Arch. Mech., 33, 4, pp. 511-522, Warszawa 1981

Application of convex analysis to the calculation of stress-state in elastic-plastic plates

D. WEICHERT (BOCHUM)

FOR AN ARBITRARY dead-load-type loading process minimum principles for stress-state in elastic-plastic plates are derived assuming infinitesimal deformations. Numerical application is given to some simple problems.

Dla dowolnego procesu obciążania typu zachowawczego wyprowadzono zasady minimum dotyczące stanów naprężenia w płytach sprężysto-plastycznych przy założeniu infinitezymalnych odkształceń. Podano przykłady zastosowań numerycznych do pewnych prostych zagadnień szczegółowych.

Для произвольного процесса нагружения консервативного типа выведен принцип минимума, касающийся напряженного состояния в упруго-пластических плитах, при предположении инфинитезимальных деформаций. Приведены примеры численных применений к некоторым простым частным задачам.

1. Introduction

IF NONPROPORTIONAL loading processes are considered, methods using only moments and curvature to describe material behaviour in plates fail in principle and can only be regarded as rough approximations. By the expansion of arbitrary smooth stress and strain distributions by Taylor series a more realistic mathematical plate model can be systematically derived and related to the general three-dimensional theory. In [1, 2] the initial boundary value problem for generalized standard elastic-visco-plastic material [3] has been solved by introducing appropriately chosen Hilbert spaces using the mathematical tool of convex analysis. In [5] the rate boundary value problem for thin elastic — ideal plastic plates was investigated. Here an application to thin plates under the assumption of infinitesimal displacements and linear hardening material behaviour is given and applied numerically to some simple problems.

2. Three-dimensional foundation

2.1. Local formulation of the problem

At every instant t_0 of the deformation process the following system of differential equations and inequalities defines the mechanical state of an elastic-plastic body (generalized standard elastic-plastic material [2]) occupying the volume $V(x_1, x_2, x_3, t)$ in the Cartesian product space of R^3 and space T of time $t \in [0, \infty)$, with the regular boundary

$B = B_s \cup B_k$, with the prescribed forces $\mathbf{p}$ on B_s and the prescribed displacements $\mathbf{u}^*$ on B_k:

$$
\begin{aligned}
2\boldsymbol{\epsilon} &= (\mathrm{Grad}\,\mathbf{u})_{\mathrm{sym}} \quad \text{in} \quad V,\\
\mathbf{u} &= \mathbf{u}^* \quad \text{on} \quad B_k,\\
\mathrm{Div}\,\boldsymbol{\sigma} &= -\mathbf{f} \quad \text{in} \quad V,\\
\boldsymbol{\sigma}\cdot\mathbf{n} &= \mathbf{p} \quad \text{on} \quad B_s,\\
&\left.\begin{aligned}\mathbf{e}^e+\mathbf{e}^p &= \mathbf{e}\\ \mathbf{e}^e &\in \partial\psi(\mathbf{S})\\ \dot{\mathbf{e}}^p &\in \partial\varphi(\mathbf{S})\end{aligned}\right\} \quad \text{in} \quad V
\end{aligned}
\tag{2.1}
$$

with $\boldsymbol{\sigma} = \boldsymbol{\sigma}(x_1, x_2, x_3, t)$ as the symmetric stress tensor, $\boldsymbol{\epsilon} = \boldsymbol{\epsilon}(x_1, x_2, x_3, t)$ as the symmetric strain tensor, $\mathbf{p}$ surface forces, $\mathbf{f}$ body forces (dead-load-type), $\mathbf{n}$ unit normal vector, Grad and Div gradient and divergence operator. Following [2], the generalized stress $\mathbf{s} = [\boldsymbol{\sigma}, \boldsymbol{\pi}]$ and generalized strain $\mathbf{e} = [\boldsymbol{\epsilon}, \mathbf{0}]$, composed of an elastic part $\mathbf{e}^e = [\boldsymbol{\epsilon}^e, \boldsymbol{\omega}]$ and plastic part $\mathbf{e}^p = [\boldsymbol{\epsilon}^p, \boldsymbol{\varkappa}]$ are introduced with $\boldsymbol{\pi}$, $\boldsymbol{\omega}$ and $\boldsymbol{\varkappa}$ as internal parameters, determined by the (linear) hardening rule [2, 3]. The superposed dot denotes the time derivative, $\psi(\mathbf{s})$ and $\varphi(\mathbf{s})$ denote the generalized elastic and plastic potential, assumed as independent of each other [1]. The elastic part of the generalized strain $\mathbf{e}^e$ and the rate of the plastic part of the generalized strain $\dot{\mathbf{e}}^p$ are assumed to be elements of the subdifferential of $\psi(\mathbf{s})$ and $\varphi(\mathbf{s})$ resp. [1].

