Regular reflection of a weak shock wave from an inclined plane isothermal wall

K. PIECHÓR (WARSZAWA)

THE PROBLEM of reflection of a weak shock wave from an oblique plane wall is analysed. Only the case of the isothermal wall is considered. The flow domain is divided into two parts: an outer domain containing both shock waves and a boundary layer close to the wall. In order to determine the outer flow, the Lighthill technique and the multiple scales method are combined. The flow in the boundary layer is described by the linearized Prandtl equations. So as to determine some unknown functions, the matching principle is used. As a result, the structure and the trajectory of the reflected shock wave are obtained. The location of the trajectory of the reflected shock wave is influenced by the boundary layer. Also a criterion of regularity of the reflection is obtained.

W pracy analizuje się odbicie słabej fali uderzeniowej od pochyłej, płaskiej ścianki. Rozważa się tylko przypadek ścianki izotermicznej. Obszar przepływu podzielony jest na dwie części: obszar zewnętrzny zawierający obie fale uderzeniowe i warstwę przyścienną. W celu wyznaczenia przepływu zewnętrznego, łączy się technikę Lighthilla z metodą wielu skal. Przepływ w warstwie przyściennej jest opisany przez zlinearyzowane równanie Prandtla. Zasada kojarzenia rozwiązań jest użyta do wyznaczenia pewnych niewiadomych funkcji. W wyniku otrzymuje się m.in. strukturę i trajektorię fali odbitej. Na położenie trajektorii odbitej fali uderzeniowej ma wpływ obecność warstwy przyściennej. Otrzymuje się również kryterium regularności odbicia.

В работе анализируется отражение слабой ударной волны от наклонной, плоской стенки. Рассматривается только случай изотермической стенки. Область течения разделена на две части: внешняя область, содержавшая обе ударные волны; и пограничный слой. С целью определения внешнего течения комбинируется техника Лайтхилла с методом многих масштабов. Течение в пограничном слое описано линеаризованным управлением Прандтля. Принцип сращивания решений используется для определения некоторых неизвестных функций. В результате получаются, между прочим, структура и траектория отраженной волны. На положение траектории отраженной ударной волны имеет влияние присутствие пограничного слоя. Получается тоже критерий регулярности отражения.

1. Introduction

INTVESTIGATIONS of the process of reflection of a shock wave from a solid obstacle have been carried out for about forty years. Now the literature on the problem is quite abundant and the references [1–4] represent some monographs (long lists of references are given there). In the present paper the same problem is considered, but the fact that the shock wave is not simply a jump discontinuity is taken into account. Hawover, we have to confine our considerations to the case of weak shock waves. Such an assumption makes it possible to apply the singular perturbation methods. The flow domain is divided into two parts: an inner domain (boundary layer) and an outer domain involving both the incident and the reflected shock waves. In order to determine the outer flow, the Lighthill technique [5] and the multiple scales method [5] are combined. Such a method was applied in the

previous paper by the present author [6] and it turned out to give the same results as those obtained by the other authors. Thus we may believe that the results of the present paper are also correct.

2. Basic assumptions

Let an oblique shock wave travel along a solid plane wall at constant velocity D^*. If we denote the angle between the shock wave and the wall by σ, then the point of intersection of the shock wave and the wall moves at constant velocity $-D^*/\sin\sigma$. Thus we are able to choose such a coordinate system that the shock wave is at rest in it, the flow becomes stationary and the wall moves in its plane with constant velocity $D^*/\sin\sigma$. We choose the origin 0 of the Cartesian coordinate system at the point of intersection (see Fig. 1). We
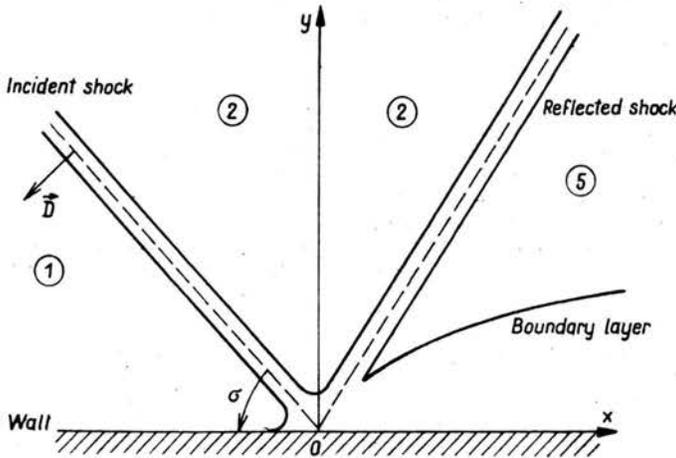

