WHARTON-
SOLUTIONS

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COMPLETE SOLUTIONS

OF

EXAMPLES IN ALGEBRA.

1875

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WHARTON'S

COMPLETE SOLUTIONS

OF EVERY CLASS OF

EXAMPLES IN ALGEBRA,

FORMING A COMPLETE COURSE ON THE SUBJECT, AND CALCULATED TO FACILITATE AND EXTEND THE STUDY OF MATHEMATICS AS A LOGICAL COURSE.

BY

J. WHARTON, B.A., M.C.P.,

LATE MATHEMATICAL EXAMINER FOR THE COLLEGE OF PRECEPTORS.

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1863.

PREFACE.

This Publication, entitled "Solutions of Examples in Algebra," is the first part of a work designed as a help to Students of that science. The first part, complete in itself, was nearly finished when the Author was, after a painful illness, removed by death. The book will be found to be a useful companion, more especially to those Students who have not access to a living Instructor to explain their difficulties.

The late James Wharton, B.A., M.C.P., was a School-master, and the author of several Elementary Books. He was educated at St. John's College, Cambridge, and his name appeared the fourteenth in the list of Senior Optimes, in the year 1834. He was one of the originators of the Royal College of Preceptors, and a most active and zealous promoter of its success. He was also, for sometime, one of its Mathematical Examiners, and he continued to labour for the interests of the College as long as his health enabled him to do so.

The Second Part of this work it is intended to complete, if the First Part should be found to accomplish the design of the Author.

CAMBRIDGE,
April, 1863.

CONTENTS.

SOLUTIONS

OF

EXAMPLES IN ALGEBRA.

SUBSTITUTIONS.

I. (1). Find the value of $\dfrac{x^2 - ax + a^2}{x^2 + ax + a^2}$; when $x = \frac{1}{2}$ and $a = \frac{1}{4}$.

By substituting for x and a the values, $\frac{1}{2}$ and $\frac{1}{4}$, we have

$$\frac{x^2 - ax + a^2}{x^2 + ax + a^2} = \frac{\frac{1}{4} - \frac{1}{8} + \frac{1}{16}}{\frac{1}{4} + \frac{1}{8} + \frac{1}{16}} = \frac{4 - 2 + 1}{4 + 2 + 1} = \frac{3}{7}.$$

(2). $\dfrac{x - y}{x + y} + \dfrac{x^2 - y^2}{x^2 + y^2}$; when $x = \frac{3}{2}$, $y = \frac{2}{3}$

$$= \frac{\frac{3}{2} - \frac{2}{3}}{\frac{3}{2} + \frac{2}{3}} + \frac{\frac{9}{4} - \frac{4}{9}}{\frac{9}{4} + \frac{4}{9}} = \frac{9 - 4}{9 + 4} + \frac{81 - 16}{81 + 16} = \frac{1330}{1261}.$$

(3). $\dfrac{x^2 - 2xy + y^2}{x^4 + x^2y^2 + y^4}$; when $x = \frac{1}{3}$, $y = \frac{1}{4}$

$$= \frac{\frac{1}{9} - \frac{1}{6} + \frac{1}{16}}{\frac{1}{81} + \frac{1}{144} + \frac{1}{256}} = \frac{(16 - 24 + 9)\,81 \times 256)}{144\,(256 + 144 + 81)} = \frac{144}{481}.$$

(4). $\sqrt[3]{(x^3 - 3x^2y + 3xy^2 - y^3)} \times \sqrt{(x^2 - 2xy + y^2)}$; when $x = \frac{3}{4}$, $y = \frac{4}{5}$

$= (x - y)^2 = (\frac{3}{4} - \frac{4}{5})^2 = \frac{1}{400}$.

(5). $4a^4 - 11a^3 + 12a^2 - 4$; when $a = 4$

$= 4 \times 256 - 11 \times 64 + 12 \times 16 - 4 = 508.$

(6). $\dfrac{a - b}{a + b} + \sqrt{\left(\dfrac{a^2 + b^2}{a^2 - b^2}\right)}$; when $a = \frac{1}{4}$, $b = \frac{1}{5}$

$$= \frac{\frac{1}{4} - \frac{1}{5}}{\frac{1}{4} + \frac{1}{5}} + \sqrt{\left(\frac{\frac{1}{16} + \frac{1}{25}}{\frac{1}{16} - \frac{1}{25}}\right)} = \frac{1}{9} + \sqrt{\left(\frac{41}{9}\right)} = \frac{1}{9} + \frac{1}{3}\sqrt{(41)}.$$

(7). $(x^3 - 3x^2 + 2x) \times (x^3 + 3x^2 - 2x)$; when $x = \frac{4}{5}$

$$= \frac{16}{25}\left(\frac{16}{25} - \frac{12}{5} + 2\right) \times \left(\frac{16}{25} + \frac{12}{5} - 2\right) = \frac{2496}{15625}.$$

(8). $x^2 - 16 + \sqrt{(x^2 - 16)} - 12$; when $x = 4\sqrt{(2)}$

$= 32 - 16 + \sqrt{(32 - 16)} - 12 = 8.$

(9). $x^3 - 2x^2 + x - 2$; when $x = 2 \pm \sqrt{(-5)}$,

$[\because \{2 \pm \sqrt{(-5)}\}^2 \times \{2 \pm \sqrt{(-5)}\} = \{-1 \pm 4\sqrt{(-5)}\} \times \{2 \pm \sqrt{(-5)}\}$

$= -22 \pm 7\sqrt{(-5)}]$

$= -22 \pm 7\sqrt{(-5)} + 2 \mp 8\sqrt{(-5)} + 2 \pm \sqrt{(-5)} - 2$

$= -20.$

(10). $\sqrt{(12x^2 - 84)} - x^2 + \sqrt{(x^2 - 7)}$; when $x = 2 - \sqrt{(3)}$,

$\{\because (2 - \sqrt{3})^2 = 7 - 4\sqrt{(3)}\}$

$= \sqrt{[12\{-4\sqrt{(3)}\}]} - 7 + 4\sqrt{(3)} + \sqrt{\{-4\sqrt{(3)}\}}$

$= 4\sqrt{(-3\sqrt{3})} - 7 + 4\sqrt{3} + 2\sqrt{\{-\sqrt{(3)}\}}.$

(11). $\frac{a}{b} - \sqrt{\left(\frac{1 + a}{1 - b}\right)}$; when $a = \frac{1}{4}$, $b = \frac{1}{5}$

$$= \frac{5}{4} - \sqrt{\left(\frac{5}{4} \times \frac{5}{4}\right)} = 0.$$

(12). $\frac{x^2 + 2xy + y^2}{x^2 - 2xy + y^2}$; when $x = \frac{1}{2}$, $y = \frac{2}{5}$

$$= \left(\frac{x + y}{x - y}\right)^2 = \left(\frac{9}{1}\right)^2 = 81.$$

(13). $\sqrt{\left(\frac{1 + x}{1 - y}\right)} + \sqrt{\left\{\frac{3(1 - 2x^2)}{1 - y^2}\right\}} + \sqrt{(x^2 - 4xy + 4y^2)}$;

when $x = \frac{1}{4}$, $y = \frac{1}{5}$

$$= \sqrt{\left(\frac{5}{4} \times \frac{5}{4}\right)} + \sqrt{\left\{3\left(\frac{7}{8} \times \frac{25}{24}\right)\right\}} + \frac{1}{4} - \frac{2}{5}$$

$$= \frac{5}{4} + \frac{5}{8}\sqrt{7} - \frac{3}{20} = \frac{11}{10} + \frac{5}{8}\sqrt{7}.$$

EQUALITIES AND INEQUALITIES.

II. (1). Which is the greater, $2^{\frac{1}{2}}$ or $3^{\frac{1}{3}}$?

$$2^{\frac{1}{2}} > \text{ or } < \text{(greater or less than) } 3^{\frac{1}{3}};$$

raise both sides to the sixth power;

$$\text{then } 2^3 > \text{ or } < 3^2;$$

$$\text{but } 3^2 = 9, \text{ and } 2^3 = 8;$$

$$\therefore\ 3^2 > 2^3; \text{ and consequently } 3^{\frac{1}{3}} > 2^{\frac{1}{2}}.$$

(2). Which is greater, $\frac{19}{20}$ or $\frac{18}{19}$?

$$\frac{19}{20} > \text{ or } < \frac{18}{19},\ \therefore\ 361 > \text{ or } < 360;$$

$$\text{but } 361 > 360,\ \therefore\ \frac{19}{20} > \frac{18}{19}.$$

(3). Show that $\frac{a}{b^2} + \frac{b}{a^2} > \frac{1}{a} + \frac{1}{b}$.

Multiplying both sides by a^2b^2,

$$a^3 + b^3 > \text{ or } < ab^2 + a^2b > \text{ or } < ab\,(a + b).$$

Dividing both sides by $a + b$,

$$a^2 - ab + b^2 > \text{ or } < ab;$$

$$\text{then } a^2 - 2ab + b^2, \text{ or } (a - b)^2, > 0,$$

since all rational quantities when squared are positive, and therefore greater than *nothing*;

$$\therefore\ \frac{a}{b^2} + \frac{b}{a^2} > \frac{1}{a} + \frac{1}{b}.$$

(4). Show that $6^{\frac{1}{2}} > 12^{\frac{1}{3}}$, $\frac{19}{20} < \frac{20}{21}$,

and that $\surd(19) + \surd(3) > \surd(10) + \surd(7)$,

$$6^{\frac{1}{2}} >< 12^{\frac{1}{3}};\ \therefore\ 216 >< 144;$$

$$\text{but } 216 > 144; \ \therefore\ 6^{\frac{1}{2}} > 12^{\frac{1}{3}},$$

$$\frac{19}{20} >< \frac{20}{21}, \text{ but } 399 > 400;$$

$$\therefore \frac{19}{20} < \frac{20}{21},$$

$$\sqrt{(19)} + \sqrt{(3)} >< \sqrt{(10)} + \sqrt{(7)};$$

$$\therefore\ 22 + 2\sqrt{(57)} >< 17 + 2\sqrt{(70)},$$

$$\text{or } 5 + 2\sqrt{(57)} >< 2\sqrt{(70)},$$

$$353 + 20\sqrt{(57)} >< 280,$$

$$\text{or } 73 + 20\sqrt{(57)} > 0,$$

$$\therefore\ \sqrt{(19)} + \sqrt{3} > \sqrt{(10)} + \sqrt{(7)}.$$

(5). Show that $a^6 + a^4b^2 + a^2b^4 + b^6 > (a^3 + b^3)^2$,

$$a^6 + a^4b^2 + a^2b^4 + b^6 >< a^6 + 2a^3b^3 + b^6,$$

$$\text{or } a^2b^2(a^2 + b^2) >< 2a^2b^2 \times ab,$$

$$\text{or } (a - b)^2 > 0;$$

$$\therefore\ a^6 + a^4b^2 + a^2b^4 + a^6 > \overline{a^3 + b^3}|^2.$$

(6). Show that $(n^3 + 1) > n(1 + n)$,

$$n^3 + 1 >< n(1 + n),$$

$$\text{or } n^2 - n + 1 >< n, \text{ but } n^2 - 2n + 1 > 0;$$

$$\therefore\ n^3 + 1 > n^2 + n.$$

(7). Show that $3(1 + a^2 + a^4) > (1 + a + a^2)^2$;
and that $a^m - b^m < ma^{m-1}(a - b)$ and $> mb^{m-1}(a - b)$, if a be $> b$

$$3(1 + a^2 + a^4) >< (1 + a + a^2)^2,$$

$$\therefore\ 3(1 - a + a^2) >< 1 + a + a^2,$$

$$2 - 4a + 2a^2 >< 0,$$

$$\text{or } 1 - 2a + a^2 > 0,$$

$$\text{or } (1 - a)^2 > 0;$$

$$\therefore\ 3(1 + a^2 + a^4) > (1 + a + a^2)^2,$$

Let $a = bx$ where $x > 1$, then $a^m - b^m = b^m (x^m - 1)$,

and $ma^{m-1} (a - b) = mx^{m-1} b^m (x - 1)$;

