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Plane strain elastoplastic consolidation of soil by finite elements
I. An elastoplastic consolidation theory

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THE SATURATED soil is treated as a multiphase material. The soil skeleton is assumed to be an elastoplastic material with isotropic hardening. A double-hardening model is proposed for which the effective stress-strain relationships are derived. The liquid phase is assumed to be linearly compressible. In the paper a finite element approach to the consolidation problem is presented and two extremes of soil behaviour, i.e. drained and undrained deformations, are discussed. The analysis is restricted to the plane strain conditions.

W przedstawionej pracy grunt traktowany jest jako ośrodek wielofazowy. Szkielet opisany jest przez model sprężysto-plastyczny z izotropowym wzmocnieniem, dla którego wyprowadzone są odpowiednie związki konstytutywne. Zakłada się ponadto, że ciecz wypełniająca pory jest liniowo ściśliwa. W pracy analizowane są dwa ekstremalne przypadki zachowań ośrodka, tj. zachowanie „z możliwością odpływu" (ang. drained) oraz „bez odpływu" (ang. undrained). Z kolei, szeroko dyskutowane jest jedno ze sformułowań metody elementów skończonych dotyczące zagadnienia konsolidacji w warunkach płaskiego stanu odkształcenia.

Грунт рассматривается как многофазная среда. Скелет описывается с помощью упруго-пластической модели с изотропным упрочнением, для которого вводятся соответствующие определяющие зависимости. Предполагается также что жидкость, заполняющая поры линейно сжимаема. В работе рассматриваются два предельных случая поведения среды: со стоком жидкости и без возможности стока. Подробно обсуждается одна из формулировок метода конечных элементов применительно к плоскому деформированному состоянию.

1. Introduction

SATURATED soil is in general a three-phase material consisting of a compressible solid phase (the skeleton of soil particles), gas and liquid phase (the air-water mixture filling the pores of the skeleton). In the special case when the pores are completely saturated one may regard the soil as a two-phase material only. The basis of theoretical analysis of such a three- (or two-) phase material is Therzaghi's effective stress concept which assumes that the total normal stress on any plane is the algebraic sum of the normal stress in the soil skeleton (called the effective normal stress) and the pore (water) pressure.

Saturated soil may exhibit two different extreme behaviours depending on the loading rate. When the load is applied very quickly, then the excess pore pressure cannot dissipate and the soil is said to behave in an undrained manner. On the other hand, when the behaviour of soil under very slow loading or after a very long time period following load application is considered, no excess pore pressure develops. In this case the soil is said to behave in a so-called drained manner. Both drained and undrained conditions represent two extremes of soil behaviour. In general, when the soil is loaded, it responds first in an undrained manner and then the excess pore pressure begins to dissipate. The last process is a time-dependent consolidation phenomenon. After a sufficiently long time period, when all excess pore pressure has dissipated, the soil behaves in a drained manner.

The aim of this paper is to present the mathematical relations of an elastoplastic consolidation theory. In Sect. 2 the discretized equilibrium statement based on the virtual work expression is derived. The obtained system of equations allows for both drained and undrained finite element effective stress analysis. Then, a finite element formulation of the consolidation problem is discussed. All the relations are derived in matrix notation which is convenient for digital computer programming. In the last section an elastoplastic isotropically hardening model of soil skeleton is proposed. The analysis presented below is restricted to the small strain theory and to the plane strain conditions.

2. Discretized equilibrium statement. Drained and undrained analysis

With the aid of imaginary cuts the porous medium can be conceived as a composition of elements. Let us denote by w_i the pore stress at the nodal points of these elements. Then the pore stress within a certain element can be represented in the following form:

$$w(x, y) = N_i w_i \quad i = 1, \ldots, n, \tag{2.1}$$

where w_i is the pore stress at the nodal point i, n is the number of nodal points of the element considered and N_i is the so-called shape function.

Similarly, the displacement field δ within a typical element can be expressed by the nodal point displacements δ_i

$$\delta(x, y) = N_i \delta_i. \tag{2.2}$$

In both formulas (2.1) and (2.2) x and y are the Cartesian coordinates in the plane of deformation.

The shape function defines the mode of interpolation over an element. ZIENKIEWICZ [1] and others have presented suitable shape functions for several kinds of elements. In this paper only the simple triangular element with three nodal points, i.e. the corner points, will be considered. For such an element the shape function is

$$N_i = (a_i x + b_i y + c_i)/D, \tag{2.3}$$

where

$$\begin{aligned} a_1 &= y_2 - y_3, & b_1 &= x_3 - x_2, & c_1 &= x_2 y_3 - x_3 y_2,\\ a_2 &= y_3 - y_1, & b_2 &= x_1 - x_3, & c_2 &= x_3 y_1 - x_1 y_3,\\ a_3 &= y_1 - y_2, & b_3 &= x_2 - x_1, & c_3 &= x_1 y_2 - x_2 y_1, \end{aligned}$$

$$D = \det \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$$

and x_i, y_i are the coordinates of the nodal point i.

In order to obtain approximate solution for the stresses and strains in a continuum, the so-called displacement method is applied. This method is based on the virtual work equation

$$\int_V \{\Delta\boldsymbol{\sigma}\}^T \boldsymbol{\epsilon} dV - \{\Delta\mathbf{P}\}^T \boldsymbol{\delta}^N = 0 \tag{2.4}$$

holding for any stress distribution in equilibrium and any virtual displacement field satisfying the boundary conditions. For an arbitrary element under consideration $\boldsymbol{\delta}^N$ denotes the nodal point displacements vector and $\{\mathbf{P}\}^T$ the transposed nodal forces vector. The nodal forces are due to distributed or concentrated load, body forces based on the bulk unit weight or due to initial strains or stresses.

The strain field ε_{ij} corresponding to the virtual displacement field is defined as

$$\varepsilon_{ij} = \frac{1}{2}(\delta_{i,j} + \delta_{j,i}). \tag{2.5}$$

Now, regarding Eqs. (2.2) and (2.3) for the plane strain conditions we can write

$$\boldsymbol{\epsilon} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \frac{1}{D} \begin{bmatrix} a_1 & 0 & a_2 & 0 & a_3 & 0 \\ 0 & b_1 & 0 & b_2 & 0 & b_3 \\ b_1 & a_1 & b_2 & a_2 & b_3 & a_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ v_1 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} = [B]\boldsymbol{\delta}^N, \tag{2.5'}$$

where u_i and v_i are the horizontal and vertical displacement components at the nodal point i, respectively.

