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Harmonic state in an elastic dielectric

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A CENTROSYMMETRIC isotropic elastic dielectric is considered under the collective influence of an external body force, an applied electric field and a charge distribution, each serving as a source function with an arbitrary spatial dependence that is, generally, anisotropic. Its time dependence is subsequently specified as being harmonic. Exact and asymptotic radiation conditioned solutions, which must normally be quasi-isotropic, are then secured simultaneously in 2- and 3-dimensions for the ultimate or harmonic state. On a wave kinematical interpretation, the group velocity (with which energy is transmitted to sustain asymptotically dominant effects) points radially outwards — a material independent outcome. The phase velocity also points radially outwards; hovewer, material parameters dictate this outcome. A detailed application is made to the problem for an axisymmetric charge distribution together with an electric field and a body force, each having axisymmetric transverse, azimuthal and axial components.

Rozważono centrosymetryczny izotropowy dielektryk sprężysty poddany wspólnemu działaniu zewnętrznej siły masowej, przyłożonego pola elektrycznego oraz rozkładu ładunków, przy czym każdy z tych czynników stanowi funkcję źródeł o dowolnym, w ogólności anizotropowym, rozkładzie przestrzennym. Zależność od czasu przyjęto jako harmoniczną. Otrzymuje się wtedy rozwiązania ścisłe i asymptotyczne rozwiązania dwu- i trójwymiarowe. W interpretacji kinematyki falowej prędkość grupowa, z którą przenoszona jest energia, skierowana jest radialnie na zewnątrz, co jest wynikiem niezależnym od własności materiałowych. Wektor prędkości fazowej jest również skierowany radialnie na zewnątrz, co jednak wynika z przyjętych parametrów materiałowych. Szczegółowo rozpatrzono zastosowanie przedstawionej metody do rozwiązania zagadnienia osiowo-symetrycznego rozkładu ładunków oraz pola elektrycznego i sił masowych, z których każde charakteryzuje się osiowo-symetrycznym rozkładem składowych poprzecznych, azymutalnych i osiowych.

Рассмотрен центральносимметричный изотропный упругий диэлектрик, подвергнутый общему действию внешней массовой силы, приложенного электрического поля и распределения зарядов, причем каждый из этих факторов составляет функцию источников с произвольным, в общем анизотропным, пространственным распределением. Зависимость от времени принята гармонической. Получаются тогда точные и асимптотические решения двух и трехмерные решения. В интерпретации волновой кинематики групповая скорость, с которой переносится энергия, направлена радиально наружу, что является результатом независящим от материальных свойств. Вектор фазовой скорости тоже направлен радиально наружу, что однако следует из принятых материальных параметров. Подробно рассмотрено применение представленного метода в решении задачи осесимметричного распределения зарядов, а также электрического поля и массовых сил, каждое из которых характеризуется осесимметричным распределением поперечных, азимутных и осевых составляющих.

1. Introduction

THIS PAPER deals with an induced harmonic state in a centrosymmetric isotropic elastic dielectric whose variations obey MINDLIN's [1] extended version of TOUPIN's [2] equations; the extension was designed to include the polarization gradient. Such a harmonic state represents an ultimate attainment of disturbances created by sources pulsating steadily with a common frequency.

The present problem, which covers both 2- and 3-dimensional cases, is tackled as follows. The governing equations are first dissociated and reorganized into two inhomogeneous matrix systems, one satisfied by a column of two solenoidal vectors and the other by a column of three scalars. The inhomogeneities arise from the sources. These correspond to an external body force, an applied electric field and a charge distribution. They are all assumed to be purely pulsatory. However, their spatial functions are arbitrary and, generally, anisotropic. Hence the disturbances they generate are expected to be not isotropic but quasi-isotropic. Both matrix systems are next related to two column equations, each involving a determinantal scalar operator and to which a technique proposed by the present author [3] is applicable. That technique, which has only recently been applied [4] to the associated harmonic state problem of micropolar elastodynamics, accommodates a radiation condition. This essentially prevents reception of any free wave from infinity. Consequently, all detectable perturbations originate from the sources.

Not all roots to characteristic equations for the determinantal operators can admit contributions into the observation field. Furthermore among those that do, real and complex roots contribute terms with different symbolic representations. They also differ within a physical context. Thus, for example, the radiation condition governs the admissibility of any real root contribution; this propagates as a wave quantity which, though normally subjected to a moderate algebraic attenuation, nevertheless dominates at long ranges. This presumably occurs through sustenance by energy transported with the group velocity [5] from the sources. On the other hand, the admissibility of any complex root contribution depends not on the radiation condition but on a stability hypothesis coupled to a convergence rule of contour integration [3]. Such a contribution decays exponentially at long ranges where it is thus negligible; note, in particular, that like the induced wave it cannot originate at infinity. It therefore becomes significant to distinguish between real and complex roots, and to question admissibility. For this purpose one can incorporate a criterion of SCHWARTZ [6] on positive definiteness pertaining to an energy density.

Schwartz's paper focusses on the static equilibrium state based again on the Mindlin-Toupin equations. Working along different lines, CHOWDHURY and GLOCKNER [7] have formulated Galerkin-type representations by means of the method of associated matrices and obtained three separate categories of 3-dimensional, harmonic state fundamental solutions corresponding to a concentrated force, a concentradet electric field and a concentrated charge. Other investigations on the elastic dielectric theory include those of TOUPIN [8], MINDLIN and TOUPIN [9] and MINDLIN [10, 11].

2. Separated matrix systems

Within a centrosymmetric isotropic elastic dielectric, the displacement $\mathbf{u}$, polarization $\mathbf{P}$ and Maxwell's potential ϕ generated by the combination of an external body force $\mathbf{F}^0$, applied electric field $\mathbf{E}^0$ and a charge with density distribution D^0 are governed by [1]

$$(2.1)\qquad c_{44}\nabla^2\mathbf{u}+(c_{12}+c_{44})\nabla\nabla\cdot\mathbf{u}+d_{44}\nabla^2\mathbf{P}+(d_{12}+d_{44})\nabla\nabla\cdot\mathbf{P}+\mathbf{F}^0=\varrho\mathbf{u}_{tt},$$

(2.2) $$d_{44}\nabla^2\mathbf{u}+(d_{12}+d_{44})\nabla\nabla\cdot\mathbf{u}+(b_{44}+b_{77})\nabla^2\mathbf{P}+(b_{12}+b_{44}-b_{77})\nabla\nabla\cdot\mathbf{P}+\mathbf{E}^0 = a\mathbf{P}+\nabla\phi,$$

(2.3) $$\nabla\cdot\mathbf{P}+\mathsf{D}^0 = \varepsilon_0\nabla^2\phi,$$

where a, b_{12}, b_{44}, b_{77}, c_{12}, c_{44}, d_{12}, d_{44} are material constants, ϱ is the density and ε_0 is an electrical permittivity.

We can extract a 2×2 system for a 2×1 column of solenoidal vectors by operating on Eqs. (2.1) and (2.2) with the curl, as well as a separate 3×3 system for a 3×1 column of scalars by operating on Eqs. (2.1) and (2.2) with the divergence and admitting Eq. (2.3).

Thus, if

(2.4) $$L_1 \equiv (\partial^2/\partial t^2, \nabla^2) = \varrho\partial^2/\partial t^2 - c_{44}\nabla^2, \quad L_2 \equiv L_2(\nabla^2) = a-(b_{44}+b_{77})\nabla^2,$$

(2.5) $$L_3 \equiv L_3(\partial^2/\partial t^2, \nabla^2) = \varrho\partial^2/\partial t^2-(c_{12}+2c_{44})\nabla^2, \quad L_4 \equiv L_4(\nabla^2) = a-(b_{12}+2b_{44})\nabla^2,$$

then

(2.6) $$\mathbf{L}_1\begin{pmatrix}\nabla\times\mathbf{u}\\ \nabla\times\mathbf{P}\end{pmatrix} = \begin{pmatrix}\nabla\times\mathbf{F}^0\\ \nabla\times\mathbf{E}^0\end{pmatrix}, \quad \mathbf{L}_2\begin{pmatrix}\nabla\cdot\mathbf{u}\\ \nabla\cdot\mathbf{P}\\ \nabla^2\phi\end{pmatrix} = \begin{pmatrix}\mathsf{D}^0\\ \nabla\cdot\mathbf{F}^0\\ \nabla\cdot\mathbf{E}^0\end{pmatrix},$$

where the matrix operators

(2.7) $$\mathbf{L}_1 \equiv \mathbf{L}_1(\partial^2/\partial t^2, \nabla^2) = \begin{pmatrix}L_1 & -d_{44}\nabla^2\\ -d_{44}\nabla^2, & L_2\end{pmatrix},$$

(2.8) $$\mathbf{L}_2 \equiv \mathbf{L}_2(\partial^2/\partial t^2, \nabla^2) = \begin{pmatrix}0 & -1 & \varepsilon_0\\ L_3 & -(d_{12}+2d_{44})\nabla^2 & 0\\ -(d_{12}+2d_{44})\nabla^2 & L_4 & 1\end{pmatrix}.$$

Their determinants are

(2.9) $$\mathsf{L}_1 \equiv \mathsf{L}_1(\partial^2/\partial t^2, \nabla^2) = \det\mathbf{L}_1 = L_1L_2-d_{44}^2\nabla^4,$$

(2.10) $$\mathsf{L}_2 \equiv \mathsf{L}_2(\partial^2/\partial t^2, \nabla^2) = \det\mathbf{L}_2 = \varepsilon_0L_5+L_3,$$

with

(2.11) $$L_5 \equiv L_5(\partial^2/\partial t^2, \nabla^2) = L_3L_4-(d_{12}+2d_{44})^2\nabla^4.$$

