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Theory of nonlocal electromagnetic fluids

B. M. McCAY and M. N. L. NARASIMHAN (OREGON)

A CONTINUUM theory of nonlocal electromagnetic viscous fluids is proposed. Nonlocal and local balance laws and jump conditions are obtained. Using a Clausius-Duhem thermodynamic inequality which encompasses nonlocal effects specific forms of the constitutive equations including the electromagnetic stress and energy are derived for dynamic nonrelativistic systems. A complete linear constitutive theory with thermodynamic restrictions on the material coefficients is developed along with the associated field equations and boundary conditions with a view to facilitate practical applications of the theory.

Zaproponowano kontynualną teorię nielokalnej, elektromagnetycznej cieczy lepkiej. Otrzymano nielokalne i lokalne warunki równowagi i warunki nieciągłości. Wykorzystując termodynamiczną nierówność Clausiusa-Duhema uzględniającą efekty nielokalne, wyprowadzono szczególne postacie równań konstytutywnych zawierające naprężenia elektromagnetyczne i energię w przypadku nierelatywistycznych układów dynamicznych. Rozwinięto pełną teorię konstytutywną wraz z ograniczeniami termodynamicznymi dotyczącymi współczynników materiałowych, a także równania pola stowarzyszonego i warunki brzegowe z myślą o ułatwieniu zastosowań praktycznych przedstawionej teorii.

Предложена континуальная теория нелокальной, электромагнитной вязкой жидкости. Получены нелокальные и локальные условия равновесия и условия разрыва. Используя термодинамическое неравенство Клаузиуса-Дюгема, учитывающее нелокальные эффекты, выведены частные виды определяющих уравнений, содержавшие электромагнитные напряжения и энергию в случае нерелятивистических динамических систем. Развита полная определяющая теория совместно с термодинамическими ограничениями, касающимися материальных коэффициентов, а также управнения ассоциированного поля и граничные условия с целью облегчения практических примений представленной теории.

1. Introduction

IN RECENT years nonlocal continuum mechanics has been rapidly emerging as a most powerful field aimed at bridging the gap between the microscopic and macroscopic theories of matter. Early developments in the field of nonlocal theories of mechanics include those due to ERINGEN [1, 2], KRÖNER [3], KUNIN [4, 5], EDELEN [6], and EDELEN and LAWS [7].

It is well known that classical continuum mechanics does not have the necessary mechanism to explain physical phenomena involving materials whose response to certain applied stimuli are dominated by their internal structures. For example, the dispersive character of plane waves such as Rayleigh surface waves in materials, the phenomena of surface tension, surface energy, optical branches of the dispersion curve, the state of stress at a crack tip in a material, secondary flows and turbulence fail to find rotational explanations through classical continuum mechanics. ERINGEN [1, 2] has shown that in order to explain such phenomena, it is not necessary to address the atomic nature of materials for such discrepancies. Instead one can fruitfully pursue the nonlocal continuum mechanical approach which is capable of meeting the challenges at the atomic scale while still remaining within the basic continuum framework.

The present status of nonlocal continuum mechanics has been enriched by the construction of a nonlocal thermodynamics and rational constitutive theory on an axiomatic foundation by Eringen [8]. A variational approach for nondissipative cases (elasticity) was provided by Edelen and Laws [7], and Eringen and Edelen [9]. Theories of polar nonlocal continua, nonlocal elasticity, nonlocal fluid dynamics, nonlocal microfluid dynamics, nonlocal continuum thermodynamics, nonlocal memory dependent materials, nonlocal electromagnetic elastic solids, nonlocal theory of fracture mechanics, especially the important problem of state of stress at the crack tip in a material, and nonlocal elasticity and waves were developed by Eringen in a series of papers [2, 9–16]. These works, apart from containing the theoretical developments, also demonstrate the applications to practical problems of interest. Furthermore, they provide for methods of determining the nonlocal material coefficients by comparing the theoretically predicted results such as those involving wave dispersion, with lattice dynamical or experimental results.

The importance of nonlocal effects in electromagnetic interactions with deformable materials has been explored for elastic solids (cf. Eringen [14]) as mentioned above. This work, to the present, develops a general constitutive theory for solids based on nonlocal continuum mechanics, with no corresponding work existing for fluids.

The aim of the present paper is to establish a nonlocal continuum theory for electromagnetic fluids. In Sects. 3 and 4 the balance laws for the nonlocal theory and the entropy inequality are presented. Upon developing the general constitutive theory and subsequent constitutive relations arising from the Clausius-Duhem inequality in Sect. 5, we then derive a linear constitutive theory and in Sect. 6, the field equations are obtained for purposes of practical applications.

A special feature of our present work is, besides developing the constitutive theory for nonlocal electromagnetic fluids, the derivation in Sect. 5, of a complete set of thermodynamic restrictions on the material coefficients characterizing the electromechanical constitution of fluids. These restrictions on the material coefficients are expected to serve as useful informations in seeking meaningful experimental verifications of theoretically predicted results. A practical application of the theory developed here to the problem of dispersion of Reyleigh surface waves in *dielectric fluids* is scheduled for a forthcoming publication.

2. Balance laws

In formulating the mechanical and thermodynamic balance laws for a nonlocal continuum capable on interacting the electromagnetic fields, we shall incorporate the electromagnetic interactions through the linear momentum density, the body force density, energy flux vector, and internal energy density (all of which, in general, include both mechanical and electromagnetic effects). The total energy density is incorporated in such a way that the rate of production of entropy is equal to the total energy flux of the material minus the Poynting vector (Grot [17]), as viewed in a frame moving with the material divided by the temperature. To these laws it is necessary to add the balance laws of Gauss, Faraday, and Ampere, the conservation of magnetic flux, and the conservation of charge. Although these laws are extremely general, it must, however, be noted that they are not adequate for a complete description of such phenomena as magnetic spin interactions. In the present work we exclude such phenomena and restrict ourselves to a nonrelativistic case valid for small material velocities, v, as compared to the speed of light in a vacuum, $(v^2/c^2 \ll 1)$. Following ERINGEN [14], the balance laws for a nonlocal electromagnetic material are written down in the following localized form:

Conservation of mass

$$
\begin{aligned}
&\frac{\partial \varrho}{\partial t}+(\varrho v^k)_{,k} = \hat{\varrho}, \quad \text{in } v-\sigma,\\
&[\varrho(v^k-u^k)-\hat{\varrho}]^k n_k = 0, \quad \text{on } \sigma.
\end{aligned}
\tag{2.1}
$$

Balance of linear momentum

$$
\begin{aligned}
&\mathbf{t}^k_{,k}+\varrho(\mathbf{f}-\dot{\mathbf{v}}-\dot{\mathbf{g}}) = \hat{\varrho}(\mathbf{v}+\mathbf{g})-\varrho\hat{\mathbf{f}}, \quad \text{in } v-\sigma,\\
&[\mathbf{t}^k-\varrho(\mathbf{v}+\mathbf{g})(v^k-u^k)+\hat{\mathbf{f}}^k]n_k = 0, \quad \text{on } \sigma.
\end{aligned}
\tag{2.2}
$$

Balance of moment of momentum

$$
\begin{aligned}
&\mathbf{p}_{,k}\times\mathbf{t}^k-\varrho\mathbf{v}\times\mathbf{g} = \varrho\mathbf{p}\times\hat{\mathbf{f}}-\varrho\hat{\mathbf{l}}, \quad \text{in } v-\sigma,\\
&[\varrho\mathbf{p}\times\{\mathbf{t}^k-(\mathbf{v}+\mathbf{g})(v^k-u^k)\}+\hat{\mathbf{l}}^k]n_k = 0, \quad \text{on } \sigma.
\end{aligned}
\tag{2.3}
$$

Conservation of energy

$$
\begin{aligned}
&\varrho\dot{\varepsilon}-\varrho\dot{\mathbf{g}}\cdot\mathbf{v}+\mathbf{t}^k\cdot\mathbf{v}_{,k}-q^k_{,k}-\varrho h = \varrho\hat{h}-\varrho\mathbf{v}\cdot\hat{\mathbf{f}}-\hat{\varrho}\left(\varepsilon+\frac{1}{2}v^2-\mathbf{v}\cdot\mathbf{g}\right), \quad \text{in } v-\sigma,\\
&\left[\mathbf{t}^k\cdot\mathbf{v}+q^k-\varrho\left(\varepsilon+\frac{1}{2}v^2\right)(v^k-u^k)+\hat{h}^k\right]n_k = 0, \quad \text{on } \sigma.
\end{aligned}
\tag{2.4}
$$

Faraday's law

$$
\begin{aligned}
&\nabla\times\underline{\mathscr{E}}+\frac{1}{c}\mathbf{B}^* = \frac{1}{c}\hat{\mathbf{b}}, \quad \text{in } v-\sigma,\\
&\left[\mathbf{E}+\frac{1}{c}\mathbf{u}\times\mathbf{B}+\hat{\mathbf{E}}\right]\times\mathbf{n} = \mathbf{0}, \quad \text{on } \sigma,
\end{aligned}
\tag{2.5}
$$

Ampere's law

$$
\begin{aligned}
&\nabla\times\underline{\mathscr{H}}-\frac{1}{c}\mathbf{D}^*-\frac{1}{c}\underline{\mathscr{J}} = \frac{1}{c}\hat{\underline{\mathscr{J}}}, \quad \text{in } v-\sigma,\\
&\left[\mathbf{H}-\frac{1}{c}\mathbf{u}\times\mathbf{D}+\hat{\mathbf{H}}\right]\times\mathbf{n} = \mathbf{0}, \quad \text{on } \sigma.
\end{aligned}
\tag{2.6}
$$

Gauss's law

$$\text{(2.7)}\qquad \begin{aligned} &\nabla\cdot\mathbf{D}-q=\hat{q}, \quad \text{in } v-\sigma,\\ &[\mathbf{D}+\hat{\mathbf{D}}]\cdot\mathbf{n}=0, \quad \text{on } \sigma. \end{aligned}$$

Conservation of magnetic flux

$$\text{(2.8)}\qquad \begin{aligned} &\nabla\cdot\mathbf{B}=\hat{m}, \quad \text{in } v-\sigma,\\ &[\mathbf{B}+\hat{\mathbf{B}}]\cdot\mathbf{n}=0, \quad \text{on } \sigma. \end{aligned}$$