Thus we have

$$\Delta\boldsymbol{\epsilon} = [B]\Delta\boldsymbol{\delta}^N \tag{2.6}$$

and, according to Eq. (2.1),

$$\Delta w = \{N_1, N_2, N_3\} \begin{Bmatrix} \Delta w_1 \\ \Delta w_2 \\ \Delta w_3 \end{Bmatrix} = \{\mathbf{N}\}^T \Delta\mathbf{w}. \tag{2.7}$$

The total stress increment can be written as

$$\Delta\boldsymbol{\sigma} = \Delta\boldsymbol{\sigma}' + \Delta w \mathbf{I}, \tag{2.8}$$

where $\Delta\boldsymbol{\sigma}'$ is the so-called effective stress increment and

$$\Delta\boldsymbol{\sigma} = \begin{Bmatrix} \Delta\sigma_x \\ \Delta\sigma_y \\ \Delta\sigma_{xy} \end{Bmatrix}, \quad \mathbf{I} = \begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix}.$$

Although, such a static decomposition is always permissible, we shall now make the assumption that the *effects* of the two stress-phases can be superimposed. Under this assumption the effective stress-strain constitutive relationship can be written in the form (see [2]):

$$\Delta\boldsymbol{\sigma}' = [D](\Delta\boldsymbol{\epsilon} - \mathbf{I}\Delta w/3K_s), \tag{2.9}$$

where K_s is the average bulk modulus of the solid phase and $\mathbf{I}\Delta w/3K_s$ represents the strain resulting from the compression of grains by pressure Δw. $[D]$ is the tangent stiffness matrix of the skeleton and is obtained from tests on the drained material.

Although in rock the strain resulting from compressibility of solid phase is significant, in soil it may be considered neglegible. Therefore, for the latter material the relationship (2.9) may be written in the simplified form

$$\Delta\boldsymbol{\sigma}' = [D]\Delta\boldsymbol{\epsilon}. \tag{2.10}$$

Now, utilizing Eqs. (2.5)–(2.8) and Eq. (2.10), the relation (2.4) can be expressed as

$$\{\Delta\mathbf{P}\}^T\boldsymbol{\delta}^N = \{\boldsymbol{\delta}^N\}^T\int_A [B]^T[D][B]dA\cdot\Delta\boldsymbol{\delta}^N + \{\boldsymbol{\delta}^N\}^T[B]^T\mathbf{I}\int_A\{\mathbf{N}\}^T dA\cdot\Delta\mathbf{w}^N, \tag{2.11}$$

where A is the area of the element considered.

Consider now the two integrals in the formula (2.11). For the first we can write simply

$$\int_A [B]^T[D][B]dA = [B]^T[D][B]A = [k], \tag{2.12}$$

where $[k]$ is the well-known stiffness matrix (6×6).

For the second, in view of Eq. (2.3) we obtain

$$\int_A\{\mathbf{N}\}^T dA = \frac{1}{D}\{(a_1A_x+b_1A_y+c_1A);\ (a_2A_x+b_2A_y+c_2A);\ (a_3A_x+b_3A_y+c_3A)\} = \{\overline{\mathbf{N}}\}^T, \tag{2.13}$$

where

$$A_x = \int_A x\,dA = \frac{A}{3}(x_1+x_2+x_3);\qquad A_y = \int_A y\,dA = \frac{A}{3}(y_1+y_2+y_3).$$

Thus the expression (2.11) can be rewritten in the form

$$\{\boldsymbol{\delta}^N\}^T[k]\Delta\boldsymbol{\delta}^N + \{\boldsymbol{\delta}^N\}^T[B]^T\mathbf{I}\{\overline{\mathbf{N}}\}^T\Delta\mathbf{w}^N - \{\boldsymbol{\delta}^N\}^T\Delta\mathbf{P} = 0 \tag{2.14}$$

or accounting for the relation

$$[B]^T\mathbf{I} = \frac{1}{D}\begin{Bmatrix} a_1\\ b_1\\ a_2\\ b_2\\ a_3\\ b_3\end{Bmatrix} = \boldsymbol{\varphi} \tag{2.15}$$

in the final form

$$\{\boldsymbol{\delta}^N\}^T([k]\Delta\boldsymbol{\delta}^N + \boldsymbol{\varphi}\{\overline{\mathbf{N}}\}^T\Delta\mathbf{w}^N - \Delta\mathbf{P}) = 0. \tag{2.16}$$

The virtual work is zero for all values of u_i and v_i which satisfy the boundary conditions. Then the virtual work equation (2.16) provides the system of equations of the type

$$[k]\Delta\boldsymbol{\delta}^N + \boldsymbol{\varphi}\{\overline{\mathbf{N}}\}^T\Delta\mathbf{w}^N = \Delta\mathbf{P}. \tag{2.17}$$

Let us now discuss two extreme cases of soil behaviour, i.e. drained and undrained deformation. In the latter case when the load is applied so quickly that the excess pore pressure cannot dissipate, the pressure rise can be directly related to strain and eliminated from Eq. (2.17). In general, the reduction of volume in an element of soil mass is due to compressibility of fluid and compressibility of solid phase. Accounting for both effects the balance equation takes the form

$$\Delta\varepsilon_v = \mathbf{I}^T\Delta\boldsymbol{\epsilon} = n\Delta w/K_p + (1-n)\Delta w/K_s + (\Delta\sigma'_x + \Delta\sigma'_y + \Delta\sigma'_z)/3K_s, \tag{2.18}$$

where n is the porosity and K_p is the compression modulus of the pore fluid (or air-water mixture in the pores).

Neglecting the compressibility of solid phase, Eq. (2.18) simplifies to the form

$$\Delta\varepsilon_v \approx n\Delta w/K_p, \quad \text{i.e.} \quad \Delta w \approx \frac{K_p}{n}\mathbf{I}^T\Delta\boldsymbol{\epsilon}. \tag{2.19}$$

Substituting Eq. (2.19) into Eq. (2.17) we obtain

$$\left([k] + \int_A [B]^T\mathbf{I}\frac{K_p}{n}\mathbf{I}^T[B]dA\right)\Delta\boldsymbol{\delta}^N = \Delta\mathbf{P} \tag{2.20}$$

or, accounting for Eq. (2.12),

$$\left(\int_A [B]^T[D_t][B]dA\right)\Delta\boldsymbol{\delta}^N = \Delta\mathbf{P}, \tag{2.21}$$

where

$$[D_t] = [D] + \mathbf{I}\frac{K_p}{n}\mathbf{I}^T.$$

Here the matrix $[D_t]$ represents the total stiffness of the material, $[D]$ the stiffness of the skeleton and $\mathbf{I}\frac{K_p}{n}\mathbf{I}^T$ the stiffness of the pore fluid.

Equation (2.21) applied directly to the whole continuum enables us to carry out the undrained effective stress analysis. Once the nodal displacements have been determined by solution of the overall "structural" type equations, the total stresses, the effective stresses and the pore pressure distribution in each element can be found from the relations (2.19) and (2.8).

It should be noted that the pore fluid compressibility is low compared to the material skeleton compressibility. Thus the completely saturated soil in undrained conditions is a nearly incompressible material. On the other hand, it is known that the displacement finite element method suffers the disadvantage that the accuracy of the stress prediction decreases with the reduction of compressibility. Therefore, the undrained deformation analysis of fully-saturated soil requires a special numerical treatment as it is discussed in detail by D. J. NAYLOR in [3].

