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One-dimensional shock waves in magnetized electrically conducting elastic materials

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IN THIS PAPER we consider the behaviour of plane shock waves in elastic media of finite electrical conductivity which are subjected to a magnetic field whose direction lies in the plane of the wave. The equation which governs the amplitude of shock waves is derived and its implications are examined in detail. Finally, the results which hold when the amplitude of the shock becomes infinitesimally small are also discussed.

W pracy rozważono zachowanie się płaskich fal uderzeniowych w ciałach sprężystych o skończonej przewodności elektrycznej poddanych działaniu pola magnetycznego, którego kierunek leży w płaszczyźnie fal. Wyprowadzono równanie rządzące amplitudą fal uderzeniowych i rozważono jego implikacje. Przedyskutowano również wyniki odpowiadające przypadkowi, gdy amplitudy fal uderzeniowych stają się infinitezymalne.

В работе рассмотрено поведение плоских ударных волн в упругих телах с конечной электропроводностью, подвергнутых действию магнитного поля, направление которого лежит в плоскости волн. Выведено уравнение описывающее амплитуды ударных волн и рассмотрены его следствия. Обсуждены тоже результаты отвечающие случаю, когда амплитуды ударных волн становятся инфинитезимальными.

1. Introduction

IN THIS PAPER we examine the behaviour of plane shock waves in elastic nonmagnetic media of finite electrical conductivity. We assume that the material is uniformly magnetized ahead of the shock wave and that the magnetic induction vector lies in the plane of the wave.

After dispensing with preliminaries, we derive an expression for the intrinsic velocity of the shock. We find that the intrinsic velocity is not influenced by the magnetic induction field. The differential equation which the amplitude of the shock must obey is derived. We find that the evolutionary behaviour of the amplitude of the shock depends on the relative magnitudes of the jump in the deformation gradient and the quality λ^* which we call the critical jump in the deformation gradient, and that λ^* depends on the shock amplitude as well as on the mechanical and electromagnetic conditions prevailing immediately ahead of the wavefront. The implications of the amplitude equation on the propagation of compressive shocks are studied in detail. Finally, we apply our results to the study of the propagation of shock waves of infinitesimal amplitude and we show that the amplitudes of such waves decay exponentially as the waves propagate.

2. Basic equations and formulae

Let (X, Y, Z) be the Cartesian coordinates of a material point of a homogeneous elastic electrically conducting body in a fixed reference configuration with mass density ϱ_0 We assume that the motion of the body is one-dimensional and may be described by a scalar function $x = \chi(X, t)$ which gives the location $\mathbf{x} = (x, Y, Z)$ at time t of the material point $\mathbf{X} = (X, Y, Z)$.

We let the vectors $\mathbf{h}, \mathbf{b}, \mathbf{e}$ and $\mathbf{j}$ denote the magnetic field, magnetic induction field, electric field and conduction current vectors, respectively, at the spatial point $\mathbf{x}$ at time t. We assume that the material is nonmagnetic so that $\mathbf{b} = \varkappa \mathbf{h}$, where $\varkappa$ is a constant. The deformation gradient at the material point $\mathbf{X}$ has the representation

$$F = \begin{bmatrix} F & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad F = \frac{\partial \chi}{\partial X}. \tag{2.1}$$

Let us now define the vectors $\mathbf{H}, \mathbf{B}, \underline{\mathscr{E}}$ and $\underline{\mathscr{J}}$ by the relations

$$\begin{aligned} \mathbf{H} &= (Fh_1, h_2, h_3), \quad \mathbf{B} = (b, Fb, Fb_3), \\ \underline{\mathscr{E}} &= (F\varepsilon_1, \varepsilon_2, \varepsilon_3), \quad \underline{\mathscr{J}} = (j_1, Fj_2, Fj_3), \end{aligned} \tag{2.2}$$

where

$$\boldsymbol{\varepsilon} = \mathbf{e} + \mathbf{v} \times \mathbf{b} \tag{2.3}$$

and $\mathbf{v} = (\dot{x}, 0, 0)$ is the velocity of the point $\mathbf{X}$. Here and in what follows a superposed dot denotes material time differentiation.