$\psi(\mathbf{s})$ is assumed to be strictly convex. In the case of differentiability of $\psi(\mathbf{s})$ for linear elastic behaviour as we shall assume in the following, we have

$$
\mathbf{e}^e = \frac{\partial\psi(\mathbf{s}^e)}{\partial\mathbf{s}^e},
\tag{2.2}
$$

$$
\begin{aligned}
2\psi(\mathbf{e}^e) &= \mathbf{e}^e \dots \mathbf{G}^{-1} \dots \mathbf{e}^e = \varepsilon^e_{ij} L_{ijkl}\, \varepsilon^e_{kl} Z + \omega_{mmn}\omega_n;\\
G^{-1} &= \begin{bmatrix} \mathbf{L} & 0\\ 0 & \mathbf{Z}\end{bmatrix}, \quad i,j = 1,2,3; \quad m,n = 1,2,\dots,r
\end{aligned}
\tag{2.3}
$$

with r as the number of internal parameters. $\mathbf{L}$ and $\mathbf{Z}$ denote coefficients of elasticity resp. hardening, both positive definit tensors.

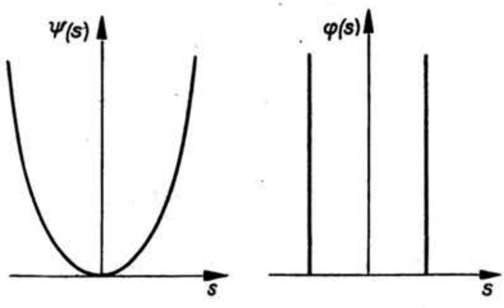

FIG. 1.

$\varphi(\mathbf{s})$ is defined as an indicator function of the domain E_t of admissible generalized stress tensors, attaining minimum at the origin $\mathbf{s} = \mathbf{0}$.

$$\varphi(\mathbf{s}) = \begin{cases} 0 & \text{if} \quad s \in E_t, \\ +\infty & \text{if} \quad s \notin E_t, \end{cases} \tag{2.4}$$

where E_t is determined by the yield condition. In [1] it is proved that the relations (2.1) describe adequately the initial boundary value problem of a three-dimensional elastic-plastic body (Fig. 1).

2.2. Global formulation of the problem

Starting from space C^∞ of the smooth tensor fields and introducing the scalar product

$$(\boldsymbol{\tau}, \boldsymbol{\tau}^*)_G = \int_V \boldsymbol{\tau} \dots \mathbf{G} \dots \boldsymbol{\tau}^* e^{-t} dV, \quad \boldsymbol{\tau}^* \in C^\infty \tag{2.5}$$

in [1] the Hilbert space H of all the generalized tensor fields $\boldsymbol{\tau}$ with the finite norm $\|\boldsymbol{\tau}\|$, induced by Eq. (2.5) is constructed by completion of C^∞. H may be decomposed into H^c and H^s of all kinematically admissible stress fields s^c defined by

$$\mathbf{s}^c = [\boldsymbol{\sigma}^c, 0] := \{\mathbf{s} \in H : \boldsymbol{\sigma} = \mathbf{L} \dots (\operatorname{Grad} \mathbf{u})_{\text{sym}} \quad \text{in} \quad V, \quad \mathbf{u} = 0 \quad \text{on} \quad B_k\} \tag{2.6}$$

and statically admissible stress fields $\mathbf{s}^s$ defined by

$$\mathbf{s}^s = [\boldsymbol{\sigma}^s, \boldsymbol{\pi}] := \{\mathbf{s} \in H : \operatorname{Div} \boldsymbol{\sigma} = 0 \quad \text{in} \quad V, \quad \boldsymbol{\sigma} \cdot \mathbf{n} = 0 \quad \text{on} \quad B_s\}. \tag{2.7}$$

From the Gauss divergence theorem it follows that all fields $\mathbf{s}^c \in H^c$ are orthogonal to all fields $\mathbf{s}^s \in H^s$ with respect to the scalar product (2.5). The definition of the global plastic potential $\phi(\mathbf{s})$ and restriction on the subspaces $H^{s\prime}$ and $H^{c\prime}$ of all time-differentiable fields allows a global definition of material behaviour [1]. For a given perfectly elastic solution $\mathbf{s}^0$ representing external loading of the body, in [1] it is proved that the functional

$$\Lambda_0(\mathbf{s}^s) = \phi_0(\mathbf{s}^0 - \mathbf{s}^s) + \phi_0^*(\dot{\mathbf{s}}^s) - (\mathbf{s}^0 - \mathbf{s}^s, \dot{\mathbf{s}}^s)_G \tag{2.8}$$

with

$$\phi_0(\mathbf{s}) = \begin{cases} \phi(\mathbf{s}) = \lim\limits_{c \to \infty} \int_V \varphi_c(\mathbf{s}) e^{-t} dV & \text{for} \quad \mathbf{s} \in \mathbf{s}^0 + H^{s\prime}, \\ +\infty & \text{for} \quad \mathbf{s} \notin \mathbf{s}^0 + H^{s\prime}, \end{cases}$$
$$\varphi_c(\mathbf{s}) = \begin{cases} 0 & \text{for} \quad \mathbf{s} \in E_t, \quad 0 < c < +\infty, \\ c & \text{for} \quad \mathbf{s} \notin E_t \end{cases} \tag{2.9}$$

and

$$\phi_0^*(\dot{\mathbf{s}}^s) = \sup_{\mathbf{s} \in H^{s\prime}} [(\mathbf{s}^0 - \mathbf{s}, \dot{\mathbf{s}}^s) - \phi_0(\mathbf{s}^0 - \mathbf{s})] \tag{2.10}$$

is strictly convex and attains minimum value equal to zero for the solution if a solution in the chosen space exists. In [1] it is proved that the solution is unique.