FIG 1.

assume that the gas is at rest unitil the shock wave arrives, and that its density and temperature is ϱ_1^* and T_1^*, respectively. Both ϱ_1^* and T_1^* are assumed to be constant. However, in our coordinate system the gas flows with constant velocity

$$u_1^* = \frac{D^*}{\sin\sigma}, \quad v_1^* = 0,$$

where u^* is the velocity component parallel to the wall, and v^* denotes the velocity component normal to the wall. The subscript 1 refers to the flow domain denoted by 1 in Fig. 1; it is the domain of the gas at rest in front of the incident shock wave.

Let a_1^* be the sound velocity in the quiescent gas. The Mach number M of the incident shock wave is

$$M = \frac{D^*}{a_1^*}.$$

The small parameter ε is defined by

$$\varepsilon = \frac{1}{\Gamma} \frac{M^2-1}{M^2},$$

where

$$\Gamma = \frac{\gamma+1}{2},$$

γ is the specific heats ratio.

The dimensionless variables are defined as follows:

(2.1) $$x^* = \frac{\Gamma}{2\beta} \frac{\varrho_1^* a_1^* \varepsilon}{\mu^*} x, \qquad y^* = \frac{\Gamma}{2\beta} \frac{\varrho_1^* a_1^* \varepsilon}{\mu^*} y,$$

where μ^* is the coefficient of viscosity assumed to be constant, and β is a constant defined by

$$\beta = \frac{4}{3} + \frac{\gamma-1}{\mathrm{Pr}},$$

where Pr denotes the Prandtl number.

Let $u_2^*, v_2^*, \varrho_2^*, T_2^*$ denote the velocity components, the density and temperature of the gas behind the incident shock wave, respectively. According to the Rankine–Hugoniot relations, they are given by

$$u_2^* = \frac{1-\varepsilon\sin^2\sigma}{\sin\sigma} D^*, \qquad v_2^* = -\varepsilon\cos\sigma D^*,$$

$$\varrho_2^* = \frac{\varrho_1^*}{1-\varepsilon},$$

$$T_2^* = \frac{\left(1+\frac{\gamma-1}{2}\varepsilon\right)(1-\varepsilon)}{1-\varepsilon\Gamma} T_1^*.$$

The dimensionless velocity, density and temperature are defined by the relations

(2.2)
$$u^* = \frac{u_2^*+u_1^*}{2} + \frac{u_2^*-u_1^*}{2} u(x, y),$$

$$v^* = \frac{v_2^*+v_1^*}{2} + \frac{v_2^*-v_1^*}{2} v(x, y),$$

$$\varrho^* = \frac{\varrho_2^*+\varrho_1^*}{2} + \frac{\varrho_2^*-\varrho_1^*}{2} \varrho(x, y),$$

$$T^* = \frac{T_2^*+T_1^*}{2} + \frac{T_2^*-T_1^*}{2} T(x, y).$$

The Navier–Stokes equations written in these variables take the form

(2.3) $$\frac{1}{\sin\sigma} \frac{\partial\varrho}{\partial x} - \sin\sigma\frac{\partial u}{\partial x} - \cos\sigma\frac{\partial v}{\partial y} - \frac{1}{2}\varepsilon\left[(1+u)\frac{\partial\varrho}{\partial x}\sin\sigma + (1+v)\frac{\partial\varrho}{\partial y}\cos\sigma \right.$$

$$\left. + (\varrho-1)\left(\frac{\partial u}{\partial x}\sin\sigma + \frac{\partial v}{\partial y}\cos\sigma\right)\right] = 0(\varepsilon^2),$$