Conservation of charge

$$\text{(2.9)}\qquad \begin{aligned} &\nabla\cdot\underline{\mathscr{J}}+\frac{\partial q}{\partial t}+\nabla\cdot(q\mathbf{v})=\hat{\sigma}, \quad \text{in } v-\sigma,\\ &[\underline{\mathscr{J}}+\hat{\boldsymbol{\Sigma}}]\cdot\mathbf{n}=0, \quad \text{on } \sigma, \end{aligned}$$

where ϱ — mass density, $\mathbf{q}$ — electromagnetic momentum density, $\mathbf{t}^k$ — stress vector, $\mathbf{q}^k$ — total energy flux vector, h — energy density supply, q — free charge density, $\mathbf{p}$ — position vector, $\mathbf{v}$ — velocity vector, $\mathbf{f}$ — total body force density (per unit mass), ε — total energy density, and

$$\text{(2.10)}\qquad \underline{\mathscr{E}}\equiv\mathbf{E}+\frac{1}{c}\mathbf{v}\times\mathbf{B}, \quad \underline{\mathscr{H}}\equiv\mathbf{H}-\frac{1}{c}\mathbf{v}\times\mathbf{D}, \quad \underline{\mathscr{J}}\equiv\mathbf{J}^f-q\mathbf{v},$$

where $\mathbf{E}$ — electric field, $\mathbf{H}$ — magnetic field, $\mathbf{D}$ — electric displacement vector, $\mathbf{B}$ — magnetic induction vector, $\mathbf{J}^f$ — free current density vector, c — speed of light in a vacuum. A vector carrying an asterisk as a superscript is given by the following expression for its convected derivative, for example,

$$\mathbf{B}^*\equiv\frac{\partial\mathbf{B}}{\partial t}+\check{\mathbf{B}},$$

where

$$\text{(2.11)}\qquad \check{\mathbf{B}}=(\nabla\cdot\mathbf{B})\mathbf{v}+\nabla\times(\mathbf{B}\times\mathbf{v}).$$

We introduce the following relationships for further reference:

$$\text{(2.12)}\qquad \begin{aligned} &\mathbf{D}\equiv\mathbf{E}+\mathbf{P}, \quad \mathbf{B}\equiv\mathbf{H}+\mathbf{M}, \quad \underline{\mathscr{D}}\equiv\mathbf{D}+\frac{1}{c}\mathbf{v}\times\mathbf{H},\\ &\underline{\mathscr{B}}\equiv\mathbf{B}-\frac{1}{c}\mathbf{v}\times\mathbf{E}, \quad \underline{\mathscr{P}}=\mathbf{P}-\frac{1}{c}\mathbf{v}\times\mathbf{M},\\ &\underline{\mathscr{M}}=\mathbf{M}+\frac{1}{c}\mathbf{v}\times\mathbf{P}, \end{aligned}$$

where $\mathbf{P}$ — polarization vector, $\mathbf{M}$ — magnetization vector.

In the above balance laws (2.1) to (2.9), v is the configuration of the body and σ a surface of discontinuity sweeping across the body with velocity $\mathbf{u}$ and whose normal is $\mathbf{n}$. The electromagnetic laws are expressed in Heaviside–Lorentz units.

The material points of the body in the undeformed state are determined by a set of rectangular coordinates X_K, $K = 1, 2, 3$. At time t the *motion* carries X_K to the spatial points x_k, $k = 1, 2, 3$ under the continuous mapping

$$x_k = x_k(\mathbf{X}, t) \tag{2.13}$$

which possesses continuous first-order partial derivatives with respect to $\mathbf{X}$ and t. Futhermore, we require that

$$\det(x_{k,K}) > 0 \tag{2.14}$$

so that Eq. (2.13) possesses a unique inverse given by

$$X_K = X_K(\mathbf{x}, t) \tag{2.15}$$

for all points of the body, except possibly a countable set of singular surfaces, lines and points.

The summation convention over repeated indices is employed and a subscript comma shall denote partial differentiation. A superposed dot denotes material time-rate, for example,

$$\mathbf{v}_{,k} = \frac{\partial \mathbf{v}(\mathbf{x}, \mathbf{t})}{\partial x_k}, \quad \dot{\varepsilon} = \frac{D\varepsilon}{Dt} = \frac{\partial \varepsilon}{Dt} + \varepsilon_{,k} v^k. \tag{2.16}$$

The nonlocal residuals $\hat{\varrho}, \hat{\varrho}^k, \hat{\mathbf{f}}, \hat{\mathbf{f}}^k, \hat{\mathbf{l}}, \hat{\mathbf{l}}^k, \hat{h}, \hat{h}^k, \hat{\mathbf{b}}, \hat{\mathbf{E}}, \hat{\mathbf{H}}, \hat{\mathbf{B}}, \hat{\mathbf{D}}, \hat{\mathbf{\Sigma}}, \hat{q}, \hat{m}$, and $\hat{\sigma}$ are introduced to account for the effects of fields at all other points of the body on the point for which the localized balance laws are written. We require that the integrals of these residuals, defined over their manifolds of definition, vanish, that is

$$\begin{aligned} &\int_{V-\sigma} (\hat{\varrho}, \varrho\hat{\mathbf{f}}, \varrho\hat{\mathbf{l}}, \varrho\hat{h}, \hat{q}, \hat{m}, \hat{\sigma}) dv = 0, \\ &\int_{\sigma} (\hat{\varrho}^k, \hat{\mathbf{f}}^k, \hat{\mathbf{l}}^k, \hat{h}^k, \hat{D}^k, \hat{B}^k, \hat{\Sigma}^k) n_k da = 0, \\ &\int_{S-\gamma} (\hat{\mathbf{b}}, \hat{\underline{\mathscr{J}}}) \cdot d\mathbf{a} = 0, \quad \int_{\gamma} (\hat{\mathbf{E}}, \hat{\mathbf{H}}) \cdot \mathbf{k} ds = 0. \end{aligned} \tag{2.17}$$

The integral in Eq. $(2.17)_3$ is taken over an open material surface S (enclosed within a closed curve $\mathscr{C}$) which is being swept by a discontinuity curve γ having the velocity $\mathbf{u}$, and $\mathbf{k}$ is the unit tangent vector on γ; da and ds are, respectively, the elements of area and length.

The (nonlocal) current density residual $\hat{\underline{\mathscr{J}}}$ is given by

$$\hat{\underline{\mathscr{J}}} = \hat{\mathbf{J}}^f + \hat{q}\mathbf{v}. \tag{2.18}$$

The physical significance of the localization residuals can be interpreted from the equations in which they appear. For example, $\hat{\varrho}$ is the mass residual which represents the rate at which mass is created or destroyed at the point $\mathbf{x}$ due to the effects of all other material points occupying v–σ. Such phenomena could occur either through chemical reactions, through the existence of quasi-particles (e.q. electrons, excitons, phonons) or through dissociation, ionization, and fracture of the subelements of the body [18]. We may interpret $\hat{\mathbf{f}}$ as the nonlocal body force at the point $\mathbf{x}$ due to the long-range intermolecular forces produced by all other points of the body (e.g. gravitational, electromagnetic attractions). The surface residual $\hat{\varrho}^k$ is associated with the production or destruction of mass in crossing the surface σ. The electromagnetic residuals introduced by Eringen [14] have similar interpretations. Here $\hat{q}$ is the induced nonlocal charge at the point $\mathbf{x}$ due to the charges throughout the body at all other points. Similarly, $\hat{m}$ (if it exists) is the magnetic pole strength induced at $\mathbf{x}$ by the rest of the body, while $\hat{\mathbf{b}}$ represents the nonlocal magnetic induction.

3. Second law of thermodynamics

The fundamental thermodynamic law can be stated in the form of a generalized Clausius–Duhem inequality. Following Grot and Eringen [19], we assume physically that the entropy production rate in the body with electromagnetic constitution is equal to the total energy flux minus the "Poynting vector", $c\underline{\mathscr{E}}\times\underline{\mathscr{H}}$ divided by the absolute temperature θ. The vector $c\underline{\mathscr{E}}\times\underline{\mathscr{H}}$ has the interpretation that it reduces to the "Poynting vector" when the body is at rest. Thus the global form of the Clausius–Duhem inequality takes the form

$$(3.1)\qquad \frac{D}{Dt}\int_{v-\sigma}\varrho\eta dv+\oint_{\mathscr{S}-\sigma}\frac{1}{\theta}(q-c\underline{\mathscr{E}}\times\underline{\mathscr{H}})\cdot d\mathbf{a}-\int_{v-\sigma}\frac{1}{\theta}(\varrho h+\underline{\mathscr{J}}_0\cdot\underline{\mathscr{E}})dv\geqslant 0.$$

Here η is the entropy density per unit mass and $\underline{\mathscr{J}}_0$ is the external current source. It must, however, be noted that although the above form of the second law is extremely general, it does not encompass such phenomena as magnetic spin interactions and polarization gradient effects. The surface integral in Eq. (3.1) is taken over the surface $\mathscr{S}$ bounding the body.

Now localizing Eq. (3.1) we obtain

$$(3.2)\qquad \begin{aligned}&\varrho\dot{\eta}+\nabla\cdot\left[\frac{1}{\theta}(\mathbf{q}-c\underline{\mathscr{E}}\times\underline{\mathscr{H}})\right]-\frac{1}{\theta}(\varrho h+\underline{\mathscr{J}}_0\cdot\underline{\mathscr{E}})+\hat{\varrho}\eta-\frac{1}{\theta}\varrho\hat{s}\geqslant 0,\quad \text{in } v-\sigma,\\ &\left[\varrho\eta(v^k-u^k)-\frac{1}{\theta}(\mathbf{q}-c\underline{\mathscr{E}}\times\underline{\mathscr{H}})^k-\hat{s}^k\right]n_k\geqslant 0,\quad \text{on } \sigma,\end{aligned}$$

where $\hat{s}$ and $\hat{s}^k$ are entropy localization residuals subject to

$$(3.3)\qquad \int_{v-\sigma}\frac{1}{\theta}\varrho\hat{s}dv\geqslant 0,\qquad \int_{\sigma}\hat{s}^k da_k=0.$$

Using Eqs. $(2.5)_1$ and $(2.6)_1$, in order to simplify the second term in Eq. $(3.2)_1$, and introducing

$$(3.4)\qquad \begin{aligned}\Psi&=\varepsilon-\theta\eta, &\quad \overline{\psi}&=\Psi-\mathbf{v}\cdot\mathbf{g},\\ \overline{\underline{\mathscr{J}}}&=\underline{\mathscr{J}}-\underline{\mathscr{J}}_0, &\quad \mathbf{Q}&=\mathbf{q}-c\underline{\mathscr{E}}\times\underline{\mathscr{H}},\end{aligned}$$

where Ψ is the Helmholtz free-energy density, Eq. $(3.2)_1$ may be written as

$$(3.5)\quad \begin{aligned} &-\frac{\varrho}{\theta}(\dot{\psi}+\dot{\mathbf{v}}\cdot\mathbf{g}+\dot{\theta}\eta)+\frac{1}{\theta}\mathbf{t}^k\cdot\mathbf{v}_{,k}+\mathbf{Q}\cdot\nabla\left(\frac{1}{\theta}\right)+\frac{1}{\theta}(\underline{\mathscr{E}}\cdot\mathbf{D}^*+\underline{\mathscr{H}}\cdot\mathbf{B}^*)\\ &-\frac{\varrho}{\theta}\hat{\mathbf{f}}\cdot\mathbf{v}-\frac{\hat{\varrho}}{\theta}\left(\bar{\psi}+\frac{1}{2}v^2\right)+\frac{1}{\theta}\underline{\mathscr{E}}\cdot\overline{\underline{\mathscr{J}}}+\frac{1}{\theta}(\underline{\mathscr{E}}\cdot\underline{\mathscr{J}}-\underline{\mathscr{H}}\cdot\hat{\mathbf{b}})+\frac{\varrho}{\theta}(\hat{h}-\hat{s})\geqslant 0, \end{aligned}$$

in $v-\sigma$

which becomes the generalized Clausius-Duhem inequality.