We assume that the electromagnetic response of the material is quasi-static and that all electromagnetic variables are functions of X and t only. It follows that Maxwell's equations assume the forms, see e.g. McCarthy [1],

$$\begin{aligned} &B_1(X_\alpha, t) = B_1(X_\beta, t), \quad \dot{B}_1(X, t) = 0, \\ &\mathscr{J}_1(X, t) = 0, \\ &H_2(X_\beta, t) - H_2(X_\alpha, t) = \int_{X_\alpha}^{X_\beta} \mathscr{J}_3(X, t)\,dX, \\ &H_3(X_\beta, t) - H_3(X_\alpha, t) = -\int_{X_\alpha}^{X_\beta} \mathscr{J}_2(X, t)\,dX, \\ &-\frac{d}{dt}\int_{X_\alpha}^{X_\beta} B_3(x, t)\,dX = \mathscr{E}_2(X_\beta, t) - \mathscr{E}_2(X_\alpha, t), \\ &\frac{d}{dt}\int_{X_\alpha}^{X_\beta} B_2(X, t)\,dX = \mathscr{E}_3(X_\beta, t) - \mathscr{E}_3(X_\alpha, t). \end{aligned} \tag{2.4}$$

Equations (2.4) hold at all times t and for all X_α, X_β and X in $\mathscr{R}$.

It results from Eq. $(2.4)_{1,2}$ that B_1 is a constant. In what follows we assume that B_1 vanishes identically at all points of the material, i.e. we assume that the magnetic induction field is transverse.

In the absence of external body force and free charge, the law of balance of momentum takes the form

$$\frac{d}{dt}\int_{X_\alpha}^{X_\beta} \varrho_0 \dot{x}\,dX = T(X_\beta, t) - T(X_\alpha, t) + \int_{X_\alpha}^{X_\beta} f(x, t)\,dX, \tag{2.5}$$

where

$$f = F^{-1}(\mathcal{G}_2 B_3 - \mathcal{G}_3 B_2) \tag{2.6}$$

is the electromagnetic body force density per unit volume and $T = T(X, t)$ is the stress.

Finally, we assume that the response of the material is described by the constitutive equations

$$\begin{aligned} T &= T(X, t) = \hat{T}(F), \\ \underline{\mathcal{G}} &= \underline{\mathcal{G}}(X, t) = \hat{\underline{\mathcal{G}}}(F, \underline{\mathcal{E}}) \end{aligned} \tag{2.7}$$

and we assume that $\hat{T}(\cdot)$ is of class C^2, while $\hat{\underline{\mathcal{G}}}(\cdot, \cdot)$ is of class C^1.

3. General properties of shock waves

We assume that the motion contains a shock wave moving with intrinsic velocity

$$U(t) = \frac{d}{dt}\hat{X}(t) > 0, \tag{3.1}$$

where $\hat{X}(t)$ is the material point at which the wave is to be found at time t. Thus we assume that $\chi(\cdot, \cdot)$ is a continuous function everywhere; while $\dot{x}$ and F and their derivatives suffer jump discontinuities across the shock; they are continuous everywhere else.

The kinematical condition of compatibility

$$\frac{d}{dt}[g] = [\dot{g}] + U\left[\frac{\partial g}{\partial X}\right] \tag{3.2}$$

with $g = x(\cdot, \cdot)$ implies that

$$[F] = -U[\dot{x}]. \tag{3.3}$$

Here $[g]$ denotes the jump in the function g, i.e.

$$[g] = g^- - g^+ \quad \text{with} \quad g^\pm = \lim_{X \to Y(t)\pm} g(X, t).$$

Since $U(t) > 0$, g^- and g^+ are, respectively, the limiting values of g immediately behind and just in front of the wave.