$\therefore\ x^m - 1 < mx^{m-1} (x - 1)$

and $> m (m - 1)$ for any value of $m \because x > 1$.

(8). Show that $xy > ac + bd$, if $x^2 = a^2 + b^2$, and $y^2 = c^2 + d^2$,

if $x^2 = a^2 + b^2,\ y^2 = c^2 + d^2$;

then $\overline{x^2 + y^2}]^2 = a^4 + b^4 + c^4 + d^2$

$+ 2a^2b^2 + 2a^2c^2 + 2a^2d^2 + 2b^2c^2 + 2b^2d^2 + 2c^2d^2$,

and $x^4 = a^4 + b^4 + 2a^2b^2$ and $y^4 = c^4 + d^4 + 2c^2d^2$;

$\therefore\ x^2y^2 = a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2$

$= (ac + bd)^2 + (bc - ad)^2$,

and $(bc - ad)^2$ is a positive quantity;

$\therefore\ xy > ac + bd$.

(9). Show that $1 + x^2 + x^4 > 3 (1 + x^3)$,

$2 (1 + x^2 + x^4) >< 3 (1 + x^3)$,

$\therefore 2 (1 + x + x^2) >< 3 (1 + x)$,

$2x^2 - x - 1 >< 0$,

or $x (x - 1) + x^2 - 1 >< 0$,

but $x + x + 1 > 0$;

$\therefore\ 2 (1 + x^2 + x^4) > 3 (1 + x^3)$.

(10). Show that $abc > (a + b - c) (a + c - b) (b + c - a)$, unless $a = b = c$,

we have $a^2 > a^2 - (b - c)^2 > (a - b + c) . (a + b - c)$,

also $b^2 > b^2 - (a - c)^2 > (b - a + c) . (b + a - c)$,

and $c^2 > c^2 - (a - b)^2 > (c + a - b) . (c - a + b)$;

$\therefore\ a^2b^2c^2 > (a + b - c)^2 \times (a + c - b) \times (b + c - a)$,

and both sides are positive;

$\therefore\ abc > (a + b - c) \times (a + c - b) \times (b + c - a)$,

let $a = 5,\ b = 4,\ c = 3$;

then we have as an illustration

$60 > 6 \times 4 \times 2 > 48$.

Let a be the least of the three quantities, and $b=a+x$, $c=a+y$,

$$\text{then } abc = a(a+x)(a+y) = a^3 + a^2(x+y) + axy \ (A),$$

$$\text{also } (a+b-c)(a+c-b)(b+c-a)$$

$$= (a+x-y)(a+y-x)(a+x+y)$$

$$= \{a^2-(x-y)^2\}(a+x+y)$$

$$= a^3 + a^2(x+y) - (a+x+y)(x-y)^2 \ (B);$$

$$\therefore (A)-(B) = (a+x+y)\overline{x-y}\rceil^2 + axy,$$

and since $(x-y)^2$ is positive;

$$\therefore (A) > (B) \text{ unless } a=b=c.$$

(11). Show that $(a+b+c)^2 < 3(a^2+b^2+c^2)$, unless $a=b=c$,

$$\text{we have } a^2+b^2+c^2+2ab+2bc+2ac >< 3(a^2+b^2+c^2),$$

$$\text{or } a^2+b^2+c^2 >< ab+bc+ac,$$

and the sum of the squares of 3 unequal quantities is greater than the sum of their products;

$$\therefore 3(a^2+b^2+c^2) > (a+b-c)^2.$$

(12). Show that $\sqrt[3]{(a)} > \sqrt[4]{(a+1)}$, if a be not less than 3,

$$\text{we have } a^4 > (a+1)^3;$$

$$\therefore a^4 - (a+1)^3 > 0,$$

$$\text{or } a^4 - a^3 - 3a^2 - 3a - 1 > 0;$$

$$\therefore a^3(a-3) + 2a^2(a-3) + 3a(a-3) + 6(a-3) + 17 > 0,$$

which is always true if n be not less than 3;

$$\text{let } a=4, \text{ then } 4^{\frac{1}{3}} >< (5)^{\frac{1}{4}},$$

$$\text{or } 4^4 >< 5^3, \text{ but } 256 > 125;$$

$$\therefore 4^{\frac{1}{3}} > 5^{\frac{1}{4}}, \text{ and } a^{\frac{1}{3}} > (a+1)^{\frac{1}{4}}.$$

(13). Show that $\dfrac{2a\sqrt{(1+x^2)}}{x+\sqrt{(1+x^2)}} = a+b$, if $x = \frac{1}{2}\sqrt{\left(\frac{a}{b}\right)} - \frac{1}{2}\sqrt{\left(\frac{b}{a}\right)}$,

$$x^2 = \frac{a}{4b} + \frac{b}{4a} - \frac{1}{2} = \frac{a^2+b^2-2ab}{4ab},$$

$$\text{and } 1+x^2 = \frac{a^2+2ab+b^2}{4ab};$$

$$\therefore \frac{2a\sqrt{(1+x^2)}}{x+\sqrt{(1+x^2)}} = \frac{2a(a+b)}{(a-b+a+b)} = a+b.$$

(14). Show that $a(b+c)^2 + b(a+c)^2 + c(a+b)^2$

$$-(a+b)(a-c)(b-c) - (a-b)(a-c)(b+c)$$
$$+(a-b)(b-c)(a+c) = 12abc,$$

$$(b+c)(2ab+2ac-bc-a^2) + (a+c)(2ab+2bc-ac-b^2)$$
$$+(a+b)(2ac+2bc-ab-c^2)$$
$$= b(2ab+2ac-bc-a^2+2ac+2bc-ab-c^2)$$
$$+c(2ab+2ac-bc-a^2-2ab+2bc-ac-b^2)$$
$$+a(2ab+2bc-ac-b^2+2ac+2bc-ab-c^2)$$
$$= 4abc + a^2b + a^2c - ab^2 - ac^2$$
$$+4abc + ac^2 + bc^2 - ca^2 - cb^2$$
$$+4abc + ab^2 + b^2c - a^2b - c^2b = 12abc.$$

(15). Show that

$$(a^2-b^2)c + (b^2-c^2)a + (c^2-a^2)b = (a-b)(b-c)(c-a),$$
$$(a-b)\{ac+bc-(b-c)(c-a)\} + (c^2-a^2)b + (b^2-c^2)a$$
$$= (a-b)(c^2+ab) + (c^2-a^2)b + (b^2-c^2)a$$
$$= a(c^2+ab+b^2-c^2) + b(c^2-a^2-c^2-ab)$$
$$= a^2b + ab^2 - a^2b - ab^2 = 0.$$

(16). Show that $\left(x+\frac{1}{x}\right)^2 - \left(y+\frac{1}{y}\right)^2 = \left(xy-\frac{1}{xy}\right) \times \left(\frac{x}{y}-\frac{y}{x}\right)$,

and prove it when $x = \frac{1}{2}$ and $y = \frac{1}{4}$,

$$\text{if } \overline{x+\frac{1}{x}}\Big]^2 - \overline{y+\frac{1}{y}}\Big]^2 = \left(xy-\frac{1}{xy}\right)\left(\frac{x}{y}-\frac{y}{x}\right),$$

$$\text{then } x^2+2+\frac{1}{x^2}-y^2-2-\frac{1}{y^2} = x^2-\frac{1}{y^2}-y^2+\frac{1}{x^2},$$

$$\text{or } 0 = 0,$$

and if $x = \frac{1}{2}$ and $y = \frac{1}{4}$;

$$\text{then } \overline{\tfrac{1}{2}+2}\,]^2 - \overline{\tfrac{1}{4}+4}\,]^2 = (\tfrac{1}{8}-8)(2-\tfrac{1}{2}),$$

$$\text{or } \frac{25}{4} - \frac{289}{16} = -\frac{63}{8} \times \frac{3}{2} = -\frac{189}{16};$$

$$\therefore \frac{25}{4} = \frac{100}{16} = \frac{25}{4}.$$

(17). If $(a^2+bc)^2(b^2+ac)^2(c^2+ab)^2=(a^2-bc)^2(b^2-ac)^2(c^2-ab)^2$, show that either

$$a^3+b^3+c^3+abc=0, \text{ or } \frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}+\frac{1}{abc}=0,$$

$$(a^2+bc)^2(b^2+ac)^2(c^2+ab)^2-(a^2-bc)^2(b^2-ac)^2(c^2-ab)^2$$
$$=\{(a^2+bc)(b^2+ac)(c^2+ab)+(a^2-bc)(b^2-ac)(c^2-ab)\}$$
$$\times\{(a^2+bc)(b^2+ac)(c^2+ab)-(a^2-bc)(b^2-ac)(c^2-ab)\}$$
$$=(2a^2b^2c^2+2abc^4+2bca^4+2acb^4)$$
$$\times(2a^3c^3+2b^3c^3+2a^3b^3+2a^2b^2c^2)=0;$$
$$\therefore abc+a^3+b^3+c^3=0, \text{ and } \frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}+\frac{1}{abc}=0.$$

(18). Show that $a^4(b^2-c^2)+b^4(c^2-a^2)+c^4(a^2-b^2)$ (A)
$=\{a^2(b-c)+b^2(c-a)+c^2(a-b)\}\times(a+b)(b+c)(a+c)$ (B),

here $a^4(b^2-c^2)+b^4(c^2-a^2)+c^4(a^2-b^2)$

$$=a^2(b^2-c^2)(a+b)(a+c)-b^2(a^2-c^2)(a+b)(b+c)$$
$$+c^2(a^2-b^2)(b+c)(a+c),$$

or $a^2(b^2-c^2)\{a^2-(a+b)(a+c)\}+b^2(a^2-c^2)\{(a+b)(b+c)-b^2\}$
$+c^2(a^2-b^2)\{c^2-(b+c)(a+c)\}$

$$=-a^2(b^2-c^2)(ab+bc+ac)+b^2(a^2-c^2)(ab+ac+bc)$$
$$-c^2(a^2-b^2)(ab+ac+bc)$$
$$=(ab+bc+ac)(a^2b^2-b^2c^2-a^2b^2+a^2c^2-a^2c^2+b^2c^2)=0,$$

or $(A)-(B)=0$.