4. Constitutive equations

In this section we develop the constitutive theory for a general class of nonlocal electromagnetic fluids in which the mass production and heat conduction are not appreciable, that is,

$$(4.1)\qquad \hat{\varrho}=0,\quad \mathbf{Q}=\mathbf{0}.$$

In order to formulate a constitutive theory for a general class of nonlocal electromagnetic fluids, an appropriate pair of independent electromagnetic quantities must be chosen as constitutive variables. In view of certain invariance (Galilean) requirements that the nonlocal electromechanical constitutive functionals are required to satisfy (which will be discussed later in our work), and under the nonrelativistic "v^2/c^2 approximation" (that is, $v^2/c^2 \ll 1$), it is convenient to choose $(\underline{\mathscr{B}}, \underline{\mathscr{D}})$ as our independent electromagnetic quantities (cf. ERINGEN [14]). For a nonlocal electromagnetic solid, the constitutive functionals depend on the following strain measures, at *all* points $\mathbf{X}'$, of the body:

$$(4.2)\quad \begin{aligned} \boldsymbol{\chi}'(\mathbf{X}') &= \mathbf{x}(\mathbf{X}',t)-\mathbf{x}(\mathbf{X},t)\equiv \mathbf{x}'-\mathbf{x},\\ \boldsymbol{\chi}'_K(\mathbf{X}') &= \mathbf{x}'_{,K}-\mathbf{x}_{,K}\equiv \partial\mathbf{x}'/\partial X^K-\partial\mathbf{x}/\partial X^K,\\ \Lambda'_K(\mathbf{X}') &= \mathbf{x}'_{,K}\cdot\boldsymbol{\chi}',\quad C'_{KL}(\mathbf{X}')=\mathbf{x}'_{,K}\cdot\boldsymbol{\chi}'_L,\\ C_{KL}(\mathbf{X}) &= \mathbf{x}_{,K}\cdot\mathbf{x}_{,L},\\ \boldsymbol{\Gamma}'_0(\mathbf{X}') &= \underline{\mathscr{B}}(\mathbf{x}',t)-\underline{\mathscr{B}}(\mathbf{x},t)\equiv\underline{\mathscr{B}}'-\underline{\mathscr{B}},\\ \boldsymbol{\Delta}'_0(\mathbf{X}') &= \underline{\mathscr{D}}(\mathbf{x}',t)-\underline{\mathscr{D}}(\mathbf{x},t)\equiv\underline{\mathscr{D}}'-\underline{\mathscr{D}}, \end{aligned}$$

where $\mathbf{x}'$ is the image point of $\mathbf{X}'$ in the deformed body, as also $\mathbf{x}$ is the image of $\mathbf{X}$ in the deformed body.

Since Stokesian fluids are defined (ERINGEN [10]) as materials which accept every frame of reference leaving density unchanged as the material frame, we need to obtain the material time-rates of the strain measures in Eq. (4.2) by applying the limit $\mathbf{X}'\to\mathbf{x}'$, $\mathbf{X}\to\mathbf{x}$ with ϱ fixed. We obtain

$$(4.3)\quad \begin{aligned} &\boldsymbol{\chi}'\to\mathbf{x}'-\mathbf{x},\quad \dot{\boldsymbol{\chi}}'\to\mathbf{v}'-\mathbf{v},\quad \boldsymbol{\chi}'_K\to\mathbf{0},\\ &\dot{\boldsymbol{\chi}}'_K\to\mathbf{v}'_k-\mathbf{v}_{,k},\quad \Lambda_K\to x'_k-x_k,\\ &\dot{\boldsymbol{\Lambda}}_K\to\beta_k(\mathbf{x}')\equiv\beta'_k\equiv\mathbf{v}'_{,k}\cdot(\mathbf{x}'-\mathbf{x})+v'_k-v_k,\\ &\mathscr{C}'_{KL}\to 0,\quad \dot{\mathscr{C}}'_{KL}\to\gamma_{kl}(\mathbf{x}')\equiv\gamma'_{kl}\equiv v_{k,l}-v_{k,l},\\ &C_{KL}\to\delta_{kl},\quad \dot{C}_{KL}\to 2d_{kl}(\mathbf{x})\equiv v_{k,l}+v_{l,k},\\ &\boldsymbol{\Gamma}'_0\to\boldsymbol{\Gamma}'\equiv\underline{\mathscr{B}}'-\underline{\mathscr{B}},\quad \dot{\boldsymbol{\Gamma}}'_0\to\boldsymbol{\Gamma}'\equiv\dot{\underline{\mathscr{B}}}'-\dot{\underline{\mathscr{B}}},\\ &\dot{\boldsymbol{\Delta}}'_0\to\boldsymbol{\Delta}'\equiv\underline{\mathscr{D}}'-\underline{\mathscr{D}},\quad \dot{\boldsymbol{\Delta}}'_0\to\dot{\boldsymbol{\Delta}}'\equiv\dot{\underline{\mathscr{D}}}'-\dot{\underline{\mathscr{D}}}, \end{aligned}$$

where δ_{kl} is the Kronecker delta, and the primes on the Eulerian field quantities indicate that they are evaluated at $\mathbf{x}'$, while those without primes denote that they are evaluated at $\mathbf{x}$. For example,

$$(4.4)\qquad \begin{gathered} \mathbf{v} = \mathbf{v}(\mathbf{x}, t), \quad \mathbf{v}' = \mathbf{v}(\mathbf{x}', t), \quad \underline{\mathscr{B}} = \underline{\mathscr{B}}(\mathbf{x}, t), \quad \underline{\mathscr{B}}' = \underline{\mathscr{B}}(\mathbf{x}', t), \\ \beta_k' = \beta_k(\mathbf{x}', t), \quad \text{etc.} \end{gathered}$$

We will follow similar notations throughout the rest of the paper, for the sake of brevity. In the above measures the point $\mathbf{x}$ is any local point of interest chosen in the body, while $\mathbf{x}'$ is any surrounding point in the same body. Thus we are motivated to define the nonlocal electromagnetic fluids by the constitutive equation of the form

$$(4.5)\qquad \bar{\psi}(\mathbf{x}, t) = \psi(\mathfrak{I}, \mathfrak{I}'),$$

where ψ is a scalar-valued function (Helmholtz free-energy density) of the variables in $\mathfrak{I}$ and a functional of the nonlocal variables in $\mathfrak{I}'$ defined below. $\mathfrak{I}$ and $\mathfrak{I}'$ are, respectively the ordered sets:

$$(4.6)\qquad \begin{gathered} \mathfrak{I} = \{\mathbf{d}, \underline{\mathscr{B}}, \dot{\underline{\mathscr{B}}}, \underline{\mathscr{D}}, \dot{\underline{\mathscr{D}}}, \varrho^{-1}, \theta\}, \\ \mathfrak{I}' = \{r'^{-1}, \boldsymbol{\beta}', \boldsymbol{\gamma}', \boldsymbol{\Gamma}', \dot{\boldsymbol{\Gamma}}', \boldsymbol{\Delta}', \dot{\boldsymbol{\Delta}}'\}, \end{gathered}$$

where

$$(4.7)\qquad r'^{-1} = \varrho'^{-1}(\mathbf{x}') - \varrho^{-1}(\mathbf{x}) \equiv \varrho'^{-1} - \varrho^{-1}.$$

The other response functionals $\mathbf{t}, \mathbf{g}, \varepsilon, \eta, \underline{\mathscr{E}}, \underline{\mathscr{H}}$ and the conduction current

$$\mathbf{J}^c = \mathbf{J}^f - \underline{\mathscr{J}}_0$$

are assumed to be of the form given by Eq. (4.5) by the axiom of determinism and equipresence. Although the mechanical variables defined in $\mathfrak{I}$ and $\mathfrak{I}'$ are objective under arbitrary time-dependent transformations of frames of reference, the electromagnetic quantities are not. However, under the group of Galilean transformations defined by

$$(4.8)\qquad \mathbf{x}^* = \mathbf{R}\mathbf{x} + \mathbf{v}_0 t + \mathbf{b}_0, \quad t^* = t - a,$$

for some constant a, where $\mathbf{R}$ is a time-independent orthogonal transformation, that is, $\mathbf{R}\mathbf{R}^T = \mathbf{R}^T\mathbf{R} = \mathbf{I}$, det $\mathbf{R} = \pm 1$, and $\mathbf{v}_0$ is a constant velocity vector and $\mathbf{b}_0$ is some arbitrary constant vector, it can be readily verified that $\underline{\mathscr{B}}$ and $\underline{\mathscr{D}}$ transform under Eq. (4.8) as

$$(4.9)\qquad \underline{\mathscr{B}}^* = \mathbf{R}\underline{\mathscr{B}}, \quad \underline{\mathscr{D}}^* = \mathbf{R}\underline{\mathscr{D}}$$

and are therefore form invariant under Eq. (4.8). In like manner, all variables in $\mathfrak{I}$ and $\mathfrak{I}'$ can be shown to be Galilean-invariant. In view of Eq. (4.1) the entropy inequality (3.5) becomes

$$(4.10)\qquad \begin{aligned} -\frac{\varrho}{\theta}(\dot{\bar{\psi}} + \dot{\mathbf{v}} \cdot \mathbf{g} + \dot{\theta}\eta) + \frac{1}{\theta}\mathbf{t}^k \cdot \mathbf{v}_{,k} + \frac{1}{\theta}(\underline{\mathscr{E}} \cdot \mathbf{D}^* + \underline{\mathscr{H}} \cdot \mathbf{B}^*) \\ + \frac{1}{\theta}\underline{\mathscr{E}} \cdot \underline{\bar{\mathscr{J}}} - \frac{\varrho}{\theta}\hat{\mathbf{f}} \cdot \mathbf{v} + \frac{1}{\theta}(\underline{\mathscr{E}} \cdot \bar{\mathscr{J}} - \underline{\mathscr{H}} \cdot \bar{\mathbf{b}}) + \frac{\varrho}{\theta}(\hat{h} - \hat{s}) \geqslant 0 \quad \text{in } v - \sigma. \end{aligned}$$

