat{\mathbf{q}} + \hat{\mathbf{q}}\mathbf{E}). \tag{3.12}$$

Now we have only the unknown constant a_9. Assuming for simplicity that $2\varrho a_9 = \nu_3$, we get the following relation:

$$\mathbf{T} = \lambda \mathrm{tr}(\mathbf{E}-\hat{\mathbf{q}})\mathbf{1} + 2\mu(\mathbf{E}-\hat{\mathbf{q}}) + 4\nu_3(\mathbf{E}-\hat{\mathbf{q}})^2. \tag{3.13}$$

In the first approximation identifying the strain tensor $\mathbf{E}$ with the infinitesimal strain tensor $\mathbf{E} = 1/2\ (\mathbf{H}+\mathbf{H}^T)$ where $\mathbf{H}$ is the displacement gradient, both relations (3.13) give the VALANIS classical stress relation [4].

4. Evolution equation for internal variable

In the derivation of the evolution equation for the internal variable $\mathbf{q}$ we have to satisfy the identity (cf. Eq. (2.10))

$$\frac{d}{dz}(\det \mathbf{q}) = \mathrm{tr}\left(\mathbf{q}^{-1}\frac{d\mathbf{q}}{dz}\right) = \frac{d\mathbf{q}}{dz}\cdot\mathbf{q}^{-1} = 0 \tag{4.1}$$

with $\det \mathbf{q}(z)_{z=0} = 1$. This means that the products $\mathbf{q}^{-1}d\mathbf{q}/dz$ and $(d\mathbf{q}/dz)\mathbf{q}^{-1}$ should be the deviatoric part of some tensor $\mathbf{s}$ treated as a function of $\mathbf{E}$ and $\hat{\mathbf{q}}$. The tensor function $s = \bar{\mathbf{s}}(\mathbf{E}, \hat{\mathbf{q}})$ must be linear in $\mathbf{E}$ and $\hat{\mathbf{q}}$ in order to be consistent with the assumptions of the second-order approximation, i.e.

$$\bar{s}(\mathbf{E}, \hat{\mathbf{q}}) = b_0\mathbf{E} + \bar{b}_1\hat{\mathbf{q}} + \bar{b}_2\mathrm{tr}(\mathbf{E})\mathbf{1},$$

where b_0, $\bar{b}_1$, $\bar{b}_2$ are material constants. The condition (2.9) results in $\bar{b}_1 = b_0$, $\bar{b}_2 = 0$. Hence the final form of the evolution equation will be

$$\frac{1}{b_0}\frac{d\hat{\mathbf{q}}}{dz} = (\mathbf{E}-\hat{\mathbf{q}}) - \frac{1}{3}\mathrm{tr}(\mathbf{E}-\hat{\mathbf{q}})\mathbf{1} + (\mathbf{E}-\hat{\mathbf{q}})\hat{\mathbf{q}} + \hat{\mathbf{q}}(\mathbf{E}-\hat{\mathbf{q}}) - \frac{2}{3}(\mathrm{tr}(\mathbf{E}-\hat{\mathbf{q}}))\hat{\mathbf{q}}. \tag{4.2}$$

The nonlinear terms in Eq. (4.2) manifest the existence of elasto-plastic coupling effects. For the small strain theory the plastic incompressibility condition reduces to $\mathrm{tr}(d\mathbf{q}/dz) = 0$ and two first linear terms in Eq. (4.2) are sufficient. Note that in the finite deformation theory there is no linear evolution equation satisfying Eq. (2.10).

5. Dual constitutive relations

As it has been stated above, in some particular problems another set of state variables may be more useful, namely the stress $\mathbf{T}$ and the internal variable $\hat{\mathbf{q}}$. Introducing the complementary energy function Φ and making the second-order approximation, we arrive at the following strain relation:

$$\mathbf{E} = [2b_{10}\mathrm{I}_T + b_{12}\mathrm{II}_T + 3b_{11}\mathrm{I}_T^2 + b_{14}\mathrm{I}_{\hat{q}} + b_{17}\mathbf{T}\cdot\hat{\mathbf{q}}]\mathbf{1} + [b_2 + b_{12}\mathrm{I}_T](\mathrm{I}_T\mathbf{1} - \mathbf{T}) + b_3(\mathrm{II}_T\mathbf{1} - \mathrm{I}_T\mathbf{T} + \mathbf{T}^2) + b_7\hat{\mathbf{q}} + b_8\hat{\mathbf{q}}^2 + b_9(\mathbf{T}\hat{\mathbf{q}} + \hat{\mathbf{q}}\mathbf{T}), \tag{5.1}$$

where b_k are material constants.