3. Application to plates

3.1. Two-dimensional representation of three-dimensional fields

For the arbitrary loading process of plates no unique relation between stress and strain distribution over the thickness of the plate can be assumed. However, every differentiable three-dimensional function $g(x_1, x_2, x_3, t)$ can be represented by a set $\{G\}$ of two-dimensional functions $G^{(k)}(x_1, x_2, t)$, $k = 0, 1, 2, \ldots, n$ by means of the Taylor-expansion of order n:

$$G^{(k)}(x_1, x_2, t) = \frac{1}{(k-1)} \frac{\partial^{(k-1)} g(x_1, x_2, x_3, t)}{(\partial x_3)^{(k-1)}}\bigg|_{x_3=0}; \quad k = 1, 2, \ldots n,$$

(3.1)

$$g(x_1, x_2, x_3, t) = \sum_{k}^{n} G^{(k)} x_3^{(k-1)} + R_{n+1}$$

with R_{n+1} as remainder. If we restrict our considerations to those fields $g(x_1, x_2, x_3, t)$ with a vanishing remainder, then the relation between $\{G\}$ and g is a one-to-one mapping and all relations of Chapter 2 can be equivalently expressed by relations between two-dimensional fields [6].

We substitute the three-dimensional generalized tensor fields $\mathbf{s} = [\boldsymbol{\sigma}, \boldsymbol{\pi}]$ and $\mathbf{e} = [\boldsymbol{\epsilon}, \boldsymbol{\omega}]$ by two-dimensional representatives $\mathbf{n}(x_1, x_2, t)$ and $\mathbf{q}(x_1, x_2, t)$:

(3.2) $$\mathbf{n} = [\mathbf{N}^q, \mathbf{\Pi}^q], \quad \mathbf{q} = [\mathbf{Q}^q, \mathbf{\Omega}^q].$$

With

(3.3)
$$\begin{aligned} \mathbf{N}^q &:= \{N_{ij}^{(1)}, N_{ij}^{(2)}, \ldots, N_{ij}^{(q)}\}, & ij &= 1, 2, 3,\\ \mathbf{\Pi}^q &:= \{\Pi_n^{(1)}, \Pi_n^{(2)}, \ldots, \Pi_n^{(q)}\}, & n &= 1, 2, \ldots, r,\\ \mathbf{Q}^q &:= \{Q_{ij}^{(1)}, Q_{ij}^{(2)}, \ldots, Q_{ij}^{(q)}\}, & q &\geqslant 1,\\ \mathbf{\Omega}^q &:= \{\Omega_n^{(1)}, \Omega_n^{(2)}, \ldots, \Omega_n^{(q)}\}. \end{aligned}$$

Here r denotes the number of internal parameters describing linear hardening [3] and q the order of two-dimensional representatives. For all three-dimensional fields characterized by a vanishing remainder in Eq. (3.1) of the Taylor-expansion we obtain equivalently to Eq. (2.5) the scalar product defined in terms of two-dimensional quantities:

(3.4) $$\langle \mathbf{n}, \mathbf{q} \rangle = \int_F (N_{ij}^q \mathbf{m} Q_{ij}^q + \Pi_n^q \mathbf{m} \Omega_n^q) e^{-t} dx_1 dx_2 dt$$

with

(3.5)
$$\begin{aligned} N_{ij}^q \mathbf{m} Q_{ij}^q &:= \sum_{k=1}^{q} \sum_{l=1}^{q} N_{ij}^{(k)} m_{kl} Q_{ij}^{(l)},\\ \Pi_n^q \mathbf{m} \Omega_n^q &:= \sum_{k=1}^{q} \sum_{l=1}^{q} \Pi_n^{(k)} m_{kl} \Omega_n^{(l)},\\ m_{kl} &:= \int_{-h}^{h} x_3^{k+l-2} dx_3. \end{aligned}$$

Respecting Eq. (3.1) by this procedure the three-dimensional problem is equivalently expressed by two-dimensional quantities.

3.2. Assumptions for plates

If we split up Eq. (3.4) into parts containing solely quantities in the x_1, x_2-direction parallel to the midspan of the plate and parts containing quantities in the x_3-direction orthogonal to the midspan, we obtain

$$(3.6) \qquad \langle n_{ij}, q_{ij} \rangle = \langle n_{\alpha\beta}, q_{\alpha\beta} \rangle + 2\langle n_{\alpha 3}, q_{\alpha 3} \rangle + \langle n_{33}, q_{33} \rangle; \quad \alpha, \beta = 1, 2.$$

For plates we assume in general:

$$(3.7) \qquad \langle n_{33}, q_{33} \rangle \ll \langle n_{ij}, q_{ij} \rangle.$$

For thin plates we assume

$$(3.8) \qquad 2\langle n_{\alpha 3}, q_{\alpha 3} \rangle + \langle n_{33}, q_{33} \rangle \ll \langle n_{ij}, q_{ij} \rangle.$$

Both assumptions are compatible with the conventional assumptions of vanishing normal stress in the x_3-direction and, additionally, vanishing of the shear deformation of the cross section in the plate-theory [4]. In the following we restrict our considerations to thin plates.