Now assuming that ψ possesses continuous first order partial derivatives with respect to $d_{kl}, \mathscr{B}_k, \dot{\mathscr{B}}_k, \mathscr{D}_k, \dot{\mathscr{D}}_k, \varrho^{-1}$ and θ and the Fréchet partial derivatives with respect to

r'^{-1}, β'_k, γ'_{kl}, Γ'_k, $\dot{\Gamma}'_k$, Δ'_k and $\dot{\Delta}'_k$, continuous of order zero, and using Eqs. (4.5)—(4.7), we calculate $\dot{\psi}$ which yields

$$(4.11)\quad \dot{\psi} = \frac{\partial\psi}{\partial d_{kl}}\dot{d}_{kl} + \frac{\partial\psi}{\partial\dot{\mathscr{B}}_k}\ddot{\mathscr{B}}_k + \frac{\partial\psi}{\partial\mathscr{B}_k}\dot{\mathscr{B}}_k + \frac{\partial\psi}{\partial\mathscr{D}_k}\dot{\mathscr{D}}_k + \frac{\partial\psi}{\partial\dot{\mathscr{D}}_k}\ddot{\mathscr{D}}_k + \frac{\partial\psi}{\partial p^{-1}}\dot{\overline{\varrho^{-1}}} + \frac{\partial\psi}{\partial\theta}\dot{\theta} + \int_{v-\sigma}\frac{\delta\psi}{\delta r'^{-1}}(\mathfrak{J},\mathfrak{J}';\boldsymbol{\lambda})\dot{\overline{r'^{-1}}}(\boldsymbol{\lambda})dv'(\boldsymbol{\lambda})$$
$$+\int_{v-\sigma}\left[\frac{\delta\psi}{\delta\beta'_k}(\mathfrak{J},\mathfrak{J}';\boldsymbol{\lambda})\dot{\beta}'_k(\boldsymbol{\lambda}) + \frac{\delta\psi}{\delta\gamma'_{kl}}(\mathfrak{J},\mathfrak{J}';\boldsymbol{\lambda})\dot{\gamma}'_{kl}(\boldsymbol{\lambda})\right]dv'(\boldsymbol{\lambda})$$
$$+\int_{v-\sigma}\left[\frac{\delta\psi}{\delta\Gamma'_k}(\mathfrak{J},\mathfrak{J}';\boldsymbol{\lambda})\dot{\Gamma}'_k(\boldsymbol{\lambda}) + \ldots + \frac{\delta\psi}{\delta\dot{\boldsymbol{\Delta}}'_k}(\mathfrak{J},\mathfrak{J}';\boldsymbol{\lambda})\ddot{\boldsymbol{\Delta}}'_k(\boldsymbol{\lambda})\right]dv'(\boldsymbol{\lambda}),$$

where $\delta/\delta\beta$ denotes Fréchet differentation and the functional gradients appearing in the integrals are also functions of a vector $\boldsymbol{\lambda}$. Using the equation of continuity $(2.1)_1$, with $\hat{\varrho} = 0$, we have

$$(4.12)\qquad \dot{\overline{\varrho^{-1}}} = -\varrho^2\dot{\varrho} = \varrho^{-1}v^k_{,k}$$

while we may also write

$$(4.13)\qquad \frac{\partial\psi}{\partial d_{kl}}\dot{d}_{kl} = \frac{1}{2}\left(\frac{\partial\psi}{\partial d_{kl}} + \frac{\partial\psi}{\partial d_{kl}}\right)\dot{\overline{v_{k,l}}}.$$

Using Eq. (4.13) along with Eqs. (4.12), $(2.10)_3$, and (2.17) in Eq. (4.11), and then substituting the result into the entropy inequality (4.10), we have

$$(4.14)\quad -\frac{\varrho}{\theta}\left(\eta + \frac{\partial\psi}{\partial\theta}\right)\dot{\theta} + \frac{\varrho}{\theta}\left[-\hat{f}^k - \frac{1}{\varrho}\mathscr{E}^k(q+\dot{q}) + \int\frac{\delta\psi}{\delta\beta'_l}v'_{k,l} - \frac{1}{\varrho c}e_{klm}(E_l\dot{H}_m\right.$$
$$\left.+\dot{E}_lH_m)\right]v_k - \frac{\varrho}{\theta}\left(g_k - \int\frac{\delta\psi}{\delta\beta'_k}\right)\dot{v}_k + \frac{1}{\theta}\left[t^{kl} - D^k\mathscr{E}^l - B^k\mathscr{H}^l - \delta^{kl}\left(\frac{\partial\psi}{\partial\varrho^{-1}} - \mathscr{E}_mD_m\right.\right.$$
$$\left.\left.-\mathscr{H}_mB_m - \int\frac{\delta\psi}{\delta r'^{-1}}\right)\right]v_{l,k} - \frac{\varrho}{\theta}\left[\frac{1}{2}\left(\frac{\partial\psi}{\partial d_{kl}} + \frac{\partial\psi}{\partial d_{lk}}\right) - \int\frac{\delta\psi}{\delta\gamma'_{kl}}\right]\dot{\overline{v_{k,l}}} - \frac{\varrho}{\theta}\left(\frac{\partial\psi}{\partial\mathscr{B}_k}\right.$$
$$\left.-\frac{\delta\psi}{\delta\Gamma'_k} - \frac{1}{\varrho}\mathscr{H}^k\right)\dot{\mathscr{B}}_k - \frac{\varrho}{\theta}\left(\frac{\partial\psi}{\partial\mathscr{D}_k} - \int\frac{\partial\psi}{\partial\Delta'_k} - \frac{1}{\varrho}\mathscr{E}^k\right)\dot{\mathscr{D}}_k - \frac{\varrho}{\theta}\left(\frac{\partial\psi}{\partial\dot{\mathscr{B}}_k} - \int\frac{\delta\psi}{\delta\dot{\Gamma}'_k}\right)\ddot{\mathscr{B}}_k$$
$$-\frac{\varrho}{\theta}\left(\frac{\partial\psi}{\partial\dot{\mathscr{D}}_k} - \int\frac{\delta\psi}{\delta\dot{\Delta}'_k}\right)\ddot{\mathscr{D}}_k - \frac{\varrho}{\theta}D^{\hat{h}} + \frac{1}{\theta}\mathscr{E}_k(J^{\prime}_k - \mathscr{J}_{0k}) + \frac{1}{\theta}(\mathscr{E}_k\hat{J}^{\prime}_k - \mathscr{H}_k\hat{b}_k)$$
$$+\frac{\varrho}{\theta}(h-\hat{s}) + O(v^2/c^2) \geqslant 0, \quad \text{in } v-\sigma,$$

where

$$(4.15)\quad D^{\hat{h}} = \int_{v-\sigma}\left[\frac{\delta\psi}{\delta r'^{-1}}\varrho'^{-1}v'^k_{,k} + \frac{\varrho\psi}{\varrho\beta'_k}\{\dot{\overline{v'_{l,k}}}(x'_l - x_l) + v'_{l,k}v'_l + \dot{v}'_k\}\right.$$
$$\left.+\frac{\delta\psi}{\delta\gamma'_{kl}}\dot{\overline{v'_{k,k}}} + \frac{\delta\psi}{\delta\Gamma'_k}\dot{\mathscr{B}}'_k + \frac{\delta\psi}{\delta\dot{\Gamma}'_k}\ddot{\mathscr{B}}'_k + \frac{\delta\psi}{\delta\Delta'_k}\dot{\mathscr{D}}'_k + \frac{\delta\psi}{\delta\dot{\Delta}'_k}\ddot{\mathscr{D}}'_k\right]dv'$$

and for brevity we have written, for example,

$$\int \frac{\delta\psi}{\delta\beta_k'} = \int_{v-\sigma} \frac{\delta\psi}{\delta\beta_k'}(\mathfrak{I}', \mathfrak{I}; \boldsymbol{\lambda}) dv'(\boldsymbol{\lambda}). \tag{4.16}$$

Multiplying Eq. (4.14) by θ and integrating over the volume $v-\sigma$ eliminates the terms $\hat{h}$ and $\hat{s}$ by applying Eqs. (2.16) and (3.3). Because of the form invariant nature of the constitutive functionals and residuals they cannot depend on the quantities $\dot{\theta}, v_k, \dot{v}_k, \dot{v}_{k,l}$, $\dot{\mathscr{B}}_k$, $\ddot{\mathscr{B}}_k$, $\dot{\mathscr{D}}_k$ and $\ddot{\mathscr{D}}_k$, which are not form invariant under Eq. (4.8). Neglecting the terms of $O(v^2/c^2)$ the volume integral of inequality (4.14) multiplied by θ is linear in these quantities, and therefore cannot be maintained for all possible variations of these quantities unless

$$\begin{gathered}
\eta = -\frac{\partial\psi}{\partial\theta}, \quad \hat{f}_k = \int_{v-\sigma} \frac{\delta\psi}{\delta\beta_l'} v_{k,l}' dv' - \frac{1}{\varrho} \mathscr{E}_k (q+\hat{q}) - \frac{1}{\varrho c} e_{klm}(E_l \dot{H}_m + \dot{E}_l H_m), \\
g = \int_{v-\sigma} \frac{\delta\psi}{\delta\beta_k'} dv', \quad \frac{1}{2}\left(\frac{\partial\psi}{\partial d_{kl}} + \frac{\partial\psi}{\partial d_{lk}}\right) = \int_{v-\sigma} \frac{\delta\psi}{\delta\gamma_{kl}'} dv', \\
\mathscr{H}_k = \varrho\left(\frac{\partial\psi}{\partial\mathscr{B}_k} - \int_{v-\sigma} \frac{\delta\psi}{\delta\Gamma_k'} dv'\right), \quad \mathscr{E}_k = \varrho\left(\frac{\partial\psi}{\partial\mathscr{D}_k} - \int_{v-\sigma} \frac{\delta\psi}{\delta\Delta_k'} dv'\right), \\
\frac{\partial\psi}{\partial\dot{\mathscr{B}}_k} = \int_{v-\sigma} \frac{\delta\psi}{\delta\dot{\Gamma}_k'} dv', \quad \frac{\partial\psi}{\partial\dot{\mathscr{D}}_k} = \int_{v-\sigma} \frac{\delta\psi}{\delta\dot{\Delta}_k'} dv',
\end{gathered} \tag{4.17}$$

and

$$\frac{1}{\theta} {}_D t_{kl} v_{l,k} + \frac{1}{\theta} \mathscr{E}_k (J_k^c - \mathscr{J}_{0k} + \frac{1}{\theta} (\mathscr{E}_k \hat{J}_k^c - \mathscr{H}_k \hat{b}_k) - \frac{\varrho}{\theta} {}_D\hat{h} + \frac{\varrho}{\theta}(\hat{h} - \hat{s}) \geqslant 0, \quad \text{in } v-\sigma, \tag{4.18}$$

where

$$\begin{aligned}
{}_D t_{kl} &= t_{kl} - {}_M t_{kl} - \delta_{kl}\left(\frac{\partial\psi}{\partial\varrho^{-1}} - \int_{v-\sigma} \frac{\delta\psi}{\delta r'^{-1}} dv' - {}_M h\right), \\
{}_M t_{kl} &= \mathscr{E}_l D_k + \mathscr{H}_l B_k - \frac{1}{2}(\underline{\mathscr{E}} \cdot \mathbf{D} + \underline{\mathscr{H}} \cdot \mathbf{B}) \delta_{kl}, \\
{}_M h &= \frac{1}{2}(\underline{\mathscr{E}} \cdot \mathbf{D} + \underline{\mathscr{H}} \cdot \mathbf{B}).
\end{aligned} \tag{4.19}$$

The above permissible shortcut leads to the analogous results as obtained by utilizing the technique employed by EDELEN and LAWS [7], (cf. ERINGEN [8], p. 254). We have thus proved:

THEOREM. *The constitutive equations of nonlocal electromagnetic fluids are thermodynamically admissible if and only if Eqs.* (4.17), (4.18) *and* (4.19) *are satisfied.*

From Eq. $(4.17)_2$ it is clear that for nonlocal electromagnetic fluids the nonlocal force $\hat{\mathbf{f}}$ *does not* vanish.