The second extreme case of soil behaviour is the drained deformation when load is applied so solwly that no excess pore pressure develops. In this case the effective stresses are equal to the total stresses and the analysis can be carried out by setting $\Delta\mathbf{w}^N = 0$ or $K_p = 0$ ($[D_t] = [D]$) in the formulas (2.17) or (2.21), respectively.

In both undrained and drained situations the pore pressure can be either directly related to strain or simply eliminated from the equation (2.17). Thus, in these extremal cases the pore pressure does not have to be treated as a nodal variable. However, in a general case the excess pore pressure can generate and, simultaneously, the flow of the pore fluid in soil mass can occur, resulting from the rise of the pressure gradients. Then the balance equation in the form (2.18) is not valid and the system (2.17) in not complete. When the prescribed as well as non-prescribed quantities $\Delta\delta_i$(i.e. Δu_i, Δv_i) and Δw_i are regarded as unknowns, the total number of unknowns in the whole discretized continuum is $3\times N$ (N — number of nodal points). $2\times N$ equations are specified by the relations (2.17) and the remaining N equations will be derived in the following.

3. Numerical analysis of consolidation

The theory of consolidation was first investigated by THERZAGHI [4] for the case of one-dimension only, and subsequently extended to three dimensions by BIOT [5]. Unfortunately, Biot's theory is so complicated that even for highly idealized boundary conditions analytical solutions are difficult to obtain. Up to now only several useful solutions have been published (see JOSSELIN DE JONG [6], MCNAMEE and GIBSON [7]).

The first finite element formulation of the two-dimensional consolidation problem was given by SANDHU and WILSON [8].

These authors used the Gurtin type of variational principle [9]; up to now many other approaches have appeared in the literature (see BOOKER [10], SMALL *et. al.* [11], HWANG *et al.* [13]). In this paper the formulation of the finite element method for the analysis of two-dimensional consolidation problems will be presented following the analysis of P. A. VERMEER [14]. Instead of index notation (as preferred in [14]) the vector notation will be used as more convenient for computer programming.

In order to have a complete mathematical description of the consolidation problem we should consider: the equilibrium equations, the constitutive law and a generalized balance equation.

Both the equilibrium equations and constitutive law were already taken into account in the formula (2.17). Thus the proper balance equation, more general than the simplified formula (2.18), is now required to obtain a complete finite element formulation. As it is shown in Appendix I, assuming that the flow of fluid through the soil is governed by simplified Darcy's law (x, y directions are said to be the principal directions of permeability), we can write

$$\frac{\partial}{\partial t}(\varepsilon_x+\varepsilon_y)-\frac{n}{K_p}\frac{\partial w}{\partial t}+\frac{k_x}{\gamma_w}\frac{\partial^2 w}{\partial x^2}+\frac{k_y}{\gamma_w}\frac{\partial^2 w}{\partial y^2}=0, \tag{3.1}$$

where t denotes the time, γ_w is the specific weight of water and k_x, k_y are the coefficients of permeability in the x and y directions, respectively.

In Eq. (3.1) the grains compressibility and both the variation of permeability and the variation of the degree of saturation during the deformation process are neglected.

In the majority of existing formulations it is additionally assumed that the pores of soil are completely saturated with water and the term $\frac{n}{K_p}\frac{\partial w}{\partial t}$ is disregarded with respect to other terms in Eq. (3.1), i.e. the water is considered as incompressible (see [11], [13]). It often occurs, however, that the pores are not completely saturated and then the air-water mixture in the pores can have a significant compressiblility. In this case K_p can be considered as the (tangential) compressibility modulus of such a mixture and the term $\frac{n}{K_p}\frac{\partial w}{\partial t}$ may not be neglected.

Sometimes it is possible to assume that the soil is isotropic with respect to permeability, i.e. $k_x = k_y = k$. Then the balance equation (3.1) has a simplified form

$$\frac{\partial}{\partial t}(\varepsilon_x+\varepsilon_y)-\frac{n}{K_p}\frac{\partial w}{\partial t}+\frac{k}{\gamma_w}\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right) \tag{3.2}$$

as discussed by P. A. VERMEER [14]. In the following however, we shall use the more general equation (3.1).

As it follows from Eq. (3.1) the pore pressure w depends on the spatial coordinates and the time t. An approximation to the time variation can be performed by the finite difference technique. In [14] the simple one-step method was chosen which for an ordinary differential equation of the type $dy/dt = f(y, t)$ is defined as

$$\Delta y = \Delta t f(y_0, t_0)+\alpha\Delta t\Delta f; \quad \Delta f = f(y_0+\Delta y, t_0+\Delta t)-f(y_0, t_0). \tag{3.3}$$

Here Δy denotes the increment of y over the time interval $t_0 < t < t_0+\Delta t$, whereas the integration constant α $(0 < \alpha < 1)$ determines the type of interpolation used ($\alpha = 1/2$ corresponds to the linear interpolation, $\alpha = 0$ to the forward differences and $\alpha = 1$ to the backward differences). In general, the numerical procedure has to be "stable", i.e. the error due to the discretization over a certain time step must not systematically grow in the course of the following time steps. As it was proved by BOOKER and SMALL [12], for a linear elastic material a sufficient condition of numerical stability is $\alpha > 1/2$. However, for an elasto-plastic medium higher values of α are more profitable.

Now, applying the procedure defined by Eq. (3.3) we can transform the differential equation (3.1) into the equation

$$\Delta\varepsilon_v-\frac{n}{K_p}\Delta w+\Delta t\left(\frac{k_x}{\gamma_w}\frac{\partial^2 w^0}{\partial x^2}+\frac{k_y}{\gamma_w}\frac{\partial^2 w^0}{\partial y^2}\right)+\alpha\Delta t\left(\frac{k_x}{\gamma_w}\frac{\partial^2(\Delta w)}{\partial x^2}+\frac{k_y}{\gamma_w}\frac{\partial^2(\Delta w)}{\partial y^2}\right) = 0, \tag{3.4}$$

where w^0 is the pore stress at the initial time and increments are denoted by Δ.