Imposing the constraint $\mathbf{E}|_{\mathbf{T}=0} = \hat{\mathbf{q}}$ we get $b_8 = b_{13} = 0$, $b_7 = 1$. Assuming that the linear part of the initial response from the natural state (i.e. $\mathbf{T} = \mathbf{0}$, $\hat{\mathbf{q}} = \mathbf{0}$) is governed by the classical elasticity coefficients, we get

$$-b_2 = \frac{1}{2\mu}, \quad 2b_{10} = \frac{1}{E_0}$$

and the final form of the strain relation will be

$$\mathbf{E} = -\frac{\nu}{E_0}\mathrm{I}_T\mathbf{1} + \frac{1+\nu}{E_0}\mathbf{T} + [(b_3 + 3b_{11})\mathrm{I}_T^2 + (b_3 + b_{12})\mathrm{II}_T]\mathbf{1} - (b_3 + b_{12})\mathrm{I}_T\mathbf{T} + b_3\mathbf{T}^2 + \hat{\mathbf{q}} + b_{17}(\mathbf{T}\cdot\hat{\mathbf{q}})\mathbf{1} + b_9(\mathbf{T}\hat{\mathbf{q}} + \hat{\mathbf{q}}\mathbf{T}), \tag{5.2}$$

where ν is the Poisson ratio, E_0 is the Young modulus. In order to determine the unknown constants b_3, b_8, b_{11}, b_{12} and b_{17}, we may use the idea presented in Sect. 3. Moreover, if one wishes to known the relationships between b's constants and a's the following procedure shuld be applied: Find the expressions for the invariants of $\mathbf{E}$ in terms of the invariants of $\mathbf{T}$ using the dual equation for the elastic part of the complementary energy Φ_e; express Φ_e in terms of the invariants of $\mathbf{E}$ and finally identify this function with

$$\Phi_e = \varrho\,\mathrm{tr}\left(\frac{\partial \Psi_e}{\partial \mathbf{E}}\mathbf{E}\right) - \varrho\Psi_e.$$

It should be pointed out that this method does not supply any expression for b_9 and b_{17} appearing in the elasto-plastic coupling terms.

To obtain a dual form of the evolution equation, we use the restrictions (2.9) and (2.10). In the same manner as in Sect. 4 we find that $\mathbf{q}^{-1}d\mathbf{q}/dz$ and $(d\mathbf{q}/dz)\mathbf{q}^{-1}$ should be the deviatoric part of some tensor $\mathbf{t}$. The tensor $\mathbf{t}$ treated as a function of $\mathbf{T}$ and $\hat{\mathbf{q}}$ must be linear (of the first order) in order to be consistent with the second-order approximation procedure, i.e.

$$\mathbf{t} = d_0\mathbf{T} + d_1\hat{\mathbf{q}} + d_2\,\mathrm{tr}(\mathbf{T})\mathbf{1}$$

with d_0, d_1 and d_2 as material constants. The condition (2.9) results in $d_1 = 0$ and $d_2 = d_0$; this leads to

$$\frac{1}{d_0}\frac{d\hat{\mathbf{q}}}{dz} = \mathbf{T}' + \mathbf{T}'\hat{\mathbf{q}} + \hat{\mathbf{q}}\mathbf{T}', \tag{5.3}$$

where $\mathbf{T}'$ is the deviator of the stress tensor $\mathbf{T}$.

It is interesting to note that in the linear case the evolution equation (5.3) reduces to

$$d\hat{\mathbf{q}} = d_0\mathbf{T}'dz, \tag{5.4}$$

that is the increment of $\hat{\mathbf{q}}$ is proportional to the stress deviator, with the factor of proportionality $d_0 dz$. This is similar to the Prandtl–Reuss equation but here we have the Lagrange measures.

The stress-based intrinsic time measure can be defined as follows:

$$\dot{\xi}_{\mathbf{T}}^2 = R_1(\mathrm{I}_{\dot{\mathbf{T}}})^2 + R_2|\mathrm{II}_{\dot{\mathbf{T}}}| + R_3|\mathrm{III}_{\dot{\mathbf{T}}}|^{2/3},$$

where the superposed dot denotes the time (material) derivative. For the time scale we propose

$$\frac{dz}{d\xi_{\mathbf{T}}} = g_1(\mathbf{T}, \hat{\mathbf{q}})g_2(\xi_{\mathbf{T}}),$$

where g_1 and g_2 are positive scalar-valued functions.