(19). Prove that

$$(a+b+c+d)(a+b-c-d)(a-b-c+d)(a-b+c-d)$$
$$+(b-a+c+d)(a-b+c+d)(a+b-c+d)(a+b+c-d)=16abcd,$$

here $\{(a+b)^2-(c+d)^2\}\{(a-b)^2-(c-d)^2\}$
$-\{(a-b)^2-(c+d)^2\}\{(a+b)^2-(c-d)^2\}$

$$=(a^2-b^2)^2+(c^2-d^2)^2-(a^2-b^2)^2-(c^2-d^2)^2$$
$$-(a+b)^2(c-d)^2-(a-b)^2(c+d)^2+(a-b)^2(c-d)^2+(a+b)^2(c+d)^2$$
$$=(c+d)^2\times 4ab-(c-d)^2\times 4ab$$
$$=4ab\times\{(c+d)^2-(c-d)^2\}=16abcd.$$

G. C. M. & L. C. M.

III. Find the Greatest Common Measure of—

(1). $6a^2 + 7ax - 3x^2$ and $6a^2 + 11ax + 3x^2$,

$$\frac{6a^2 + 7ax - 3x^2}{6a^2 + 11ax + 3x^2} = \frac{(3a - x)(2a + 3x)}{(2a + 3x)(3a + x)} = \frac{3a - x}{3a + x};$$

$\therefore$ G.C.M. $= 2a + 3x$.

(2). $x^4 + a^2x^2 + a^4$ and $x^4 + ax^3 - a^3x - a^4$,

$$\frac{x^4 + a^2x^2 + a^4}{(x^4 - a^4) + ax(x^2 - a^2)} = \frac{x^4 + a^2x^2 + a^4}{(x^2 - a^2)(x^2 + a^2 + ax)} = \frac{x^2 - ax + a^2}{x^2 - a^2};$$

$\therefore$ G.C.M. $= x^2 + ax + a^2$.

(3). $6x^4 - 25a^2x^2 - 9a^4$ and $3x^3 - 15ax^2 + a^2x - 5a^3$,

$$\frac{6x^4 - 25a^2x^2 - 9a^4}{3x^2(x - 5a) + a^2(x - 5a)} = \frac{(3x^2 + a^2)\,2x^2 - 9a^2(3x^2 + a^2)}{(3x^2 + a^2)(x - 5a)} = \frac{2x^2 - 9a^2}{x - 5a};$$

$\therefore$ G.C.M. $= 3x^2 + a^2$.

(4). $x^3 - 19x^2 + 119x - 245$ and $3x^2 - 38x + 119$,

$$\frac{x^3 - 19x^2 + 119x - 245}{3x^2 - 38x + 119} = \frac{(x - 7)x^2 - 12x(x - 7) + 35(x - 7)}{3x(x - 7) - 17(x - 7)}$$

$$= \frac{x^2 - 12x + 35}{3x - 17}; \quad \therefore \text{G.C.M.} = x - 7.$$

(5). $3x^3 - 22x - 15$ and $5x^4 - 17x^3 + 18x$,

$$\frac{3x^3 - 22x - 15}{5x^4 - 17x^3 + 18x} = \frac{3x^2(x - 3) + 9x(x - 3) + 5(x - 3)}{5x^3(x - 3) - 2x^2(x - 3) - 6x(x - 3)}$$

$$= \frac{3x^2 + 9x + 5}{5x^3 - 2x^2 - 6x}; \quad \therefore \text{G.C.M.} = x - 3.$$

(6). $x^3 - 3x^2 + 7x - 21$ and $2x^4 + 19x^2 + 35$,

$$\frac{x^3 - 3x^2 + 7x - 21}{2x^4 + 19x^2 + 35} = \frac{x(x^2 + 7) - 3(x^2 + 7)}{2x^2(x^2 + 7) + 5(x^2 + 7)} = \frac{x - 3}{2x^2 + 5};$$

$\therefore$ G.C.M. $= x^2 + 7$.

(7). $20x^4 + x^2 - 1$ and $25x^4 + 5x^3 - x - 1$,

$$\frac{20x^4 + x^2 - 1}{25x^4+5x^3-x-1} = \frac{4x^2(5x^2-1)+(5x^2-1)}{5x^2(5x^2-1)+x(5x^2-1)+5x^2-1} = \frac{4x^2+1}{5x^2+x+1};$$

$$\therefore \text{ G.C.M.} = 5x^2 - 1.$$

(8). $a^4 - x^4$ and $a^4 + a^3x - ax^3 - x^4$,

$$\frac{a^4 - x^4}{a^4 - x^4 + ax(a^2 - x^2)} = \frac{a^2 + x^2}{a^2 + ax + x^2};$$

$$\therefore \text{ G.C.M.} = a^2 - x^2.$$

(9). $x^6 + x^2y - x^4y^2 - y^3$ and $x^4 - x^2y - x^2y^2 + y^3$,

$$\frac{x^4(x^2 - y^2) + y(x^2 - y^2)}{x^2(x^2 - y^2) - y(x^2 - y^2)} = \frac{x^4 + y}{x^2 - y};$$

$$\therefore \text{ G.C.M.} = x^2 - y^2.$$

(10). $6a^4x^3 - 10a^2x^4y - 9a^3x^2y^2 + 15ax^3y^3$

and $10a^4xy^2 - 15a^3y^4 + 8a^2x^2y^3 - 12axy^5$,

$$\frac{(2ax - 3y^2)\,3a^3x^2 - (2ax - 3y^2)\,5ax^3y}{(2ax - 3y^2)\,5a^3y^2 + (2ax - 3y^2)\,4axy^3} = \frac{3a^2x^2 - 5x^3y}{5a^2y^2 + 4xy^3};$$

$$\therefore \text{ G.C.M.} = a(2ax - 3y^2).$$

(11). $x^5 - x^4 - x + 1$ and $5x^4 - 4x^3 - 1$,

$$\frac{x^4(x-1) - (x-1)}{5x^3(x-1) + x^3 - 1} = \frac{x^4 - 1}{5x^3 + x^2 + x + 1};$$

$$\therefore \text{ G.C.M.} = x - 1.$$

(12). $x^3 - 8x + 3$ and $x^6 + 3x^5 + x + 3$,

$$\frac{x^3 - 8x + 3}{x^6 + 3x^5 + x + 3} = \frac{x^2(x+3) - 3x(x+3) + x + 3}{x^5(x+3) + x + 3} = \frac{x^2 - 3x + 1}{x^5 + 1};$$

$$\therefore \text{ G.C.M.} = x + 3.$$

(13). $x^3 + xy^2 + x^2y + y^3$ and $x^4 - y^4$,

$$\frac{x^3 + x^2y + xy^2 + y^3}{(x - y)(x^3 + x^2y + xy^2 + y^3)} = \frac{1}{x - y};$$

$$\therefore \text{ G.C.M.} = x^3 + x^2y + xy^2 + y^3.$$

(14). $x^4 + ax^3 - 9a^2x^2 + 11a^3x - 4a^4$ and $x^4 - ax^3 - 3a^2x^2 + 5a^3x - 2a^4$,

$$\frac{x^3(x-a) + 2ax^2(x-a) - 7a^2x(x-a) + 4a^3(x-a)}{x^3(x-a) - 3a^2x(x-a) + 2a^3(x-a)}$$

and by reduction

$$= \frac{x^2(x-a) + 3ax(x-a) - 4a^2(x-a)}{x^2(x-a) + ax(x-a) - 2a^2(x-a)} = \frac{x(x-a) + 4a(x-a)}{(x^2-a^2) + a(x-a)}$$

$$= \frac{x+4a}{x+2a}; \;\therefore \text{G.C.M.} = (x-a)^3.$$

Find the Least Common Multiple of—

(15). $a^3 + x^3$ and $a^2 - x^2$,

$$a^3 + x^3 = (a^2 - ax + x^2)(a + x);$$

$$\therefore \text{L.C.M.} = \frac{(a^3 + x^3)(a^2 - x^2)}{a + x} = (a^3 + x^3)(a - x).$$

(16). $2x - 1$, $4x^2 - 1$, and $4x^2 + 1$,

$$\text{L.C.M.} = (4x^2 - 1)(4x^2 + 1) = 16x^4 - 1.$$

(17). $x - 1$, $x - 2$, $x^2 - 4$, and $x + 1$,

$$\text{L.C.M.} = (x^2 - 1)(x^2 - 4) = x^4 - 5x^2 + 4.$$

(18). $4(1 - x)^2$, $8(1 - x)$, $8(1 + x)$, and $4(1 + x^2)$,

$$\text{L.C.M.} = 8(1 - x)^2(1 + x^2) \times (1 + x) = 8(1 - x^4)(1 - x).$$

(19). $a^3 - x^3$, $a^2 - x^2$, and $a^3 + x^3$,

$$\text{L.C.M.} = (a^3 - x^3)(a^3 + x^3) = a^6 - x^6.$$

(20). $x - a$, $x + a$, $x^2 - a^2$, and $x^2 + a^2$,

$$\text{L.C.M.} = (x^2 - a^2)(x^2 + a^2) = x^4 - a^4.$$

(21). $x^5 + x^4y + x^3y^2 + x^2y^3 + xy^4 + y^5$ and $x^5 - x^4y + x^3y^2 - x^2y^3 + xy^4 - y^5$

here $x^4(x + y) + x^2y^2(x + y) + y^4(x + y) = (x^4 + x^2y^2 + y^4)(x + y)$,

also $x^4(x - y) + x^2y^2(x - y) + y^4(x - y) = (x^4 + x^2y^2 + y^4)(x - y)$;

$$\therefore \text{G.C.M. is } (x^4 + x^2y^2 + y^4),$$

and $\text{L.C.M.} = (x^4 + x^2y^2 + y^4)(x^2 - y^2) = x^6 - y^6$.