In Eqs. (2.6) and (2.7) kinematical and statical admissibility was defined for three-dimensional bodies. For application to the plate theory we introduce the displacement representatives $\mathbf{U}^q$

$$(3.9) \qquad U_\alpha^q(x_1, x_2, t) := \{\tilde{U}_\alpha^{(1)}, \tilde{U}_\alpha^{(2)}, \ldots, \tilde{U}_\alpha^{(q)}\}$$

with the corresponding strain representatives $\mathbf{Q}^c$:

$$(3.10) \qquad Q_{\alpha\beta}^c(x_1, x_2, t) = \{(\tilde{U}_{\alpha,\beta}^{(1)})_{\text{sym}}, (\tilde{U}_{\alpha,\beta}^{(2)})_{\text{sym}} \ldots (\tilde{U}_{\alpha,\beta}^{q})_{\text{sym}}\}$$

related to three-dimensional strain distribution by Eq. (3.1)

$$\varepsilon_{\alpha\beta}^c(x_1, x_2, t) \doteq (\tilde{U}_{\alpha,\beta}^{(1)})_{\text{sym}} + (\tilde{U}_{\alpha,\beta}^{(2)})_{\text{sym}} x_3 + \ldots (\tilde{U}_{\alpha,\beta}^{(k)})_{\text{sym}} x_3^{k-1} + \ldots (\tilde{U}_{\alpha,\beta}^{(q)})_{\text{sym}} x_3^{q-1}.$$

From definition $\boldsymbol{\epsilon}^c$ fulfills the condition of compatibility in the volume of the plate. If we introduce the displacement representatives $U_\alpha^{(i)}$ such that

$$(3.11) \qquad \tilde{U}_{\alpha,\beta}^{(i)} = U_{\alpha,\beta}^{(i)}, \quad i = 1, 3, \ldots, q, \quad \tilde{U}_{\alpha,\beta}^{(2)} = U_{,\alpha\beta}^{(2)},$$

then $U^{(2)}$ can be identified with the deflection of the considered plate. The kinematically admissible generalized strain representatives $\mathbf{q}^c$ will be called every set $\mathbf{q} = [\mathbf{Q}^c, \mathbf{\Omega}]$ for which $\mathbf{u}^q$ vanishes on the boundary Z_k and $\mathbf{\Omega} \equiv \mathbf{0}$. By the generalized elastic coefficients $\mathbf{G}$, $\mathbf{q}^c$ is then uniquely related to $\mathbf{s}^c$ in Eq. (2.6). Statical admissibility will be defined by the orthogonality condition; every set $\mathbf{n}$ fulfilling

$$(3.12) \qquad \langle \mathbf{n}, \mathbf{q}^c \rangle = 0$$

will be called the statically admissible stress representative $\mathbf{n}^s$.

EXAMPLE

We consider a thin plate of arbitrary shape with a regular boundary and elastic-ideal plastic material behaviour. We choose the order of Taylor expansion $q = 4$, corresponding to the shape-functions for stress and strain distribution in the x_3-direction of Fig. 2.

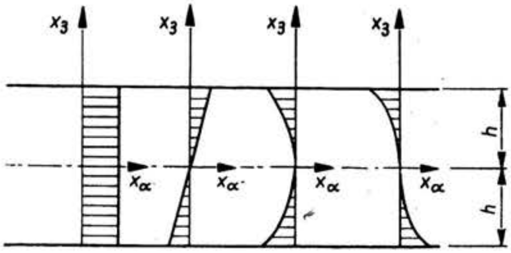

FIG. 2.

With

(3.13) $m_{kl} =$

$k \backslash l$	1	2	3	4
1	$2h$	0	$\frac{2h^3}{3}$	0
2	0	$\frac{2h^3}{3}$	0	$\frac{2h^5}{5}$
3	$\frac{2h^3}{3}$	0	$\frac{2h^5}{5}$	0
4	0	$\frac{2h^5}{5}$	0	$\frac{2h^7}{7}$

We obtain for Eq. (3.12)

$$(3.14)\quad \langle \mathbf{n}, \mathbf{q}^c \rangle = -\int_F \Big[2hN^{(1)}_{\alpha\beta} U^{(1)}_{\alpha,\beta} + \frac{2h^3}{3}(N^{(2)}_{\alpha\beta} U^{(2)}_{,\alpha\beta} + N^{(1)}_{\alpha\beta} U^{(3)}_{\alpha,\beta} + N^{(3)}_{\alpha\beta} U^{(1)}_{\alpha,\beta})$$

$$\frac{2h^5}{5}(N^{(3)}_{\alpha\beta} U^{(3)}_{\alpha,\beta} + N^{(2)}_{\alpha\beta} U^{(4)}_{\alpha,\beta} + N^{(4)}_{\alpha\beta} U^{(2)}_{,\alpha\beta}) + \frac{2h^7}{7} N^{(4)}_{\alpha\beta} U^{(4)}_{\alpha,\beta}\Big] e^{-t} dx_1 dx_2 dt$$