6. Plastic anisotropy in the endochronic theory

In this section we are going to show an important feature of the constitutive equation proposed. Let us assume that at the intrinsic time $\xi = \xi^*$ the values of $\mathbf{C}$ and $\mathbf{q}$ are equal, i.e. $\mathbf{C} = \mathbf{q} = \mathbf{q}^*$ and the material is at an unloaded state. Then, in view of our assumptions (2.8) and (2.9)

$$\left.\frac{d\hat{\mathbf{q}}}{dz}\right|_{z=z(\xi^*)} = \mathbf{0}, \quad \mathscr{T}(\mathbf{C}(\xi^*), \mathbf{q}(\xi^*)) = \mathbf{0}. \tag{6.1}$$

Let us make a small increment $\Delta\xi$ and find the linear part of the increment in the stress $\mathbf{T}$. We have

$$\mathbf{T}(\xi^* + \Delta\xi) = \mathbf{T}(\xi^*) + \frac{d\mathbf{T}}{d\xi}(\xi^*)\Delta\xi + o(\Delta\xi^2).$$

For $d/d\xi\,\mathbf{T}(\xi^*)$ we have

$$\frac{d\mathbf{T}}{d\xi}(\xi^*) = \frac{\partial\mathscr{T}}{\partial\mathbf{C}}\frac{d\mathbf{C}}{d\xi}\bigg|_{\xi=\xi^*} + \frac{\partial\mathscr{T}}{\partial\mathbf{q}}\frac{d\mathbf{q}}{d\xi}\bigg|_{\xi=\xi^*} = \frac{\partial\mathscr{T}}{\partial\mathbf{C}}\frac{d\mathbf{C}}{d\xi}\bigg|_{\xi=\xi^*},$$

because of Eq. (6.1). Using the constitutive equation (3.10), we obtain

$$\mathbf{T}(\xi^* + \Delta\xi) = \lambda\mathrm{tr}(\Delta\mathbf{E})\mathbf{1} + 2\mu\Delta\mathbf{E} + (4\varrho a_9 - 2\nu_3)(\hat{\mathbf{q}}^*\Delta\mathbf{E} + \Delta\mathbf{E}\hat{\mathbf{q}}^*) + o(\Delta\xi^2), \tag{6.2}$$

where

$$2\Delta\mathbf{E} = \left.\frac{d\mathbf{C}}{d\xi}\right|_{\xi=\xi^*}\Delta\xi.$$

This equation shows the existence of elasticity in the linear approximation without introducing explicitly the yield criterion concept. It should be noted that VALANIS showed in his recent paper [11] the existence of elasticity in ET; however, he used a new definition of intrinsic time measure. Moreover, even in the linear part of the material response from any unloaded but prestrained state some kind of "plastic" anisotropy occurs. This is seen in the last term of Eq. (6.2), where the previous permanent deformation $\hat{\mathbf{q}}^*$ influences the linear elastic properties of the material. At virgin state $\xi = 0$ and $\mathbf{q} = 0$, the linear response is purely elastic and isotropic. This is a consequence of our first assumption made in Sect. 3.

7. Steady axial stress and cyclic shear strain ratcheting

In this section we analyse by means of the derived equation the representative ratcheting test: a steady normal stress superposed on cyclic shear strain.

Under this simple loading condition the Cauchy stress $\overset{c}{\mathbf{T}}$ must have the form

$$[\overset{c}{\mathbf{T}}] = \begin{bmatrix} \overset{c}{T}{}_1^0 & \overset{c}{T}(t)_{12} & 0 \\ \overset{c}{T}(t)_{12} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{7.1}$$

in the Cartesian coordinate system. $\overset{c}{T}{}_1^0$ is constant throughout the experiment while $\overset{c}{T}(t)_{12}$ is a time dependent. The deformation gradient $\mathbf{F}$, the strain $\mathbf{E}$ and the internal state variable $\hat{\mathbf{q}}$ related to $\overset{c}{\mathbf{T}}$ are

$$[\mathbf{F}] = \begin{bmatrix} a & d & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}, \quad [\mathbf{E}] = \begin{bmatrix} E_1 & E_{12} & 0 \\ E_{12} & E_2 & 0 \\ 0 & 0 & E_3 \end{bmatrix}, \quad [\hat{\mathbf{q}}] = \begin{bmatrix} \hat{q}_1 & q_{12} & 0 \\ q_{12} & \hat{q}_2 & 0 \\ 0 & 0 & \hat{q}_3 \end{bmatrix}. \tag{7.2}$$