(2.4) $-\dfrac{\partial u}{\partial x} + \dfrac{1}{\gamma}\dfrac{\partial \varrho}{\partial x} + \dfrac{\gamma-1}{\gamma}\dfrac{\partial T}{\partial x} + \dfrac{1}{2}\varepsilon\Big\{[(1+u)\sin^2\sigma-(\varrho-1)]\dfrac{\partial u}{\partial x}$

$$+(1+v)\dfrac{\partial u}{\partial y}\cos\sigma\sin\sigma+\dfrac{(\gamma-1)T-2}{\gamma}\dfrac{\partial\varrho}{\partial x}+\dfrac{(\gamma-1)(\varrho-2)}{\gamma}\dfrac{\partial T}{\partial x}$$

$$+\dfrac{\Gamma}{\beta}\Big(\dfrac{4}{3}\dfrac{\partial^2 u}{\partial x^2}\sin\sigma+\dfrac{\partial^2 u}{\partial y^2}\sin\sigma+\dfrac{1}{3}\dfrac{\partial^2 v}{\partial x\partial y}\cos\sigma\Big)\Big\} = 0(\varepsilon^2),$$

(2.5) $-\dfrac{\cos\sigma}{\sin\sigma}\dfrac{\partial v}{\partial x} + \dfrac{1}{\gamma}\dfrac{\partial\varrho}{\partial y} + \dfrac{\gamma-1}{\gamma}\dfrac{\partial T}{\partial y} + \dfrac{1}{2}\varepsilon\Big\{-\Big[-(1+u)\sin\sigma+\dfrac{\varrho-1}{\sin\sigma}\Big]\dfrac{\partial v}{\partial x}\cos\sigma$

$$+(1+v)\dfrac{\partial v}{\partial y}\cos^2\sigma+\dfrac{(\gamma-1)T-2}{\gamma}\dfrac{\partial\varrho}{\partial y}+\dfrac{(\gamma-1)(\varrho-2)}{\gamma}\dfrac{\partial T}{\partial y}$$

$$+\dfrac{\Gamma}{\beta}\Big(\dfrac{\partial^2 v}{\partial x^2}\cos\sigma+\dfrac{4}{3}\dfrac{\partial^2 v}{\partial y^2}\cos\sigma+\dfrac{1}{3}\dfrac{\partial^2 u}{\partial x\partial y}\sin\sigma\Big)\Big\} = 0(\varepsilon^2),$$

(2.6) $\dfrac{1}{\sin\sigma}\Big(\dfrac{\partial T}{\partial x}-\dfrac{\partial\varrho}{\partial x}\Big)-\dfrac{1}{2}\varepsilon\Big[(1+u)\Big(\dfrac{\partial T}{\partial x}-\dfrac{\partial\varrho}{\partial x}\Big)\sin\sigma+(1+v)\Big(\dfrac{\partial T}{\partial y}-\dfrac{\partial\varrho}{\partial y}\Big)\cos\sigma$

$$-\dfrac{\varrho-2}{\sin\sigma}\dfrac{\partial T}{\partial x}+\dfrac{(\gamma-1)T-2}{\sin\sigma}\dfrac{\partial\varrho}{\partial x}+\dfrac{\gamma}{Pr}\dfrac{\Gamma}{\beta}\Big(\dfrac{\partial^2 T}{\partial x^2}+\dfrac{\partial^2 T}{\partial y^2}\Big)\Big] = 0(\varepsilon^2).$$

In the above equations, the pressure is eliminated with the help of the perfect gas equation

$$p^* = R^*\varrho^* T^*,$$

where R^* is the gas constant.