In order to be specific let us assume that the soil occupies a region V. The stress and strain boundary conditions are already satisfied by introducing the virtual work equation (2.4). Assume now that at the instant t^0 the pore pressure disribution w^0 in the whole region V is known. Moreover, on a certain part of the boundary (S_w) the pore stress w is prescribed and on the remaining part of the boundary (S_0) the outflow of the fluid is prevented. Hence

$$\begin{aligned} &w^0 = f^0, \quad \Delta w = \Delta f \quad \text{on} \quad S_w,\\ &w^0_{,i}n_i = \Delta w_{,i}n_i = 0 \quad \text{on} \quad S_0, \end{aligned} \tag{3.5}$$

where n_i is the outward normal upon the boundary at the point considered, while f is a function of coordinates. As it is proved in [14] Eqs. (3.1) and (3.5) are satisfied when $\partial F/\partial(\Delta w) = 0$ where F is a functional defined as

$$(3.6) \qquad F = \int_V \left\{ -\Delta\varepsilon_v \Delta w + \frac{n}{2K_p}\Delta w^2 + \Delta t \left(\frac{k_x}{\gamma_w}\frac{\partial w^0}{\partial x}\frac{\partial(\Delta w)}{\partial x} + \frac{k_y}{\gamma_w}\frac{\partial w^0}{\partial y}\frac{\partial(\Delta w)}{\partial y} \right) + \alpha\frac{\Delta t}{2}\left[\frac{k_x}{\gamma_w}\left(\frac{\partial(\Delta w)}{\partial x^2}\right)^2 + \frac{k_y}{\gamma_w}\left(\frac{\partial(\Delta w)}{\partial y^2}\right)^2 \right] \right\} dV.$$

Consider now the contribution of an arbitrary triangular element to the functional F. In order to simplify our analysis let us write this functional in the form

$$(3.7) \qquad F^e = F_1^e + \frac{1}{2}F_2^e + \frac{1}{2}F_3^e + F_4^e,$$

where

$$F_1^e = -\int_A \Delta\varepsilon_v \Delta w dA,$$

$$F_2^e = \int_A \frac{n}{K_p}\Delta w^2 dA,$$

$$F_3^e = \int_A \alpha\Delta t \left[\frac{k_x}{\gamma_w}\left(\frac{\partial(\Delta w)}{\partial x}\right)^2 + \frac{k_y}{\gamma_w}\left(\frac{\partial(\Delta w)}{\partial y}\right)^2 \right] dA,$$

$$F_4^e = \int_A \Delta t \left(\frac{k_x}{\gamma_w}\frac{\partial w^0}{\partial x}\frac{\partial(\Delta w)}{\partial x} + \frac{k_y}{\gamma_w}\frac{\partial w^0}{\partial y}\frac{\partial(\Delta w)}{\partial y} \right) dA$$

and A is the area of the element considered.

In the plane strain conditions the volumetric strain of the porous medium can be written, according to Eqs. (2.5) and (2.14), as

$$(3.8) \qquad \Delta\varepsilon_v = \Delta\varepsilon_x + \Delta\varepsilon_y = \frac{1}{D}\{a_1, b_1, a_2, b_2, a_3, b_3\}\Delta\boldsymbol{\delta}^N = \{\boldsymbol{\varphi}\}^T \Delta\boldsymbol{\delta}^N.$$

Now the sub-functional F_1^e, in view of Eqs. (3.8) and (2.7), will have the form

$$(3.9) \qquad F_1^e = -\int_A \{\boldsymbol{\varphi}\}^T \Delta\boldsymbol{\delta}^N \{\mathbf{N}\}^T \Delta\mathbf{w}^N dA = -\{\varphi\}^T \Delta\boldsymbol{\delta}^N \int_A \{\mathbf{N}\}^T dA \Delta\mathbf{w}^N$$

or, according to Eq. (2.12),

$$(3.10) \qquad F_1^e = -\{\boldsymbol{\varphi}\}^T \Delta\boldsymbol{\delta}^N \{\overline{\mathbf{N}}\}^T \Delta\mathbf{w}^N.$$

For the second sub-functional F_2^e, in view of Eq. (2.6), after some transformations we obtain

$$(3.11) \qquad F_2^e = \frac{n}{K_p}\{\Delta\mathbf{w}^N\}^T \int_A \mathbf{N}\{\mathbf{N}\}^T dA \Delta\mathbf{w}^N.$$

Performing the multiplication of both vectors $\mathbf{N}$, the integral in Eq. (3.11) can be expressed as

$$(3.12) \qquad \int_A \mathbf{N}\{\mathbf{N}\}^T dA = [S],$$

where $[S]$ is a symmetric matrix (3×3) defined as

$$S_{ij}=\frac{1}{D^2}\{a_ia_jA_{xx}+(a_ib_j+b_ia_j)A_{xy}+b_ib_jA_{yy}+(c_ia_j+a_ic_j)A_x+(b_ic_j+c_ib_j)A_y+c_ic_jA\},\qquad i,j=1,2,3$$

and

$$A_{xx}=\int_A x^2dA=\frac{A}{12}(x_1^2+x_2^2+x_2^2)+\frac{A}{12}(x_1+x_2+x_3)^2,$$

$$A_{yy}=\int_A y^2dA=\frac{A}{12}(y_1^2+y_2^2+y_3^2)+\frac{A}{12}(y_1+y_2+y_3)^2,$$

$$A_{xy}=\int_A xydA=\frac{A}{12}(x_1y_1+x_2y_2+x_3y_3)+\frac{A}{12}(x_1+x_2+x_3)(y_1+y_2+y_3),$$

A_x and A_y are defined in Eq. (2.12).

Finally, the sub-functional F_2^e can be written in the form

$$F_2^e=\frac{n}{K_p}\{\Delta\mathbf{w}^N\}^T[S]\Delta\mathbf{w}^N. \tag{3.13}$$

For F_3^e, according to Eq. (2.7), we have

$$F_3^e=\int_A \alpha\Delta t\{\Delta\mathbf{w}^N\}^T\left[\frac{k_x}{\gamma_w}\left(\frac{\partial}{\partial x}\mathbf{N}\right)\left(\frac{\partial}{\partial x}\{\mathbf{N}\}^T\right)+\frac{k_y}{\gamma_w}\left(\frac{\partial}{\partial y}\mathbf{N}\right)\left(\frac{\partial}{\partial y}\{\mathbf{N}\}^T\right)\right]\Delta\mathbf{w}^N dA \tag{3.14}$$

or, regarding Eq. (2.3),

$$F_3^e=\alpha\Delta t\{\Delta\mathbf{w}^N\}^T[C]\Delta\mathbf{w}^NA, \tag{3.15}$$

where

$$[C]=\frac{1}{D^2}\left(\frac{k_x}{\gamma_w}\begin{bmatrix}a_1^2, & a_1a_2, & a_1a_3\\ a_2a_1, & a_2^2, & a_2a_3\\ a_3a_1, & a_3a_2, & a_3^2\end{bmatrix}+\frac{k_y}{\gamma_w}\begin{bmatrix}b_1^2, & b_1b_2, & b_1b_3\\ b_2b_1, & b_2^2, & b_2b_3\\ b_3b_1, & b_3b_2, & b_3^2\end{bmatrix}\right). \tag{3.15a}$$

In the particular case, when the soil is isotropic with respect to permeability (i.e. $k_x=$ $=k_y=k$), the matrix $[C]$ will have the form

$$[C]=\frac{1}{D^2}\frac{k}{\gamma_w}\begin{bmatrix}(a_1^2+b_1^2) & (a_1a_2+b_1b_2) & (a_1a_3+b_1b_3)\\ & (a_2^2+b_2^2) & (a_2a_3+b_2b_3)\\ \text{symmetry} & & (a_3^2+b_3^2)\end{bmatrix}. \tag{3.15b}$$