(22). $a^2 - b^2$, $a^2 + b^2$, $(a - b)^2$, $(a + b)^2$, $a^3 - b^3$, and $a^3 + b^3$,

L.C.M. of $(a - b)^2$ and $(a + b)^2 = (a^2 - b^2)^2$,

L.C.M. of $(a^3 - b^3)$ and $(a^3 + b^3) = a^6 - b^6$,

also $a^2 - b^2$ is a factor of the latter;

$\therefore$ L.C.M. $(a^2-b^2)^2$ and $(a^6-b^6) = \dfrac{\overline{a^2-b^2}|^2 (a^6-b^6)}{a^2 - b^2} = (a^2-b^2)(a^6-b^6)$;

$\therefore$ L.C.M. of all the quantities $= (a^2 + b^2)(a^2 - b^2)(a^6 - b^6)$

$= (a^4 - b^4)(a^6 - b^6)$.

(23). $x^3 - 3x^2 + 3x - 1$, $x^3 - x^2 - x + 1$, $x^4 - 2x^3 + 2x - 1$,

$x^4 - 2x^3 + 2x^2 - 2x + 1$,

here $x^3 - x^2 - x + 1 = x^2(x - 1) - (x - 1) = (x + 1)(x - 1)^2$;

$\therefore$ L.C.M. of $(x - 1)^3$ and $(x^3 - x^2 - x + 1) = (x - 1)^3(x + 1)$ (A),

also $(x^4 - 2x^3 + 2x - 1) = x^4 - 1 - 2x(x^2 - 1) = (x^2 - 2x + 1)(x^2 - 1)$

$= (x - 1)^3(x + 1)$,

and $x^4 - 2x^3 + 2x^2 - 2x + 1 = (x^2 - x)^2 + (x - 1)^2 = (x^2 + 1)(x - 1)^2$;

$\therefore$ L.C.M. of these two $= (x - 1)^3(x + 1)(x^2 + 1)$ (B);

$\therefore$ L.C.M. of (A) and $(B) = (x - 1)^3(x + 1)(x^2 + 1)$.

In the preceding examples the method of inspection has been used throughout, but if a factor cannot thus be easily found, it is requisite to divide one quantity by the other, and the first divisor by the remainder, till a divisor be found which leaves no remainder, and this last divisor will be the G.C.M.; thus, to find the G.C.M. of $9x^3 + 53x^2 - 9x - 18$ and $x^2 + 11x + 30$,

$$
\begin{array}{r|l}
x^2 + 11x + 30) & 9x^3 + 53x^2 - 9x - 18\,(9x - 46 \\
 & \underline{9x^3 + 99x^2 + 270x} \\
 & \quad - 46x^2 - 279x - 18 \\
 & \quad \underline{- 46x^2 - 506x - 1380} \\
 & \qquad - 227x + 1362 = 227\,(x + 6)
\end{array}
$$

227 not being a common divisor of the two proposed quantities may be neglected.

Then, dividing the $x^2 + 11x + 30$ by $x + 6$, we have

$$
\begin{array}{r|l}
x + 6) & x^2 + 11x + 30 \, (x + 5 \\
& \underline{x^2 + \;\; 6x} \\
& \quad\;\; 5x + 30 \\
& \quad\;\; \underline{5x + 30}
\end{array}
$$

and $x + 6$ is the G.C.M. required.

Or, to find the G.C.M. of $7a^2 - 23ab + 6b^2$ and $5a^3 - 18a^2b + 11ab^2 - 6b^3$.

Making the first term the divisor, the first term of the quotient would be $\frac{5a}{7}$; to avoid, therefore, this fraction, multiply the second by 7; thus

$$
\begin{array}{rl}
& 5a^3 - 18a^2b + 11ab^2 - 6b^3 \\
& \underline{7} \\
7a^2 - 23ab + 6b^2) & 35a^3 - 126a^2b + 77ab^2 - 42b^3 \, (5a \\
& \underline{35a^3 - 115a^2b + 30ab^2} \\
& \quad - 11a^2b + 47ab^2 - 42b^3 \\
\text{Multiply by} & \quad\quad \underline{7} \\
& \quad - 77a^2b + 329ab^2 - 294b^3 \, (- 11b \\
& \quad \underline{- 77a^2b + 253ab^2 - \;\; 66b^3} \\
& \quad\quad\quad\quad 76ab^2 - 288b^3 \\
& \quad\quad\quad\quad \text{or } 76b^2 \, (a - 3b)
\end{array}
$$

and since neither 76 nor b^2 are divisors of the proposed quantities, take $a - 3b$, as a divisor; then

$$
\begin{array}{r|l}
a - 3b) & 7a^2 - 23ab + 6b^2 \, (7a - 2b \\
& \underline{7a^2 - 21ab} \\
& \quad - \;\; 2ab + 6b^2 \\
& \quad \underline{- \;\; 2ab + 6b^2} \\
& \quad\quad\;\; * \quad\;\; *
\end{array}
$$

and $a - 3b$ is the G.C.M.

INVOLUTION AND EVOLUTION.

IV. Find the square of—

(1). $2a + 3b$, $\frac{a}{2} \pm \frac{b}{3}$, and $a - 3b$,

$$(2a + 3b)^2 = 4a^2 + 12ab + 9b^2,$$

$$\left(\frac{a}{2} \pm \frac{b}{3}\right)^2 = \frac{a^2}{4} \pm \frac{ab}{3} + \frac{b^2}{9},$$

$$(a - 3b)^2 = a^2 - 6ab + 9b^2.$$

(2). $1 + x + x^2$, $1 + \frac{x}{2} + \frac{x^2}{4}$, and $\frac{x^2}{4} + \frac{xy}{3} + \frac{y^2}{4}$,

$$(1 + x + x^2)^2 = (1 + x)^2 + 2x^2 (1 + x) + x^4$$
$$= 1 + 2x + 3x^2 + 2x^3 + x^4,$$

$$\left(1 + \frac{x}{2} + \frac{x^2}{4}\right)^2 = \overline{1 + \frac{x}{2}}\Big|^2 + \frac{x^2}{2}\left(1 + \frac{x}{2}\right) + \frac{x^4}{16}$$
$$= 1 + x + \frac{3x^2}{4} + \frac{x^3}{4} + \frac{x^4}{16},$$

$$\left(\frac{x^2}{4} + \frac{xy}{3} + \frac{y^2}{4}\right)^2 = \left(\frac{x^2}{4} + \frac{xy}{3}\right)^2 + \frac{y^2}{2}\left(\frac{x^2}{4} + \frac{xy}{3}\right) + \frac{y^6}{16}$$
$$= \frac{x^4}{16} + \frac{x^3y}{6} + \frac{17x^2y^2}{72} + \frac{xy^3}{6} + \frac{y^4}{16}.$$

(3). $x - 1 + \frac{1}{x}$, $x + 1 - \frac{1}{x}$, and $\frac{x^2}{a^2} - \frac{xy}{ab} + \frac{y^2}{b^2}$,

$$\left(x - 1 + \frac{1}{x}\right)^2 = \left(x + \frac{1}{x}\right)^2 - 2\left(x + \frac{1}{x}\right) + 1$$
$$= x^2 + \frac{1}{x^2} - 2\left(x + \frac{1}{x}\right) + 3,$$

$$\left(x + 1 - \frac{1}{x}\right)^2 = \left(x - \frac{1}{x}\right)^2 + 2\left(x - \frac{1}{x}\right) + 1$$
$$= x^2 + \frac{1}{x^2} + 2\left(x - \frac{1}{x}\right) - 1,$$

$$\left(\frac{x^2}{a^2} - \frac{xy}{ab} + \frac{y^2}{b^2}\right)^2 = \left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)^2 - \frac{2xy}{ab}\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right) + \frac{x^2y^2}{a^2b^2}$$
$$= \frac{x^4}{a^4} - \frac{2x^3y}{a^3b} + \frac{3x^2y^2}{a^2b^2} - \frac{2xy^3}{ab^3} + \frac{y^4}{b^4}.$$

Find the cube of—

(4). $a + b$, $a + 2b$, and $\frac{a}{2} + \frac{b}{3}$,

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3,$$

$$\overline{a+2b}|^3 = a^3 + 6a^2b + 12ab^2 + 8b^3,$$

$$\left(\frac{a}{2}+\frac{b}{3}\right)^3 = \frac{a^3}{8} + \frac{a^2b}{4} + \frac{ab^2}{6} + \frac{b^3}{27}.$$

(5). $x^2 + ax + a^2$, and $1 - \frac{x}{2} + \frac{x^2}{4}$,

$$(x^2+ax+a^2)^3 = (x^2+a^2)^3 + 3ax\,(x^2+a^2)^2 + 3a^2x^2\,(x^2+a^2) + a^3x^3$$
$$= x^6 + 3ax^5 + 6a^2x^4 + 7a^3x^3 + 6a^4x^2 + 3a^5x + a^6,$$

$$\left(1-\frac{x}{2}+\frac{x^2}{4}\right)^3 = 1 - \frac{3x}{2} + \frac{3x^2}{2} - \frac{7x^3}{8} + \frac{3x^4}{8} - \frac{3x^5}{32} + \frac{x^6}{64}.$$

(6). $1 + 2x + 3x^2$, $x^{\frac{1}{2}} - y^{\frac{1}{3}}$, and $x^{\frac{1}{3}} - 2y^{\frac{1}{3}}$,

$$(1+2x+3x^2)^3 = 1 + 6x + 21x^2 + 44x^3 + 63x^4 + 54x^5 + 27x^6,$$

$$(x^{\frac{1}{2}} - y^{\frac{1}{3}})^3 = x^{\frac{3}{2}} - 3xy^{\frac{1}{3}} + 3x^{\frac{1}{2}}y^{\frac{2}{3}} - y,$$

$$(x^{\frac{1}{3}} - 2y^{\frac{1}{3}})^3 = x - 6x^{\frac{2}{3}}y^{\frac{1}{3}} + 12x^{\frac{1}{3}}y^{\frac{2}{3}} - 8y.$$