Using Eqs. (4.17) and (4.19) the energy equation (2.4) may be written as

$$-({}_D t_{(kl)} + {}_M t_{(kl)} - {}_M h \delta_{kl}) d_{kl} + \varrho(\theta\dot{\eta} - \mathbf{g} \cdot \dot{\mathbf{v}}) + \underline{\mathscr{H}} \cdot \mathbf{B} + \underline{\mathscr{E}} \cdot \mathbf{D} + \varrho\mathbf{v} \cdot \hat{\mathbf{f}} + ({}_D\hat{h} - h - \hat{h}) = 0, \tag{4.20}$$

where parentheses enclosing subscript-indices indicate symmetrization, for example,

$$_{D}t_{(kl)} \equiv \frac{1}{2}\left({}_{D}t_{kl}+{}_{D}t_{kl}\right).$$

We now require the energy equation (4.20) to be invariant under all rigid motions (4.8). Since ψ, and hence η and $\dot{\eta}$ are invariant as can be seen from Eq. $(4.17)_1$, and ${}_{D}t_{kl}$, ${}_{M}t_{kl}$ and d_{kl} are invariant, so are ${}_{D}t_{(kl)}d_{kl}$ and ${}_{M}t_{(kl)}d_{kl}$ as well as the scalar products in Eq. (4.20). Moreover, since the energy source h is considered invariant, we must have

$$\hat{h} = \hat{h}_0 + {}_{D}\hat{h} + \mathbf{v}\cdot\hat{\mathbf{f}}, \tag{4.21}$$

where $\hat{h}_0$ is a scalar-valued constitutive functional of the same type as ψ, satisfying the same invariance requirements under Eq. (4.8). Thus we may write the energy equation as

$$\varrho\theta\dot{\eta} - ({}_{D}t_{(kl)} + {}_{M}t_{(kl)} - {}_{M}h\delta_{kl})d_{kl} + \underline{\mathscr{H}}\cdot\dot{\mathbf{B}} + \underline{\mathscr{E}}\cdot\mathbf{D} - \varrho(\dot{\mathbf{v}}\cdot\mathbf{g} + h - \hat{h}_0) = 0. \tag{4.22}$$

The nonlocal residues $\hat{f}_k$ and $\hat{h}$ are thus determined by Eqs. $(4.17)_2$ and (4.21) to within a class of scalar-valued functions $\hat{h}_0$ obeying the constitutive axioms of the theory. These and other residuals are further restricted by the vanishing of their integrals over the body (Eqs. (2.16)). We do not pursue the discussion of these restrictions any further.

For a linear constitutive theory the dissipative stress may be written as

$$\begin{aligned} {}_{D}t^{kl} &= a_1^{klmn}d_{mn} + a_2^{klm}\mathscr{B}_m + a_3^{klm}\dot{\mathscr{B}}_m + a_4^{klm}\mathscr{D}_m + a_5^{klm}\dot{\mathscr{D}}_m \\ &\quad + \int_{v-\sigma}(b_1^{klmn}\gamma_{mn} + b_2^{klm}\beta_m + b_3^{klm}\Gamma_m + b_4^{klm}\dot{\Gamma}_m + b_5^{klm}\Delta_m + b_6^{klm}\dot{\Delta}_m)dv', \end{aligned} \tag{4.23}$$

where the term involving $r^{-1}(x')$ has been dropped since it may be absorbed in the expression for the total stress tensor which follows later in Eq. (4.26). The tensors $\mathbf{a}_i$, $i = 1, 2, 3, 4, 5$ are functions of ϱ^{-1} and θ while all $\mathbf{b}_j, j = 1, 2, 3, 4, 5, 6$ depend on ϱ'^{-1}, θ', and $||\mathbf{x}'-\mathbf{x}||$.

The invariance of Eq. (4.23) under the full orthogonal group of transformations $\{\mathbf{R}\}$ and translations (4.8) dictates that $\mathbf{a}_i$ and $\mathbf{b}_j$ are isotropic.

Now we will examine the axiom of attenuating neighborhoods in relation to materials with electromechanical constitution. It has been shown by ERINGEN [18] that the axiom of attenuating neighborhoods is one of the basic axioms of nonlocality governing thermomechanical materials. This axiom is based on the fact that for a material with thermomechanical constitution the nonlocal interactions at a material point $\mathbf{x}$ due to all surrounding points $\mathbf{x}'$ of the material die out rapidly as the distance $||\mathbf{x}'-\mathbf{x}||$ increases. In our case we have a fluid with electromechanical constitution. At first sight it might seem that the axiom of attenuating neighborhoods for materials with electromechanical constitution may be violated in view of possible distant electromagnetic influences that may occur. But if one recalls that there are no external electromagnetic fields acting on the fluid in our case, then one finds that there is no physical reason to suppose that the mechanical and electromagnetic constituents of the medium would individually exhibit uncoupled responses. Thus it seems logical to expect that the overall behavior of the material is dictated only by the resultant coupled electromechanical constitution and not by mechanical responses separated from the electromagnetic responses. Now if one invokes the axiom of attenuating neighborhoods to such materials, the interpretation of this axiom would lie in the fact that it is not the electromagnetic effects that are confined to small neighborhoods but it is the resultant *electromechanical* interactions that are confined to small neighborhoods of the local point $\mathbf{x}$. This will permit one to ignore all of the higher order small terms $O(\|\mathbf{x}'-\mathbf{x}\|^2)$ to obtain a linear constitutive theory, both in the nonlocal material coefficients and the constitutive equations. As a consequence in the linear theory the nonlocal material coefficient tensors $\mathbf{a}_i$ and $\mathbf{b}_j$ reduce to:

$$(4.24)\qquad \begin{aligned} a_i^{klm} &= 0 = b_j^{klm}, \quad 2 \leqslant i \leqslant 5, \quad 2 \leqslant j \leqslant 6,\\ a_1^{klmn} &= \lambda_1 \delta^{kl}\delta^{mn} + \mu_1(\delta^{km}\delta^{ln}+\delta^{kn}\delta^{lm}),\\ b_1^{klmn} &= \lambda_1' \delta^{kl}\delta^{mn} + \mu_1'(\delta^{km}\delta^{ln}+\delta^{kn}\delta^{lm}),\end{aligned}$$

and hence the electomagnetic quantities drop out from the stress constitutive equation in the linear theory.

Letting

$$(4.25)\qquad \lambda_v \equiv \lambda_1 - \int\limits_{v-\sigma} \lambda_1' dv', \quad \mu_v \equiv \mu_1 - \int\limits_{v-\sigma} \mu_1' dv'$$

and using Eq. $(4.19)_1$, it follows from Eq. (4.23) that

$$(4.26)\qquad t_{kl} = (-\pi - {}_M h + \lambda_v d_{rr})\delta_{kl} + 2\mu_v d_{kl} + {}_M t_{kl} + \int\limits_{v-\sigma} [(\sigma' + \lambda_v' d_{rr}')\delta_{kl} + 2\mu_v' d_{kl}']dv',$$

where

$$(4.27)\qquad \pi \equiv -\frac{\partial\psi}{\partial\varrho^{-1}}, \quad \sigma' \equiv \frac{\delta\psi}{\delta r'^{-1}}.$$

To obtain the linear constitutive expressions for $\underline{\mathscr{H}}$ and $\underline{\mathscr{E}}$ we assume that the free-energy ψ to have a constitutive make-up involving linear functionals. To this end we take

$$(4.28)\qquad \psi = \psi_0 + \int\limits_{v-\sigma} (\psi_1^k \Gamma_k' + \psi_2^k \dot{\Gamma}_k' + \psi_k^3 \Delta_k' + \psi_4^k \dot{\Delta}_k' + \psi_k^5 \beta_k' + \psi_6^{kl} \gamma_{kl}')dv',$$

where

$$(4.29)\qquad \begin{aligned} \psi_0 &= \psi_0(\mathfrak{J}) = \psi_0(\underline{\mathscr{B}}, \underline{\dot{\mathscr{B}}}, \underline{\mathscr{D}}, \underline{\dot{\mathscr{D}}}, \mathbf{d}, \varrho^{-1}, \theta),\\ \psi_i^k &= \psi_i^k(\underline{\mathscr{B}}', \underline{\dot{\mathscr{B}}}', \underline{\mathscr{D}}', \underline{\dot{\mathscr{D}}}', \mathbf{d}', \varrho'^{-1}, \theta', \|\mathbf{x}'-\mathbf{x}\|), \quad i = 1, 2, \dots, 5,\\ \psi_6^{kl} &= \psi_6^{kl}(\underline{\mathscr{B}}', \underline{\dot{\mathscr{B}}}', \underline{\mathscr{D}}', \underline{\dot{\mathscr{D}}}', \mathbf{d}', \varrho'^{-1}, \theta', \|\mathbf{x}'-\mathbf{x}\|).\end{aligned}$$

or a completely linear theory ψ_i^k and ψ_6^{kl} are taken in the form

$$(4.30)\qquad \begin{aligned} \psi_0 &= \alpha_1^k \mathscr{B}_k + \alpha_2^k \dot{\mathscr{B}}_k + \alpha_3^k \mathscr{D}_k + \alpha_4^k \dot{\mathscr{D}}_k + \alpha_5^{kl} d_{kl}\\ &\qquad + \left(\frac{1}{2}\beta_1^{kl}\mathscr{B}_l + \beta_2^{kl}\dot{\mathscr{B}}_l + \beta_3^{kl}\mathscr{D}_l + \beta_4^{kl}\dot{\mathscr{D}}_l + \beta_5^{klm} d_{lm}\right)\dot{\mathscr{B}}_k,\\ \psi_i^k &= \gamma_{i1}^{kl}\mathscr{B}_l' + \gamma_{i2}^{kl}\dot{\mathscr{B}}_l' + \gamma_{i3}^{kl}\mathscr{D}_l' + \gamma_{i4}^{kl}\dot{\mathscr{D}}_l' + \gamma_{i5}^{klm} d_{lm}', \quad i = 1, 2, 3, 4, 5,\\ \psi_6^{kl} &= \gamma_{61}^{klm}\mathscr{B}_m' + \gamma_{62}^{klm}\dot{\mathscr{B}}_m' + \gamma_{63}^{klm}\mathscr{D}_m' + \gamma_{64}^{klm}\dot{\mathscr{D}}_m' + \gamma_{65}^{klmn} d_{mn}',\end{aligned}$$

where in ψ_0 only the pertinent quadratic terms have been included. Using Eq. (4.30) in Eq. (4.28) and letting

$$(4.31)\qquad \mathscr{H}'^k \equiv \mathscr{H}^k - \varrho\alpha_1^k,$$

while dropping the prime in $\mathscr{H}'^k$ for convenience and setting

$$(4.32)\quad \begin{aligned} \sigma_i^{kl} &\equiv \beta_{1i}^{kl} - \int_{v-\sigma} (\gamma_{1i}^{kl} + \gamma_{i1}^{kl}) dv', \quad i = 1, 2, 3, 4, \\ \sigma_5^{klm} &\equiv \beta_5^{klm} - \int_{\sigma-v} \gamma_{15}^{klm} dv', \end{aligned}$$

the linear constitutive equation for $\underline{\mathscr{H}}$ is given by Eq. $(4.17)_5$:

$$(4.33)\quad \mathscr{H}^k = \varrho\Big[\sigma_1^{kl}\mathscr{B}_l + \sigma_2^{kl}\dot{\mathscr{B}}_l + \sigma_3^{kl}\mathscr{D}_l + \sigma_4^{kl}\dot{\mathscr{D}}_l + \sigma_5^{klm} d_{lm} + \int_{v-\sigma} (\gamma_{11}^{kl}\mathscr{B}'_l + \gamma_{21}^{kl}\dot{\mathscr{B}}'_l + \gamma_{31}^{kl}\mathscr{D}'_l + \gamma_{41}^{kl}\dot{\mathscr{D}}'_l + \gamma_{51}^{kl}\beta'_l + \gamma_{61}^{klm} d'_{lm}) dv'\Big].$$