$$= \int_F \Big[\Big(2hN^{(1)}_{\alpha\beta} + \frac{2h^3}{3} N^{(3)}_{\alpha\beta}\Big), \Big(\frac{2h^3}{3} N^{(2)}_{\alpha\beta} + \frac{2h^5}{5} N^{(4)}_{\alpha\beta}\Big), \Big(\frac{2h^3}{3} N^{(1)}_{\alpha\beta} + \frac{2h^5}{5} N^{(3)}_{\alpha\beta}\Big),$$

$$\Big(\frac{2h^5}{5} N^{(2)}_{\alpha\beta} + \frac{2h^7}{7} N^{(4)}_{\alpha\beta}\Big)\Big] [U^{(1)}_{\alpha,\beta}, U^{(2)}_{,\alpha\beta}, U^{(3)}_{\alpha,\beta}, U^{(4)}_{\alpha,\beta}]^T e^{-t} dx_1 dx_2 dt = 0.$$

Applying the divergence theorem twice gives:

$$(3.15)\quad \langle \mathbf{n}, \mathbf{q}^c \rangle = -\int_F \Big[\Big(2hN^{(1)}_{\alpha\beta,\beta} + \frac{2h^3}{3} N^{(3)}_{\alpha\beta,\beta}\Big), \Big(\frac{2h^3}{3} N^{(2)}_{\alpha\beta,\beta\alpha} + \frac{2h^5}{5} N^{(4)}_{\alpha\beta,\beta\alpha}\Big),$$

$$\Big(\frac{2h^3}{3} N^{(1)}_{\alpha\beta,\beta} + \frac{2h^5}{5} N^{(3)}_{\alpha\beta,\beta}\Big), \Big(\frac{2h^5}{5} N^{(2)}_{\alpha\beta,\beta} + \frac{2h^7}{7} N^{(4)}_{\alpha\beta,\beta}\Big)\Big]$$

$$\times [U^{(1)}_{\alpha}, U^{(2)}, U^{(3)}_{\alpha}, U^{(4)}_{\alpha}]^T e^{-t} dx_1 dx_2 dt$$

$$+\int_{Z}\Bigg[\left(2hN^{(1)}_{\alpha n}+\frac{2h^3}{3}N^{(3)}_{\alpha n}\right),\ (V+M_{ns,s}),\ M_{nn},\ \left(\frac{2h^3}{3}N^{(1)}_{\alpha n}+\frac{2h^5}{5}N^{(3)}_{\alpha n}\right),$$

$$\left(\frac{2h^5}{5}N^{(2)}_{\alpha n}+\frac{2h^7}{7}N^{(4)}_{\alpha n}\right)\Bigg][U^{(1)}_{\alpha},\ U^{(2)},\ U^{(2)}_{,n},\ U^{(3)}_{\alpha},\ U^{(4)}_{\alpha}]^T e^{-t}ds\,dt = 0.$$

With

$$\frac{\partial}{\partial x_1} = n_1\frac{\partial}{\partial n} - n_2\frac{\partial}{\partial s}, \qquad \frac{\partial}{\partial x_2} = n_2\frac{\partial}{\partial n} + n_1\frac{\partial}{\partial s},$$

$$n_1 = \cos(x_1, n), \qquad n_2 = \cos(x_2, n),$$

(3.16)
$$V = n_\alpha\left(\frac{2h^3}{3}N^{(2)}_{\alpha\beta}+\frac{2h^5}{5}N^{(4)}_{\alpha\beta}\right)_{,\beta}, \qquad N^{(i)}_{\alpha n} = N^{(i)}_{\alpha\beta}n_\beta,$$

$$M_{ns} = -n_2 n_\alpha\left(\frac{2h^3}{3}N^{(1)}_{\alpha 1}+\frac{2h^5}{5}N^{(4)}_{\alpha 1}\right)+n_1 n_\alpha\left(\frac{2h^3}{3}N^{(2)}_{\alpha 2}+\frac{2h^5}{5}N^{(4)}_{\alpha 2}\right),$$

$$M_{nn} = n_\alpha n_\beta\left(\frac{2h^3}{3}N^{(2)}_{\alpha\beta}+\frac{2h^5}{5}N^{(4)}_{\alpha\beta}\right).$$

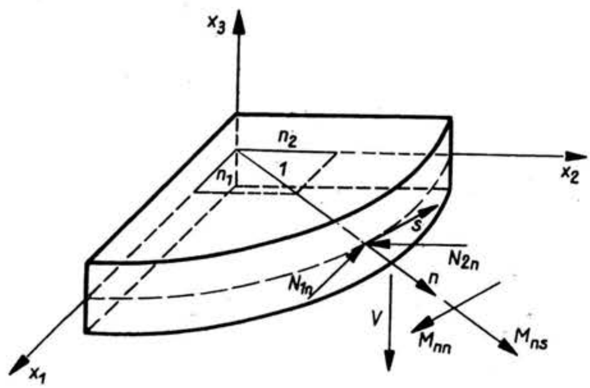

FIG. 3.