We next introduce a column $\mathbf{X}_1^0$ of two vectors and another column $\mathbf{X}_0^2$ of one scalar and two vectors. Suppose these satisfy

(2.12) $$\mathsf{L}_\nu\mathbf{X}_\nu^0 = \mathbf{Y}_\nu^0 \quad (\nu = 1, 2),$$

with

(2.13) $$\mathbf{Y}_1^0 = \begin{pmatrix}\mathbf{F}^0\\ \mathbf{E}^0\end{pmatrix}, \quad \mathbf{Y}_0^2 = \begin{pmatrix}\mathsf{D}^0\\ \mathbf{F}^0\\ \mathbf{E}^0\end{pmatrix}.$$

Then it follows from Eq. (2.6) that if

(2.14) $$\operatorname{adj}\mathbf{L}_1 = \begin{pmatrix}L_2 & d_{44}\nabla^2\\ d_{44}\nabla^2 & L_1\end{pmatrix},$$

$$\text{(2.15)} \qquad \operatorname{adj}\mathsf{L}_2 = \begin{pmatrix} -(d_{12}+2d_{44})\nabla^2 & 1+\varepsilon_0 L_4 & \varepsilon_0(d_{12}+2d_{44})\nabla^2 \\ -L_3 & \varepsilon_0(d_{12}+2d_{44})\nabla^2 & \varepsilon_0 L_3 \\ L_5 & (d_{12}+2d_{44})\nabla^2 & L_3 \end{pmatrix},$$

these being, respectively, the adjoints of L_1 and L_2, we have

$$\text{(2.16)} \qquad \begin{pmatrix} \nabla\times\mathbf{u} \\ \nabla\times\mathbf{P} \end{pmatrix} = \operatorname{adj}\mathsf{L}_1 \begin{pmatrix} \nabla\times & \mathbf{0}\times \\ \mathbf{0}\times & \nabla\times \end{pmatrix} \mathbf{X}_1^0,$$

$$\text{(2.17)} \qquad \begin{pmatrix} \nabla\cdot\mathbf{u} \\ \nabla\cdot\mathbf{P} \\ \nabla^2\phi \end{pmatrix} = \operatorname{adj}\mathsf{L}_2 \begin{pmatrix} 1 & \mathbf{0}\cdot & \mathbf{0}\cdot \\ 0 & \nabla\cdot & \mathbf{0}\cdot \\ 0 & \mathbf{0}\cdot & \nabla\cdot \end{pmatrix} \mathbf{X}_2^0,$$

which will enable the left columns to be computed from the solutions to Eq. (2.12).

3. The harmonic state

Throughout this paper we are solely concerned with the ultimate state induced by purely harmonic anisotropic sources:

$$\text{(3.1)} \qquad \mathsf{D}^0 = \mathsf{D}(\mathbf{x})\exp(-i\omega t), \quad \mathbf{E}^0 = \mathbf{E}(\mathbf{x})\exp(-i\omega t), \quad \mathbf{F}^0 = \mathbf{F}(\mathbf{x})\exp(-i\omega t),$$

ω being a real frequency. Thus if

$$\text{(3.2)} \qquad \mathbf{Y}_1(\mathbf{x}) = \begin{pmatrix} \mathbf{F} \\ \mathbf{E} \end{pmatrix}, \quad \mathbf{Y}_2(\mathbf{x}) = \begin{pmatrix} \mathsf{D} \\ \mathbf{F} \\ \mathbf{E} \end{pmatrix},$$

then Eq. (2.12) becomes

$$\text{(3.3)} \qquad \mathsf{L}_\nu(\partial^2/\partial t^2, \nabla^2)\mathbf{X}_\nu^0 = \mathbf{Y}_\nu(\mathbf{x})\exp(-i\omega t) \quad (\nu = 1, 2),$$

a quasi-isotropic equation within a class studied by CHEE-SENG [3] in n-dimensions with $n \geqslant 2$; as such, its ultimate or harmonic state solution which must satisfy a radiation condition is

$$\text{(3.4)} \qquad \mathbf{X}_\nu^0 = \exp(-i\omega t)\,\mathsf{M}_\nu^{-1}[\mathbf{Y}_\nu],$$

with

$$\text{(3.5)} \qquad \mathsf{M}_\nu^{-1}[\mathbf{Y}_\nu] = \lim_{\varepsilon\to 0_+} \mathbf{X}_{\nu,\varepsilon} = \lim_{\varepsilon\to 0_+} \mathsf{M}_{\nu,\varepsilon}^{-1}[\mathbf{Y}_\nu]$$

denoting a limit of the inverse to the equation

$$\text{(3.6)} \qquad \mathsf{M}_{\nu,\varepsilon}\mathbf{X}_{\nu,\varepsilon} = \mathbf{Y}_\nu(\mathbf{x}),$$

whose operator $\mathsf{M}_{\nu,\varepsilon} = \mathsf{L}_\nu(-(\omega+i\varepsilon)^2, \nabla^2)$.

The results of [3] have been recently applied to the harmonic problem of micropolar elastodynamics [4]. To apply those results to our present problem, we first need to consider the polynomials $\mathsf{L}_\nu(-\omega^2, -\alpha^2)(\nu = 1, 2)$. These are algebraic transforms of the operators expressed by Eqs. (2.9) and (2.10), accompanied by Eqs. (2.4), (2.5) and (2.11), viz.

$$\text{(3.7)} \qquad \mathsf{L}_1(\partial^2/\partial t^2, \nabla^2) = ac_{44}[A_1\nabla^4 + B_1(-\partial^2/\partial t^2)\nabla^2 + C_1(-\partial^2/\partial t^2)],$$

(3.8) $$\mathsf{L}_2(\partial^2/\partial t^2, \nabla^2) = (1+a\varepsilon_0)(c_{12}+2c_{44})[A_2\nabla^4+B_2(-\partial^2/\partial t^2)\nabla^2+C_2(-\partial^2/\partial t^2)],$$

where

(3.9) $$A_1 = (ac_{44})^{-1}[c_{44}(b_{44}+b_{77})-d_{44}^2],$$

(3.10) $$B_1(\omega^2) = \varrho\omega^2(ac_{44})^{-1}(b_{44}+b_{77})-1,$$

(3.11) $$C_1(\omega^2) = -\varrho\omega^2 c_{44}^{-1},$$

(3.12) $$A_2 = \varepsilon_0(1+a\varepsilon_0)^{-1}[b_{12}+2b_{44}-(c_{12}+2c_{44})^{-1}(d_{12}+2d_{44})^2],$$

(3.13) $$B_2(\omega^2) = \varrho\omega^2\varepsilon_0(b_{12}+2b_{44})(c_{12}+2c_{44})^{-1}(1+a\varepsilon_0)^{-1}-1,$$

(3.14) $$C_2(\omega^2) = -\varrho\omega^2(c_{12}+2c_{44})^{-1}.$$

Therefore, after factorization the polynomials

(3.15) $$\mathsf{L}_1(-\omega^2, -\alpha^2) = ac_{44}A_1(\alpha^2-\alpha_{1+}^2)(\alpha^2-\alpha_{1-}^2),$$

(3.16) $$\mathsf{L}_2(-\omega^2, -\alpha^2) = (1+a\varepsilon_0)(c_{12}+2c_{44})A_2(\alpha^2-\alpha_{2+}^2)(\alpha^2-\alpha_{2-}^2),$$

with

(3.17) $$\alpha_{\nu\pm}^2 = \frac{1}{2}A_\nu^{-1}\{B_\nu(\omega^2)\pm[B_\nu^2(\omega^2)-4A_\nu C_\nu(\omega^2)]^{\frac{1}{2}}\} \quad (\nu = 1, 2).$$

Now, by SCHWARTZ's [6] argument on the energy density it is necessary that

(3.18) $$A_1 > 0, \quad A_2 > 0.$$

Furthermore, from Eq. (2.1) $c_{44}\varrho^{-1}$ and $(c_{12}+2c_{44})\varrho^{-1}$ are in the classical elastodynamic theory squares of the equivoluminal and dilatational wave speeds, respectively, so that

(3.19) $$C_1(\omega^2) < 0, \quad C_2(\omega^2) < 0.$$

By Eq. (3.17), then, $\alpha_{\nu+}^2 > 0$ while $\alpha_{\nu-}^{2\ 2} < 0$. Hence the polynomial equation

(3.20) $$\mathsf{L}_\nu(-\omega^2, -\alpha^2) = 0 \quad (\nu = 1, 2)$$

has two distinct, symmetric real roots at $\alpha = |\alpha_{\nu+}|, -|\alpha_{\nu+}|$ plus two distinct, purely imaginary conjugate roots at $\alpha = i|a_{\nu-}|, -i|\alpha_{\nu-}|$.