3. Outer expansion

Let Q be any of the variables u, v, ϱ or T. Following [6], we look for the solutions of Eqs. (2.3)–(2.6) in the form

(3.1) $Q = Q(\xi, \eta, \zeta; \varepsilon),$

where ξ, η and ζ are new independent variables related to x and y through the equations

(3.2) $x = \Phi(\xi, \eta, \zeta; \varepsilon),$

(3.3) $y = \Psi(\xi, \eta, \zeta; \varepsilon),$

(3.4) $x^1 = \zeta,$

where

(3.5) $x^1 = \varepsilon x$

is treated as a new independent variable (see [5]).

The functions Φ and Ψ in Eqs. (3.2) and (3.3), respectively, are unknown and they will be determined according to the principles of the strained coordinate method. However, this method does not define them uniquely and therefore we may impose some additional

conditions on them: first, we assume that the Jacobian of Φ and Ψ in respect to any pair of their independent variables is different from zero; second, we assume that

$$(3.6) \qquad \Phi(\xi, \xi, \zeta; \varepsilon) = \frac{\xi}{\sin \sigma},$$

$$(3.7) \qquad \Psi(\xi, \xi, \zeta; \varepsilon) = 0.$$

We assume also that all unknown functions, including Φ and Ψ are analytical functions of ε, and therefore they can be represented in the form

$$(3.8) \qquad Q(\xi, \eta, \zeta; \varepsilon) = \sum_{n=0}^{\infty} \varepsilon^n Q_n(\xi, \eta, \zeta),$$

$$(3.9) \qquad \Phi(\xi, \eta, \zeta; \varepsilon) = \frac{\xi + \eta}{2 \sin \sigma} + \sum_{n=1}^{\infty} \varepsilon^n \Phi_n(\xi, \eta, \zeta),$$

$$(3.10) \qquad \Psi(\xi, \eta, \zeta; \varepsilon) = \frac{\xi - \eta}{2 \cos \sigma} + \sum_{n=1}^{\infty} \varepsilon^n \Psi_n(\xi, \eta, \zeta).$$

From the assumptions about Φ and Ψ it follows that the wall location in the (ξ, η) plane is given by the equation $\xi = \eta$ and the flow domain consists of all points (ξ, η) such that $\xi \geq \eta$.

Now we assume that

$$(3.11) \qquad \lim_{\substack{\xi \to -\infty \\ \eta \to -\infty \\ \xi - \eta - \text{fixed}}} Q_0(\xi, \eta, \zeta) = -1,$$

$$(3.12) \qquad \lim_{\substack{\xi \to -\infty \\ \eta \to -\infty \\ \xi - \eta - \text{fixed}}} Q_n(\xi, \eta, \zeta) = 0 \qquad (n = 1, 2, 3, \ldots).$$

The above conditions express mathematically our earlier physical assumption that the gas in front of the incident shock wave is at rest. The conditions (3.11) and (3.12) are sufficient for the time being, although they do not constitute the complete set of boundary conditions.

Now the expansion (3.8)–(3.10) is used to obtain equations for every Q_n. The procedure is standard (see [5]; also many particulars may be found in [6]) and we will not go into details.