Using the known relation the second Piola–Kirchhoff stress $\mathbf{T} = (\det\mathbf{F})\ \mathbf{F}^{-1}\overset{c}{\mathbf{T}}(\mathbf{F}^{-1})^T$ can be expressed as follows:

$$[\mathbf{T}] = \begin{bmatrix} T_1 & T_{12} & 0 \\ T_{12} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{c}{a}(b\overset{c}{T}{}_1^0 - 2d\overset{c}{T}_{12}) & c\overset{c}{T}_{12} & 0 \\ c\overset{c}{T}_{12} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{7.3}$$

while the relations between the components of $\mathbf{E}$ and $\mathbf{F}$ are the following:

$$\begin{aligned} a &= \sqrt{1+2E_1}, & d &= 2E_{12}/\sqrt{1+2E_1}, \\ b &= \sqrt{1+2E_2-d^2}, & c &= \sqrt{1+2E_3}. \end{aligned} \tag{7.4}$$

In the further investigation the evolution equation (4.2) with the nonlinear coupling terms are used together with the constitutive equation (5.2). For the axial strain E_1 we get

$$(7.5)\quad E_1 = \frac{1}{E_0}\left(\frac{bc}{a}\overset{c}{T}{}_1^0 - \frac{2cd}{a}\overset{c}{T}_{12}\right) + (b_3 - b_{11} + 3b_{12})\left(\frac{bc}{a}\overset{c}{T}{}_1^0 - \frac{2cd}{a}\overset{c}{T}_{12}\right)^2 - b_{11}c(\overset{c}{T}_{12})^2$$
$$+\hat{q}_1 + (2b_9 + b_{17})\frac{bc}{a}\overset{c}{T}{}_1^0\hat{q}_1 - (b_{17} + 2b_9)\frac{2cd}{a}\overset{c}{T}_{12}\hat{q}_1 + 2(b_{17} + b_9)c\overset{c}{T}_{12}\hat{q}_{12}.$$

One can see here the nonlinear dependence of E_1 on the steady state $\overset{c}{T}{}_1^0$. This follows not only from the second quadratic term in Eq. (7.5), but also in the coupled term $\overset{c}{T}{}_1^0\hat{q}_1$, for the component $\hat{q}_1$ depends throughout the evolution equation (cf. Eq. (5.3)) on $\overset{c}{T}{}_1^0$.

In the VALANIS and WU theory [8] the axial ratcheting strain ε_1 is a linear function of the axial steady stress σ_0. On the other hand most experiments report nonlinear dependence of the ratchet strain on the steady stress. In his thesis NASHIRO [12] has calculated the theoretical prediction of Eq. (7.5) of the presented theory and compared with the experimental data reported by IKEGAMI *et al.* [13]. These data were based on observations of a thin-walled tube of annealed 6:4 brass under steady axial stress and strain-controlled cyclic shear strain. For numerical calculations Nashiro used the following material constants:

$$E_0 = 10960\ \text{kg/mm}^2,\quad E_1 = 4248\ \text{kg/mm}^2,\quad c_1 = 0.29,\quad b_0 = 1300,$$
$$c_2 = 0.2,\quad c_3 = 0.6,\quad c_4 = 0.4,\quad c_5 = 2,\quad b_9 = 0.0001\ \text{mm}^2/\text{kg},$$
$$b_{17} = 0.0001\ \text{mm}^2/\text{kg},\quad b_3 = b_{11} = b_{12} = 0.$$

Figure 1 shows two steady axial stress $\overset{c}{T}{}_1^0 = 6.88\ \text{kg/mm}^2$ and $\overset{c}{T}{}_1^0 = 9.65\ \text{kg/mm}^2$ superposed shear strain amplitude $\gamma^a = 1.73\%$ (engineering strain) ratcheting curves. The marks "▲" and "●" denote experimental data, and the solid lines calculated values. In