According to [3] (§§ 2 and 4), among those four roots only that real root denoted by $\alpha_{\nu+} = = \alpha_{\nu+}(\omega)$ whose derivative

(3.21) $$\alpha_{\nu+}'(\omega) > 0,$$

together with the upper imaginary root $i|\alpha_{\nu-}| = \alpha_{\nu-}$, say, can contribute to the radiation conditioned solution for M_ν^{-1} $[\mathbf{Y}_\nu]$; precisely, if in n-dimensions with $n = 2$ or 3,

(3.22) $$S(\mathbf{x}; \alpha; \mathbf{Y}_\nu) = \left(\frac{\alpha}{2\pi}\right)^{\frac{1}{2}n-1} \int_{R_n} \mathbf{Y}_\nu(\mathbf{v}) \frac{H_{\frac{1}{2}n-1}^{(1)}(\alpha|\mathbf{x}-\mathbf{y}|)}{|\mathbf{x}-\mathbf{y}|^{\frac{1}{2}n-1}} d\mathbf{y},$$

wherein the integral ranges with $\mathbf{y}$ over the infinite n-space R_n and $H_{\frac{1}{2}n-1}^{(1)}(z)$ is a Hankel function, then

$$\text{(3.23)} \qquad \mathsf{M}_\nu^{-1}[\mathbf{Y}_\nu] = \frac{1}{2}\, i \sum_{\alpha=\alpha_{\nu_+},\,\alpha_{\nu_-}} \frac{\alpha S(\mathbf{x};\,\alpha;\,\mathbf{Y}_\nu)}{\partial \mathsf{L}\,(-\omega^2,\,-\alpha^2)/\partial\alpha},$$

$$\text{(3.24)} \qquad = \frac{1}{2}\, i\, \mathsf{A}_\nu [S(\mathbf{x};\,\alpha_{\nu_+};\,\mathbf{Y}_\nu) - S(\mathbf{x};\,\alpha_{\nu_-}\mathbf{Y}_\nu)] \quad (\nu = 1, 2),$$

$$\text{(3.25)} \qquad \mathsf{A}_1 \equiv \mathsf{A}_1(\omega^2) = \frac{1}{2}\,(ac_{44})^{-1}[B_1^1(\omega^2) - 4A_1\,C_1(\omega^2)]^{-\frac{1}{2}},$$

$$\text{(3.26)} \qquad \mathsf{A}_2 \equiv \mathsf{A}_2(\omega^2) = \frac{1}{2}\,(1+a\varepsilon_0)^{-1}(c_{12}+2c_{44})^{-1}[B_2^2(\omega^2) - 4A_2\,C_2(\omega^2)]^{-\frac{1}{2}}.$$

For computational purposes the expression (3.22) can be expanded into

$$\text{(3.27)} \qquad S(\mathbf{x};\,\alpha;\,\mathbf{Y}_\nu) = (\pi x)^{1-\frac{1}{2}n} \sum_{k=0}^{\infty} \Big\{ H^{(1)}_{\frac{1}{2}n+k-1}(\alpha x) \int_0^x S_k(\hat{\mathbf{x}};\,y;\,\mathbf{Y}_\nu) J_{\frac{1}{2}n+k-1}(\alpha y)\, y^{\frac{1}{2}n} dy$$

$$+ J_{\frac{1}{2}n+k-1}(\alpha x) \int_0^x S_k(\hat{\mathbf{x}};\,y;\mathbf{Y}_\nu) H^{(1)}_{\frac{1}{2}n+k-1}(\alpha y)\, y^{\frac{1}{2}n} dy \Big\},$$

an infinite series of Hankel and Bessel functions coupled to Hankel-type transforms of spherical integrals of the form

$$\text{(3.28)} \qquad S_k(\hat{\mathbf{x}};\,y;\,\mathbf{Y}_\nu) = \left(\frac{1}{2}n+k-1\right)\Gamma\left(\frac{1}{2}n-1\right) \int_\Omega \mathbf{Y}_\nu(y\boldsymbol{\xi})\, C_k^{\frac{1}{2}n-1}(\hat{\mathbf{x}}\cdot\boldsymbol{\xi})\, d\Omega,$$

which ranges with the unit position $\boldsymbol{\xi}$ over the surface Ω of the n-dimensional unit sphere (circle if $n = 2$); here $x = |\mathbf{x}|$, $\hat{\mathbf{x}} = \mathbf{x}x^{-1}$, and $C_k^{\frac{1}{2}n-1}$ denotes a Gegenbauer function. Regarding Eq. (3.24), $S(\mathbf{x};\,\alpha_{\nu_-};\,\mathbf{Y}_\nu)$, the contribution from that real root which satisfies Eq. (3.21) is determined by direct substitution of $\alpha = \alpha_{\nu_+}$ into Eq. (3.22) or Eq. (3.27). On the other hand, the complementary contribution from the upper imaginary root α_{ν_-} takes the forms

(3.29)

$$S(\mathbf{x};\,\alpha_{\nu_-};\,\mathbf{Y}_\nu) = \frac{2}{\pi i}\left(\frac{|\alpha_{\nu_-}|}{2\pi}\right)^{\frac{1}{2}n-1} \int_{R_n} \mathbf{Y}_\nu(\mathbf{y})\, \frac{K_{\frac{1}{2}n-1}(|\alpha_{\nu_-}||\mathbf{x}-\mathbf{y}|)}{|\mathbf{x}-\mathbf{y}|^{\frac{1}{2}n-1}}\, d\mathbf{y},$$

$$\text{(3.30)} \qquad = \frac{2}{\pi i}\,(\pi x)^{1-\frac{1}{2}n} \int_{k=0}^{\infty} \Big\{ K_{\frac{1}{2}n+k-1}(|\alpha_{\nu_-}|x) \int_0^x S_k(\hat{\mathbf{x}};\,y;\,\mathbf{Y}_\nu) I_{\frac{1}{2}n+k-1}(|\alpha_{\nu_-}|y)\, y^{\frac{1}{2}n} dy$$

$$+ I_{\frac{1}{2}n+k+1}(|\alpha_{\nu_-}|x) \int_x^\infty S_k(\hat{\mathbf{x}};\,y;\,\mathbf{Y}_\nu) K_{\frac{1}{2}n+k-1}(|\alpha_{\nu_-}|y)\, y^{\frac{1}{2}n} dy \Big\},$$

which follow from Eqs. (3.22) and (3.27) via the relations between $J_\mu(iz)$, $H_\mu^{(1)}(iz)$ and the respective modified Bessel functions $I_\mu(z)$, $K_\mu(z)$.

A general asymptotic representation ([3], Eq. (4.26)) is also applicable to $\mathsf{M}_\nu^{-1}[\mathbf{Y}_\nu]$. Thus, if the column

(3.31) $$\mathbf{Y}_\nu(\mathbf{x}) \equiv \mathbf{0} \text{ outside some finite region } \mathsf{R}_\nu,$$

and its Fourier transform

(3.32) $$\mathsf{F}(\boldsymbol{\alpha}; \mathbf{Y}_\nu) = (2\pi)^{-n} \int_{\mathsf{R}_\nu} \mathbf{Y}_\nu(\mathbf{y}) \exp(-i\boldsymbol{\alpha}\cdot\mathbf{y}) d\mathbf{y},$$

then far from R_ν,

(3.33) $$\mathsf{M}_\nu^{-1}[\mathbf{Y}_\nu] = \mathsf{A}_\nu (2\pi)^{\frac{1}{2}(n+1)} x^{-\frac{1}{2}(n-1)} \alpha_{\nu_+}^{\frac{1}{2}(n-3)} \mathsf{F}(\alpha_{\nu_+}\hat{\mathbf{x}}; \mathbf{Y}_\nu) \exp\left\{i\left[\alpha_{\nu_+} x - \frac{1}{4}(n-3)\pi\right]\right\} + \mathbf{0}\left(x^{-\frac{1}{2}(n+1)}\right).$$

This is dominated by an α_ν -independent term which attenuates like $x^{-\frac{1}{2}(n-1)}$. The net contribution from the upper imaginary root α_{ν_-} is negligible by virtue of an exponentially decaying factor.

3.1. 2-and 3-dimensional forms. Suppose $n = 3$. Then using the oscillatory and exponential forms for $H^{(1)}_{\frac{1}{2}}(z)$ and $K_{\frac{1}{2}}(z)$, we obtain from Eqs. (3.24), (3.22) and (3.29),

(3.34) $$\mathsf{M}_\nu^{-1}[\mathbf{Y}_\nu] = \frac{\mathsf{A}_\nu}{2\pi} \int_{R_3} \frac{\mathbf{Y}_\nu(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|} [\exp(i\alpha_{\nu_+}|\mathbf{x}-\mathbf{y}|) - \exp(-|\alpha_{\nu_-}||\mathbf{x}-\mathbf{y}|)] d\mathbf{y}.$$

Alternatively, $\mathsf{M}_\nu^{-1}[Y_\nu]$ can be evaluated from Eqs. (3.24), (3.27) and (3.30) together with Eq. (3.28). The latter can be made more explicit. Suppose the unit vectors

(3.35) $$\hat{\mathbf{x}} = (\sin\Theta\cos\theta, \sin\Theta\sin\theta, \cos\Theta) \quad (0 \leqslant \theta < 2\pi, 0 \leqslant \Theta \leqslant \pi),$$

(3.36) $$\boldsymbol{\xi} = (\sin\Theta'\cos\theta', \sin\Theta'\sin\theta', \cos\Theta') \quad (0 \leqslant \theta' < 2\pi, 0 \leqslant \Theta' \leqslant \pi).$$

Now, $C_k^{\frac{1}{2}}(\hat{\mathbf{x}}\cdot\boldsymbol{\xi}) = P_k(\hat{\mathbf{x}}\cdot\boldsymbol{\xi})$, a Legendre polynomial which $\equiv 1$ when $k = 0$; for $k \geqslant 1$, however, it can be expanded into a finite series by the addition rule ([12], 4.3). Thus, from Eq. (3.28),

(3.37) $$S_k(\hat{\mathbf{x}}; y; \mathbf{Y}_\nu) = \pi^{\frac{1}{2}}\left(k+\frac{1}{2}\right) \int_0^{2\pi} d\theta' \int_0^{\pi} \mathbf{Y}_\nu(y\boldsymbol{\xi}) P_k(\hat{\mathbf{x}}\cdot\boldsymbol{\xi}) \sin\Theta' d\Theta',$$

(3.38) $$= \frac{1}{2}\pi^{\frac{1}{2}} \int_0^{\pi} \sin\Theta' d\Theta' \int_0^{2\pi} \mathbf{Y}_\nu(y\boldsymbol{\xi}) d\theta' \quad (k = 0),$$

(3.39) $$= \frac{1}{2}\pi^{\frac{1}{2}}(2k+1) P_k(\cos\Theta) \int_0 P_k(\cos\Theta')\sin\Theta' d\Theta' \int_0^{2\pi} \mathbf{Y}_\nu(y\boldsymbol{\xi}) d\theta'$$
$$+ \pi^{\frac{1}{2}}(2k+1) \sum_{s=1}^{k} \frac{(k-s)!}{(k+s)!} P_k^s(\cos\Theta) \int_0^{\pi} P_k^s(\cos\Theta')\sin\Theta' d\Theta'$$
$$\times \int_0^{2\pi} \mathbf{Y}_\nu(y\boldsymbol{\xi}) \cos[s(\theta-\theta')] d\theta' \quad (k \geqslant 1),$$

P_k^s denoting an associated Legendre function.