It can be shown that the functions u_0, v_0, ϱ_0 and T_0 satisfy the following equations:

$$(3.14) \qquad \left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right)(-\varrho_0 + u_0 \sin^2 \sigma) + \left(\frac{\partial}{\partial \xi} - \frac{\partial}{\partial \eta}\right) v_0 \cos^2 \sigma = 0,$$

$$(3.15) \qquad \left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right)\left(-u_0 + \frac{\varrho_0 + (\gamma - 1)T_0}{\gamma}\right) = 0,$$

$$(3.16) \qquad \left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right) v_0 - \frac{1}{\gamma}\left(\frac{\partial}{\partial \xi} - \frac{\partial}{\partial \eta}\right)(\varrho_0 + (\gamma - 1)T_0) = 0,$$

$$(3.17) \qquad \left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right)(T_0 - \varrho_0) = 0.$$

This is a system of linear homogeneous partial differential equations of the hyperbolic type. We solve it, subject to the conditions (3.11), and obtain

(3.18) $u_0 = \varrho_0 = T_0 = g_0(\xi, \zeta) + h_0(\eta, \zeta).$

(3.19) $v_0 = g_0(\xi, \zeta) - h_0(\eta, \zeta),$

where g_0 and h_0 are arbitrary bounded functions such that

(3.20) $\lim_{\xi \to -\infty} g_0(\xi, \zeta) = -1,$

(3.21) $\lim_{\eta \to -\infty} h_0(\eta, \zeta) = 0.$

A similar, but rather lengthy procedure, gives u_1, v_1, ϱ_1, T_1 and further approximations. Although we are not interested in them, we have to analyse the next approximation in order to find equations for g_0, h_0 and Φ_1, Ψ_1. It can be shown in a similar way as in [6] (see also [5]) that the functions u_1, v_1, ϱ_1 and T_1 are bounded if and only if the following relations hold:

(3.22) $\dfrac{\partial^2 g_0}{\partial \xi^2} + 2g_0 \dfrac{\partial g_0}{\partial \xi} = \dfrac{4\cos^2 \sigma}{\Gamma \sin \sigma} \dfrac{\partial g_0}{\partial \xi},$

(3.23) $\dfrac{\partial^2 h_0}{\partial \eta^2} + 2(h_0 - 1) \dfrac{\partial h_0}{\partial \eta} = \dfrac{4\cos^2 \sigma}{\Gamma \sin \sigma} \dfrac{\partial h_0}{\partial \zeta},$

and

(3.24) $\Phi_1 \sin \sigma - \Psi_1 \cos \sigma = -\dfrac{4\sin^2 \sigma + \gamma - 3}{8\cos^2 \sigma} \displaystyle\int_\eta^\xi [g_0(\tau, \zeta) + 1] d\tau,$

(3.25) $\Phi_1 \sin \sigma + \Psi_1 \cos \sigma = -\dfrac{4\sin^2 \sigma + \gamma - 3}{8\cos^2 \sigma} \displaystyle\int_\xi^\eta h_0(\tau, \zeta) d\tau.$

The relations (3.22) and (3.23) are partial differential equations from which the unknown functions g_0 and h_0 can be found.

Until now, only the boundary conditions (3.11) and (3.12) have been used. Let us now assume that the incident shock wave has the classical Taylor structure. Mathematically, it can be expressed as

(3.26) $\lim_{\eta \to -\infty} \varrho_0(\xi, \eta, \zeta) = \operatorname{th}\xi.$

This condition as well as Eqs. (3.20) and (3.21), when applied to Eqs. (3.18) and (3.19), give

(3.27) $g_0(\xi, \zeta) = \operatorname{th}\xi.$

It is a matter of simple calculation to check that this function satisfies Eq. (3.22); therefore the function g_0 is found.

Our final group of assumptions takes the form

(3.28) $Q_0(\xi, \xi, \zeta) = -1,$

where now Q_0 is u_0, v_0 or T_0.

If $Q_0 = u_0$, then the condition (3.28) means that the gas particle "sticks" to the wall; if $Q_0 = v_0$, then Eq. (3.28) says that the wall is impermeable; finally, if $Q_0 = T_0$, then Eq. (3.28) expresses two physical assumptions, namely that the wall is isothermal and the gas is in the thermal equilibrium with the wall.