Equations $(2.4)_3$–$(2.4)_6$ imply that for all $X \neq \hat{X}(t)$

$$\begin{aligned} \frac{\partial H_2}{\partial X} &= \mathcal{J}_3, & \frac{\partial H_3}{\partial H} &= -\mathcal{G}_2, \\ \frac{\partial \mathcal{E}_2}{\partial X} &= -\dot{B}_3, & \frac{\partial \mathcal{E}_3}{\partial X} &= \dot{B}_2, \end{aligned} \tag{3.4}$$

and across the shock

$$(3.5)\qquad \begin{aligned} [H_2] &= 0, & [H_3] &= 0, \\ [\mathscr{E}_2] &= U[B_3], & [\mathscr{E}_3] &= -U[B_2], \\ \left[\frac{\partial H_2}{\partial X}\right] &= [\mathscr{G}_3], & \left[\frac{\partial H_3}{\partial X}\right] &= -[\mathscr{I}_2], \\ \left[\frac{\partial \mathscr{E}_2}{\partial X}\right] &= -g[\dot{B}_3], & \left[\frac{\partial \mathscr{E}_3}{\partial X}\right] &= [\dot{B}_2]. \end{aligned}$$

When Eqs. $(2.2)_{1,2}$ and $(3.4)_{1,2}$ are used in Eq. (2.6), we find that Eq. (2.5) may be rewritten in the form

$$(3.6)\qquad \frac{d}{dt}\int\limits_{X_\alpha}^{X_\beta} \varrho_0 \dot{x}\, dX = S(X_\beta, t) - S(X_\alpha, t),$$

where

$$(3.7)\qquad S = T - \frac{\varkappa}{2}(H_2^2 + H_3^2).$$

It follows from Eq. (3.6) that when $X \neq \hat{X}(t)$

$$(3.8)\qquad \frac{\partial S}{\partial X} = \varrho_0 \ddot{x}$$

and, when Eqs. $(3.5)_{1,2}$ are used, it follows that across the shock we have

$$(3.9)\qquad \begin{aligned} [S] = [T] &= -\varrho_0 U[\dot{x}], \\ \left[\frac{\partial S}{\partial X}\right] &= \varrho_0 [\ddot{x}]. \end{aligned}$$

Equation $(3.9)_1$ with Eq. (3.3) yields the well-known formula

$$(3.10)\qquad U^2 = \frac{[T]}{\varrho_0 [F]}$$

for the intrinsic velocity of the shock. Equation $(3.9)_2$, together with the condition (3.2) with $g = \dot{x}$ and $g = F$, respectively, implies that

$$(3.11)\qquad 2U\frac{d}{dt}[F] + [F]\frac{dU}{dt} = U^2\left[\frac{\partial F}{\partial X}\right] - \frac{1}{\varrho_0}\left[\frac{\partial S}{\partial X}\right].$$

This is the equation which the amplitude of the shock must obey.

4. The shock amplitude equation

Here we shall derive the equation which governs the amplitude of a shock wave propagating in a magnetised electrically conducting elastic material. It follows from Eqs. (3.7) and $(3.4)_{1,2}$ and the definition (3.7) that

$$(4.1)\qquad \frac{\partial S}{\partial X} = E\frac{\partial F}{\partial X} - \frac{1}{\varkappa}\{H_2 \mathscr{G}_3 - H_3 \mathscr{G}_2\},$$

for all $X \neq \check{X}(t)$, where

$$E = \hat{E}(F) = \frac{\partial \hat{T}(F)}{\partial F}. \tag{4.2}$$

Since the jump in the product $\Phi\Psi$ may be written in the form

$$[\Phi\Psi] = \Phi^- [\Psi] + \Psi^+ [\Phi], \tag{4.3}$$

it follows from Eqs. (4.1) and $(3.5)_{1,2}$ that

$$\left[\frac{\partial S}{\partial X}\right] = E^- \left[\frac{\partial F}{\partial X}\right] + [E]\frac{\partial F^+}{\partial X} - \varkappa\{H_2[\mathscr{G}_3] - H_3[\mathscr{G}_2]\}. \tag{4.4}$$