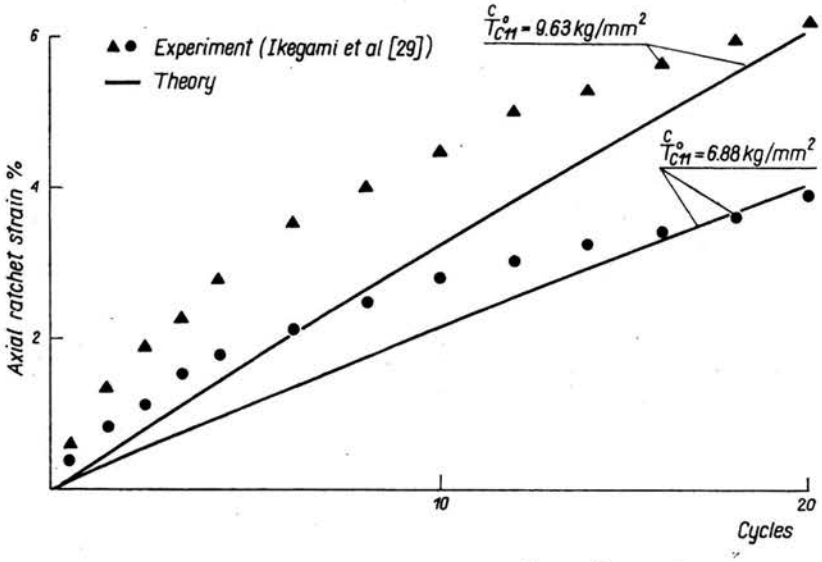

FIG. 1. Axial ratchet strain accumulation. Here $\overset{c}{T}{}_{c11}^0 \equiv \overset{c}{T}{}_1^0$ and [29] ≡ [13].

spite of the fact that in small cycle number agreement is not good, as a whole it can describe the ratchet strain accumulating behaviour in a cyclic process.

Figure 2 shows the influence of the steady stress on accumulated axial strain. A very good agreement is seen in the description of the nonlinear influence of the steady stress on the ratchet strain.

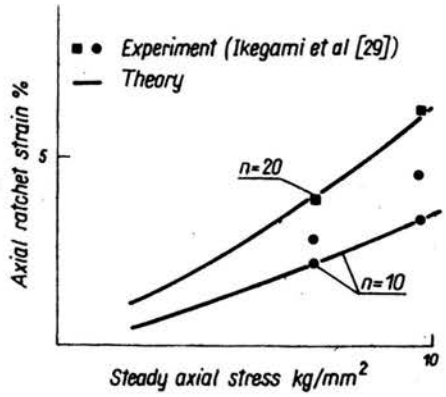

FIG. 2. Influence of steady stress. Here [29] ≡ [13].

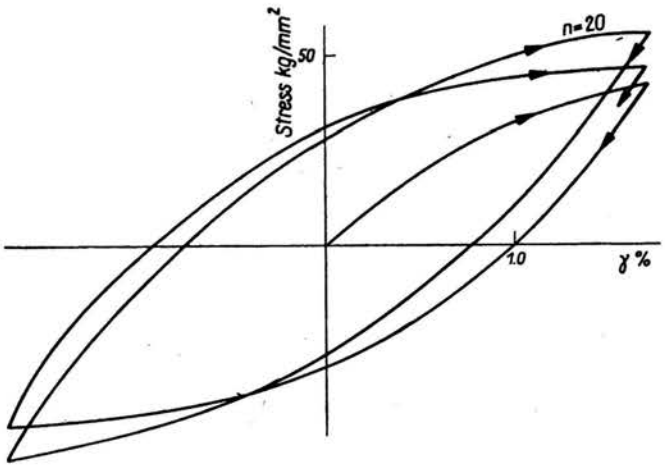

FIG. 3. Cyclic shear stress-strain curve.

Figure 3 represents the cyclic shear stress-strain curves. In a large cycle number the hysteresis loop does not tend to elastic response as Valanis' original theory predicted.

8. Description of cyclic hardening and cyclic softening

When metallic materials are under strain-controlled large cyclic straining accompanying plastic strains, then they (a) cyclically harden, (b) soften or (c) harden at first and soften later, depending upon the initial internal structure of the material. The cyclic

hardening and cyclic softening can be characterized by two factors, that is the peak stress and the shape of the hysteresis loop at each cycle number. Both phenomena, cyclic hardening and cyclic softening, have transient and steady states. Our objective is to describe consistently the peak stress and the hysteresis loop, and both phenomena using our endochronic theory proposed above.