Finally, for the last sub-functional F_4^e we obtain analogously to F_3^e

$$F_4^e=\Delta t\{\mathbf{w}^{0N}\}^T[C]\Delta\mathbf{w}^NA. \tag{3.16}$$

Hence, according to Eqs. (3.7), (3.10), (3.13), (3.15) and (3.16) we can write

$$F^e=-\{\boldsymbol{\varphi}\}^T\Delta\boldsymbol{\delta}^N\{\overline{\mathbf{N}}\}^T\Delta\mathbf{w}^N+\frac{n}{2K_p}\{\Delta\mathbf{w}^N\}^T[S]\Delta\mathbf{w}^N+\alpha\Delta tA\{\Delta\mathbf{w}^N\}^T[C]\Delta\mathbf{w}^N+\Delta tA\{\mathbf{w}^{0N}\}^T[C]\Delta\mathbf{w}^N. \tag{3.17}$$

The functional F should have a stationary value with respect to the pore pressure. This means that for an arbitrary element the partial derivatives $\partial F/\partial \Delta w_i$ $(i = 1, 2, 3)$ should be zero. Thus we finally obtain

$$\boldsymbol{\varphi}\{\overline{\mathrm{N}}\}^T \Delta\boldsymbol{\delta}^N - \left(\frac{n}{K_p}[S] + \alpha\Delta t A[C]\right)\Delta\mathbf{w}^N = \Delta t A[C]\mathbf{w}^{0N}. \tag{3.18}$$

The equation (3.18) can be written in a more general form

$$[X]\Delta\boldsymbol{\delta}^N + [Y]\Delta\mathbf{w}^N = \mathbf{W}^0, \tag{3.19}$$

where the matrices $[X]$ and $[Y]$ ((3×6) and (3×3), respectively) and vector $\mathbf{W}^0$ are defined according to Eq. (3.18). Now the generalization of the system (3.19) to the whole continuum will provide N additional equations (N — number of nodal points) which together with the generalized equations (2.17) enable us to solve a time step in a consolidation problem by a digital computer.

For an arbitrary element we finally have Eq. (2.17), i.e.

$$[k]\Delta\boldsymbol{\delta}^N + [X]\Delta\mathbf{w}^N = \Delta\mathbf{P}$$

and the relation (3.19). The formulas (2.17) and (3.19) can be written in a combined matrix form

$$[k']\Delta\overline{\boldsymbol{\delta}}^N = \Delta\overline{\mathbf{P}} \tag{3.20}$$

that is

$$\begin{bmatrix} \underset{(6\times 6)}{[k]} & \underset{(3\times 6)}{[X]} \\ \underset{(6\times 3)}{[X]^T} & \underset{(3\times 3)}{[Y]} \end{bmatrix} \begin{Bmatrix} \Delta u_1 \\ \Delta v_1 \\ \Delta u_2 \\ \Delta v_2 \\ \Delta u_3 \\ \Delta v_3 \\ \Delta w_1 \\ \Delta w_2 \\ \Delta w_3 \end{Bmatrix} = \begin{Bmatrix} \Delta P_1^u \\ \Delta P_1^v \\ \Delta P_2^u \\ \Delta P_2^v \\ \Delta P_3^u \\ \Delta P_3^v \\ w_1^0 \\ w_2^0 \\ w_3^0 \end{Bmatrix}.$$

Here P_i^u and P_i^v denote the nodal forces; the superscripts refer to the component (horizontal or vertical, respectively) and subscripts stand for the number of the nodal points.

In the formulation presented above the stiffness matrix $[k']$ is a (9×9) matrix and its symmetry depends only on the symmetry of the stress-strain relation. It is also worthy to note that only the components of the matrix $[Y]$ depend on the applied time increments.

The matrix $[k]$ refers to mechanical properties of the soil skeleton. It is known that the assumption of a purely elastic skeleton is not very realistic. Fortunately, the system (3.20) remains valid for any incremental stress-strain law of the form (2.10). The constitutive matrix does not have to be necessarily symmetric and may depend upon the current stress state and the previous history of the body. In the next section we shall propose an elastoplastic model for the soil skeleton.

4. Elastoplastic model of soil skeleton behaviour

The idea presented below was originally derived by J. H. PREVOST and K. HOEG in [15]. In what follows we shall restrict our considerations to the plane strain conditions and adopt this concept after some substantial modifications.

Introducing the plane strain invariants of the effective stress and incremental plastic strain tensors,

$$t = \frac{1}{2}|\sigma_3' - \sigma_1'| = \left[\left(\frac{\sigma_x' - \sigma_y'}{2}\right)^2 + \sigma_{xy}^2\right]^{1/2},$$

(4.1)
$$S = -\frac{1}{2}(\sigma_3' + \sigma_1') = -\frac{1}{2}(\sigma_x' + \sigma_y'),$$

$$\Delta\varepsilon_v^P = -(\Delta\varepsilon_1^P + \Delta\varepsilon_3^P); \quad \Delta\varepsilon_q^P = \Delta\varepsilon_3^P - \Delta\varepsilon_1^P,$$

we assume that the yield surface in the s, t-plane is represented by two straight lines:

(4.2)
$$f_1 = s - N(\varepsilon_v^P) = 0,$$
$$f_2 = t - H(\varepsilon_v^P, \varepsilon_q^P) = 0.$$

The surface $f_1 = 0$ (Fig. 1) is the so-colled volumetric yield surface. The position of this surface depends on the plastic volumetric strains, whereas the shape remains the same

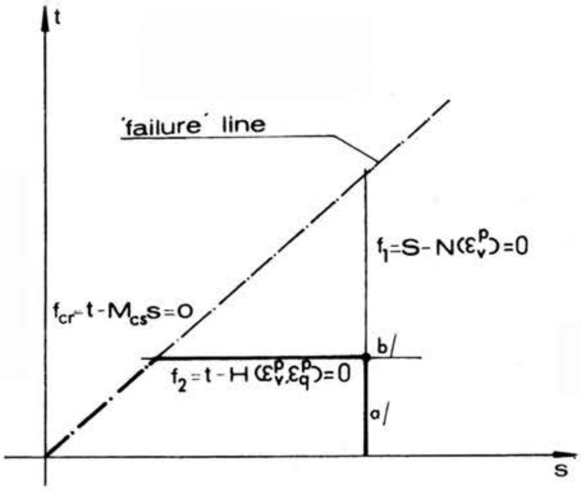

FIG. 1. Yield surface in the s, t-plane: a) volumetric yield surface, b) shear yield surface.

during the deformation process. The yield surface $f_2 = 0$ (called the shear yield surface) may be considered as the surface of the Tresca type. Because the soil may undergo strain hardening as well as strain softening, both expansion and contraction of this surface are possible depending on the form of the function H (see [15]). In general the function H depends on both the plastic volumetric strains and plastic shear distortions.