(7). $x - x^{-1}$, $x - 1 + x^{-1}$, and $x^{\frac{1}{3}} + 1 - x^{-\frac{1}{3}}$,

$$\left(x-\frac{1}{x}\right)^3 = x^3 - 3\left(x-\frac{1}{x}\right) - \frac{1}{x^3},$$

$$\left(x+\frac{1}{x}-1\right)^3 = \overline{x+\frac{1}{x}}\Big|^3 - 3\left(x+\frac{1}{x}\right)^2 + 3\left(x+\frac{1}{x}\right) - 1$$
$$= x^3 + \frac{1}{x^3} + 6\left(x+\frac{1}{x}\right) - 3\left(x^2+\frac{1}{x^2}\right) - 7,$$

$$\left(x^{\frac{1}{3}} - \frac{1}{x^{\frac{1}{3}}} + 1\right)^3 = (x^{\frac{1}{3}} - x^{-\frac{1}{3}})^3 + 3\,(x^{\frac{1}{3}} - x^{-\frac{1}{3}})^2 + 3\,(x^{\frac{1}{3}} - x^{-\frac{1}{3}}) + 1$$
$$= x - \frac{1}{x} - 3\left(x^{\frac{2}{3}} - \frac{1}{x^{\frac{2}{3}}}\right) - 5.$$

Find the square root of—

(8). $9a^4b^2c^2$, $64a^6b^4c^6$, and $576a^{2m}b^{2n}c^{4n}$,

$$\sqrt{(9a^4b^2c^2)} = 3a^2bc,\quad \sqrt{(64a^6b^4c^6)} = 8a^3b^2c^3,$$

$$\sqrt{(576a^{2m}b^{2n}c^{4n})} = 24a^mb^nc^{2n}.$$

(9). $16a^2 - 40ab + 25b^2$, and $16a^4 - 48a^3 + 100a^2 - 96a + 64$,

$$\sqrt{(16a^2 - 40ab + 25b^2)} = 4a - 5$$

$$\begin{array}{r|l} 8a - 5b & -40ab + 25b^2 \\ \hline \end{array}$$

$$\sqrt{(16a^4 - 48a^3 + 100a^2 - 96a + 64)} = 4a^2 - 6a + 8$$

$$\begin{array}{r|l} 8a^2 - 6a & -48a^3 + 36a^2 \\ \hline 8a^2 - 12a + 8 & 64a^2 - 96a + 64 \\ & 64a^2 - 96a + 64 \\ \hline \end{array}$$

(10). $a^2 - 2ab + c^2 + b^2 + 2ac - 2bc$,

$$(a^2 - 2ab - 2bc + 2ac + b^2 + c^2)^{\frac{1}{2}} = a - b + c.$$

(11). $x^6 - 6x^5y + 15x^4y^2 - 20x^3y^3 + 15x^2y^4 - 6xy^5 + y^6$,

$$(x^6 - 6x^5y + 15x^4y^2 - 20x^3y^3 + 15x^2y^4 - 6xy^5 + y^6)^{\frac{1}{2}}$$
$$= x^3 - 3x^2y + 3xy^2 - y^3.$$

(12). $x^4 - \frac{x}{2} + \frac{3x^2}{2} + \frac{1}{16} - 2x^3$,

$$\sqrt{\left(x^4 - 2x^3 + \frac{3x^2}{2} - \frac{x}{2} + \frac{1}{16}\right)} = x^2 - x + \frac{1}{4}$$

$$\begin{array}{r|l} 2x^2 - x & -2x^3 + x^2 \\ \hline 2x^2 - 2x + \frac{1}{4} & + \frac{x^2}{2} - \frac{x}{2} + \frac{1}{16} \\ & + \frac{x^2}{2} - \frac{x}{2} + \frac{1}{16} \\ \hline \end{array}$$

(13). $\frac{a^4}{9} + \frac{2a^3x}{3} + ax^3 + \frac{x^4}{4} + \frac{4a^2x^2}{3}$,

$$\sqrt{\left(\frac{a^4}{9} + \frac{2a^3x}{3} + \frac{4a^2x^2}{3} + ax^3 + \frac{x^4}{4}\right)} = \frac{a^2}{3} + ax + \frac{x^2}{2}$$

$$\begin{array}{r|l} \frac{2a^2}{3} + ax & \frac{2a^3x}{3} + a^2x^2 \\ \hline \frac{2a^2}{3} + 2ax + \frac{x^2}{2} & \frac{a^2x^2}{3} + ax^3 + \frac{x^4}{4} \\ & \frac{a^2x^2}{3} + ax^3 + \frac{x^4}{4} \\ \hline \end{array}$$

(14). $x^2+\frac{1}{x^2}+2\left(x-\frac{1}{x}\right)-1$,

$$\sqrt{\left(x^2+2x-\frac{2}{x}-1+\frac{1}{x^2}\right)}=x+1-\frac{1}{x}$$

$$\begin{array}{r|l} 2x+1 & 2x+1 \\ \hline 2x+2-\frac{1}{x} & -2\ -\frac{2}{x}+\frac{1}{x^2} \\ & -2\ -\frac{2}{x}+\frac{1}{x^2} \\ \hline \end{array}$$

(15). $25\frac{3}{7}-\frac{20x}{7y}+\frac{9y^2}{16x^2}-\frac{15y}{2x}+\frac{4x^2}{49y^2}$,

$$\sqrt{\left(\frac{4x^2}{49y^2}-\frac{20x}{7y}+\frac{178}{7}-\frac{15y}{2x}+\frac{9y^2}{16x^2}\right)}=\frac{2x}{7y}-5+\frac{3y}{4x}$$

$$\begin{array}{r|l} \frac{4x}{7y}-5 & -\frac{20x}{7y}+25 \\ \hline \frac{4x}{7y}-10+\frac{3y}{4x} & +\ \frac{3}{7}-\frac{15y}{2x}+\frac{9y^2}{16x^2} \\ & +\ \frac{3}{7}-\frac{15y}{2x}+\frac{9y^2}{16x^2} \\ \hline \end{array}$$

(16). $4x^4-8x^3+6x^2-2x+\frac{1}{4}$, and $\frac{a^2}{b^2}+\frac{2a}{b}+\frac{2b}{a}+3+\frac{b^2}{a^2}$,

$$\sqrt{(4x^4-8x^3+6x^2-2x+\tfrac{1}{4})}=2x^2-2x+\tfrac{1}{2}$$

$$\begin{array}{r|l} 4x^2-2x & -8x^3+4x^2 \\ \hline 4x^2-4x+\frac{1}{2} & 2x^2-2x+\frac{1}{4} \\ & 2x^2-2x+\frac{1}{4} \\ \hline \end{array}$$

$$\sqrt{\left(\frac{a^2}{b^2}+\frac{2a}{b}+\frac{2b}{a}+3+\frac{b^2}{a^2}\right)}=\frac{a}{b}+1+\frac{b}{a}$$

$$\begin{array}{r|l} \frac{2a}{b}+1 & \frac{2a}{b}+1 \\ \hline \frac{2a}{b}+2+\frac{b}{a} & +2+\frac{2b}{a}+\frac{b^2}{a^2} \\ & +2+\frac{2b}{a}+\frac{b^2}{a^2} \\ \hline \end{array}$$

(17). $\frac{x^2}{4} - \frac{bx}{4} - \frac{ab}{12} + \frac{ax}{6} + \frac{b^2}{16} + \frac{a^2}{36}$,

$$\sqrt{\left(\frac{x^2}{4} - \frac{bx}{4} + \frac{ax}{6} - \frac{ab}{12} + \frac{b^2}{16} + \frac{a^2}{36}\right)} = \frac{x}{2} - \frac{b}{4} + \frac{a}{6}$$

$$\begin{array}{l|l} x - \frac{b}{4} & -\frac{bx}{4} + \frac{b^2}{16} \\ \hline x - \frac{b}{2} + \frac{a}{6} & \frac{ax}{6} - \frac{ab}{12} + \frac{a^2}{36} \\ & \frac{ax}{6} - \frac{ab}{12} + \frac{a^2}{36} \\ \hline \end{array}$$

(18). $\frac{1051x^2}{25} - \frac{6x}{5} - \frac{14x^3}{5} + 9 + 49x^4$,

$$\sqrt{\left(49x^4 - \frac{14x^3}{5} + \frac{1051x^2}{25} - \frac{6x}{5} + 9\right)} = 7x^2 - \frac{x}{5} + 3$$

$$\begin{array}{l|l} 14x^2 - \frac{x}{5} & -\frac{14x^3}{5} + \frac{x^2}{25} \\ \hline 14x^2 - \frac{2x}{5} + 3 & 42x^2 - \frac{6x}{5} + 9 \\ & 42x^2 - \frac{6x}{5} + 9 \\ \hline \end{array}$$

(19). $\frac{x^2}{y^2} + \frac{y^2}{x^2} - \sqrt{(2)}\left(\frac{x}{y} + \frac{y}{x}\right) + \frac{5}{2}$,

$$\left\{\frac{x^2}{y^2} + \frac{y^2}{x^2} - \sqrt{2}\left(\frac{x}{y} + \frac{y}{x}\right) + \frac{5}{2}\right\}^{\frac{1}{2}} = \frac{x}{y} + \frac{y}{x} - \frac{1}{\sqrt{2}}.$$

(20). $a^2 + x$,

$$\sqrt{(a^2 + x)} = a + \frac{x}{2a} - \frac{x^2}{8a^3} + \&c.$$

$$\begin{array}{l|l} 2a + \frac{x}{2a} & x + \frac{x^2}{4a^2} \\ \hline & -\frac{x^2}{4a^2} \end{array}$$

(21). $\dfrac{a^4}{a^2 - x}$,

$$\begin{array}{r|l} a^2 & a^4 \\ \hline -x & -a^4 - a^2x - x^2 \\ & \qquad + a^2x + x^2 + \dfrac{x^3}{a^2} \\ \hline & a^2 + x + \dfrac{x^2}{a^2} + \&c. \end{array}$$

$$\therefore \sqrt{\left(\frac{a^4}{a^2 - x}\right)} = \sqrt{\left(a^2 + x + \frac{x^2}{a^2} + \&c.\right)} = a + \frac{x}{2a} + \frac{3x^2}{8a^3} +$$

$$\begin{array}{r|l} 2a + \dfrac{x}{2a} & x + \dfrac{x^2}{4a^2} \\ \hline 2a + \dfrac{x}{a} + \dfrac{3x^2}{8a^3} & + \dfrac{3x^2}{4a^2} \\ & + \dfrac{3x^2}{4a^2} + \dfrac{3x^3}{8a^4} + \dfrac{9x^4}{64a^6} \\ \hline \end{array}$$

(22). $\dfrac{a + x}{a - x}$,

$$\frac{a + x}{a - x} = 1 + \frac{2x}{a} + \frac{2x^2}{a^2} + \&c.;$$

$$\therefore \sqrt{\left(\frac{a + x}{a - x}\right)} = 1 + \frac{x}{a} + \frac{x^2}{2a^2} + \&c.$$