When we apply only shear stress T_{12}, we can assume that $\mathbf{E}$ and $\hat{\mathbf{q}}$ have the following forms:

$$[\mathbf{E}] = \begin{bmatrix} E_1 & E_{12} & 0 \\ E_{12} & E_2 & 0 \\ 0 & 0 & E_3 \end{bmatrix}, \quad [\hat{\mathbf{q}}] = \begin{bmatrix} \hat{q}_1 & \hat{q}_{12} & 0 \\ \hat{q}_{12} & \hat{q}_2 & 0 \\ 0 & 0 & \hat{q}_3 \end{bmatrix}$$

in some Cartesian coordinate system. We cannot write in general $E_1 = E_2 = E_3 = \hat{q}_1 = \hat{q}_2 = \hat{q}_3 = 0$ in this shear stress loading condition because this theory is not of infinitesimal strain. The linear part of the constitutive equation (3.12) has the following form:

$$\begin{aligned} T_{12} &= 2\mu(E_{12}-\hat{q}_{12}), \\ 0 &= (\lambda+2\mu)(E_1-\hat{q}_1)+\lambda(E_2-\hat{q}_2)+\lambda(E_3-\hat{q}_3), \\ 0 &= \lambda(E_1-\hat{q}_1)+(\lambda+2\mu)(E_2-\hat{q}_2)+\lambda(E_3-\hat{q}_3), \\ 0 &= \lambda(E_1-\hat{q}_1)+\lambda(E_2-q_2)+(\lambda+2\mu)(E_3-\hat{q}_3). \end{aligned} \tag{8.1}$$

The last three equations give the relation $E_1-\hat{q}_1 = E_2-\hat{q} = E_3-\hat{q}_3 = 0$. But it does not mean that $E_1 = E_2 = E_3$. The evolution equation (4.2) under cyclic torsion is

$$\begin{aligned} &\frac{d\hat{q}_1}{dz} = \frac{d\hat{q}_2}{dz} = 2b_0(E_{12}-\hat{q}_{12})\hat{q}_{12}, \quad \frac{d\hat{q}_3}{dz} = \frac{d\hat{q}_{13}}{dz} = \frac{d\hat{q}_{23}}{dz} = 0, \\ &\frac{d\hat{q}_{12}}{dz} = b_0(E_{12}-\hat{q}_{12})+b_0(E_{12}-\hat{q}_{12})(q_1+q_2) = b_0(E_{12}-\hat{q}_{12})(1+2q_1). \end{aligned} \tag{8.2}$$

The relations $d\hat{q}_1/dz = d\hat{q}_2/dz$, $d\hat{q}_3/dz = d\hat{q}_{13}/dz = d\hat{q}_{23}/dz = 0$ mean that the tensor $d\hat{q}/dz$ can be expressed by two components $d\hat{q}_1/dz$ and $d\hat{q}_{12}/dz$ in this case. Note that $\hat{q}_1 = \hat{q}_2$, $\hat{q}_3 = 0$, $E_1 = \hat{q}_1$, $E_2 = \hat{q}_2$. The invariants of $\dot{\mathbf{E}}$ are

$$\mathrm{I}_{\dot{\mathbf{E}}} = 2\frac{d\hat{q}_1}{dt}, \quad \mathrm{II}_{\dot{\mathbf{E}}} = \left(\frac{d\hat{q}_1}{dt}\right)^2 - \left(\frac{dE_{12}}{dt}\right)^2, \quad \mathrm{III}_{\dot{\mathbf{E}}} = 0. \tag{8.3}$$

Then the time measure is

$$d\xi = k_1|2d\hat{q}_1|+k_2|(d\hat{q}_1)^2-(dE_{12})^2|^{1/2}. \tag{8.4}$$

For the function f_1 Eqs. (2.13) and (8.1) give $|\mathrm{II}_{(\mathbf{E}-\hat{\mathbf{q}})}|^{c_2} = |E_{12}-\hat{q}_{12}|^{2c_2}$; then the time scale (2.12) is given by

$$\frac{dz}{d\xi} = |E_{12}-\hat{q}_{12}|^{2c_2}(c_3+c_4e^{-c_5\xi}). \tag{8.5}$$

Numerical calculation was made with material constants of annealed copper

$$\begin{aligned} &E_0 = 16700 \text{ ksi}, \ \mu = 5600 \text{ ksi}, \\ &\nu = 0.33 \text{ and } b_0 = 14270, \ k_1 = k_2 = k_3 = 1, \ c_2 = 0.125, \\ &c_3 = 0.4, \ c_4 = 0.6, \ c_5 = 20.0 \end{aligned} \tag{8.6}$$

and compared with the experimental data by LAMBA [14].