Assuming $\underline{\mathscr{H}}$ to be isotropic under Eq. (4.8) and again applying the axiom of attenuating neighborhoods, the coefficients in Eq. (4.33) take the form

$$(4.34)\quad \begin{aligned} \sigma_i^{kl} &= b_i \delta^{kl}, \quad i = 1, 2, 3, 4, \\ \gamma_{i1}^{kl} &= b'_i \delta^{kl}, \quad i = 1, 2, 3, 4, 5, \\ \sigma_5^{klm} &= 0 = \gamma_{61}^{klm}, \end{aligned}$$

where b_i depend on ϱ^{-1} and θ, and b'_i depend on ϱ'^{-1}, θ, and $||\mathbf{x}'-\mathbf{x}||$. Using Eq. (4.34) in Eq. (4.33) yields the linear constitutive equation for $\underline{\mathscr{H}}$,

$$(4.35)\quad \mathscr{H}^k = \varrho\Big[b_1\mathscr{B}_k + b_2\dot{\mathscr{B}}_k + b_3\mathscr{D}_k + b_4\dot{\mathscr{D}}_k + \int_{v-\sigma} (b'_1\mathscr{B}'_k + b'_2\dot{\mathscr{B}}'_k + b'_3\mathscr{D}'_k + b'_4\dot{\mathscr{D}}'_k + b'_5\beta'_k) dv'\Big].$$

Similarly, using Eq. $(4.17)_6$ we obtain the constitutive equation for $\underline{\mathscr{E}}$:

$$(4.36)\quad \mathscr{E}_k = \varrho\Big[s_1\mathscr{B}_k + s_2\dot{\mathscr{B}}_k + s_3\mathscr{D}_k + s_4\dot{\mathscr{D}}_k + \int_{v-\sigma} (s'_1\mathscr{B}'_k + s'_2\dot{\mathscr{B}}'_k + s'_3\mathscr{D}'_k + s'_4\dot{\mathscr{D}}'_k + s'_5\beta'_k) dv'\Big],$$

where s_i depend on ϱ^{-1} and θ, and s'_i depend on ϱ'^{-1}, θ', and $||\mathbf{x}'-\mathbf{x}||$.

The linear constitutive expression for the conduction current $\mathbf{J}^c \equiv \mathbf{J}^f - \underline{\mathscr{J}}_0$ cannot be derived from Eq. (4.17). As dictated by the axiom of equipresence $\mathbf{J}^c$ is assumed to be a vector-valued functional given by

$$(4.37)\quad J_k^c = \phi_k(\mathfrak{J}, \mathfrak{J}').$$

Following a procedure analogous to the one for the dissipative stress yields

$$(4.38)\quad J_k^c = \alpha_1\mathscr{B}_k + \alpha_2\dot{\mathscr{B}}_k + \alpha_3\mathscr{D}_k + \alpha_4\dot{\mathscr{D}}_k + \int_{v-\sigma} (\alpha'_1\mathscr{B}'_k + \alpha'_2\dot{\mathscr{B}}'_k + \alpha'_3\mathscr{D}'_k + \alpha'_4\dot{\mathscr{D}}'_k + \alpha'_5\beta'_k) dv',$$

where

$$(4.39)\quad \begin{aligned} \alpha_i &= \alpha_i(\varrho^{-1}, \theta), \\ \alpha'_j &= \alpha'_j(\varrho'^{-1}, \theta', ||\mathbf{x}'-\mathbf{x}||), \end{aligned}$$

for all $1 \leqslant i \leqslant 5$, $1 \leqslant j \leqslant 6$.

This completes the linear constitutive theory for nonlocal electromagnetic fluids.

5. Thermodynamic restrictions

All of the material coefficients appearing in Eqs. (4.26), (4.35), (4.36), and (4.38) are subject to the restrictions arising from the entropy inequality (4.18). Integrating Eq. (4.18) over $v-\sigma$ yields the classical dissipation inequality for the entire body:

$$\int_{v-\sigma} \frac{1}{\theta}\, {}_D t^{kl} d_{kl}\, dv + \int_{v-v} \frac{1}{\theta}\, \mathscr{E}^k J^c_k\, dv + \int_{v-\sigma} \frac{1}{\theta}\, [\mathscr{E}^k \hat{J}_k - \mathscr{H}^k \hat{b}_k + \varrho(\hat{h} - {}_D\hat{h} - \hat{s})]\, dv \geqslant 0. \tag{5.1}$$

The above inequality is postulated to be valid for all independent mechanical as well as electromagnetic processes, and as a consequence the last integral in Eq. (5.1) may be set equal to zero without loss of generality. Through Eqs. (4.26), (4.36) and (4.38), the remaining two integrals in Eq. (5.1) vary independently of each other.

Considering the case where we require (point-wise satisfaction)

$$\int_{v-\sigma} \frac{1}{\theta}\, {}_D t^{kl} d_{kl}\, dv \geqslant 0, \tag{5.2}$$

where we assume $\theta > 0$, yields analogous results as obtained by ERINGEN [14], namely, the "mechanical" material moduli must satisfy

$$\mu_v \geqslant 0, \quad \mu'_v \geqslant 0, \quad 3\lambda_v + 2\lambda_v + 2\mu_v \geqslant 0, \quad 3\lambda'_v + 2\mu_v \geqslant 0. \tag{5.3}$$

We now take the first integral in Eq. (5.1) to be zero and require

$$\int_{v-\sigma} \frac{1}{\theta}\, \mathscr{E}^k J^c_k\, dv \geqslant 0. \tag{5.4}$$

Demanding point-wise satisfaction of Eq. (4.5) a sufficient condition is

$$\mathscr{E}^k J^c_k \geqslant 0. \tag{5.5}$$

Using Eqs. (4.36) and (4.38), Eq. (5.5) is satisfied if and only if

$$A_{mn} y^{(k)}_m y^{(k)}_n + \int_{v-\sigma} P'_{mn} y^{(k)}_m y'^{(k)}_n dv' + \Big(\int_{v-\sigma} s'_m y'^{(k)}_m dv'\Big)\Big(\int_{v-\sigma} \alpha'_m y'^{(k)}_m dv'\Big) \geqslant 0, \tag{5.6}$$

where the summation convention is applied over the repeated indices m and n, for $m, n = 1, 2, 3, 4, 5$ and the index k which is not to be summed is placed inside the parenthesis for example, (k) which appears in Eq. (5.6) as a superscript in each term. The various terms in Eq. (5.6) have the following expressions:

$$\begin{gathered}
y^{(k)}_1 = \mathscr{B}_k, \quad y^{(k)}_2 = \dot{\mathscr{B}}_k, \quad y^{(k)}_3 = \mathscr{D}_k, \quad y^{(k)}_4 = \dot{\mathscr{D}}_k, \quad y^{(k)}_5 = 0,\\
y'^{(k)}_1 = \mathscr{B}'_k, \quad y'^{(k)}_2 = \dot{\mathscr{B}}'_k, \quad y'^{(k)}_3 = \mathscr{D}'_k, \quad y'^{(k)}_4 = \dot{\mathscr{D}}'_k, \quad y'^{(k)}_5 = \beta'_k,\\
A_{mn} = A_{nm} = \frac{1}{2}(s_m \alpha_n + s_n \alpha_m), \quad \text{for} \quad m, n = 1, 2, 3, 4, 5,\\
A_{m5} = 0 = A_{5m}, \quad m = 1, 2, 3, 4, 5,\\
P'_{mn} = s_m \alpha'_n + \alpha_m s'_n \neq P'_{nm}, \quad m, n = 1, 2, 3, 4, 5,\\
P'_{5m} = 0, \quad m = 1, 2, 3, 4, 5.
\end{gathered} \tag{5.7}$$

The inequality (5.6) must be satisfied for all independent processes, which implies that each of the following terms must separately be nonnegative:

(5.8) $$A_{mn} y_m^{(k)} y_n^{(k)} \geqslant 0,$$

(5.9) $$\int_{v-\sigma} P'_{mn} y_m^{(k)} y_n'^{(k)} dv' \geqslant 0,$$

(5.10) $$\left(\int_{v-\sigma} s'_m y_m'^{(k)} dv'\right)\left(\int_{v-\sigma} \alpha'_m y_m'^{(k)} dv'\right) \geqslant 0.$$

First, we consider the thermodynamic restrictions arising from the inequality (5.8) which is free from the nonlocal terms. Equation (5.8) can be satisfied for all independent electromagnetic processes if the coefficients A_{mn} of the quadratic form satisfy the following restrictions:

(5.11) $$A_{\underline{mm}} \geqslant 0, \quad m = 1, 2, 3, 4 \text{ (no sum on } m);$$

(5.12) $$A_{12} \leqslant 0, \quad A_{34} \leqslant 0;$$

(5.13) $$A_{13} = 0 = A_{23}, \quad A_{14} = 0 = A_{24},$$

where the underscored subscripts in Eq. (5.11) suspend the summation convention. The inequality (5.11) readily follows from the well-known fact that if a quadratic form with a symmetric matrix (A_{mn}) is positive definite, then its diagonal elements $A_{\underline{mm}}$ must be nonnegative. Equation (5.12) can be established as follows. It is readily seen that A_{12} and A_{34} are, respectively, the coefficients of the material time-rates of $\frac{1}{2}\mathscr{B}_k^2$ and $\frac{1}{2}\mathscr{D}_k^2$ in the quadratic form in Eq. (5.8), which represent the magnetic and electric energies stored in the medium. Under appropriate external electromagnetic sources, consistent with given physical situations, the electromagnetic energy stored within the medium must be dissipating so that

(5.14) $$\dot{\overline{\left(\frac{1}{2}\mathscr{B}_k^2\right)}} \leqslant 0, \quad \dot{\overline{\left(\frac{1}{2}\mathscr{D}_k^2\right)}} \leqslant 0.$$

Hence the quadratic form in Eq. (5.8) cannot be maintained positive definite for *all* independent variations of the electromagnetic variables present unless the coefficients of the terms in Eq. (5.14), namely A_{12} and A_{34} are negative or zero which yields Eq. (5.12). The remaining coefficients $A_{13}, A_{23}, A_{14}, A_{24}$ must clearly vanish for otherwise one can, by choosing $\mathscr{B}_k$ and $\mathscr{D}_k$ and their material timerates arbitrarily, show that the nonnegative character of the quadratic form in Eq. (5.8) is violated. Hence Eq. (5.13) is established.