Find the cube root of—

(23). $x^6 - 3a^4x^4 + 3a^8x^2 - a^{12}$,

If from the cube of $(a + b)^3$ we take a^3, we have

$$(a + b)^3 - a^3 = b\,(3a^2 + 3ab) + b^3,$$

by using this formula, therefore

$$\sqrt[3]{(x^6 - 3a^4x^4 + 3a^8x^2 - a^{12})} = x^2 - a^4$$

$$\begin{array}{r|l} -a^4\,(3x^4 - 3a^4x^2 + a^8) & -3a^4x^4 + 3a^8x^2 - a^{12} \\ \hline \end{array}$$

(24). $x^6 - 6x^5 + 15x^4 - 20x^3 + 15x^2 - 6x + 1$,

$$\sqrt[3]{(x^6 - 6x^5 + 15x^4 - 20x^3 + 15x^2 - 6x + 1)} = x^2 - 2x + 1.$$

(25). $\frac{a^3}{8}+\frac{8}{27a^6}+\frac{2}{3a^3}+\frac{1}{2}$,

$$\sqrt[3]{\left(\frac{a^3}{8}+\frac{1}{2}+\frac{2}{3a^3}+\frac{8}{27a^6}\right)}=\frac{a}{2}+\frac{2}{3a^2}$$

$$\frac{2}{3a^2}\left(\frac{3a^2}{4}+\frac{1}{a}+\frac{4}{9a^4}\right)\ \Big|\ \frac{1}{2}+\frac{2}{3a^3}+\frac{8}{27a^6}$$

(26). $\frac{x^3}{y^6}-3\frac{x}{y^2}-\frac{y^6}{x^3}+3\frac{y^2}{x}$,

$$\sqrt[3]{\left\{\frac{x^3}{y^6}-3\left(\frac{x}{y^2}-\frac{y^2}{x}\right)-\frac{y^6}{x^3}\right\}}=\frac{x}{y^2}-\frac{y^2}{x}.$$

(27). $x^3-12x^2+54x-112+\frac{108}{x}-\frac{48}{x^2}+\frac{8}{x^3}$,

$$\sqrt[3]{\left(x^3-12x^2+54x-112+\frac{108}{x}-\frac{48}{x^2}+\frac{8}{x^3}\right)}$$
$$=x-4+\frac{2}{x}$$

$$-4\,(3x^2-12x+16)\ \Big|\ -12x^2+48x-64$$

$$\frac{2}{x}\times\left\{3\,(x-4)^2+3\,(x-4)\times\frac{2}{x}+\frac{4}{x^2}\right\}\ \Big|\ 6x-48$$

$$\text{or }\frac{2}{x}\times\left(3x^2-24x+54-\frac{24}{x}+\frac{4}{x^2}\right)\ \Big|\ 6x-48+\frac{108}{x}-\frac{48}{x^2}+\frac{8}{x^3}$$

(28). $\frac{a^3c^3}{b^3}-\frac{3a^2c}{bx}+\frac{3ab}{cx^2}-\frac{b^3}{c^3x^3}$,

$$\sqrt[3]{\left(\frac{a^3c^3}{b^3}-\frac{3a^2c}{bx}+\frac{3ab}{cx^2}-\frac{b^3}{c^3x^3}\right)}=\frac{ac}{b}-\frac{b}{cx}.$$

(29). $e^{3x}-e^{-3x}-3\,(e^x-e^{-x})$ and $\frac{x^6}{a^6}-\frac{3x^3}{a^2}+\frac{3a^2}{x^2}-\frac{a^6}{x^6}$,

$$\sqrt[3]{(e^{3x}-3e^x+3e^{-x}-e^{-3x})}=e^x-e^{-x}$$

$$-e^{-x}\,(3e^{2x}-3+e^{-2x})\ \Big|\ -3e^x+3e^{-x}-e^{-3x}$$

$$\sqrt[3]{\left(\frac{x^6}{a^6}-\frac{3x^2}{a^2}+\frac{3a^2}{x^2}-\frac{a^6}{x^6}\right)}=\frac{x^2}{a^2}-\frac{a^2}{x^2}.$$

(30). $x^3-\frac{1}{x^3}+3\left(x-\frac{1}{x}\right)$ and $x^3-\frac{1}{x^3}-3x^2-\frac{3}{x^2}+5.$

$$\sqrt[3]{(x^3 - 3x + 3x^{-1} - x^{-3})} = x - x^{-1}$$
$$-x^{-1}(3x^2 - 3 + x^{-2}) \mid -3x + 3x^{-1} - x^{-3}$$

$$\left\{x^3 - \frac{1}{x^3} - 3\left(x^2 + \frac{1}{x^2}\right) + 5\right\}^{\frac{1}{3}} = x - 1 - \frac{1}{x}.$$

(31). Find the 4th root of

$$16x^4 - 96x^3y + 216x^2y^2 - 216xy^3 + 81y^4,$$

by taking the square root of the square root,

$$\therefore (16x^4 - 96x^3y + 216x^2y^2 - 216xy^3 + 81y^4)^{\frac{1}{4}}$$

$$= (4x^2 - 12xy + 9y^2)^{\frac{1}{2}} = 2x - 3y.$$

(32). Find the 4th root of

$$x^8 - \frac{3a^2x^7}{b} + \frac{27a^4x^6}{8b^2} - \frac{27a^6x^5}{16b^3} + \frac{81a^8x^4}{256b^4},$$

$$\left(x^8 - \frac{3a^2x^7}{b} + \frac{27a^4x^6}{8b^2} - \frac{27a^6x^5}{16b^3} + \frac{81a^8x^4}{256b^4}\right)^{\frac{1}{4}} = x^2 - \frac{3a^2x}{4b}.$$

(33). Find the square root of

$$(a-b)^2\{(a-b)^2 - 2(a^2+b^2)\} + 2(a^4+b^4),$$

$$\overline{a-b}|^2\{(a-b)^2 - 2(a^2+b^2)\} + 2(a^4+b^4)\ (A)$$
$$= (a-b)^2\{a^2 - 2ab + b^2 - 2(a^2+b^2)\} + 2(a^4+b^4)$$
$$= (a-b)^2(-a^2-2ab-b^2) + 2(a^4+b^4) = -(a+b)^2(a-b)^2 + 2(a^4+b^4)$$
$$= a^4 + 2a^2b^2 + b^4;$$

$$\therefore A^{\frac{1}{2}} = (a^4 + 2a^2b^2 + b^4)^{\frac{1}{2}} = a^2 + b^2.$$

(34). Find the square root of $m^2n^2 + m^2r^2 + n^2r^2$,

when $m = \frac{1}{ab^2} + \frac{1}{a^2b}$, $n = \frac{2}{ab^2} + \frac{1}{a^2b}$, $r = \frac{3}{ab^2} + \frac{2}{a^2b}$,

by substituting $\frac{a+b}{a^2b^2}$ for m, $\frac{2a+b}{a^2b^2}$ for n, and $\frac{3a+2b}{a^2b^2}$ for r, we have

$$m^2n^2 + (m^2+n^2)r^2$$
$$= \frac{(a+b)^2}{a^4b^4} \times \frac{(2a+b)^2}{a^4b^4} + \left\{\frac{(a+b)^2}{a^4b^4} + \frac{(2a+b)^2}{a^4b^4}\right\}\frac{(3a+2b)^2}{a^4b^4}$$
$$= \frac{4a^4+12a^3b+13a^2b^2+6ab^3+b^4+45a^4+114a^3b+110a^2b^2+48ab^3+8b^4}{a^8b^8}$$
$$= \frac{49a^4 + 126a^3b + 123a^2b^2 + 54ab^3 + 9b^4}{a^8b^8};$$

$$\therefore \text{ the square root} = \frac{7a^2 + 9ab + 3b^2}{a^4b^4} = \frac{7}{a^2b^4} + \frac{9}{a^3b^3} + \frac{3}{a^4b^2}.$$

(35). Find the 6th root of

$$x^6 + \frac{1}{x^6} - 6\left(x^4 + \frac{1}{x^4}\right) + 15\left(x^2 - \frac{1}{x^2}\right) - 20,$$

$$\left(x^6 - 6x^4 + 15x^2 + \frac{15}{x^2} - \frac{6}{x^4} + \frac{1}{x^6} - 20\right)^{\frac{1}{2}} = x^3 - 3x + \frac{3}{x} - \frac{1}{x^3}$$

$$\begin{array}{r|l}
2x^3 - 3x & \underline{- 6x^4 + 9x^2} \\
2x^3 - 6x + \frac{3}{x} & 6x^2 \\
 & \underline{6x^2 - 18 + \frac{9}{x^2}} \\
2x^3 - 6x + \frac{6}{x} - \frac{1}{x^3} & - 2 + \frac{6}{x^2} \\
 & \underline{- 2 + \frac{6}{x^2} - \frac{6}{x^4} + \frac{1}{x^6}}
\end{array}$$

(36). Find the 5th root of

$$32x^5 - 80x^4 + 80x^3 - 40x^2 + 10x - 1,$$

the expansion of

$$(a + b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5,$$

therefore, after taking away the 5th power of the first root, we have remaining

$$b\,(5a^4 + 10a^3b + 10a^2b^2 + 5ab^3 + b^4),$$

and by applying this formula, we obtain

$$\sqrt[5]{\{(2x)^5 - 5\,(2x)^4 + 10\,(2x)^3 - 10\,(2x)^2 + 5\,(2x) - 1\}} = 2x - 1.$$

FRACTIONS.