For $n = 2$ we find instead

$$\mathsf{M}_\nu^{-1}[\mathbf{Y}_\nu] = \mathsf{A}_\nu \int_{R_2} \mathbf{Y}_\nu(\mathbf{y}) \left[\frac{1}{2} i H_0^{(1)}(\alpha_{\nu_+}|\mathbf{x}-\mathbf{y}|) - \pi^{-1} K_0(|\alpha_{\nu_-}||\mathbf{x}-\mathbf{y}|) \right] d; \tag{3.40}$$

moreover, writing

$$\hat{\mathbf{x}} = (\cos\theta, \sin\theta) \quad (0 \leqslant \theta < 2\pi), \qquad \boldsymbol{\xi} = (\cos\theta', \sin\theta') \quad (0 \leqslant \theta' < 2\pi), \tag{3.41}$$

we have ([3], Eqs. (4.11) (4.12))

$$S_0(\hat{\mathbf{x}}; y; \mathbf{Y}_\nu) = \int_0^{2\pi} \mathbf{Y}_\nu(y\boldsymbol{\xi}) d\theta', \tag{3.42}$$

$$S_k(\hat{\mathbf{x}}; y; \mathbf{Y}_\nu) = 2\int_0^{2\pi} \mathbf{Y}_\nu(y\boldsymbol{\xi}) \cos[k(\theta-\theta')] d\theta' \quad (k = 1, 2, \ldots). \tag{3.43}$$

4. Choice of α_{ν_+}

Now $\alpha_{\nu_+} (\neq 0)$ represents one of the two real roots to Eq. (3.20) and its choice must comply with Eq. (3.21) or, equivalently,

$$\frac{\omega}{\alpha_{\nu_+}} \frac{d\alpha_{\nu_+}^2}{d\omega^2} > 0. \tag{4.1}$$

Such a criterion actually follows from an incorporated radiation condition. From Eq. (3.17) we derive

$$\frac{d\alpha_{\nu_+}^2}{d\omega^2} = \frac{\alpha_{\nu_+}^2 B_\nu'(\omega^2) - C_\nu'(\omega^2)}{[B_\nu^2(\omega^2) - 4A_\nu C_\nu(\omega^2)]^{\frac{1}{2}}}, \tag{4.2}$$

wherein primes denote ω^2-derivatives. In particular, Eqs. (3.9)–(3.14) give

$$B_1'(\omega^2) = \varrho a^{-1} d_{44}^2 c_4^{-2} - A_1 C_1(\omega^2)\omega^{-2}, \tag{4.3}$$

$$B_2'(\omega^2) = \varrho\varepsilon_0(1+a\varepsilon_0)^{-1}(d_{12}+2d_{44})^2(c_{12}+2c_{44})^{-2} - A_2 C_2(\omega^2)\omega^{-2}, \tag{4.4}$$

$$C_\nu'(\omega^2) = C_\nu(\omega^2)\omega^{-2} \quad (\nu = 1, 2). \tag{4.5}$$

Normally $\varrho > 0$, and we assume that $\varepsilon_0 > 0$, $a > 0$. It then follows from Eqs. (4.2)–(4.5) together with Eqs. (3.18) and (3.19) that

$$\frac{d\alpha_{\nu_+}^2}{d\omega^2} > 0 \quad (\nu = 1, 2). \tag{4.6}$$

So, by Eq. (4.1) α_{ν_+} should be *that real root with the same sign as* ω.

We shall next attach a wave kinematical interpretation associated with α_{ν_+}. According to LIGHTHILL [5], there is a correspondence between the operator $\mathsf{L}_\nu(\partial^2/\partial t^2, \nabla^2)$ and a travelling wave configuration about a real frequency ω and a real wave vector $\boldsymbol{\alpha}$ necessarily governed by $\mathsf{L}_\nu(-\omega^2, -\boldsymbol{\alpha}^2) = 0$. The eikonal is then

$$\boldsymbol{\alpha}\cdot\mathbf{x} - \omega t, \tag{4.7}$$

while the phase velocity

$$\mathbf{v} = \omega\hat{\boldsymbol{\alpha}}|\boldsymbol{\alpha}|^{-1} \qquad (\hat{\boldsymbol{\alpha}} = \boldsymbol{\alpha}|\boldsymbol{\alpha}|^{-1}), \tag{4.8}$$

and the group velocity

$$\mathbf{V} = \nabla_{\alpha}\omega, \tag{4.9}$$

∇_{α} being the gradient operator in $\boldsymbol{\alpha}$-space. Evidently, $|\boldsymbol{\alpha}| = |\alpha_{\nu_+}|$. Whence we can show from Eq. (4.9) that

$$\mathbf{V} = \hat{\boldsymbol{\alpha}}\,\mathrm{sgn}\,\alpha'_{\nu_+}(\omega). \tag{4.10}$$

A travelling wave constituent of $\mathbf{X}^0_{\nu}$ is clearly discernible through the long range estimate (3.33) of its spatial factor; comparison of its eikonal with Eq. (4.7) discloses

$$\hat{\boldsymbol{\alpha}} \cdot \hat{\mathbf{x}} = \mathrm{sgn}\,\alpha_{\nu_+}. \tag{4.11}$$

Whereupon,

$$\mathbf{V} \cdot \hat{\mathbf{x}} = 1/\alpha'_{\nu_+}(\omega) = |\mathbf{V}|\,\mathrm{sgn}[\alpha'_{\nu_+}(\omega)]. \tag{4.12}$$

Also,

$$\mathbf{v} \cdot \hat{\mathbf{x}} = \omega\alpha_{\nu_+}^{-1} = |\mathbf{v}|\,\mathrm{sgn}(\omega\alpha_{\nu_+}^{-1}). \tag{4.13}$$

Therefore, the *criterion* (3.21) *requires the group velocity* $\mathbf{V}$ *to be directed radially outwards*; furthermore, since $\omega\alpha_{\nu_+}^{-1} > 0$, *the phase velocity* $\mathbf{v}$ *is likewise directed radially outwards.* Note that while the *outward* radial orientation of $\mathbf{V}$ is never influenced by material parameters and may, consequently, be envisaged as a fundamental principle, that of $\mathbf{v}$ relies on the inequality (4.6) which holds through the material coefficients. Such a wave configuration is quasi-isotropic- "quasi" because of the normally anisotropic factors $S_k(\hat{\mathbf{x}}; y; \mathbf{Y}_{\nu})$ and $\mathsf{F}(\alpha_{\nu_+}\hat{\mathbf{x}}; \mathbf{Y}_{\nu})$ involved with the respective exact and asymptotic representations (3.24) and (3.33). The anisotropy is imparted by a directionally dependent $\mathbf{Y}_{\nu}(\mathbf{x})$. Wave energy, released by the sources, gets transported with the group velocity to sustain the travelling wave constituent. Observe from Eqs. (4.6), (4.8) and (4.10) the relationship

$$(\mathbf{V} \cdot \mathbf{v})^{-1} = |\mathbf{V}|^{-1}|\mathbf{v}|^{-1} = \frac{d\alpha_{\nu_+}^2}{d\omega^2}, \tag{4.14}$$

the right side being given by Eq. (4.2).

5. Displacement, polarization and potential in a charge-free material

Let us next determine in the absence of a charge distribution

$$\mathsf{D}^0 \equiv 0, \tag{5.1}$$

the displacement and polarization vectors as well as the Maxwell's scalar potential for the ultimate state in terms of established harmonic inverses. First we can, by virtue of Eqs. (2.12) and (2.13), express

$$\text{(5.2)} \qquad \mathbf{X}_1^0 = \begin{pmatrix} \mathbf{X}_{11}^0 \\ \mathbf{X}_{12}^0 \end{pmatrix}, \quad \mathbf{X}_2^0 = \begin{pmatrix} 0 \\ \mathbf{X}_{22}^0 \\ \mathbf{X}_{23}^0 \end{pmatrix},$$

whose elements satisfy

$$\text{(5.3)} \qquad \mathsf{L}_1 \mathbf{X}_{11}^0 = \mathbf{F}^0 = \mathsf{L}_2 \mathbf{X}_{22}^0, \quad \mathsf{L}_1 \mathbf{X}_{12}^0 = \mathbf{E}^0 = \mathsf{L}_2 \mathbf{X}_{23}^0.$$