Besides the volumetric and shear yield surfaces the "failure" line

(4.3)
$$f_{cr} = t - M_{cs}s = 0$$

is introduced. Here M_{cs} denotes a material constant. Along this line a perfectly plastic behaviour satisfying the normality rule is assumed. Moreover, the hardening functions $N(\varepsilon_v^P)$ and $H(\varepsilon_v^P, \varepsilon_q^P)$ are proposed in the form

$$N(\varepsilon_v^P) = \alpha(\varepsilon_v^P)^\beta,$$

(4.4)
$$H(\varepsilon_v^P, \varepsilon_q^P) = t_{cr}\frac{\varepsilon_q^P}{a+\varepsilon_q^P} = M_{cs}\alpha(\varepsilon_v^P)^\beta\frac{\varepsilon_q^P}{a+\varepsilon_q^P},$$

where

$$t_{cr} = M_{cs}\alpha(\varepsilon_v^P)^\beta$$

and a, α, β are the material constants. The function H in Eq. (4.4) describes the hardening of the material.

Let us note that the presented approach differs from the formulations based on the concept of the initial yield surface $f^0 = 0$. According to Eq. (4.4) the initial stress state at any point of the soil mass is always located at the corner A (Fig. 1) and, accounting for the previous history of the soil, the localization of this corner is known.

In the following we will derive the effective stress-strain relations for both the volumetric and shear yield surface and for the corner A (see Fig. 1). In general, the stress-strain relation for the "failure" line may also be derived in a similar way to the one presented below.

The major assumption is that the associated flow rule in the form

(4.5)
$$\Delta\boldsymbol{\epsilon}^P = \Delta\lambda\frac{\partial f}{\partial\boldsymbol{\sigma}'}$$

is postulated in which $\Delta\lambda$ is a positive multiplier and $\boldsymbol{\sigma}'$ is the current effective stress vector. Such a flow rule requires that the plastic strain rates be directed along the outward normal to the surface $f = 0$.

4.1. Volumetric yield surface

This surface is given by Eq. (4.2a) and the associated flow rule may be written in terms of stress and strain invariants as

(4.6)
$$\Delta\varepsilon_v^P = \Delta\lambda\frac{\partial f_1}{\partial s} = \Delta\lambda, \quad \Delta\varepsilon_q^P = \Delta\lambda\frac{\partial f_1}{\partial t} = 0.$$

Now, writing an increment of the effective stress vector in the form

(4.7)
$$\Delta\boldsymbol{\sigma}' = [D^{ep}]\Delta\boldsymbol{\epsilon} = [D^e](\Delta\boldsymbol{\epsilon}-\Delta\boldsymbol{\epsilon}^P) = [D^e]\left(\Delta\boldsymbol{\epsilon}-\Delta\lambda\frac{\partial f_1}{\partial\boldsymbol{\sigma}'}\right),$$

($[D^e]$ and $[D^{ep}]$ denote the elastic and elastoplastic constitutive matrices, respectively), and satisfying the consistency condition

(4.8)
$$\left\{\frac{\partial f_1}{\partial\boldsymbol{\sigma}'}\right\}^T\Delta\boldsymbol{\sigma}' + \frac{\partial f_1}{\partial\varepsilon_v^P}\Delta\varepsilon_v^P = 0,$$

we obtain after some transformations

$$\Delta\lambda = \frac{\left\{\frac{\partial f_1}{\partial \boldsymbol{\sigma}'}\right\}^T [D^e]\Delta\boldsymbol{\epsilon}}{\left\{\frac{\partial f_1}{\Delta \boldsymbol{\sigma}'}\right\}^T [D^e]\frac{\partial f_1}{\partial \boldsymbol{\sigma}'} + \frac{\partial N}{\partial \varepsilon_v^P}}. \tag{4.9}$$

Thus, denoting $\Delta\boldsymbol{\sigma}^* = [D^e]\Delta\boldsymbol{\epsilon}$ we have finally

$$\Delta\varepsilon^P = \Delta\lambda \frac{\partial f_1}{\partial \boldsymbol{\sigma}'} = \frac{\left\{\frac{\partial f_1}{\partial \boldsymbol{\sigma}'}\right\}^T \Delta\boldsymbol{\sigma}^*}{\left\{\frac{\partial f_1}{\partial \boldsymbol{\sigma}'}\right\}^T [D^e]\frac{\partial f_1}{\partial \boldsymbol{\sigma}'} + \frac{\partial N}{\partial \varepsilon_v^P}} \frac{\partial f_1}{\partial \boldsymbol{\sigma}'}. \tag{4.10}$$

Such a form of the constitutive relation is already convenient for use in the "initial stress" finite element approach. A similar expression relating $\Delta\boldsymbol{\epsilon}^p$ to $\Delta\boldsymbol{\sigma}^*$ will be derived below for both the shear yield surface and the corner A.

4.2. Shear yield surface

Here the associated flow rule may be written in terms of stress and strain invariants as

$$\begin{aligned} \Delta\varepsilon_v^P &= \Delta\lambda \frac{\partial f_2}{\partial s} = 0, \\ \Delta\varepsilon_q^P &= \Delta\lambda \frac{\partial f_2}{\partial t} = \Delta\lambda. \end{aligned} \tag{4.11}$$

Moreover, the consistency condition has the form

$$\left\{\frac{\partial f}{\partial \boldsymbol{\sigma}'}\right\}^T \Delta\boldsymbol{\sigma}' + \frac{\partial f_2}{\partial \varepsilon_v^P}\Delta\varepsilon_v^P + \frac{\partial f_2}{\partial \varepsilon_q^P}\Delta\varepsilon_q^P = 0. \tag{4.12}$$

Thus, finally

$$\Delta\lambda = \frac{\left\{\frac{\partial f_2}{\partial \boldsymbol{\sigma}'}\right\}^T [D^e]\Delta\boldsymbol{\epsilon}}{\left\{\frac{\partial f_2}{\partial \boldsymbol{\sigma}'}\right\}^T [D^e]\frac{\partial f_2}{\partial \boldsymbol{\sigma}'} + \frac{\partial H}{\partial \varepsilon_q^P}} \tag{4.13}$$

and

$$\Delta\boldsymbol{\epsilon}^P = \Delta\lambda \frac{\partial f_2}{\partial \boldsymbol{\sigma}'} = \frac{\left\{\frac{\partial f_2}{\partial \boldsymbol{\sigma}'}\right\}^T \Delta\boldsymbol{\sigma}^*}{\left\{\frac{\partial f_2}{\partial \boldsymbol{\sigma}'}\right\}^T [D^e]\frac{\partial f_2}{\partial \boldsymbol{\sigma}'} + \frac{\partial H}{\partial \varepsilon_q^P}} \frac{\partial f_2}{\partial \boldsymbol{\sigma}'}. \tag{4.14}$$