V. Find the sum of—

(1). $\frac{b}{12a} + \frac{3b}{4a} + \frac{5b}{8a} - \frac{13b}{16a} = \frac{4b + 36b + 30b - 39b}{48a} = \frac{31b}{48a}.$

(2). $\frac{3}{2x} + \frac{5}{4x} + \frac{6}{15x} - \frac{11}{20x} = \frac{90 + 75 + 24 - 33}{60x} = \frac{156}{60x} = \frac{26}{10x} = \frac{13}{5x}.$

(3). $\frac{4x-30}{15a} - \frac{3x-15}{5a} + \frac{10x-11}{30a}$

$$= \frac{8x - 60 - 18x + 90 + 10x - 11}{30a} = \frac{19}{30a}.$$

(4). $\frac{4x-3y}{3(1-y)} - \frac{x+3y}{3(1-y)} + \frac{2y}{1-y} - \frac{x}{1-y} = \frac{3x - 6y + 6y - 3x}{3(1-y)} = 0.$

(5). $\frac{3m-4n}{7} - \frac{2m-n-1}{3} + \frac{15m-4}{12} - \frac{85m-20n}{84}$

$$= \frac{36m - 48n - 56m + 28n + 28 + 105m - 28 - 85m + 20n}{84} = 0$$

(6). $\frac{x^4-a^4}{(x-a)^2} \times \frac{x-a}{x(x+a)} \times \frac{x}{x^2+a^2} = \frac{x^2-a^2}{(x-a)(x+a)} = 1.$

(7). $\frac{a}{c} - \frac{(ad-bc)x}{c(c+dx)} - \frac{a}{c+dx} - \frac{bx}{c+dx}$

$$= \frac{ac + adx - adx + bcx - ac - bcx}{c(c+dx)} = 0.$$

(8). $\frac{1}{2} \times \frac{3x+2y}{3x-2y} - \frac{1}{2} \times \frac{3x-2y}{3x+2y} = \frac{12xy}{9x^2-4y^2}.$

(9). $\frac{x^2+xy}{x-y} \times \frac{(x-y)^2}{x^4-y^4} \times \frac{x^2+y^2}{x} = \frac{(x+y)(x-y)}{x^2-y^2} = 1.$

(10). $\left(\frac{1}{1+x} + \frac{x}{1-x}\right) \div \left(\frac{1}{1-x} - \frac{x}{1+x}\right) = \frac{1+x^2}{1-x^2} \times \frac{1-x^2}{1+x^2} = 1.$

(11). $\frac{a-b}{a+b} + \frac{2ab}{a^2-b^2} - \frac{a^2+b^2}{a^2-b^2} = \frac{a^2 - 2ab + b^2 + 2ab - a^2 - b^2}{a^2-b^2} = 0.$

(12). $\frac{b}{a-b} + \frac{a}{a+b} - \frac{a}{a-b} + \frac{b}{a+b} = \frac{a+b}{a+b} - \frac{a-b}{a-b} = 0.$

(13). $\frac{1}{x-1} - \frac{1}{2(x+1)} - \frac{x+3}{2(x^2+1)} - \frac{x+3}{x^4-1}$

$$= \frac{2x+2-x+1}{2(x^2-1)} - \frac{(x+3)(x^2-1+2)}{2(x^4-1)} = \frac{x+3}{2(x^2-1)} - \frac{(x+3)(x^2+1)}{2(x^4-1)} = 0.$$

(14). $\dfrac{1}{2(x-1)} - \dfrac{1}{2(x+1)} - \dfrac{1}{x^2} - \dfrac{1}{x^2(x^2-1)}$

$$= \frac{1}{x^2-1} - \frac{x^2-1+1}{x^2(x^2-1)} = \frac{1}{x^2-1} - \frac{1}{x^2-1} = 0.$$

(15). $\dfrac{x+y}{y} - \dfrac{2x}{x+y} + \dfrac{x^3-x^2y}{y^3-x^2y}$

$$= \frac{(x+y)^2-2xy}{y(x+y)} - \frac{x^2(x-y)}{y(x^2-y^2)} = \frac{x^2+y^2-x^2}{y(x+y)} = \frac{y}{x+y}.$$

(16). $\dfrac{x}{1-x} - \dfrac{x^2}{(1-x)^2} + \dfrac{x^3}{(1-x)^3} - x$

$$= \frac{x-2x^2+x^3-x^2+x^3+x^3-x(1-3x+3x^2-x^3)}{(1-x)^3} = \frac{x^4}{(1-x)^3}.$$

(17). $\dfrac{1}{4(1+x)} + \dfrac{1}{4(1-x)} + \dfrac{1}{2(1+x^2)} = \dfrac{1}{2(1-x^2)} + \dfrac{1}{2(1+x^2)} = \dfrac{1}{1-x^4}.$

(18). $\dfrac{3}{4(1-x)^2} + \dfrac{3}{8(1-x)} + \dfrac{1}{8(1+x)} - \dfrac{1-x}{4(1+x^2)}$

$$= \frac{3+3x+(x+2)(1-x)}{4(1-x^2)(1-x)} - \frac{1-x}{4(1+x^2)}$$

$$= \frac{5+2x+4x^2+2x^3-x^4-(1-2x+2x^3-x^4)}{4(1-x^4)(1-x)}$$

$$= \frac{1+x+x^2}{(1-x^4)(1-x)}.$$

(19). $\dfrac{a+c}{(a-b)(x-a)} - \dfrac{b+c}{(a-b)(x-b)} - \dfrac{x+c}{(x-a)(x-b)}$

$$= \frac{ax-ab+cx-bc-(bx-ab+cx-ac)}{(a-b)(x-a)(x-b)} - \frac{x+c}{(x-a)(x-b)}$$

$$= \frac{(a-b)x+c(a-b)}{(a-b)(x-a)(x-b)} - \frac{x+c}{(x-a)(x-b)} = 0.$$

(20). $\dfrac{1}{(x+1)(x+2)} - \dfrac{1}{(x+1)(x+2)(x+3)} - \dfrac{1}{(x+1)(x+3)}$

$$= \frac{x+3-1-x-2}{(x+1)(x+2)(x+3)} = 0.$$

(21). $$\frac{1}{(x+1)(x+2)} - \frac{3}{(x+1)(x+2)(x+3)} - \frac{x}{(x+1)(x+2)(x+3)}$$

$$= \frac{1}{(x+1)(x+2)} - \frac{1}{(x+1)(x+2)} = 0.$$

(22). $$\frac{1}{1+x} \div \frac{x}{1+x} - \frac{1}{x} = \frac{1}{x} - \frac{1}{x} = 0.$$

(23). $$(a-x)^2 + \frac{6a^2x + 2x^3}{a-x} - \frac{(a+x)^3}{a-x}$$

$$= \frac{a^3 - 3a^2x + 3ax^2 - x^3 + 6a^2x + 2x^3}{a-x} - \frac{(a+x)^3}{a-x} = 0.$$

(24). $$1 - \frac{a^2+b^2-c^2}{2ab} - \frac{(a+c-b)(b+c-a)}{2ab}$$

$$= \frac{c^2-(a-b)^2}{2ab} - \frac{(a+c-b)(b+c-a)}{2ab}$$

$$= \frac{(c-a+b)(c+a-b)-(a+c-b)(b+c-a)}{2ab} = 0.$$

(25). $$b^2 - \frac{(b^2+c^2-a^2)^2}{4c^2} = \frac{\overline{2bc}|^2 - (b^2+c^2-a^2)^2}{4c^2} - (\quad)$$

$$= \frac{(2bc-b^2-c^2+a^2)(2bc+b^2+c^2-a^2)}{4c^2} - (\quad)$$

$$= \frac{\{a^2-(b-c)^2\}\{(b+c)^2-a^2\}}{4c^2} - (\quad)$$

$$= \frac{(a-b+c)(a+b-c)(a+b+c)(b+c-a)}{4c^2} - (\quad) = 0.$$

(26). $$1 - \frac{a^2+b^2-c^2-d^2}{2(ab+cd)} - (\quad)$$

$$= \frac{2ab+2cd-a^2-b^2+c^2+d^2}{2ab+cd} - (\quad)$$

$$= \frac{(c+d)^2-(a-b)^2}{2(ab+cd)} - \frac{(c+d)^2-(a-b)^2}{2(ab+cd)} = 0.$$

(27). $$\left(\frac{y}{1+y} + \frac{1-y}{y}\right) \div \frac{y^2-(1-y^2)}{y(1+y)} = \frac{1}{y(1+y)} \times \frac{y(1+y)}{2y^2-1} = \frac{1}{2y^2-1}$$

C

(28). $\dfrac{a+2x}{a-x}\times\dfrac{(a-x)^2}{(a+2x)^2}-\dfrac{a-x}{(a+2x)}=\dfrac{a-x}{a+2x}-\dfrac{a-x}{a+2x}=0.$

(29). $\dfrac{a^2+ax+x^2}{a^3-a^2x+ax^2-x^3}\times\dfrac{a^2-ax+x^2}{a+x}\div\dfrac{a^4+a^2x^2+x^4}{a^4-x^4}$

$$=\frac{a^4+a^2x^2+x^4}{a^3-x^3-ax(a-x)}\times\frac{1}{a+x}\times\frac{(a^2-x^2)(a^2+x^2)}{a^4+a^2x^2+x^4}$$

$$=\frac{(a-x)(a^2+x^2)}{(a^2+ax+x^2-ax)(a-x)}=1.$$

(30). $\dfrac{pr+(pq+qr)x+q^2x^2}{p-qx}\times\dfrac{ps+(pt-qs)x-qtx^2}{p+qx}$

$$=\frac{(p+qx)r+qx(p+qx)}{p+qx}\times\frac{s(p-qx)+tx(p-qx)}{p-qx}$$

$$=(r+qx)(s+tx)=rs+rtx+qsx+tqx^2.$$

(31). $\dfrac{a^2+(2ac-b^2)x^2+c^2x^4}{a^3+2abx+(2ac+b^2)x^2+2bcx^3+c^2x^4}\times\dfrac{a^2+(ac-b^2)x^2-bcx^3}{a^2+(ac-b^2)x^2+bcx^3}$

$$=\frac{(a+bx+cx^2)(a-bx+cx^2)}{(a+bx+cx^2)(a+bx+cx^2)}\times\frac{(a-bx)(a+bx+cx^2)}{(a+bx)(a-bx+cx^2)}$$

$$=\frac{a-bx}{a+bx}.$$

(32). $\dfrac{1}{2(1-x+x^2)}+\dfrac{1}{2(1+x+x^2)}-\dfrac{x^2+1}{1+x^2+x^4}$

$$=\frac{x^2+1}{1+x^2+x^4}-\frac{x^2+1}{1+x^2+x^4}=0.$$

(33). $\dfrac{1-x}{1+x}+\dfrac{(1-x)(1-x^2)}{(1+x)(1+x^2)}-\dfrac{2(1-x)}{(1+x)(1+x^2)}$

$$=\frac{1+x^2-x-x^3+1-x-x^2+x^3-2(1-x)}{(1+x)(1+x^2)}=0.$$

(34). $\dfrac{x}{x-3}-\dfrac{x^2-9}{x(x+3)}+\dfrac{x}{x+3}-\dfrac{x+3}{x}$

$$=\frac{x^2+3x+x^2-3x}{x^2-9}-\frac{x-3+x+3}{x}$$

$$=\frac{2x^2}{x^2-9}-2=\frac{2x^2-2x^2+18}{x^2-9}=\frac{18}{x^2-9}.$$

(35). $\frac{1}{2(x-1)} + \frac{9}{2(x-3)} - \frac{4}{x-2}$

$= \frac{5x-6}{(x-1)(x-3)} - \frac{4}{x-2} = \frac{5x^2-16x+12-4(x^2-4x+3)}{(x-1)(x-2)(x-3)}$

$= \frac{x^2}{(x-1)(x-2)(x-3)}$.