From Eq. (3.15) follow immediately the conditions for statical admissibility. In the interior F of the plate we have

(3.17)
$$N^{(1)}_{\alpha\beta,\beta}+\frac{h^2}{3}N^{(3)}_{\alpha\beta,\beta} = 0,$$

$$N^{(2)}_{\alpha\beta,\alpha\beta}+\frac{3h^2}{5}N^{(4)}_{\alpha\beta,\alpha\beta} = 0,$$

$$N^{(1)}_{\alpha\beta,\beta}+\frac{3h^2}{5}N^{(3)}_{\alpha\beta,\beta} = 0,$$

$$N^{(2)}_{\alpha\beta,\beta}+\frac{5h^2}{7}N^{(4)}_{\alpha\beta,\beta} = 0.$$

On the boundary Z the conditions of statical admissibility depend on the support of the plate. In the line integral of Eq. (3.15) either the statical or the dual kinematical quantity must vanish. On a free boundary, for example, we have:

$$
\begin{aligned}
N^{(1)}_{\alpha n} + \frac{h^2}{3} N^{(3)}_{\alpha n} &= 0,\\
V + M_{ns,s} &= 0,\\
M_{nn} &= 0,\\
N^{(1)}_{\alpha n} + \frac{3h^2}{5} N^{(3)}_{\alpha n} &= 0,\\
N^{(2)}_{\alpha n} + \frac{5h^2}{7} N^{(4)}_{\alpha n} &= 0.
\end{aligned}
\tag{3.18}
$$

3.3. Minimum principles for thin plates

From the preceding derivations minimum principles for stresses as functions of place and time can be directly obtained. In the case of elastic-plastic material behaviour the domain E_t of admissible stress-states remains constant during the deformation process and the functional (2.8) may be reduced to:

$$
\Lambda^1_0(s^s_{\alpha\beta}) = \sup_{s^0_{\alpha\beta} - s^{s*}_{\alpha\beta} \in E_t} (s^s_{\alpha\beta} - s^{s*}_{\alpha\beta}, \dot{s}^s_{\alpha\beta})_G; \qquad s^0_{\alpha\beta} - s^{s*}_{\alpha\beta} \in E_t. \tag{3.19}
$$

The test function $s^s_{\alpha\beta}(x_1, x_2, x_3, t)$ minimizing Eq. (3.19) is then the solution of the problem in the chosen subspace of approximation.

If we use $n^s_{\alpha\beta}(x_1, x_2, t)$ instead of $s^s_{\alpha\beta}$, then we obtain from Eqs. (2.8)–(2.10) and (3.6)–(3.8) the minimization functional

$$
\Lambda^2_0(n^s_{\alpha\beta}) = \sup_{n^0_{\alpha\beta} - n^{s*}_{\alpha\beta} \in \tilde{E}_t} \langle n^s_{\alpha\beta} - n^{s*}_{\alpha\beta}, \dot{n}^s_{\alpha\beta} \rangle_G; \qquad n^0_{\alpha\beta} - n^{s*}_{\alpha\beta} \in \tilde{E}_t, \tag{3.20}
$$

where $\tilde{E}_t$ denotes the domain of admissible stress-states in terms of two-dimensional representatives. Minimizing the test function $n^s_{\alpha\beta}(x_1, x_2, t)$ is then the solution of the problem in the chosen subspace of approximation.

4. Numerical example

A simply supported elastic-ideal plastic square plate is proportionally loaded by a vertical sinusoidal distributed load $\tilde{q}$. The von Mises and Tresca yield criteria are used parallely to determine the domain of admissible stress. We determine the state of stress at the end of the loading process (Fig. 4).

$$
\tilde{\mathbf{q}}(x_1, x_2) = \tilde{\mathbf{q}}_0 \cos\left(\frac{\pi}{2a} \tilde{x}_1\right) \cos\left(\frac{\pi}{2a} \tilde{x}_2\right). \tag{4.1}
$$

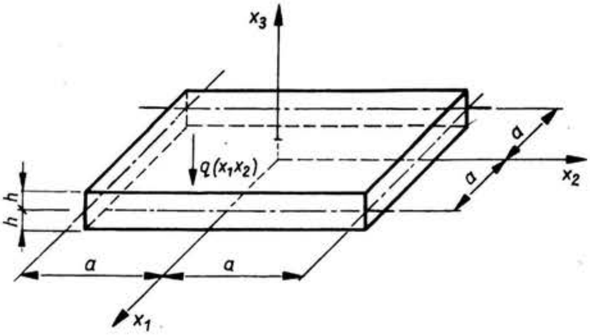

FIG. 4.

We introduce the dimensionless quantities

$$x_\alpha = \frac{\tilde{x}_\alpha}{2a}, \quad \mathbf{q} = \frac{\tilde{\mathbf{q}}}{E}\left(\frac{a}{2h}\right)^4,$$

(4.2)
$$x_3 = \frac{\tilde{x}_3}{2h}, \quad \sigma_s = \frac{\tilde{\sigma}_s}{E},$$

$$N_{\alpha\beta} = \frac{\tilde{N}_{\alpha\beta}}{Ea^2}\left(\frac{a}{2h}\right)^4,$$

with E as the elastic modulus. The Poisson's ratio ν and the dimensionless uniaxial yield-limit σ_s are chosen as $\nu = 0.3$ and $\sigma_s = 1.8 \cdot 10^{-3}$ respectively in our calculations.