However, the solutions (3.18), (3.19) and (3.27) cannot satisfy the boundary condition (3.28), what means that close to the wall a boundary layer exists. It is studied in the next paragraph.

4. Boundary layer

In order to determine the inner flow, we introduce new independent variables (r, s) defined as follows:

$$(4.1) \qquad s = \frac{\xi+n}{2\sin\sigma} = x+0(\varepsilon), \qquad r = \frac{\zeta-\eta}{2\sqrt{\varepsilon}\cos\sigma} = \frac{y}{\sqrt{\varepsilon}} +0(\varepsilon).$$

Also, new flow variables are introduced:

$$(4.2) \qquad \hat{u} = u+1, \quad \hat{\varrho} = \varrho+1, \quad \hat{T} = T+1$$

and

$$(4.3) \qquad \hat{v} = \frac{v+1}{\sqrt{\varepsilon}}.$$

If the new unknown functions $\hat{u}, \hat{v}, \hat{\varrho}$ and $\hat{T}$ are expressed as series expansions of the small parameter ε, then it follows from the Navier–Stokese quations that these functions satisfy (in the first approximation) linearized Prandtl equations of the boundary layer.

We solve these equations subject to the boundary conditions following Eq. (3.28)

$$\hat{u}_0(0, s) = 0, \quad \hat{v}_0(0, s) = 0, \quad \hat{T}_0(0, s) = 0.$$

It can be shown (see [6, 8, 9]), that

$$(4.4) \quad \hat{u}_0 = \varphi(s) - \frac{1}{\sqrt{\pi}} \int_0^\infty \varphi\left(s - \frac{\beta r^2}{2\Gamma w \sin\sigma}\right) \frac{e^{-w}}{\sqrt{w}} dw,$$

$$(4.5) \quad \hat{v}_0 = \frac{\cos\sigma}{\sin\sigma} \varphi'(s)r + \frac{1}{\sqrt{\pi}\cos\sigma} \int_0^r \int_0^\infty \left[\frac{\gamma-1}{\sin\sigma} \varphi'\left(s - \frac{\beta \Pr z^2}{\sin\sigma}\right) \right.$$
$$\left. + \varphi'\left(s - \frac{\beta z^2}{2\Gamma w \sin\sigma}\right)\sin\sigma\right] \frac{e^{-w}}{\sqrt{w}} dz dw,$$

$$(4.6) \qquad \hat{\varrho}_0 = \varphi(s) + \frac{\gamma-1}{\sqrt{\pi}} \int_0^\infty \varphi\left(s - \frac{\beta \Pr r^2}{2\Gamma w \sin\sigma}\right) \frac{e^{-w}}{\sqrt{w}} dw,$$

$$(4.7) \qquad \hat{T}_0 = \varphi(s) - \frac{1}{\sqrt{\pi}} \int_0^\infty \varphi\left(s - \frac{\beta \Pr r^2}{2\Gamma w \sin\sigma}\right) \frac{e^{-w}}{\sqrt{w}} dw,$$

2*.

where $\varphi(s)$ is an arbitrary bounded functions (to be found) and $\varphi'(s)$ is its derivative (also assumed to be bounded).

The unknown function φ is determined by means of the matching principle [5]. We do not go into details because all of them can be found in [6].

The matching leads to the following equations:

$$(4.8) \quad h_0(s\sin\sigma, 0) = 1 + g_0(s\sin\sigma)$$

$$- \frac{\sqrt{\varepsilon}}{\cos\sigma} \sqrt{\frac{\Gamma}{2\pi\beta\sin\sigma}} \left(\frac{\gamma-1}{\sqrt{\mathrm{Pr}}} + \sin^2\sigma\right) \int_0^\infty \frac{\varphi'(s-\alpha)}{\sqrt{\alpha}} \, d\alpha,$$

$$(4.9) \quad \varphi(s) = 2 + 2g_0(s\sin\sigma) - \frac{\sqrt{\varepsilon}}{\cos\sigma} \sqrt{\frac{\Gamma}{2\pi\beta\sin\sigma}} \left(\frac{\gamma-1}{\sqrt{\mathrm{Pr}}} + \sin^2\sigma\right) \int_0^\infty \frac{\varphi'(s-\alpha)}{\sqrt{\alpha}} \, d\alpha.$$