Figure 4 is Lamba's figure of the torsional cyclic straining of amplitude $\gamma = 1.1\%$ (engineering strain) of annealed copper. Figure 5 shows a theoretical description using the above material constants of the same material under the same loading condition.

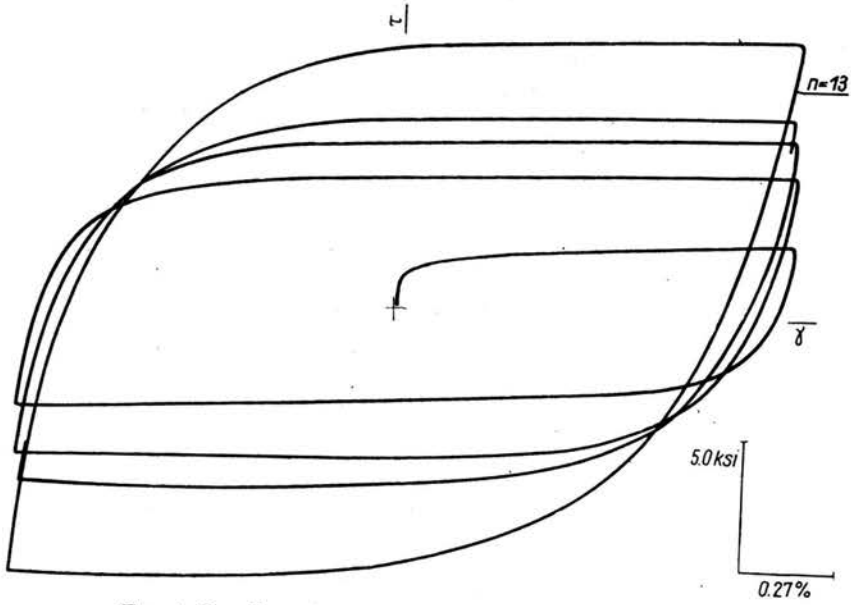

FIG. 4. Reording of torsional cyclic hardening by LAMBA [14].

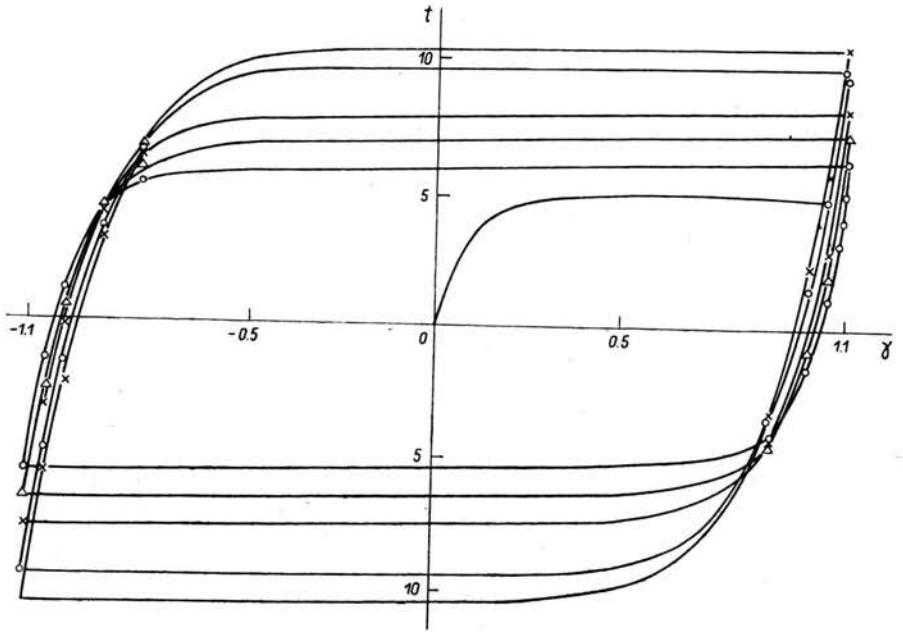

FIG. 5. Description of torsional cyclic hardening.