4.3. Corner behaviour

The total plastic deformations may be written as the sum of the contributions from the two yield functions

$$\Delta\boldsymbol{\epsilon}^P = \Delta\lambda \frac{\partial f_1}{\partial \boldsymbol{\sigma}'} + \Delta\gamma \frac{\partial f_2}{\partial \boldsymbol{\sigma}'} \tag{4.15}$$

or in terms of stress and strain invariants:

$$\Delta\varepsilon_v^P = \Delta\lambda\frac{\partial f_1}{\partial s}+\Delta\gamma\frac{\partial f_2}{\partial s} = \Delta\lambda, \tag{4.16}$$

$$\Delta\varepsilon_q^P = \Delta\lambda\frac{\partial f_1}{\partial t}+\Delta\gamma\frac{\partial f_2}{\partial t} = \Delta\gamma.$$

Thus the consistency conditions will provide a system of two equations

$$\left\{\frac{\partial f_1}{\partial \boldsymbol{\sigma}'}\right\}^T \Delta\boldsymbol{\sigma}' + \frac{\partial f_1}{\partial \varepsilon_v^P}\Delta\varepsilon_v^P = 0, \tag{4.17}$$

$$\left\{\frac{\partial f_2}{\partial \boldsymbol{\sigma}'}\right\}^T \Delta\boldsymbol{\sigma}' + \frac{\partial f_2}{\partial \varepsilon_v^P}\Delta\varepsilon_v^P + \frac{\partial f_2}{\partial \varepsilon_q^P}\Delta\varepsilon_q^P = 0.$$

By solving this system we obtain the expressions for $\Delta\lambda$ and $\Delta\gamma$. Hence the constitutive relation may be written in the form

$$\Delta\boldsymbol{\epsilon}^P = \frac{a_{12}\left\{\frac{\partial f_2}{\partial \boldsymbol{\sigma}'}\right\}^T\Delta\boldsymbol{\sigma}^* - \left(a_{22}+\frac{\partial H}{\partial \varepsilon_q^P}\right)\left\{\frac{\partial f_1}{\partial \boldsymbol{\sigma}'}\right\}^T\Delta\boldsymbol{\sigma}^*}{a_{12}\left(a_{12}+\frac{\partial H}{\partial \varepsilon_v^P}\right) - \left(a_{11}+\frac{\partial N}{\partial \varepsilon_v^P}\right)\left(a_{22}+\frac{\partial H}{\partial \varepsilon_q^P}\right)}\frac{\partial f_1}{\partial \boldsymbol{\sigma}'} \tag{4.18}$$

$$+\frac{\left(a_{12}\frac{\partial H}{\partial \varepsilon_v^P}\right)\left\{\frac{\partial f_1}{\partial \boldsymbol{\sigma}'}\right\}^T\Delta\boldsymbol{\sigma}^* - \left(a_{11}+\frac{\partial N}{\partial \varepsilon_v^P}\right)\left\{\frac{\partial f_2}{\partial \boldsymbol{\sigma}'}\right\}^T\Delta\boldsymbol{\sigma}^*}{a_{12}\left(a_{12}+\frac{\partial H}{\partial \varepsilon_v^P}\right) - \left(a_{11}+\frac{\partial N}{\partial \varepsilon_v^P}\right)\left(a_{22}+\frac{\partial H}{\partial \varepsilon_q^P}\right)}\frac{\partial f_2}{\partial \boldsymbol{\sigma}'},$$

where

$$a_{11} = \left\{\frac{\partial f_1}{\partial \boldsymbol{\sigma}'}\right\}^T [D^e]\frac{\partial f_1}{\partial \boldsymbol{\sigma}'}, \qquad a_{22} = \left\{\frac{\partial f_2}{\partial \boldsymbol{\sigma}'}\right\}^T [D^e]\frac{\partial f_2}{\partial \boldsymbol{\sigma}'},$$

$$a_{12} = \left\{\frac{\partial f_1}{\partial \boldsymbol{\sigma}'}\right\}^T [D^e]\frac{\partial f_2}{\partial \boldsymbol{\sigma}'},$$

For both yield surfaces $f_1 = 0$ and $f_2 = 0$, given by Eq. (4.2), we have

$$\frac{\partial f_1}{\partial \boldsymbol{\sigma}'} = \begin{Bmatrix} -1/2 \\ -1/2 \\ 0 \end{Bmatrix}; \qquad \frac{\partial f_2}{\partial \boldsymbol{\sigma}'} = \frac{1}{2t}\begin{Bmatrix} 1/2(\sigma_x'-\sigma_y') \\ -1/2(\sigma_x'-\sigma_y') \\ 2\sigma_{xy}' \end{Bmatrix} \tag{4.19}$$

and $\quad a_{11} = K+1/3G, \quad a_{12} = 0, \quad a_{22} = G.$

where K and G are the elastic compressibility and shear modulus of soil skeleton, respectively.

A more detailed analysis of the corner behaviour, including some remarks on the validity of the above derived constitutive relation, is given in [16].

5. Final remark

We have presented above the mathematical relations of a certain elastoplastic consolidation theory. In the second part of the present paper [17] this theory is applied to the finite element analysis of an initial-boundary-value problem of application in engineering.

Appendix I. Derivation of the balance equation

The simplified relation (2.18) is not always valid since in the general case the flow of fluid through porous soil may occur. Accounting for this phenomena, the formula (2.18) should be written in the form

$$\Delta\varepsilon_v - n\Delta w/K_p - (1-n)\Delta w/K_s - (\Delta\sigma'_x + \Delta\sigma'_y + \Delta\sigma'_z)/3K_s = \Delta V,$$

where ΔV denotes the reduction of the fluid volume in the soil mass due to the flow of fluid during a time interval. Considering an elementary volume V within a mass of soil, we have $\Delta V = \frac{\Delta V_w}{V}$ where ΔV_w is directly the volume of fluid which dissipated (in a positive or negative sense) from this element.

In order to combine the value of ΔV with the pore pressure variations, let us assume that the motion of the pore fluid is governed by the simplified Darcy's law

$$v_x = -k_x\frac{\partial h}{\partial x}, \qquad v_y = -k_y\frac{\partial h}{\partial y}, \qquad v_z = -k_z\frac{\partial h}{\partial z}.$$

Here v_x, v_y and v_z are the components of specific discharge (i.e. the discharge per unit of cross-section area) vector; k_x, k_y and k_z are the coefficients of permeability and h is the so-called groundfluid head. It is assumed that the x, y, z-directions are the principal directions of permeability and $k_{xy} = k_{xz} = k_{yz} = k_{yx} = k_{zx} = k_{zy} = 0$.