We can see that, conversely to our assumptions, both functions φ and h_0 do depend on $\sqrt{\varepsilon}$. To avoid this contradiction we admit following Lesser and Seebas [8], φ and h_0 to depend on ε. Thus we take as a solution of Eqs. (4.8) and (4.9)

$$(4.10) \qquad\qquad h_0(\eta, 0) = 1 + \mathrm{th}\,\eta - 2\sqrt{\varepsilon}\,Y(\eta),$$

$$(4.11) \qquad\qquad \varphi(s) = 2[1 + \mathrm{th}\,s] - 2\sqrt{\varepsilon}\,Y(s),$$

where

$$(4.12) \qquad Y(\eta) = \frac{1}{\cos\sigma} \sqrt{\frac{2\Gamma}{\pi\beta}} \left(\frac{\gamma-1}{\sqrt{\mathrm{Pr}}} + \sin^2\sigma\right) \int_0^\infty \mathrm{sech}^2(\eta - \alpha) \frac{d\alpha}{\sqrt{\alpha}}.$$

In Eqs. (4.10) and (4.11), terms of order of $0(\varepsilon)$ have been neglected because they contribute to higher order approximations of the flow parameters.

The equality (4.10) is an initial condition for Eq. (3.23); of course Eq. (4.11) defines the function φ. Substituting Eq. (4.11) into Eqs. (4.4)–(4.7), we obtain explicit expressions for $\hat{u}_0, \hat{v}_0, \hat{\varrho}_0$ and $\hat{T}_0$.

5. Reflected shock wave

We are interested mainly in the trajectory and the structure of the reflected shock wave. Consequently, we confine ourselves to the case of positive values of x (and ζ). Equation (3.23) is the Burgers equation, and its theory is presented in [7] (for many details see [6, 8]). The Burgers equation can be solved explicitly and in our case the solution is

$$(5.1) \quad h_0 = 1 + \frac{2\sinh\eta - \sqrt{\varepsilon}\left[e^\eta I(V_5) - e^{-\eta} I(V_2) + e^\eta \dfrac{\partial I(V_5)}{\partial V_5} + e^{-\eta} \dfrac{\partial I(V_2)}{\partial V_2}\right]}{2\cosh\eta - \sqrt{\varepsilon}\left[e^\eta I(V_5) + e^{-\eta} I(V_2)\right]},$$

where

$$(5.2) \quad I(V) = \int_{-\infty}^\infty Y(\alpha) e^{-2\sqrt{\varepsilon}X(\alpha)} \mathrm{erfc}\left[\frac{(\alpha-V)\cos\sigma}{\sqrt{\Gamma\zeta\sin\sigma}}\right] d\alpha,$$

(5.3) $V_2 = \eta - \dfrac{\Gamma\zeta\sin\sigma}{2\cos^2\sigma},$

(5.4) $V_5 = \eta + \dfrac{\Gamma\zeta\sin\sigma}{2\cos^2\sigma},$

(5.5) $X(\eta) = \displaystyle\int_{-\infty}^{\eta} Y(\alpha)d\alpha = \frac{2}{\cos\sigma}\sqrt{\frac{2\Gamma}{\pi\beta}\left(\frac{\gamma-1}{\sqrt{\text{Pr}}}+\sin^2\sigma\right)}\int_{0}^{\infty}\text{sech}^2(\eta-\alpha)\sqrt{\alpha}\,d\alpha.$