In view of Eq. (5.11), Eq. (5.13) is satisfied if either

(5.15) $$\begin{aligned} \alpha_1 &= 0 = s_1, \\ \alpha_2 &= 0 = s_2, \end{aligned} \quad \text{or} \quad \begin{aligned} s_3 &= 0 = \alpha_3, \\ s_4 &= 0 = \alpha_4. \end{aligned}$$

Utilizing Ohm's law and the fact that $\underline{\mathscr{E}}$ and $\underline{\mathscr{D}}$ are related to each other in a linear fashion in the classical constitutive theory, we take

(5.16) $$\alpha_1 = 0 = s_1, \quad \alpha_2 = 0 = s_2$$

and hence α_3, s_3, α_4 and s_4 cannot be zero in Eq. (5.15), thus dropping the local dependency of $\mathbf{J}^c$ and $\underline{\mathscr{E}}$ on $\underline{\mathscr{B}}$ and $\dot{\underline{\mathscr{B}}}$.

Next, in order to analyze the integral terms in Eq. (5.9) and obtain the consequent thermodynamic restrictions arising from nonlocality, we need only appeal to the continuity of the electromagnetic field quantities and the axiom of attenuating neighborhoods. First we note that each of the terms in the quadratic form in Eq. (5.9) consists of a product of two fields, one a function purely of a given local point $\mathbf{x}$ and the other a function of an arbitrary surrounding point $\mathbf{x}'$, in the body. Thus the total fields can vary independently of each other. Hence the inequality (5.9) cannot be maintained for all independent variations of these "local" and "nonlocal" field quantities and their material time-rates unless each term in the quadratic form is separately non-negative. Thus, for example, the term

$$\int_{v-\sigma} P'_{33} y_3^{(k)} y_3'^{(k)} dv' \equiv \int_{v-\sigma} (s_3 \alpha'_3 + \alpha_3 s'_3) \mathscr{D}_k \mathscr{D}'_k dv', \tag{5.17}$$

draws its main contribution from a small neighborhood about $\mathbf{x}$ since α'_3 and s'_3 both satisfy the axiom of attenuating neighborhoods. Also within a small neighborhood the continuity of $\underline{\mathscr{D}}$ requires that $\underline{\mathscr{D}} \cdot \underline{\mathscr{D}}' \geqslant 0$ so that Eq. (5.17) will be nonnegative if

$$P'_{33} = s_3 \alpha'_3 + \alpha_3 s'_3 \geqslant 0. \tag{5.18}$$

For such a terms as

$$\int_{v-\sigma} P'_{34} y_3^{(k)} y_4'^{(k)} dv' \equiv \int_{v-\sigma} (s_3 \alpha'_4 + \alpha_3 s'_4) \mathscr{D}_k \dot{\mathscr{D}}'_k dv', \tag{5.19}$$

we expand $\dot{\underline{\mathscr{D}}}'$ into a Taylor series about $\mathbf{x}$,

$$\dot{\underline{\mathscr{D}}}' = \dot{\underline{\mathscr{D}}} + (\mathbf{x}' - \mathbf{x}) \cdot \nabla \dot{\underline{\mathscr{D}}} + O(\|\mathbf{x}' - \mathbf{x}\|^2) \tag{5.20}$$

and ignore terms of $0(\|\mathbf{x}' - \mathbf{x}\|^2)$ or higher. Hence

$$\dot{\underline{\mathscr{D}}}' \cdot \underline{\mathscr{D}} \approx \dot{\underline{\mathscr{D}}} \cdot \underline{\mathscr{D}} + [(\mathbf{x}' - \mathbf{x}) \cdot \nabla \dot{\underline{\mathscr{D}}}] \cdot \underline{\mathscr{D}}. \tag{5.21}$$

It is physically reasonable to assume under nonrelativistic conditions that for a wide class of materials, the spatial gradients of material rates of fields such as $\underline{\mathscr{D}}$ within a small neighoorhood of $\mathbf{x}$, are small enough to permit the neglect of their products with $\|\mathbf{x}' - \mathbf{x}\|$ in comparison with $\dot{\underline{\mathscr{D}}}$. Thus using this assumption we have, for example,

$$\dot{\underline{\mathscr{D}}}' \cdot \underline{\mathscr{D}} \approx [(\mathbf{x}'\mathbf{x}) \cdot \nabla \dot{\underline{\mathscr{D}}}] \cdot \underline{\mathscr{D}} \ll \dot{\underline{\mathscr{D}}} \cdot \underline{\mathscr{D}} = \overline{\left(\frac{1}{2} \underline{\mathscr{D}}^2\right)}^{\bullet} \leqslant 0, \tag{5.22}$$

which implies that Eq. (5.19) will be nonnegative if

$$P'_{34} = s_3 \alpha'_4 + \alpha_3 s'_4 \leqslant 0.$$

Applying similar analysis to all of the integral terms in Eq. (5.9), we obtain

$$P'_{33} \geqslant 0, \quad P'_{44} \geqslant 0, \quad P'_{34} \leqslant 0, \quad P'_{43} \leqslant 0,$$
$$P'_{31} = 0 = P'_{32}, \quad P'_{41} = 0 = P'_{42}, \quad P'_{35} = 0 = P'_{45}.$$

Since s_3, α_3, s_4 and α_4 cannot be zero as established earlier, we have

$$s'_1 = 0 = \alpha'_1, \quad s'_2 = 0 = a'_2, \quad s'_5 = 0 = \alpha'_5. \tag{5.23}$$

Thus in view of Eqs. (5.16) and (5.23), we have the necessary thermodynamic restrictions on P'_{mn}:

$$P'_{33} \geqslant 0, \quad P'_{44} \geqslant 0, \quad P'_{34} \leqslant 0, \quad P'_{43} \leqslant 0 \tag{5.24}$$

and all other P'_{mn} are zero. Equation (5.24) implies the dropping of the dependency on the magnetic field as well as the dependency on the nonlocal mechanical measure β'.

Now to study the implications of Eq. (5.10), the product of the two integrals can be reduced to an iterated double integral and again applying the axiom of attenuating neighborhoods as well as the continuity of the electromagnetic variables, we find after somewhat lengthy, but simple calculations, that

$$s'_3\alpha'_3 \geqslant 0, \quad s'_4\alpha'_4 \geqslant 0, \quad s'_3\alpha'_4 + \alpha'_3 s'_4 \leqslant 0. \tag{5.25}$$

Thus, finally the electromagnetic constitutive equations in the linear theory can be written as

$$\mathscr{E}_k = \varrho\Big[s_3\mathscr{D}_k + s_4\dot{\mathscr{D}}_k + \int\limits_{v-\sigma} (s'_3\mathscr{D}'_k + s'_4\dot{\mathscr{D}}'_k)dv'\Big], \tag{5.26}$$

$$J^c_k = \alpha_3\mathscr{D}_k + \alpha_4\dot{\mathscr{D}}_k + \int\limits_{v-\sigma} (\alpha'_3\mathscr{D}'_k + \alpha'_4\dot{\mathscr{D}}'_k)dv' \tag{5.27}$$

subject to the thermodynamic restrictions (5.11)—(5.13), (5.16), (5.23) and (5.24), which finally reduce to

$$s_3\alpha_3 \geqslant 0, \quad s'_3\alpha'_3 \geqslant 0, \quad s_4\alpha_4 \geqslant 0, \quad s'_4\alpha'_4 \geqslant 0. \tag{5.28}$$

Thus Eqs. (5.3) and (5.28) give the complete set of thermodynamic restrictions on the material coefficients of the nonlocal Stokesian fluids with electromechanical constitution.

6. Field equations

In order to obtain the field equations of the nonlocal fluid mechanics governing the flow of electromagnetic fluids, we must compute the stress divergence $t_{kl,k}$ using Eq. (4.26) and substitute the result into the balance of linear momentum $(2.2)_1$. For this purpose, from Eq. (4.26) we have

$$t_{kl,k} = -\pi_{,l} - {}_Mh_{,l} + \lambda_v d_{kk,l} + 2\mu_v d_{kl,k} + {}_Mt_{kl,k} + \int\limits_{v-\sigma} (\sigma'_{,l} + \lambda'_{v,l}d'_{kk} + 2\mu'_{v,k}d'_{kl})dv', \tag{6.1}$$

where, using Eqs. (4.35) and (4.36) in Eq. $(4.19)_2$ gives

$$\begin{aligned} {}_Mt_{kl,k} = \varrho\Big[&s_3(\mathscr{D}_k\mathscr{D}_l)_{,k} + s_4(\dot{\mathscr{D}}_k\mathscr{D}_l)_{,k} + (r_k\mathscr{B}_l)_{,k} + \tau_{kl,k} + \mathscr{D}_{l,k}\int\limits_{v-\sigma} (s'_2\mathscr{D}'_k + s'_4\dot{\mathscr{D}}'_k)dv' \\ &+ \mathscr{D}_l\int\limits_{v-\sigma} (s'_{3,k}\mathscr{D}'_k + s'_{4,k}\dot{\mathscr{D}}'_k)dv' + \mathscr{B}_{l,k}\int\limits_{v-\sigma} r'_k dv' + \mathscr{B}_l\int\limits_{v-\sigma} r'_{k,k}dv'\Big], \end{aligned} \tag{6.2}$$

where

$$\begin{aligned} \tau_{kl} &= \frac{1}{c}[\mathscr{H}_k(\mathbf{v}\times\mathbf{E})_l - \mathscr{E}_k(\mathbf{v}\times\mathbf{H})_l], \\ r_k &= b_1\mathscr{B}_k + b_2\dot{\mathscr{B}}_k + b_3\mathscr{D}_k + b_4\dot{\mathscr{D}}_k, \\ r'_k &= b'_1\mathscr{B}'_k + b'_2\dot{\mathscr{B}}'_k + b'_3\mathscr{D}'_k + b'_4\dot{\mathscr{D}}'_k + b'_5\beta'_k. \end{aligned} \tag{6.3}$$

Since σ', λ'_v, μ'_v, s'_3, s'_4, and b'_l are functions of $||\mathbf{x}'-\mathbf{x}||$, we may write

$$\sigma'_{,l} = \frac{\partial \sigma'}{\partial x^l} = -\frac{\partial \sigma'}{\partial x'^l} \equiv -\sigma_{,l'}, \quad \lambda'_{v,l} \equiv -\lambda_{vl'}, \quad \text{etc.} \tag{6.4}$$