Then from Eqs. (2.15) and (2.17),

$$\text{(5.4)} \qquad \nabla^2 \phi = (d_{12}+2d_{44}) \nabla^2 \nabla \cdot \mathbf{X}_{22}^0 M L_3 \nabla \cdot \mathbf{X}_{23}^0,$$

$$\text{(5.5)} \qquad \begin{pmatrix} \nabla \cdot \mathbf{u} \\ \nabla \cdot \mathbf{P} \end{pmatrix} = \mathbf{N}(\partial^2/\partial t^2, \nabla^2) \begin{pmatrix} \nabla \cdot \mathbf{X}_{22}^0 \\ \nabla \cdot \mathbf{X}_{23}^0 \end{pmatrix},$$

where

$$\text{(5.6)} \qquad \mathbf{N}(\partial^2/\partial t^2, \nabla^2) = \begin{pmatrix} 1+\varepsilon_0 L_4 & \varepsilon_0 (d_{12}+2d_{44})\nabla^2 \\ \varepsilon_0 (d_{12}+2d_{44})\nabla^2 & \varepsilon_0 L_3 \end{pmatrix}.$$

Also, by taking the curl of each vectorial element in Eq. (2.16) and then incorporating Eq. (5.5), we get

$$\text{(5.7)} \qquad \begin{pmatrix} \nabla^2 \mathbf{u} \\ \nabla^2 \mathbf{P} \end{pmatrix} = \mathbf{N}(\partial^2/\partial t^2, \nabla^2) \begin{pmatrix} \nabla\nabla \cdot \mathbf{X}_{21}^0 \\ \nabla\nabla \cdot \mathbf{X}_{23}^0 \end{pmatrix} - \operatorname{adj} \mathbf{L}_1(\partial^2/\partial t^2, \nabla^2) \begin{pmatrix} \nabla \times (\nabla \times \mathbf{X}_{11}^0) \\ \nabla \times (\nabla \times \mathbf{X}_{12}^0) \end{pmatrix}.$$

We now resolve the spatial source factors $\mathbf{E}$ and $\mathbf{F}$ into irrotational and solenoidal constituents:

$$\text{(5.8)} \qquad \mathbf{E}(\mathbf{x}) = \nabla E_1(\mathbf{x}) + \nabla \times \mathbf{E}_2(\mathbf{x}), \quad \mathbf{F}(\mathbf{x}) = \nabla F_1(\mathbf{x}) + \nabla \times \mathbf{F}_2(\mathbf{x}).$$

It then follows from Eqs. (3.1), (3.3) and (3.4) that the harmonic state radiation conditioned solutions to Eq. (5.3) are

$$\text{(5.9)} \qquad \mathbf{X}_{11}^0 = \exp(-i\omega t)\{\nabla \mathsf{M}_1^{-1}[F_1] + \nabla \times \mathsf{M}_1^{-1}[\mathbf{F}_2]\},$$

$$\text{(5.10)} \qquad \mathbf{X}_{12}^0 = \exp(-i\omega t)\{\nabla \mathsf{M}_1^{-1}[E_1] + \nabla \times \mathsf{M}_1^{-1}[\mathbf{E}_2]\},$$

$$\text{(5.11)} \qquad \mathbf{X}_{22}^0 = \exp(-i\omega t)\{\nabla \mathsf{M}_2^{-1}[F_1] + \nabla \times \mathsf{M}_2^{-1}[\mathbf{F}_2]\},$$

$$\text{(5.12)} \qquad \mathbf{X}_{23}^0 = \exp(-i\omega t)\{\nabla \mathsf{M}_2^{-1}[E_1] + \nabla \times \mathsf{M}_2^{-1}[\mathbf{E}_2]\}.$$

Note, for example, that the scalar $\mathsf{M}_\nu^{-2}[F_1]$ and the vector $\mathsf{M}_\nu^{-1}[\mathbf{F}_2]$ are derived from the column $\mathsf{M}_\nu^{-1}[\mathbf{Y}_\nu]$ by substituting F_1 and $\mathbf{F}_2$ respectively for $\mathbf{Y}_\nu$. The solutions we seek are, from Eqs. (5.4), (5.7) and (5.9)–(5.12),

$$\text{(5.13)} \qquad \phi = \exp(-i\omega t)\{(d_{12}+2d_{44})\nabla^2 \mathsf{M}_2^{-1}[F_1] + L_3(-\omega^2, \nabla^2)\mathsf{M}_2^{-1}[E_1]\},$$

$$\text{(5.14)} \qquad \begin{pmatrix} \mathbf{u} \\ \mathbf{P} \end{pmatrix} = \exp(-i\omega t)\left\{\mathbf{N}(-\omega^2, \nabla^2) \begin{pmatrix} \nabla \mathsf{M}_2^{-1}[F_1] \\ \nabla \mathsf{M}_2^{-1}[E_1] \end{pmatrix} + \operatorname{adj} \mathbf{L}_1(-\omega^2, \nabla^2) \begin{pmatrix} \nabla \times \mathsf{M}_1^{-1}[\mathbf{F}_2] \\ \nabla \times \mathsf{M}_1^{-1}[\mathbf{E}_2] \end{pmatrix}\right\}.$$

6. Introduction by axisymmetric elements

Suppose the charge distribution is axisymmetric, and that the applied electric and external body force act with axisymmetric components along the transverse, azimuthal and axial directions of $\mathbf{i}_r = (\cos\theta, \sin\theta, 0)$, $\mathbf{i}_\theta = (-\sin\theta, \cos\theta, 0)$ and $\mathbf{i}_3 =$

$= (0, 0, 1)$, with reference to the invariant Cartesian frame in R_3. Precisely, if $r = (x_1^2+x_2^2)^{\frac{1}{2}}$, then

$$\mathsf{D}(\mathbf{x}) = \mathsf{D}(r, x_3), \tag{6.1}$$

$$\mathbf{E}(\mathbf{x}) = E_r(r, x_3)\mathbf{i}_r + E_\theta(r, x_3)\mathbf{i}_\theta + E_3(r, x_3)\mathbf{i}_3, \tag{6.2}$$

$$\mathbf{F}(\mathbf{x}) = F_r(r, x_3)\mathbf{i}_r + F_\theta(r, x_3)\mathbf{i}_\theta + F_3(r, x_3)\mathbf{i}_3. \tag{6.3}$$

To apply Eqs. (3.38) and (3.39), we account for Eq. (3.2) and note, in particular, that

$$\begin{aligned} \mathbf{E}(y\boldsymbol{\xi}) = (&E_r(y\sin\Theta', y\cos\Theta')\cos\theta' - E(y\sin\Theta', y\cos\Theta')\sin\theta', \\ =\ &E_r(y\sin\Theta', y\cos\Theta')\sin\theta' + E_\theta(y\sin\Theta', y\cos\Theta')\cos\theta', \\ =\ &E_3(y\sin\Theta', y\cos\Theta')), \end{aligned} \tag{6.4}$$

with a similar expression holding for $\mathbf{F}(y\boldsymbol{\xi})$. Thus, if the column $\mathbf{Y}_\nu$ is now substituted in turn by the vectors $\mathbf{E}$, $\mathbf{F}$ and the scalar D, Eqs. (3.38) and (3.39) lead to

$$S_k(\hat{\mathbf{x}}; y; \mathbf{E}) = S_k^1(\Theta; y; E_r)\mathbf{i}_r + S_k^1(\Theta; y; E_\theta)\mathbf{i}_\theta + S_k^0(\Theta; y; E_3)\mathbf{i}_3, \tag{6.5}$$

$$S_k(\hat{\mathbf{x}}; y; \mathbf{F}) = S_k^1(\Theta; y; F_r)\mathbf{i}_r + S_k^1(\Theta; y; F_\theta)\mathbf{i}_\theta + S_k^0(\Theta; y; F_3)\mathbf{i}_3, \tag{6.6}$$

$$S_k(\hat{\mathbf{x}}; y; \mathsf{D}) = S_k^0(\Theta; y; \mathsf{D}), \tag{6.7}$$

where, for an axisymmetric function $X = X(r, x_3)$, the transforms

$$S_k^0(\Theta; y; X) = \pi^{3/2}\int_0^\pi X(y\sin\Theta', y\cos\Theta')\sin\Theta' d\Theta' \quad (k=0), \tag{6.8}$$

$$= \pi^{3/2}(2k+1)P_k(\cos\Theta)\int_0^\pi X(y\sin\Theta', y\cos\Theta')P_k(\cos\Theta') \times \sin\Theta' d\Theta' \quad (k \geqslant 1), \tag{6.9}$$

$$S_k^1(\Theta; y; X) = 0 \quad (k=0), \tag{6.10}$$

$$= \pi^{3/2}\frac{(2k+1)}{k(k+1)}P_k^1(\cos\Theta)\int_0^\pi X(y\sin\Theta', y\cos\Theta')P_k^1(\cos\Theta') \times \sin\Theta' d\Theta' \quad (k \geqslant 1). \tag{6.11}$$