We use the two-dimensional representatives $N^s_{\alpha\beta}(x_1, x_2)$ up to order two. As we consider a special case of proportional loading, time does not appear as a parameter [1]. As test functions fulfilling statical boundary conditions we choose

(4.3)
$$N^{s(2)}_{11} = c_1(1-x_1^2)(1-x_2^2)+c_2(1-x_1^2)(1-x_2^4),$$
$$N^{s(2)}_{22} = c_3(1-x_1^2)(1-x_2^2)+c_4(1-x_1^4)(1-x_2^2),$$
$$N^{s(2)}_{12} = N^{s(2)}_{21} = c_5\left[2x_1x_2-\frac{1}{3}(x_1^3x_2+x_2^3x_1)\right]+c_6\left[2x_1x_2-\frac{1}{5}(x_1^5x_2+x_2^5x_1)\right].$$

After fulfilling the conditions of statical admissibility in F (3.17) and using the symmetry of load and geometry of the plate, we obtain for the analytically given purely elastic solution $\mathbf{N}^{0(2)}$ [4] the minimization functional Λ_0^2 as a function of c_1 and c_2:

(4.4)
$$\Lambda_0^2(c_1, c_2) = \sup_{N^{0(2)}-N^{s(2)}(c_1^*, c_2^*)\in\tilde{E}_t} [(c_1^2-c_1c_1^*)\cdot 4.01468+(c_2^2-c_2c_2^*)\cdot 6.01351$$
$$+(2c_1c_2-c_1c_2^*-c_2c_1^*)\cdot 5.21133].$$

$\tilde{E}_t$ is determined either by the Tresca or von Mises yield criteria expressed by $N^{(2)}_{\alpha\beta}$. Minimizing Λ_0^2, we obtain the unknown coefficients c_1, c_2 as result:

(4.5)

q_0	c_1	c_2	yield cond.
1.5	0.1933	−0.2708	v. Mises
	0.1575	−0.2351	Tresca
2.5	−0.3327	0.1234	v. Mises
	−0.3673	0.1580	Tresca

Figures 5 and 6 show the domain of admissible stresses defined in the c_1, c_2-plane. Figure 7 shows the qualitative distribution of the purely elastic solution $\mathbf{N}^{0(2)}$, statically admissible representative $\mathbf{N}^{s(2)}$ minimizing Λ_0^2 and superposition $\mathbf{N}^{0(2)}+\mathbf{N}^{s(2)}$ as the researched solution $\mathbf{N}^{(2)}$ using the von Mises yield criterion for $q_0 = 2.5$.

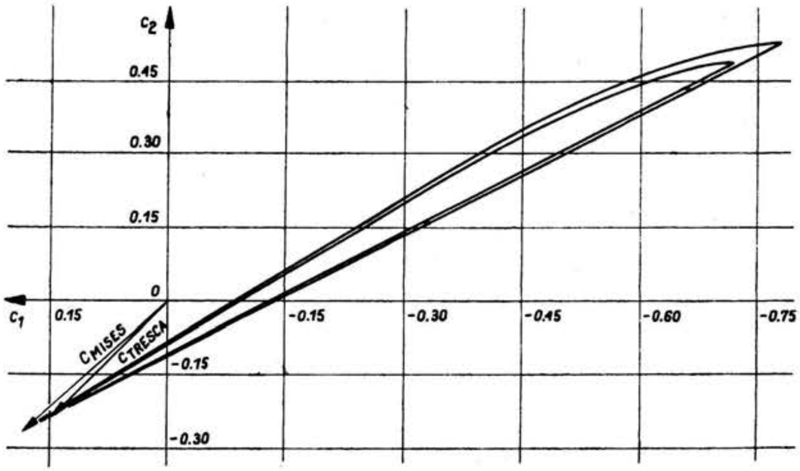

FIG. 5. Domain of admissible parameters c_1, c_2 for load parameter $q_0 = 1.5$; inner domain: Tresca yield-condition, outer domain: Mises yield-condition.

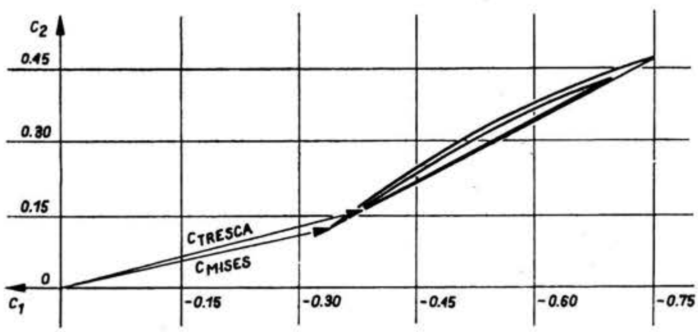

FIG. 6. Domain of admissible parameters c_1, c_2 for load parameter $q_0 = 2.5$; inner domain: Tresca yield-condition; outer domain: Mises yield-condition.