The groundfluid head at any point of the soil mass can be expressed as

$$h = h_e + h_p,$$

where h_e is called the geometric or elevation head (Fig. 2) and h_p is the pressure head. The last is defined as

$$h_p = \int_0^w \frac{dw}{\gamma_w},$$

where γ_w is the specific weight of the pore fluid and w is the pore pressure. In general case $\gamma_w = \gamma_w(w)$, however, for slightly compressible fluids (such as water) we may simply assume $\gamma_w =$ const and thus

$$h_p = w/\gamma_w.$$

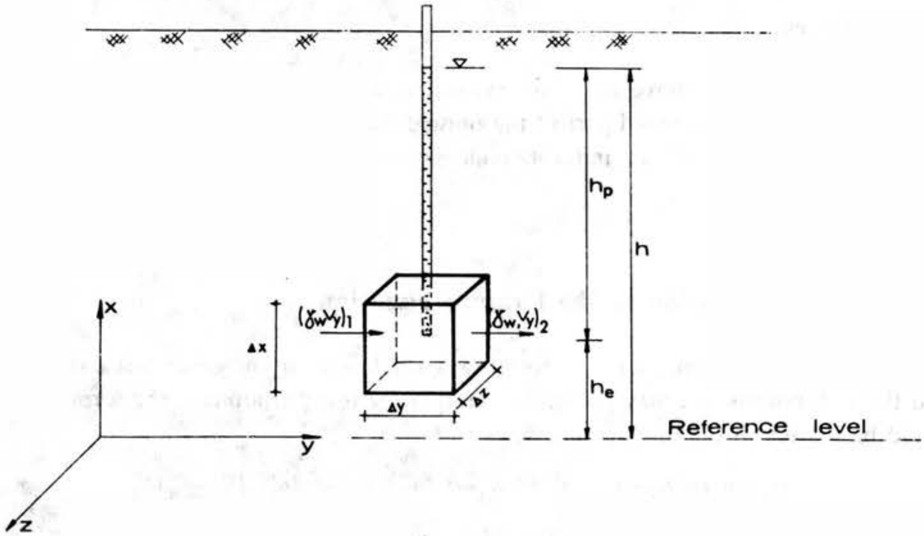

FIG. 2.

Consider now an elementary volume within a mass of soil through which fluid flow is taking place (Fig. 2). The discussion may be confined to the consideration of a flow of water which is the fluid of most interest in soil engineering. In general, water may flow into or out of the element as well as may be stored (positively or negatively) within this element during a time interval.

Consider first the flow in the y-direction. The weight flux through the left face into the element is $(\gamma_w v_y)_1 \Delta x \Delta z$, whereas the weight flux through the right face out of the element is $(\gamma_w v_y)_2 \Delta x \Delta z$. If it is now assumed that $\gamma_w v_y$ is a continuous, differentiable function of the coordinates, we may use Taylor's expansion in order to express $(\gamma_w v_y)_2$ into $(\gamma_w v_y)_1$ and its derivatives. Taking only the first two terms of this expansion we obtain

$$(\gamma_w v_y)_2 = (\gamma_w v_y)_1 + \frac{\partial(\gamma_w v_y)}{\partial y} \Delta y + \ldots .$$

Thus the storage (or loss) of weight due to the flow in the y-direction in a time interval is given by

$$[(\gamma_w v_y)_2 - (\gamma_w v_y)_1] \Delta x \Delta z = \frac{\partial(\gamma_w v_y)}{\partial y} \Delta x \Delta y \Delta z .$$

Similar expressions can be found when considering the flow in x and z-directions. Hence the total rate of storage or loss of water in the element $\partial W/\partial t$ can be mathematically expressed as

$$\frac{\partial W}{\partial t} = \frac{\partial(\gamma_w v_x)}{\partial x} + \frac{\partial(\gamma_w v_y)}{\partial y} + \frac{\partial(\gamma_w v_z)}{\partial z},$$

where W is the weight of water stored in the element per unit volume of this element.

Now, substituting Darcy's law into the above derived equation we have

$$\frac{\partial W}{\partial t} = -\frac{\partial}{\partial x}\left(\gamma_w k_x \frac{\partial h}{\partial x}\right) - \frac{\partial}{\partial y}\left(\gamma_w k_y \frac{\partial h}{\partial y}\right) - \frac{\partial}{\partial z}\left(\gamma_w k_z \frac{\partial h}{\partial z}\right)$$

or with the assumption of constant coefficients of permeability and by expanding the derivatives

$$\frac{\partial W}{\partial t} = -\left(\gamma_w k_x \frac{\partial^2 h}{\partial x^2} + \gamma_w k_y \frac{\partial^2 h}{\partial y^2} + \gamma_w k_z \frac{\partial^2 h}{\partial z^2}\right) - \left(k_x \frac{\partial \gamma_w}{\partial x}\frac{\partial h}{\partial x} + k_y \frac{\partial \gamma_w}{\partial y}\frac{\partial h}{\partial y} + k_z \frac{\partial \gamma_w}{\partial z}\frac{\partial h}{\partial z}\right).$$

The second group of terms of the right-hand side is very small compared with the first group. This follows from the low compressibility of water which results in $\gamma_w \approx$ const for the majority of soil mechanics problems. Hence the above equation can be written as

$$\frac{\partial W}{\partial t} = -\left(\gamma_w k_x \frac{\partial^2 h}{\partial x^2} + \gamma_w k_y \frac{\partial^2 h}{\partial y^2} + \gamma_w k_z \frac{\partial^2 h}{\partial z^2}\right)$$

or, regarding the definition of the groundwater head h,

$$\frac{\partial W}{\partial t} = -\left(k_x \frac{\partial^2 w}{\partial x^2} + k_y \frac{\partial^2 w}{\partial y^2} + k_z \frac{\partial^2 w}{\partial z^2}\right) = \frac{1}{\gamma_w}\frac{\partial V}{\partial t},$$

where $\partial V/\partial t$ denotes the rate of change of water volume in an elementary volume of the soil mass.

Now the general balance equation, considered at the beginning of this Section, can be completed by the relation derived above. It is worthy to note that the time cannot be directly eliminated from our considerations. Dividing both sides of the balance equation by Δt, we obtain

$$\frac{\partial \varepsilon_v}{\partial t} - \frac{n}{K_p}\frac{\partial w}{\partial t} - (1-n)/K_s \frac{\partial w}{\partial t} - \frac{\partial}{\partial t}(\sigma'_x + \sigma'_y + \sigma'_z)/3K_s = \frac{\partial V}{\partial t}$$

and when the solid phase compressibility is negligible

$$\frac{\partial \varepsilon_v}{\partial t} - \frac{n}{K_p}\frac{\partial w}{\partial t} = \frac{\partial V}{\partial t}.$$

Finally, accounting for the expression derived above for $\partial V/\partial t$, in the plane strain conditions we obtain

$$\frac{\partial}{\partial t}(\varepsilon_x + \varepsilon_y) - \frac{n}{K_p}\frac{\partial w}{\partial t} + \frac{k_x}{\gamma_w}\frac{\partial^2 w}{\partial x^2} + \frac{k_y}{\gamma_w}\frac{\partial^2 w}{\partial y^2} = 0$$

i.e. the relation (3.1) discussed further in Sect. 3.

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