Differentiation of Eq. (3.10) yields the formula

$$\frac{dU}{dt} = \frac{1}{2\varrho_0 U[F]}(E^- - \varrho_0 U^2) + \frac{[E]}{U[F]}\frac{dF^+}{dt}. \tag{4.5}$$

Finally, when Eqs. (3.11), (4.4) and (4.5) are combined, we have

THEOREM. *The amplitude of a shock wave propagating in a magnetised electrically conducting elastic body obeys the equation*

$$\frac{d}{dt}[F] = \frac{(E^- - \varrho_0 U^2)}{2\varrho_0 U\left\{+1\,\dfrac{E^- - \varrho_0 U^2}{4\varrho_0 U_2}\right\}}\left\{\lambda^* - \left[\frac{\partial F}{\partial X}\right]\right\}, \tag{4.6}$$

where

$$\lambda^* = \frac{-1}{2(E - \varrho_0 U^2)}\left\{[E]\frac{\dot{F}^+}{U} + 3[E]\left(\frac{\partial F}{\partial X}\right)^+ - 2\chi(H_2[\mathscr{G}_3] - H_3[\mathscr{G}_2])\right\}. \tag{4.7}$$

Equation (4.6) has the same form as that which is satisfied by the amplitude of shock waves in various other theories [2–4]. However, the similarity between the equation given here and that which arises in the aforementioned works is purely superficial. In its present form, the growth equation is quite complicated; and, in general, it is not possible to deduce any information from it regarding the behaviour of a shock without adopting additional assumptions. In the following section we consider the implications of Eq. (4.7) in a particular case.

5. The behaviour of some particular shock waves

We now consider the implications of Eq. (4.9) on the behaviour of certain types of shock waves. Consider a compressive shock wave propagating in a material which is initially in compression so that

$$F^+ < 1, \quad [F] < 0. \tag{5.1}$$

We assume that the stress-strain law (2.7) in compression is such that

$$\tilde{E} = \tilde{E}(F) = \frac{\partial^2 \hat{T}(F)}{\partial F^2} < 0 \quad \text{for all} \quad F < 1, \tag{5.2}$$

and that

$$E = \hat{E}(F) > 0. \tag{5.3}$$

It follows from Eqs. (3.10) and (5.1)—(5.3) that

$$E^- > \varrho_0 U^2, \tag{5.4}$$

$$[E] > 0. \tag{5.5}$$

Hence by Eqs. (4.17) and (5.4) we have

THEOREM. *Consider a compression shock propagating in an elastic material of finite electrical conductivity which is initially in compression and is subjected to a magnetic induction field which is initially in the plane of the wave. If the stress-strain law is such that* Eqs. (5.1)—(5.3) *hold, then at any instant*

$$\begin{aligned} \left[\frac{\partial F}{\partial X}\right] &< \lambda^* \leftrightarrow \frac{d}{dt}\,|[F]| < 0, \\ \left[\frac{\partial F}{\partial X}\right] &> \lambda^* \leftrightarrow \frac{d}{dt}\,|[F]| > 0, \\ \left[\frac{\partial F}{\partial X}\right] &= \lambda^* \leftrightarrow \frac{d}{dt}\,|[F]| = 0. \end{aligned} \tag{5.6}$$

Clearly, the formulae (5.6) have the usual form in that they state that whether a shock grows or decays depends on the relative values of the jump $[\partial F/\partial X]$ and the parameter λ^*, defined by Eq. (4.7). We call λ^* the critical jump in the deformation gradient for shock waves in transversely magnetized elastic media of finite electrical conductivity. It is clear from Eq. (4.7) that λ^* depends on the strength of the shock wave as well as on the mechanical and electromagnetic conditions prevailing immediately ahead of the wavefront.