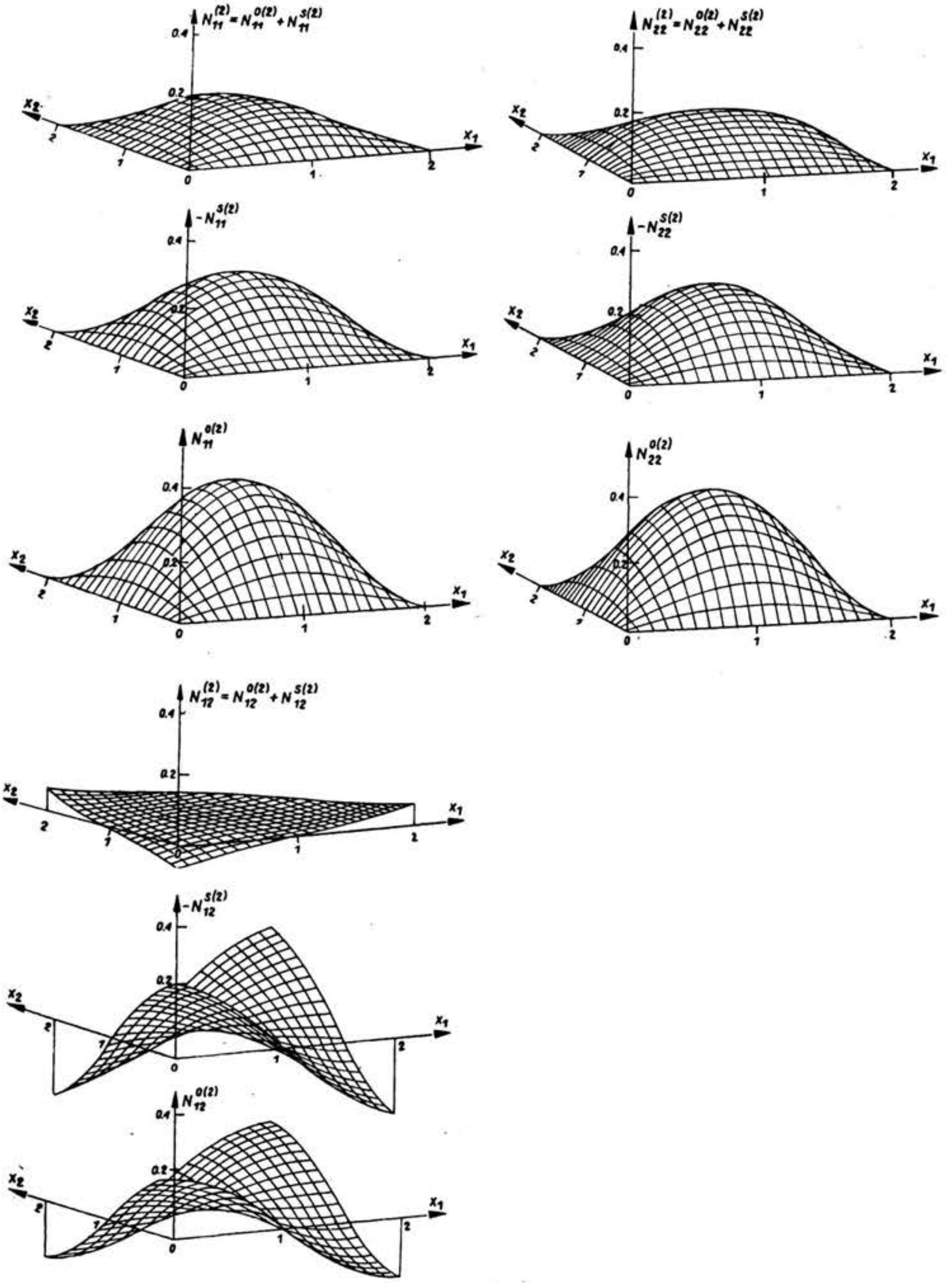

FIG. 7.

References

1. P. RAFALSKI, *Minimum principles in plasticity*, Mitteilungen aus dem Institut für Mechanik, Ruhr-Universität, 13, Bochum 1978.

2. P. RAFALSKI, *Solution of the elastic-viscoplastic boundary value problem*, Int. J. Engng. Sci., **15**, 193, 1977.
3. B. HALPHEN, Q. S. NGUYEN, *Sur les matériaux standards géneralisés*, J. Mec., **14**, 39–63, 1975.
4. S. TIMOSHENKO, S. KRIEGER-WOINOWSKI, *Theory of plates and shells*, McGraw-Hill Book Co., New York 1959.
5. P. RAFALSKI, D. WEICHERT, *On minimum principles for elastic-plastic plates*, Bull. Acad. Pol. Sci., Série Sci. Tech., **26**, 11, 1978.
6. D. WEICHERT, *Variationelle Formulierung und Lösung von Randwertproblemen in der Plastizitätstheorie und ihre Anwendung auf Plattenprobleme*, Dissertation, Ruhr Universität, Bochum 1980.

LEHRSTUHL FÜR MECHANIK II,
RUHR-UNIVERSITÄT, BOCHUM, FRG.

Received October 18, 1980.