From the considerations of Sect. 3 and from Eq. (5.1) it follows that

(5.6) $x\sin\sigma+y\cos\sigma = \xi-\varepsilon\dfrac{4\sin^2\sigma+\gamma-3}{8\cos^2\sigma}\ln\left\{1+e^{2\eta}-\sqrt{\varepsilon}\,[e^{2\eta}I(V_5)+I(V_2)]\right\},$

(5.7) $x\sin\sigma-y\cos\sigma = \eta-\varepsilon\dfrac{4\sin^2\sigma+\gamma-3}{8\cos^2\sigma}\ln\dfrac{1+e^{2\xi}}{1+e^{2\eta}}.$

Now we can derive the equation of the trajectory of the reflected shock wave.

Qualitatively, the reflected shock wave location is given by

$$\eta-\text{fixed}, \quad V_5 >> 1, \quad -V_2 >> 1.$$

But then (see [6, 8]),

(5.8) $$h_0 = \left[1-\sqrt{\varepsilon}\,Y(V_5)\right]\left[1+\text{th}\big(\eta-\sqrt{\varepsilon}X(V_5)\big)\right]$$

and

$$x\sin\sigma+y\cos\sigma = \xi,$$

$$x\sin\sigma-y\cos\sigma = \eta-\varepsilon\frac{4\sin^2\sigma+\gamma-3}{4\cos^2\sigma}(\xi-\eta)$$

or

(5.9) $$x\sin\sigma-y\cos\sigma+\frac{4\sin^2\sigma+\gamma-3}{2\cos^2\sigma}\zeta\sin\sigma = \left(1+\varepsilon\frac{4\sin^2\sigma+\gamma-3}{2\cos^2\sigma}\right)\eta.$$

From Eq. (5.8) we see, on the other hand, that the shock wave location is given by the equation

$$\eta = \sqrt{\varepsilon}X\left(\eta+\frac{\Gamma\zeta\sin\sigma}{2\cos^2\sigma}\right).$$

Using here the asymptotic expression for $X(\eta)$ (see [8])

$$X(\eta) = \frac{4}{\cos\sigma}\sqrt{\frac{2\Gamma}{\pi\beta}\left(\frac{\gamma-1}{\sqrt{\text{Pr}}}+\sin^2\sigma\right)}\sqrt{\eta}+0(\eta^{-\frac{1}{2}}), \quad \eta\to\infty,$$

we obtain

$$\eta = \frac{4\sqrt{\varepsilon}\Gamma}{\cos^2\sigma}\sqrt{\frac{\sin\sigma}{\pi\beta}\left(\frac{\gamma-1}{\sqrt{\text{Pr}}}+\sin^2\sigma\right)}\sqrt{\zeta}.$$

If this equality is substituted into Eq. (5.9), then the reflected shock wave trajectory results

(5.10) $y = x\left(1+\varepsilon\dfrac{4\sin^2\sigma+\gamma-3}{2\cos^2\sigma}\right)\text{tg}\,\sigma - \dfrac{4\Gamma\sqrt{\varepsilon x}}{\cos^3\sigma}\sqrt{\dfrac{\sin\sigma}{\pi\beta}\left(\dfrac{\gamma-1}{\sqrt{\text{Pr}}}+\sin^2\sigma\right)}.$

The first term on the right hand side gives the trajectory of the reflected shock wave in the case of the ideal gas, the second term is induced by the boundary layer, and it causes some shift of the trajectory of the reflected shock wave. This result is in a good agreement with the experiment [10].

From Eqs. (5.6) and (5.7) some limitation of the theory follows. We can see that the ratio

$$\delta = \frac{\varepsilon}{\cos^2\sigma}$$

should be actually treated as the small parameter. Thus the equality

$$\frac{\pi}{2} - \sigma = 0(\varepsilon)$$

gives the upper limit of validity of the theory. For angles σ satisfying this equality, the reflection is no more regular.

Acknowledgements

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INSTITUTE OF MECHANICS
UNIVERSITY OF WARSAW.