Next, it follows from Eqs. $(3.5)_{2,3}$ and the definitions (2.2) that

$$[\varepsilon_2] = U[F]b_3, \quad [\varepsilon_3] = -U[F]b_2 \tag{5.7}$$

so that

$$[\boldsymbol{\epsilon}]\cdot\mathbf{b} = 0. \tag{5.8}$$

Thus the jump $[\boldsymbol{\epsilon}]$ in $\boldsymbol{\epsilon}$ lies in the plane of the wave and its direction is orthogonal to the original direction of the magnetic induction field. Of course, in view of Eqs. $(2.2)_1$ and $(3.5)_{1,2}$, neither the magnitude nor directions of the magnetic induction field is altered by the shock wave.

Let us now consider the implications of Eq. (4.6) when the material ahead of the shock wave is at rest in a uniform state of magnetization and deformation. In this case the critical jump in the deformation gradient, defined by Eq. (4.7), has the reduced form

$$\lambda^* = \frac{\varkappa}{(E^- - \varrho_0 U_2)}\{H_2[\mathscr{G}_3] - H_3[\mathscr{G}_2]\}. \tag{5.9}$$

The expression (5.9) clearly illustrates the critical role played by the applied magnetic induction field and the electrical conduction properties of the material on the evolutionary behaviour of shock waves propagating into a material which is initially at rest in a uniform state. If it is further assumed that the material is a homogeneous isotropic ohmic conductor of electricity in its natural state so that $\mathbf{j} = \sigma\boldsymbol{\epsilon}$ where $\sigma(>0)$ is the electrical conductivity of the material, then the expression (5.9) may be simplified still further.

6. Shock waves of infinitesimal amplitude

Here we consider the implications of Eq. (4.9) in the limit as $F^- \to F^+$, i.e. we consider the behaviour of a shock wave of infinitesimal amplitude. We assume that the material ahead of the wave is at rest in a uniform state of deformation and that the magnetic induction field is uniform and lies in the plane of the wave.

It follows from Eqs. $(2.7)_1$, (4.2) and (5.2) that

$$[T] = E_0[F] + \frac{1}{2}\tilde{E}_0[F]^2 + o(|[F]|)^2 \tag{6.1}$$

and

$$E^- = E_0 + \tilde{E}_0[F], \tag{6.2}$$

where

$$E_0 = \hat{E}(F^+), \quad \tilde{E}_0 = \tilde{E}(F^+). \tag{6.3}$$

The formulae (3.10) and (6.1) together imply that

$$\varrho_0 U^2 = \varrho_0 U_0^2 + \frac{1}{2}\tilde{E}_0[F], \tag{6.4}$$

where

$$\varrho_0 U_0^2 = E_0. \tag{6.5}$$

In view of Eqs. (6.2) and (6.4), we have

$$E^- - \varrho_0 U^2 = \frac{1}{2}E_0[F] + o(|[F]|). \tag{6.6}$$

Next we assume that $h_2 = h$ and $h_3 = 0$ and that the material is an isotropic ohmic conductor of electricity in its natural state. Thus, by Eqs. (5.9) and (6.6) we have

$$\lambda^* = \frac{-2U_0\sigma b^2 F^+}{\tilde{E}} \tag{6.7}$$

If we further assume that $|(\partial F/\partial X)^-| = 0([F])$, then the governing equation of the amplitude (4.6) reduces to

$$\frac{d}{dt}[F] = \frac{-\sigma b^2}{2\varrho_0}[F], \tag{6.8}$$

which may be readily integrated to give

$$[F](t) = [F](0)\exp\left\{\frac{-\sigma b^2}{2\varrho_0}t\right\}. \tag{6.9}$$

Since $\sigma > 0$, we see that the amplitude of an infinitesimal shock wave decays exponentially as it traverses the material. We observe that the rate of decay of the amplitude of an infinitesimal shock wave is precisely the same as that of an acceleration wave of infinitesimal amplitude or that of a plane progressive wave with high frequency harmonic time dependence and infinitesimal amplitude (see e.g. McCarthy, [5]).

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