If these are used to define, for $\mu = 0, 1$ and $\nu = 1, 2$,

$$\begin{aligned} S^\mu(x, \Theta; \alpha_{\nu+}; X) = (\pi x)^{-\frac{1}{2}}\sum_{k=0}^{\infty}\Big\{&H_{k+\frac{1}{2}}^{(1)}(\alpha_{\nu+}x)\int_0^x S_k^\mu(\Theta; y; X)J_{k+\frac{1}{2}}(\alpha_{\nu+}y)y^{3/2}dy \\ &+ J_{k+\frac{1}{2}}(\alpha_{\nu+}x)\int_x^\infty S_k^\mu(\Theta; y; X)H_{k+\frac{1}{2}}^{(1)}(\alpha_{\nu+}y)y^{3/2}dy\Big\}, \end{aligned} \tag{6.12}$$

$$\begin{aligned} S^\mu(x, \Theta; \alpha_{\nu-}; X) = \frac{2}{\pi i}(\pi x)^{-\frac{1}{2}}\sum_{k=0}^{\infty}\Big\{&K_{k+\frac{1}{2}}(|\alpha_{\nu-}|x)\int_0^x S_k^\mu(\Theta; y; X)I_{k+\frac{1}{2}}(|\alpha_{\nu-}|y)\, y^{3/2}dy \\ &+ I_{k+\frac{1}{2}}(|\alpha_{\nu-}|x)\int_x^\infty S_k^\mu(\Theta; y; X)K_{k+\frac{1}{2}}(|\alpha_{\nu-}|y)\, y^{3\ 2}dy\Big\}. \end{aligned} \tag{6.13}$$

and if

$$T_\nu^\mu(x, \Theta; X) = \frac{1}{2} i \mathsf{A}_\nu [S^\mu(x, \Theta; \alpha_{\nu_+}; X) - S^\mu(x, \Theta; \alpha_{\nu_-}; X)], \tag{6.14}$$

then we deduce from Eqs. (3.2), (3.24), (3.27), (3.30) and (6.5)–(6.7) that

$$\mathsf{M}_1^{-1}[\mathbf{Y}_1] = \begin{pmatrix} T_1^1(x, \Theta; F_r)\mathbf{i}_r + T_1^1(x, \Theta; F_\theta)\mathbf{i}_\theta + T_1^0(x, \Theta; F_3)\mathbf{i}_3 \\ T_1^1(x, \Theta; E_r)\mathbf{i}_r + T_1^1(x, \Theta; E_\theta)\mathbf{i}_\theta + T_1^0(x, \Theta; E_3)\mathbf{i}_3 \end{pmatrix}, \tag{6.15}$$

$$\mathsf{M}_2^{-1}[\mathbf{Y}_2] = \begin{pmatrix} T_2^0(x, \Theta; \mathsf{D}) \\ T_2^1(x, \Theta; F_r)\mathbf{i}_r + T_2^1(x, \Theta; F_\theta)\mathbf{i}_\theta + T_2^0(x, \Theta; F_3)\mathbf{i}_3 \\ T_2^1(x, \Theta; E_r)\mathbf{i}_r + T_2^1(x, \Theta; E_\theta)\mathbf{i}_\theta + T_2^0(x, \Theta; E_3)\mathbf{i}_3 \end{pmatrix} \tag{6.16}$$

Clearly, $T_\nu^\mu(x, \theta, X)$ is axisymmetric, in which event

$$\begin{pmatrix} \nabla \times \mathbf{0} \times \\ \mathbf{0} \times \nabla \times \end{pmatrix} \mathsf{M}_1^{-1}[\mathbf{Y}_1] = \begin{pmatrix} U_r(r, x_3; F_\theta)\mathbf{i}_r + U_\theta(r, x_3; F_r, F_3)\mathbf{i}_\theta + U_3(r, x_3; F_\theta)\mathbf{i}_3 \\ U_r(r, x_3; E_\theta)\mathbf{i}_r + U_\theta(r, x_3; E_r, E_3)\mathbf{i}_\theta + U_3(r, x_3; E_\theta)\mathbf{i}_3 \end{pmatrix}, \tag{6.17}$$

where

$$U_r(r, x_3; X) = -\frac{\partial T_2^1}{\partial x_3}(x, \Theta; X), \tag{6.18}$$

$$U_3(r, x_3; X) = \frac{1}{r}\frac{\partial}{\partial r}[r T_1^1(x, \Theta; X)], \tag{6.19}$$

$$U_\theta(r, x_3; X, Y) = \frac{\partial T_1^1}{\partial x_3}(x, \Theta; X) - \frac{\partial T_1^0}{\partial r}(x, \Theta; Y), \tag{6.20}$$

with $Y = Y(r, x_3)$, another axisymmetric function. Likewise, defining

$$U(r, x_3; X, Y) = \frac{1}{r}\frac{\partial}{\partial r}[r T_2^1(x, \Theta; X)] + \frac{\partial T_2^0}{\partial x_3}(x, \Theta; Y), \tag{6.21}$$

we obtain

$$\begin{pmatrix} 1 & \mathbf{0}\cdot\mathbf{0}\cdot \\ 0 & \nabla\cdot\mathbf{0}\cdot \\ \mathbf{0} & \mathbf{0}\cdot\nabla\cdot \end{pmatrix} \mathsf{M}_2^{-1}[\mathbf{Y}_2] = \begin{pmatrix} T_2^0(x, \Theta; \mathsf{D}) \\ U(r, x_3; F_r, F_3) \\ U(r, x_3; E_r, E_3) \end{pmatrix}, \tag{6.22}$$

a column of axisymmetric elements.

Consider the operation by ∇^2 on a vector, such as our present $\mathbf{E}$, whose transverse, azimuthal and axial components are axisymmetric. In this case,

$$\nabla^2\mathbf{E} = \mathbf{i}_r(\nabla^2 - r^{-2})E_r + \mathbf{i}_\theta(\nabla^2 - r^{-2})E_\theta + \mathbf{i}_3\nabla^2 E_3; \tag{6.23}$$

on the right side, effectively,

$$\nabla^2 = \partial^2/\partial r^2 + r^{-1}\partial/\partial r + \partial^2/\partial x_3^2 \tag{6.24}$$

for operation on each axisymmetric scalar. Whereupon, by Eqs. (2.16), (3.4) and (6.17),

$$\begin{pmatrix} \nabla \times \mathbf{u} \\ \nabla \times \mathbf{P} \end{pmatrix} = \exp(-i\omega t) \begin{pmatrix} V_r\mathbf{i}_r + V_\theta\mathbf{i}_\theta + V_3\mathbf{i}_3 \\ W_r\mathbf{i}_r + W_\theta\mathbf{i}_\theta + W_3\mathbf{i}_3 \end{pmatrix}, \tag{6.25}$$

where

$$\begin{pmatrix} V_r & V_\theta \\ W_r & W_\theta \end{pmatrix} = \mathrm{ad}_j \mathbf{L}_1(-\omega^2, \nabla^2 - r^{-2}) \begin{pmatrix} U_r(r, x_3; F_\theta) U_\theta(r, x_3; F_r, F_3) \\ U_r(r, x_3; E_\theta) U_\theta(r, x_3; E_r, E_3) \end{pmatrix}, \tag{6.26}$$

while

$$\begin{pmatrix} V_3 \\ W_3 \end{pmatrix} = \mathrm{adj}\, \mathbf{L}_1(-\omega^2, \nabla^2) \begin{pmatrix} U_3(r, x_3; F_\theta) \\ U_3(r, x_3; E_\theta) \end{pmatrix}. \tag{6.27}$$

Evidently,

$$V_r = V_r(r, x_3; E_\theta, F_\theta), \quad W_r = W_r(r, x_3; E_\theta, F_\theta), \tag{6.28}$$

$$V_\theta = V_\theta(r, x_3; E_r, E_3, F_r, F_3), \quad W_\theta = W_\theta(r, x_3; E_r, E_3, F_r, F_3), \tag{6.29}$$

$$V_3 = V_3(r, x_3; E_\theta, F\ ), \quad W_3 = W_3(r, x_3; E_\theta, F_\theta); \tag{6.30}$$

i.e. the transverse and axial components of $\nabla \times \mathbf{u}$ and $\nabla \times \mathbf{P}$ are axisymmetric and are each induced by the azimuthal components of **E** and **F**; on the other hand, the azimuthal components of $\nabla \times \mathbf{u}$ and $\nabla \times \mathbf{P}$, which are also axisymmetric, are each induced by the transverse and axial components of **E** and **F**.

Similarly, from Eqs. (2.17), (3.4) and (6.22), we have

$$\begin{pmatrix} \nabla \cdot \mathbf{u} \\ \nabla \cdot \mathbf{P} \\ \nabla^2 \phi \end{pmatrix} = \exp(-i\omega t) \mathrm{adj}\, \mathbf{L}_2(-\omega^2, \nabla^2) \begin{pmatrix} T_2^0(x, \Theta; \mathsf{D}) \\ U(r, x_3; F_r, F_3) \\ U(r, x_3; E_r, E_3) \end{pmatrix}, \tag{6.31}$$

which, in particular, indicates that $\nabla \cdot \mathbf{u}$, $\nabla \cdot \mathbf{P}$ and $\nabla^2 \phi$ are all axisymmetric and are each induced by D together with the transverse and axial components of **E** and **F**.

7. Axisymmetric asymptotics

The exact solutions expressed by Eqs. (6.25)–(6.27) and (6.31) involve, through Eqs. (6.18)–(6.21), the axisymmetric $T^{\mu}_{\nu_s}$ whose values must be determined from Eq. (6.14) by computing infinite series of the types (6.12) and (6.13). Suppose D, **E** and **F**, defined again by Eqs. (6.1)–(6.3), are confined within finite concentric cylindrical regions:

$$\mathsf{D} \equiv 0 \text{ outside } r_0(\mathsf{D}) \leqslant r \leqslant r_1(\mathsf{D}), \quad l_0(\mathsf{D}) \leqslant x_3 \leqslant l_1(\mathsf{D}), \tag{7.1}$$

$$E_\xi \equiv 0 \text{ outside } r_0(E_\xi) \leqslant r \leqslant r_1(E_\xi), \quad l_0(E_\xi) \leqslant x_3 \leqslant l_1(E_\xi), \tag{7.2}$$

$$F_\xi \equiv 0 \text{ outside } r_0(F_\xi) \leqslant r \leqslant r_1(F_\xi), \quad l_0(F_\xi) \leqslant x_3 \leqslant l_1(F_\xi), \tag{7.3}$$

with $\xi = r, \theta, 3$, and where $r_0(\mathsf{D}) \geqslant 0$, $r_0(E_\xi) \geqslant 0$, $r_0(F_\xi) \geqslant 0$. Generally, some or all of these regions intersect; moreover, two or more of them may be identical. Far from these regions those infinite series of the type (6.13) are insignificant by comparison with those of the type (6.12) which therefore dominate corresponding T^{μ}_{ν}. Furthermore, such infinite expansions for T^{μ}_{ν} can be replaced by single-term asymptotic approximations which can be formulated through Eqs. (3.31)–(3.33).