Using Eqs. (6.2) and (6.4) in Eq. (6.1) we have

$$\begin{aligned} t_{kl,k} = & -\pi_{,l} - {}_M h_{,l} + \lambda_v d_{kk,l} + 2\mu_v d_{kl,k} + \varrho[s_3(\mathscr{D}_k\mathscr{D}_l)_{,k} + s_4(\dot{\mathscr{D}}_k\mathscr{D}_l)_{,k} \\ & + (r_k\mathscr{B}l)_{,k} + \tau_{kl,k}] + \int_{v-\sigma} \{\lambda'_v d_{kk,l} + 2\mu'_v d_{kl,k} + \varrho[\mathscr{D}_l(s'_3\mathscr{D}_{k,k} + s'_4\dot{\mathscr{D}}_{k,k'}) \\ & + \mathscr{D}_{l,k}(s'_3\mathscr{D}'_k + s'_4\dot{\mathscr{D}}'_k) + \mathscr{B}_{k,k}r'_k]\}dv' - \int_{v-\varrho} [\sigma'\delta_{kl} + \lambda'_v d'_{rr}\delta_{kl} + 2\mu'_v d'_{kl} \\ & + \varrho(\mathscr{D}_l s'_3\mathscr{D}'_k + \mathscr{D}_l s_4\mathscr{D}'_k + \mathscr{B}_l r'_k)]_{,k} dv'. \end{aligned} \tag{6.5}$$

Applying the generalized Green-Gauss theorem to the last integral in Eq. (6.5), we may write this term as

$$-\oint_{\mathscr{S}-\sigma} s'_{kl}da'_k + \int_{\sigma} [s'_{kl}]da'_k, \tag{6.6}$$

where $\mathscr{S}$ is the bounding surface of the body and

$$s'_{kl} = s'_{kl}(\mathbf{x}', \mathbf{x}) = \sigma'\delta_{kl} + \lambda'_v v_{r,r'}\delta_{kl} + \mu'_v(v_{k,l'} + v_{l,k'}) + \varrho(\mathscr{D}_l s'_3\mathscr{D}'_k + \mathscr{D}_l s'_4\mathscr{D}'_k + \mathscr{B}_l r'_k). \tag{6.7}$$

Recalling that $\hat{\varrho} = 0$ in Eq. (2.1), and substituting Eq. (6.5) into Eq. (2.2) yields the equations of motion for nonlocal electromagnetic fluids:

$$\frac{\partial \varrho}{\partial t} + (\varrho v_k)_{,k} = 0, \tag{6.8}$$

$$\begin{aligned} & -(\pi + {}_M h)_{,l} + (\lambda_v + \mu_v)v_{k,kl} + \mu_v v_{l,kk} + \varrho[s_3(\mathscr{D}_k\mathscr{D}_l)_{,k} + s_4(\mathscr{D}_k\mathscr{D}_l)_{,k} + (r_k\mathscr{B}_l)_{,k} + \tau_{kl,k}] \\ & + \int_{v-\sigma} \{(\lambda'_v + \mu'_v)v_{k',k'l'} + \mu' v_{l',k'k'} + \varrho[\mathscr{D}_l(s'_3\mathscr{D}_{k,k'} + s'_4\dot{\mathscr{D}}_{k,k'}) + \mathscr{D}_{l,k}(s'_3\mathscr{D}'_k + s'_4\dot{\mathscr{D}}'_k) \\ & + \mathscr{B}_{l,k}r'_k]\}dv' - \oint_{\mathscr{S}-\sigma} s'_{kl}da'_k + \int_{\sigma} [s'^{kl}_{r}]da'_k + \varrho(f_k - \dot{v}_k - \dot{g}_k) - \varrho\hat{f}_k = 0. \end{aligned} \tag{6.9}$$

It is worth noting that the surface effects, including surface tension, surface viscosities, and surface energies, are included in the expression (6.7) and hence in Eq. (6.9).

In order to derive the electromagnetic field equations, the constitutive equations relating $(\underline{\mathscr{H}}, \underline{\mathscr{B}})$ and $(\underline{\mathscr{E}}, \underline{\mathscr{D}})$ must be used. From Eqs. (4.35) and (5.26) we have

$$\begin{aligned} \underline{\mathscr{H}} &= \varrho\Big[b_1\underline{\mathscr{B}} + \underline{\mathscr{A}} + \int_{v-\sigma} \mathbf{r}'dv'\Big], \\ \underline{\mathscr{E}} &= \varrho\Big[s_3\underline{\mathscr{D}} + s_4\underline{\dot{\mathscr{D}}} + \int_{v-\sigma} \underline{\mathscr{A}}'dv'\Big], \end{aligned} \tag{6.10}$$

where $\mathbf{r}'$ is given by Eq. $(6.3)_3$ and

$$\underline{\mathscr{A}} = b_2\underline{\dot{\mathscr{B}}} + b_3\underline{\mathscr{D}} + b_4\underline{\dot{\mathscr{D}}}, \quad \underline{\mathscr{A}}' = s'_3\underline{\mathscr{D}}' + s'_4\underline{\mathscr{D}}'. \tag{6.11}$$

Now with the use of the notation (2.11), applying the "v^2/c^2 approximation" to each of the equations $(2.5)_1$, $(2.6)_1$, and (2.7) wherein Eq. $(2.12)_3$ is used to substitute for $\mathbf{D}$ in terms of $\underline{\mathscr{D}}$, we obtain

$$\nabla \times \underline{\mathscr{H}} = \frac{1}{c}(\underline{\mathscr{I}} + \underline{\hat{\mathscr{I}}}) + \frac{1}{c}\left(\frac{\partial \underline{\mathscr{D}}}{\partial t} + \underline{\check{\mathscr{D}}}\right), \tag{6.12}$$

(6.13) $$\nabla\times\underline{\mathscr{E}} = \frac{1}{c}\,\hat{\mathbf{b}} - \frac{1}{c}\left[\frac{\partial\underline{\mathscr{B}}}{\partial t} + \check{\underline{\mathscr{B}}}\right],$$

(6.14) $$\nabla\cdot\underline{\mathscr{D}} = q + \hat{q} - \frac{1}{c}\,\nabla\cdot(\mathbf{v}\times\underline{\mathscr{H}}).$$

Taking the curl of Eq. (6.13), substituting for $\underline{\mathscr{E}}$ from Eq. $(6.10)_2$, and using Eq. (6.14), we obtain

(6.15) $$\varrho s_3\nabla(q+\hat{q}) - \frac{\varrho s_3}{c}\,\nabla[\nabla\cdot(\mathbf{v}\times\underline{\mathscr{H}})] - \varrho s_3\nabla^2\underline{\mathscr{D}} + \varrho s_4\nabla\times\nabla\times\dot{\underline{\mathscr{D}}}$$
$$+\varrho\int_{v-\sigma}\nabla\times\nabla\times\underline{\mathscr{A}}'dv = \frac{1}{c}\,\nabla\times\hat{\mathbf{b}} - \frac{1}{c}\,\frac{\partial}{\partial t}\,(\nabla+\underline{\mathscr{B}}) - \frac{1}{c}\,\nabla\times\check{\underline{\mathscr{B}}}.$$

The expression for the second term on the right side of Eq. (6.15) can be obtained by first substituting for $\underline{\mathscr{H}}$ from Eq. $(6.10)_1$ into Eq. (6.12) and taking $\partial/\partial t$ of the result. Then upon substituting the resulting expression for $\frac{\partial}{\partial t}(\nabla\times\underline{\mathscr{B}})$ into Eq. (6.15), we obtain

(6.16) $$\nabla^2\underline{\mathscr{D}} - \alpha^2\,\frac{\partial^2\underline{\mathscr{D}}}{\partial t^2} = \nabla(q+\hat{q}) + \frac{s_4}{s_3}\,\nabla\times\nabla\times\dot{\underline{\mathscr{D}}} + \frac{1}{\varrho c s_3}\,\nabla + \check{\underline{\mathscr{B}}} - \frac{1}{\varrho c s_3}\,\nabla\times\hat{\mathbf{b}}$$
$$+\frac{1}{\varrho s_3}\,\nabla + \nabla\times\int_{v-\sigma} + \underline{\mathscr{A}}'dv' - c\alpha^2\frac{\partial}{\partial t}\left[\nabla\times\underline{\mathscr{A}}\times\nabla\times\int_{v-\sigma}\mathbf{r}'dv' - \frac{v}{c}\,\check{\underline{\mathscr{D}}} - \frac{1}{c}\,\underline{\mathscr{J}} - \frac{1}{c}\,\hat{\underline{\mathscr{J}}}\right].$$

This is the governing equation for the field $\underline{\mathscr{D}}$ where

(6.17) $$\alpha^2 = 1/\varrho^2c^2b_1s_3.$$

By following an analogous procedure one can show that the governing equation for the field $\underline{\mathscr{B}}$ is

(6.18) $$\nabla^2\underline{\mathscr{B}} - \alpha^2\,\frac{\partial^2\underline{\mathscr{B}}}{\partial t^2} = \nabla\hat{m} + \frac{1}{\varrho b_1}\,\nabla\times\nabla\times\underline{\mathscr{A}} - \frac{1}{\varrho c b_1}\,\nabla\times\check{\underline{\mathscr{D}}} - \frac{1}{\varrho c b_1}\,\nabla\times\underline{\mathscr{J}}$$
$$-\frac{1}{\varrho c b_1}\,\nabla\times\hat{\underline{\mathscr{J}}} + \frac{1}{\varrho b_1}\,\nabla\times\nabla\times\int_{v-\sigma}\mathbf{r}'dv'$$
$$+c\alpha^2\,\frac{\partial}{\partial t}\left[\varrho s_4\nabla\times\dot{\underline{\mathscr{D}}} + \nabla\times\int_{v-\sigma}\underline{\mathscr{A}}'dv + \frac{1}{c}\,\check{\underline{\mathscr{B}}} - \frac{1}{c}\,\hat{\mathbf{b}}\right].$$

Thus along with Eqs. (6.8) and (6.9), Eqs. (6.16) and (6.18) yield the complete set of field equations governing the flow of nonlocal electromagnetic fluids.

The boundary conditions are obtained by setting $\varrho = 0$ in Eq. $(2.1)_2$ and using Eqs. $(2.5)_2$ through $(2.9)_2$. Thus, on the moving surface of discontinuity σ we require the following jump conditions:

(6.19) $$[\varrho(\mathbf{v}-\mathbf{u})]\cdot\mathbf{n} = 0, \quad \left[\mathbf{E} + \frac{1}{c}\,\mathbf{u}\times\mathbf{B} + \hat{\mathbf{E}}\right]\times\mathbf{n} = \mathbf{0},$$
$$\left[\mathbf{H} - \frac{1}{c}\,\mathbf{u}\times\mathbf{D} + \hat{\mathbf{H}}\right]\times\mathbf{n} = \mathbf{0}, \quad [\mathbf{D}+\hat{\mathbf{D}}]\cdot\mathbf{n} = 0,$$
$$[\mathbf{B}+\hat{\mathbf{B}}]\cdot\mathbf{n} = 0, \quad [\mathscr{J}+\hat{\mathbf{\Sigma}}]\cdot\mathbf{n} = 0.$$

We note that if all the electromagnetic terms and nonlocal terms are set to zero, Eq. (6.9) is nothing more than the classical Navier–Stokes equations. Furthermore, we obtain the classical Maxwell's equations for free space, in the rest frame, from Eqs. (6.16) and (6.18) by setting all of the nonlocal terms to zero and using the classical constitutive equations for (**H**, **B**) and (**E**, **D**).

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DEPARTMENT OF MATHEMATICS
OREGON STATE UNIVERSITY, CORVALLIS, OREGON, USA.

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