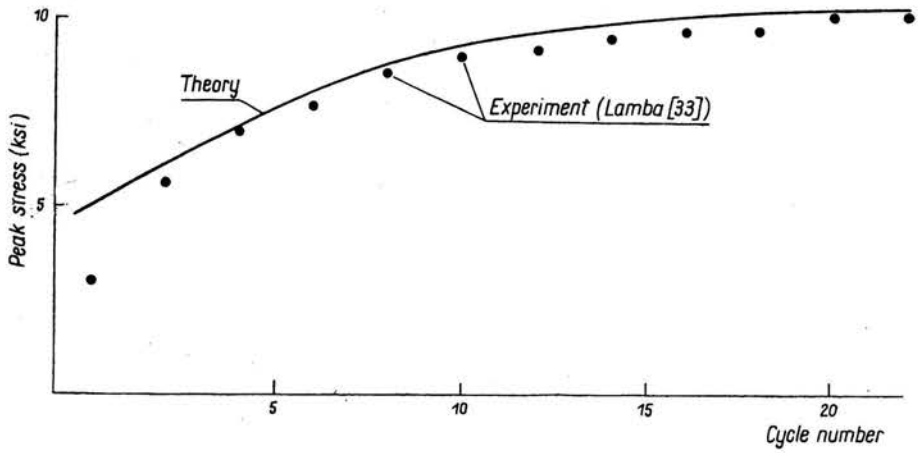

FIG. 6. Change of peak stress. Here [33] ≡ [14].

Figure 6 shows the change of peak stress with respect to the cycle number n, where the solid line is the theoretical prediction. Figures 5 and 6 are in good accordance with the experiment. It can also describe properties in both the transient state and the steady state.

Note added in proof: Recently K. C. Valanis and the second author have described that phenomenon using different procedure.

References

1. B. D. COLEMAN and M. E. GURTIN, *Thermodynamics with internal state variables*, J. Chem. Phys., **47**, 597–613, 1967.
2. K. C. VALANIS, *Unified theory of thermomechanical behavior of viscoelastic materials*, Symp. Mech. Behav. Mater. Dyn. Loads, San Antonio 1967, Springer Verlag, 343–364, New York 1968.
3. P. PERZYNA and W. WOJNO, *Thermodynamics of a rate sensitive plastic material*, Arch. Mech., **20**, 499–511, 1968.
4. K. C. VALANIS, *A theory of viscoplasticity without a yield surface, Parts I and II*, Arch. Mech., **23**, 517–551, 1971.
5. H. C. WU, K. C. VALANIS and A. R.-F. YAO, *Application of the endochronic theory of plasticity in the Gibbs free energy form*, Lett. Appl. Engng. Sci., **4**, 127–136, 1976.
6. A. R.-F. YAO, *A nonlinear form of the endochronic theory and its application to dynamic plasticity*, Ph. D. Thesis, University of Iowa, Iowa City 1976.
7. H. C. WU and M. C. YIP, *Strain rate and strain rate history effects on the dynamic behavior of metallic materials*, Int. J. Solids Struct., **16**, 515–536, 1980.
8. K. C. VALANIS and H. C. WU, *Endochronic representation of cyclic creep and relaxation in metals*, J. Appl. Mech., (Trans. ASME), **42**, 67–73, 1975.
9. Z. P. BAZANT, *Endochronic inelasticity and incremental plasticity*, Int. J. Solids Struct., **14**, 671–714, 1978.
10. R. S. RIVLIN, *Some topics in finite elasticity*, Proc. Symp. Naval Struct. Mech., Stanford Univ., 1958, Pergamon Press, Oxford-London-New York-Paris 1960.

11. K. C. VALANIS, *Fundamental consequences of a new intrinsic time measure. Plasticity as a limit of the endochronic theory*, Arch. Mech., **32**, 171-191, 1980.
12. T. NASHIRO, *Evaluation of mechanical ratcheting using endochronic theory*, Ph. D. Thesis, Institute of Fundamental Technological Research, Warsaw 1979.
13. K. IKEGAMI, F. YOSHIDA, T. MURAKAMI, K. KANEKO and E. SHIRATORI, *Plastic behavior in combined loading along straight stress path with a bend. Investigation on the stress-strain relation by an anisotropic hardening plastic potential*, Bull. JSME, **20**, 1549–1556, 1977.
14. H. S. LAMBA, *Nonproportional cyclic plasticity*, T.A.M. Report, 413, U-Eng 76, University of Illinois, Urbana 1976.

POLISH ACADEMY OF SCIENCES
INSTITUTE OF FUNDAMENTAL TECHNOLOGICAL RESEARCH.

Received August 15, 1980.