Now, in order that Eq. (3.31) holds for $\nu = 1, 2$, it is sufficient to consider R_ν as a finite cylindrical region, say,

$$r_0 \leqslant r \leqslant r_1, \quad l_0 \leqslant x_3 \leqslant l_1,$$

which contains all seven cylindrical regions represented in Eqs. (7.1)–(7.3). Then, using Eqs. (3.32), (3.35) and (3.36),

$$(7.4)\qquad \mathsf{F}(\alpha_{\nu_+}\hat{\mathbf{x}};\mathbf{Y}_\nu) = (2\pi)^{-3}\int_{l_0}^{l_1}\exp(-iy_3\alpha_{\nu_+}\cos\Theta)dy_3\int_{r_0}^{r_1}sds$$
$$\times\int_{\theta}^{\theta+2\pi}\mathbf{Y}_\nu(\mathbf{y})\exp[-is\alpha_{\nu_+}\sin\Theta\cos(\theta'-\theta)]d\theta'$$

in cylindrical coordinates with $s = |\mathbf{y}|\sin\theta'$ and $y_3 = |\mathbf{y}|\cos\theta'$. Each $\mathbf{Y}_\nu(\mathbf{y})$ involves the $\mathbf{E}$-vector given within the invariant Cartesian frame by Eq. (6.4). To evaluate the innermost θ'-integral for the Fourier transform of $\mathbf{E}$, we need the three following results. First ([12]), 3.5)

$$(7.5)\qquad \int_0^{2\pi}\exp(-iz\cos\beta)d\beta = 2\pi J_0(z),$$

so that

$$(7.6)\qquad \int_0^{2\pi}\exp(-iz\cos\beta)\cos\beta d\beta = 2\pi iJ_0'(z) = -2\pi iJ_1(z);$$

also,

$$(7.7)\qquad \int_0^{2\pi}\exp(-iz\cos\beta)\sin\beta d\beta \equiv 0,$$

owing to an antisymmetric integrand. Consequently, according to Eq. (7.1)–(7.7) it is seen that if we define, for $\mu = 0, 1$ and $\nu = 1, 2$,

$$(7.8)\qquad \mathsf{F}_\nu^\mu(\Theta;X) = i^{-\mu}(2\pi)^{-2}\int_{l_0(X)}^{l_1(X)}\exp(-iy_3\alpha_{\nu_+}\cos\Theta)dy_3\int_{r_0(X)}^{r_1(X)}X(s,y_3)J_\mu(s\alpha_{+\nu}\sin\Theta)sds,$$

a Fourier-Hankel transform of the axisymmetric scalar $X = X(r, x_3)$ which vanishes identically outside $r_0(X) \leqslant r \leqslant r_1(X)$, $l_0(X) \leqslant x_3 \leqslant l_1(X)$, then

$$(7.9)\qquad \mathsf{F}(\alpha_{1_+}\hat{\mathbf{x}};\mathbf{Y}_1) = \begin{pmatrix}\mathsf{F}_1^1(\Theta;F_r)\mathbf{i}_r + \mathsf{F}_1^1(\Theta;F_\theta)\mathbf{i}_\theta + \mathsf{F}_1^0(\Theta;F_3)\mathbf{i}_3\\ \mathsf{F}_1^1(\Theta;E_r)\mathbf{i}_r + \mathsf{F}_1^1(\Theta;E_\theta)\mathbf{i}_\theta + \mathsf{F}_1^0(\Theta;E_3)\mathbf{i}_3\end{pmatrix},$$

$$(7.10)\qquad \mathsf{F}(\alpha_{2_+}\hat{\mathbf{x}};\mathbf{Y}_2) = \begin{pmatrix}\mathsf{F}_2^0(\Theta;\mathsf{D})\\ \mathsf{F}_2^1(\Theta;F_r)\mathbf{i}_r + \mathsf{F}_2^1(\Theta;F_\theta)\mathbf{i}_\theta + \mathsf{F}_2^0(\Theta;F_3)\mathbf{i}_3\\ \mathsf{F}_2^1(\Theta;E_r)\mathbf{i}_r + \mathsf{F}_2^1(\Theta;E_\theta)\mathbf{i}_\theta + \mathsf{F}_2^0(\Theta;E_3)\mathbf{i}_3\end{pmatrix}.$$

Whereupon we deduce from Eqs. (3.33), (6.15) and (6.16) that at sufficiently long ranges,

$$(7.11)\qquad T_\nu^\mu(x,\Theta;X) \sim \mathsf{A}_\nu(2\pi)^2x^{-1}\,\mathsf{F}_\nu^\mu(\Theta;X)\exp(i\alpha_{\nu_+}x).$$

This formula can then be applied to Eqs. (6.18)–(6.21), (6.25)–(6.27) and (6.31) to approximate the curls and divergences of $\mathbf{u}$ and $\mathbf{P}$, as well as the scalar $\nabla^2\phi$.

8. Radial action

Consider a charge-free material upon which both the applied electric field and the external body force act radially with spherically symmetric magnitudes:

$$\mathsf{D}(\mathbf{x}) \equiv 0, \quad \mathbf{E}(\mathbf{x}) = E(x)\hat{\mathbf{x}}, \quad \mathbf{F}(\mathbf{x}) = F(x)\hat{\mathbf{x}}. \tag{8.1}$$

In this case,

$$\nabla \times \mathbf{E} \equiv \mathbf{0} \equiv \nabla \times \mathbf{F}, \tag{8.2}$$

so that Eq. (5.8) is applicable with

$$\mathbf{E}_2 \equiv \mathbf{F}_2 \equiv \mathbf{0}. \tag{8.3}$$

Furthermore, as $\mathbf{E}$ and $\mathbf{F}$ are radial, E_1 and F_1 must be spherically symmetric: $E_1 = E_1(x)$, $F_1 = F_1(x)$, and we may take

$$E_1(x) = \int^{x} E(y)dy, \quad F_1(x) = \int^{x} F(y)dy. \tag{8.4}$$

Now, if $\mathbf{Y}_\nu$ is substituted by the spherically symmetric scalar $Z = Z(x)$, say, then Eqs. (3.38) and (3.39) reduce to, respectively,

$$S_0(\hat{\mathbf{x}}; y; Z) = 2\pi^{3/2}Z(y), \tag{8.5}$$

$$S_k(\hat{\mathbf{x}}; y; Z) = (2k+1)\pi^{3/2}Z(y)P_k(\cos\Theta) \int_{-1}^{1} P_k(z)P_0(z)dz \equiv 0 \quad (k \geqslant 1), \tag{8.6}$$

with vanishment following from an orthogonality law governing the Legendre polynomials. On applying Eqs. (8.5) and (8.6) to Eq. (3.27) and replacing $J_{\frac{1}{2}}$ and $H^{(1)}_{\frac{1}{2}}$ by their oscillatory forms, we obtain

$$S(\mathbf{x}; \alpha; Z) = 2(x\alpha)^{-1} \int_{0}^{\infty} \{\exp(i\alpha|x-y|) - \exp[i\alpha(x+y)]\} Z(y)ydy, \tag{8.7}$$

which is spherically symmetric and holds for both real and complex α. According to Eq. (3.24) then,

$$\mathsf{M}_2^{-1}[Z] = \frac{1}{2} i\mathsf{A}_2[S(\mathbf{x}; \alpha_{2_+}; Z) - S(\mathbf{x}; \alpha_{2_-}; Z)] = R(x; Z), \tag{8.8}$$

a spherically symmetric function derived from Z.

Hence, by Eq. (5.13) Maxwell's potential

$$\phi = \exp(-i\omega t)[(d_{12}+2d_{44})\nabla^2 R(x; F_1) + L_3(-\omega^2, \nabla^2)R(x; E_1)], \tag{8.9}$$

wherein, effectively, the Laplacian

$$\nabla^2 = \partial^2/\partial x^2 + 2x^{-1}\partial/\partial x \tag{8.10}$$

for operation on each spherically symmetric scalar. Clearly, ϕ is spherically symmetric and is induced by both $\mathbf{E}$ and $\mathbf{F}$. Moreover, since $\mathsf{M}_1^{-1}[\mathbf{E}_2] \equiv \mathsf{M}_1^{-1}[\mathbf{F}_2] \equiv 0$, and in view of the fact that

$$\nabla^2[\hat{\mathbf{x}}Z(x)] = \hat{\mathbf{x}}(\nabla^2 - 2x^{-2})Z(x), \tag{8.11}$$

the Laplacian on the right side being given by Eq. (8.10), we deduce from Eq. (5.14) that

$$\begin{pmatrix}\mathbf{u}\\ \mathbf{P}\end{pmatrix} = \exp(-i\omega t)\begin{pmatrix}\chi_1\hat{\mathbf{x}}\\ \chi_2\hat{\mathbf{x}}\end{pmatrix}, \tag{8.12}$$

with

$$\begin{pmatrix}\chi_1\\ \chi_2\end{pmatrix} = \mathbf{N}(-\omega^2, \nabla^2-2x^{-2})\begin{pmatrix}\partial R(x;F_1)/\partial x\\ \partial R(x;E_1)/\partial x\end{pmatrix}. \tag{8.13}$$

Obviously,

$$\chi_\nu = \chi_\nu(x; E_1, F_1); \tag{8.14}$$

i.e. the displacement $\mathbf{u}$ and polarization $\mathbf{P}$ are each induced by both $\mathbf{E}$ and $\mathbf{F}$; furthermore, like $\mathbf{E}$ and $\mathbf{F}$ they are radially directed and possess spherically symmetric magnitudes.

9. Fundamental solutions

To extract fundamental solutions, say, in the absence of an applied electric field $\mathbf{E}^0 \equiv \mathbf{0}$, let us suppose that the charge and external force are singularly concentrated about $\mathbf{x} = \boldsymbol{\beta}$ and $\mathbf{x} = \boldsymbol{\varkappa}$ respectively, viz.

$$\mathsf{D}(\mathbf{x}) = \mathsf{D}_0\,\delta(\mathbf{x}-\boldsymbol{\beta}), \quad \mathbf{F}(\mathbf{x}) = \mathbf{F}_0\,\delta(\mathbf{x}-\boldsymbol{\varkappa}), \tag{9.1}$$

D_0 being a constant scalar and $\mathbf{F}_0$ — a constant vector; here, δ denotes the Dirac delta function. For the ($n =$) 2-dimensional problem: $\mathbf{x} = (x_1, x_2)$, $\boldsymbol{\varkappa} = (\varkappa_1, \varkappa_2)$, $\boldsymbol{\beta} = (\beta_1, \beta_2)$, and we take $\mathbf{F}_0 = (F_{01}, F_{02}, 0)$ in 3-space; in particular then, $\mathbf{F}(\mathbf{x})$ is singularly concentrated along and acts perpendicularly to the line $x_1 = \varkappa_1$, $x_2 = \varkappa_2$; likewise $\mathsf{D}(x)$ is singularly concentrated along the line $x_1 = \beta_1$, $x_2 = \beta_2$. The 2- and 3-dimensional problems will be resolved simultaneously. Recently, CHOWDHURY and GLOCKNER [7] have employed another technique to secure from first principles 3-dimensional fundamental solutions for each of the following separate cases: (i) a concentrated force, (ii) a concentrated electric field, (iii) a concentrated charge. However, they ignored the radiation condition, the distinction associated with real and complex roots to characteristic equations of the type (3.20), and the admissibility of the contribution from each such root. While this remains consistent in a formal treatment, it obscures some contrasting features (summarized under general terms in § 1) and restricts the scope of interpretation, e.g. the interpretation attempted in § 4.

On adopting Eq. (9.1) and $\mathbf{E} \equiv \mathbf{0}$, Eqs. (3.2), (3.34) and (3.40) lead to

$$\mathsf{M}_1^{-1}[\mathbf{Y}_1] = \begin{pmatrix}\mathbf{F}_0 G_1(|\mathbf{x}-\boldsymbol{\varkappa}|)\\ \\ \mathbf{0}\end{pmatrix}, \quad \mathsf{M}_2^{-1}[\mathbf{Y}_2] = \begin{pmatrix}\mathsf{D}_0 G_2(|\mathbf{x}-\boldsymbol{\beta}|)\\ \mathbf{F}_0 G_2(|\mathbf{x}-\boldsymbol{\varkappa}|)\\ \mathbf{0}\end{pmatrix}, \tag{9.2}$$

where

$$G_\nu(x) = \mathsf{A}_\nu(2\pi x)^{-1}[\exp(i\alpha_{\nu_+}x)-\exp(-|\alpha_{\nu_-}|x)] \quad (n = 3), \tag{9.3}$$

$$= \mathsf{A}_\nu\left[\frac{1}{2}\,iH_0^{(1)}(\alpha_{\nu_+}x)-\pi^{-1}K_0(|\alpha_{\nu_-}|x)\right] \quad (n = 2). \tag{9.4}$$

Note that with reference to Eq. (9.2), $G_\nu(|\mathbf{x}-\boldsymbol{\varkappa}|)$ is symmetric about $\mathbf{x} = \boldsymbol{\varkappa}$, while $G_2(|\mathbf{x}-\boldsymbol{\beta}|)$ is symmetric about $\mathbf{x} = \boldsymbol{\beta}$.

We shall now proceed to derive explicit versions of Eqs. (2.16) and (2.17). Defining

$$\mathbf{f}_1(\mathbf{x}) = (\mathbf{x}-\boldsymbol{\varkappa})\times\mathbf{F}_0 \qquad (n=3), \tag{9.5}$$

$$= (x_1-\varkappa_1, x_2-\varkappa_2, 0)\times\mathbf{F}_0 \qquad (n=2), \tag{9.6}$$

we have

$$\nabla\times[\mathbf{F}_0 G_1(|\mathbf{x}-\boldsymbol{\varkappa}|)] = \mathbf{f}_1(\mathbf{x})G_1'(|\mathbf{x}-\boldsymbol{\varkappa}|)|\mathbf{x}-\boldsymbol{\varkappa}|^{-1}, \tag{9.7}$$

from which it can be established that

$$\nabla^2\nabla\times[\mathbf{F}_0 G_1(|\mathbf{x}-\boldsymbol{\varkappa}|)] = \mathbf{f}_1(\mathbf{x}_\varkappa)\mathsf{P}_\varkappa[G_1'(|\mathbf{x}-\boldsymbol{\varkappa}|)|\mathbf{x}-\boldsymbol{\varkappa}|^{-1}], \tag{9.8}$$

where the operator

$$\mathsf{P}_\varkappa = \nabla_\varkappa^2 + \frac{2}{|\mathbf{x}-\boldsymbol{\varkappa}|}\frac{d}{d|\mathbf{x}-\boldsymbol{\varkappa}|}, \tag{9.9}$$

with

$$\nabla_\varkappa^2 = \frac{d^2}{d|\mathbf{x}-\boldsymbol{\varkappa}|^2} + \frac{n-1}{|\mathbf{x}-\boldsymbol{\varkappa}|}\frac{d}{d|\mathbf{x}-\boldsymbol{\varkappa}|}. \tag{9.10}$$

When operating on a function which is symmetric about $\mathbf{x} = \boldsymbol{\varkappa}$, $\nabla_\varkappa^2$ is essentially the n-dimensional Laplacian. Likewise, if

$$f_2(\mathbf{x}) = \mathbf{F}_0\cdot(\mathbf{x}-\boldsymbol{\varkappa}) \qquad (n=3), \tag{9.11}$$

$$= \mathbf{F}_0\cdot(x_1-\varkappa_1, x_2-\varkappa_2, 0) \qquad (n=2), \tag{9.12}$$

then

$$\nabla\cdot[\mathbf{F}_0 G_2(|\mathbf{x}-\boldsymbol{\varkappa}|)] = f_2(\mathbf{x})G_2(|\mathbf{x}-\boldsymbol{\varkappa}|)|\mathbf{x}-\boldsymbol{\varkappa}|^{-1}, \tag{9.13}$$

$$\nabla^2\nabla\cdot[\mathbf{F}_0 G_2(|\mathbf{x}-\boldsymbol{\varkappa}|)] = f_2(\mathbf{x})\mathsf{P}_\varkappa[G_2(|\mathbf{x}-\boldsymbol{\varkappa}|)|\mathbf{x}-\boldsymbol{\varkappa}|^{-1}]. \tag{9.14}$$

Whereupon, from Eqs. (2.4), (2.5), (2.11), (2.14)–(2.17), (3.4), (9.2), (9.7), (9.8), (9.13) and (9.14), we deduce

$$\nabla\times\mathbf{u} = \exp(-i\omega t)\mathbf{f}_1(\mathbf{x})\, L_2(\mathsf{P}_\varkappa)[G_1(|\mathbf{x}-\boldsymbol{\varkappa}|)|\mathbf{x}-\boldsymbol{\varkappa}|^{-1}], \tag{9.15}$$

$$\nabla\times\mathbf{P} = \exp(-i\omega t)\mathbf{f}_1(\mathbf{x})d_{44}\mathsf{P}_\varkappa[G_1(|\mathbf{x}-\boldsymbol{\varkappa}|)|\mathbf{x}-\boldsymbol{\varkappa}|^{-1}], \tag{9.16}$$

$$\begin{pmatrix}\nabla\cdot\mathbf{u}\\ \nabla\cdot\mathbf{P}\\ \nabla^2\phi\end{pmatrix} = \exp(-i\omega t)\begin{pmatrix}-\mathsf{D}_0(d_{12}+2d_{44})\nabla_\beta^2 f_2(\mathbf{x})\left(1+\varepsilon_0 L_4(\mathsf{P}_\varkappa)\right)\\ -\mathsf{D}_0 L_3(-\omega^2,\nabla_\beta^2)\, f_2(\mathbf{x})\,\varepsilon_0(d_{12}+2d_{44})\mathsf{P}_\varkappa\\ \mathsf{D}_0 L_5(-\omega^2,\nabla_\beta^2)\quad f_2(\mathbf{x})(d_{12}+2d_{44})\mathsf{P}_\varkappa\end{pmatrix}\begin{pmatrix}G(|\mathbf{x}-\boldsymbol{\varkappa}|)\\ \dfrac{G_2'(|\mathbf{x}-\boldsymbol{\varkappa}|)}{|\mathbf{x}-\boldsymbol{\varkappa}|}\end{pmatrix}. \tag{9.17}$$

Observe from Eqs. (9.5), (9.6), (9.15) and (9.16) that $\nabla\times\mathbf{u}$ and $\nabla\times\mathbf{P}$ are each perpendicular to $\mathbf{F}_0$ and an observation vectors; thus, in the case: $n = 2$, $\nabla\times\mathbf{u}$ and $\nabla\times\mathbf{P}$ are both normal to the x_1-x_2 plane.

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