(36). $\frac{a-b}{ab} + \frac{c-a}{ac} + \frac{b-c}{bc} = \frac{1}{b} - \frac{1}{a} + \frac{1}{a} - \frac{1}{c} + \frac{1}{c} - \frac{1}{b} = 0.$

(37). $\frac{2a+b}{a-b} - \frac{a-2b}{a+b} - \frac{a^2-3ab+b^2}{a^2+b^2}$

$= \frac{2a^2+3ab+b^2-(a^2-3ab+2b^2)}{a^2-b^2} - \frac{a^2-3ab+b^2}{a^2+b^2}$

$= \frac{a^2+6ab-b^2}{a^2-b^2} - \frac{a^2-3ab+b^2}{a^2+b^2}$

$= \frac{a^4-b^4-(a^4-b^4)+3ab(2a^2+2b^2+a^2-b^2)}{a^4-b^4}$

$= \frac{3ab(3a^2+b^2)}{a^4-b^4}$.

(38). $\frac{a^3}{(a+b)^3} - \frac{ab}{(a+b)^2} + \frac{b}{a+b}$

$= \frac{a^3-a^2b-ab^2+b(a^2+2ab+b^2)}{(a+b)^3} = \frac{a^3+ab^2+b^3}{(a+b)^3}$.

(39). $\frac{x(x-2y)^3}{(x+y)^3} + \frac{y(2x-y)^3}{(x+y)^3}$

$= \frac{x^4-6x^3y+12x^2y^2-8xy^3+8x^3y-12x^2y^2+6xy^3-y^4}{(x+y)^3}$

$= \frac{x^4+2x^3y-2xy^3-y^4}{(x+y)^3} = \frac{(x^2-y^2)(x^2+2xy+y^2)}{(x+y)^3} = x-y.$

(40). $\frac{(1+x)^2}{\overline{3-x}|^2} + \frac{1-x^2}{(3-x)^2} - \frac{1+x^2}{(3-x)^2}$

$= \frac{2+2x-1-x^2}{(3-x)^2} = \frac{1+2x-x^2}{(3-x)^2}$.

(41). $\frac{x-a}{x+a}+\frac{x+a}{x-a}-\frac{4ax}{x^2-a^2}$

$$=\frac{2x^2+2a^2-4ax}{x^2-a^2}=\frac{2(x^2-2ax+a^2)}{x^2-a^2}=\frac{2(x-a)}{x+a}.$$

(42). $\frac{1-2x}{3(x^2-x+1)}+\frac{1+x}{2(x^2+1)}+\frac{1}{6(x+1)}$

$$=\frac{2-2x-4x^2+x^2-x+1}{6(x^3+1)}+\frac{x+1}{2(x^2+1)}$$

$$=\frac{1-x-x^2}{2(x^3+1)}+\frac{x+1}{2(x^2+1)}$$

$$=\frac{1-x^4-x^3-x+x^4+x^3+x+1}{2(x^3+1)(x^2+1)}=\frac{1}{(x^3+1)(x^2+1)}.$$

(43). $\frac{12}{5(x+3)}-\frac{1}{15(x-2)}-\frac{4}{3(x+1)}$

$$=\frac{36x-72-x-3}{15(x+3)(x-2)}-\frac{4}{3(x+1)}$$

$$=\frac{7x^2-8x-15-4(x^2+x-6)}{3(x+1)(x-2)(x+3)}=\frac{x^2-4x+3}{(x+1)(x-2)(x+3)}.$$

(44). $\frac{3}{4(1-x^2)}+\frac{3}{8(1-x)}+\frac{1}{8(1+x)}-\frac{1-x}{4(1+x^2)}$

$$=\frac{3}{4(1-x^2)}+\frac{2+x}{4(1-x^2)}-\frac{1-x}{4(1+x^2)}$$

$$=\frac{5+5x^2+x+x^3-(1-x-x^2+x^3)}{4(1-x^4)}=\frac{2+x+3x^2}{2(1-x^4)}.$$

(45). $\frac{1}{2(x+1)}-\frac{4}{x+2}+\frac{9}{2(x+3)}$

$$=\frac{x+3+9x+9}{2(x+1)(x+3)}-\frac{4}{x+2}$$

$$=\frac{5x+6}{(x+1)(x+3)}-\frac{4}{x+2}=\frac{5x^2+16x+12-4(x^2+4x+3)}{(x+1)(x+2)(x+3)}$$

$$=\frac{x^2}{(x+1)(x+2)(x+3)}.$$

(46). $\frac{1}{1+x}+\frac{1}{1-x}+\frac{2}{1-2x}+\frac{2}{1+2x}$

$$=\frac{2}{1-x^2}+\frac{4}{1-4x^2}$$

$$=\frac{2-8x^2+4-4x^2}{(1-x^2)(1-4x^2)}=\frac{6(1-2x^2)}{(1-x^2)(1-4x^2)}.$$

(47). $\frac{3}{(1+x)^3}-\frac{1}{1+x}-\frac{1}{1-x}+\frac{2x}{(1-x)(1+x)^2}$

$$=\frac{3-3x+2x+2x^2}{(1-x)(1+x)^3}-\frac{2}{1-x^2}$$

$$=\frac{3-x+2x^2-2(1+2x+x^2)}{(1-x)(1+x)^3}=\frac{1-5x}{(1-x)(1+x)^3}.$$

(48). $\frac{1}{5(x-1)}-\frac{1}{4x^2}-\frac{3}{16x}-\frac{1}{80(x+4)}$

$$=\frac{16x+64-x+1}{80(x+4)(x-1)}-\frac{4+3x}{16x^2}$$

$$=\frac{3x+13}{16(x+4)(x-1)}-\frac{3x+4}{16x^2}$$

$$=\frac{3x^3+13x^2-(3x^3+13x^2-16)}{16x^2(x+4)(x-1)}$$

$$=\frac{1}{x^2(x-1)(x+4)}.$$

(49). $\frac{1}{(1+x)^2}+\frac{3}{1+x}+\frac{5}{(1+2x)^2}-\frac{6}{1+2x}$

$$=\frac{4+3x}{(1+x)^2}-\frac{1+12x}{(1+2x)^2}$$

$$=\frac{4+19x+28x^2+12x^3-(1+14x+25x^2+12x^3)}{(1+x)^2(1+2x)^2}$$

$$=\frac{3+5x+3x^2}{(1+x)^2(1+2x)^2}.$$

(50). $\frac{1}{x^3}+\frac{1}{x^2}+\frac{2}{x}+\frac{1}{2(1-x)^2}+\frac{7}{4(1-x)}-\frac{1}{4(1+x)}$

$$=\frac{1+x+2x^2}{x^3}+\frac{2+2x+7-7x^2-1+2x-x^2}{4(1-x)(1-x^2)}$$

$$=\frac{1+x+2x^2}{x^3}+\frac{2+x-2x^2}{(1-x)(1-x^2)}$$

$$=\frac{1-2x^3-x^4+2x^5+2x^3+x^4-2x^5}{x^3(1-x)(1-x^2)}=\frac{1}{x^3(1-x)(1-x^2)}.$$

(51). $\frac{4}{x^2+1}-\frac{2x+1}{(x^2+1)^2}+\frac{1}{(x^2+x+1)^2}-\frac{4}{x^2+x+1}$

$$=\frac{4(x^2+x+1-x^2-1)}{(x^2+1)(x^2+x+1)}+\frac{x^4+2x^2+1-(2x^5+5x^4+8x^3+7x^2+4x+1)}{(x^2+1)^2(x^2+x+1)^2}$$

$$=\frac{4x(x^4+x^3+2x^2+x+1)-2x^5-4x^4-8x^3-5x^2-4x}{(x^2+1)^2(x^2+x+1)^2}$$

$$=\frac{2x^5-x^2}{(x^2+1)^2(x^2+x+1)^2}.$$

(52). $\frac{1}{x^3}+\frac{1}{x^2}-\frac{2}{x}+\frac{x-1}{(x^2+1)^2}+\frac{2x-1}{x^2+1}$

$$=\frac{1+x-2x^2}{x^3}+\frac{x-1+2x^3-x^2+2x-1}{(x^2+1)^2}$$

$$=\frac{-2x^6+x^5-3x^4+2x^3+x+1+2x^6-x^5+3x^4-2x^3}{x^3(x^2+1)^2}$$

$$=\frac{x+1}{x^3(x^2+1)^2}.$$

(53). $\frac{x+y}{y}-\frac{x}{x+y}-\frac{x^2(x-y)}{y(x^2-y^2)}-\frac{(x-y)^2}{2xy}$

$$=\frac{x^2+xy+y^2}{y(x+y)}-\frac{x^2}{y(x+y)}+\frac{(x-y)^2}{2xy}$$

$$=1+\frac{(x-y)^2}{2xy}=\frac{1}{2}\frac{x^2+y^2}{xy}=\frac{1}{2}\left(\frac{x}{y}+\frac{y}{x}\right).$$

(54). $\frac{8}{5(y-2)} - \frac{13}{80(y+3)} - \frac{5}{4(y-1)^2} - \frac{23}{16(y-1)}$

$$= \frac{128y + 384 - 13y + 26}{80(y-2)(y+3)} - \frac{20 + 23y - 23}{16(y-1)^2}$$

$$= \frac{23y + 82}{16(y-2)(y+3)} - \frac{23y - 3}{16(y-1)^2}$$

$$= \frac{23y^3 + 36y^2 - 141y + 82 - (23y^3 + 20y^2 - 141y + 18)}{16(y-2)(y+3)(y-1)^2}$$

$$= \frac{y^2 + 4}{(y-2)(y+3)(y-1)^2}.$$

(55). $\frac{x^{3n} - 1}{x^n - 1} - \frac{x^{2n} - 1}{x^n + 1} = x^{2n} + x^n + 1 - x^n + 1 = x^{2n} + 2.$

(56). $\left(\frac{1+x}{1-x} + \frac{4x}{1+x^2} + \frac{8x}{1-x^4} - \frac{1-x}{1+x}\right) \div \left(\frac{1+x^2}{1-x^2} + \frac{4x^2}{1-x^4} - \frac{1-x^2}{1+x^2}\right)$

$$= \left(\frac{4x}{1-x^2} + \frac{4x(1+x^2) + 8x}{1-x^4}\right) \div \frac{4x^2 + 4x^2}{1-x^4}$$

$$= \frac{16}{1-x^4} \times \frac{1-x^4}{8x} = \frac{2}{x}.$$

(57). $\frac{1}{3(x-1)} - \frac{1}{x+1} - \frac{x-1}{2(x^2-x+1)} + \frac{7x-1}{6(x^2+x+1)}$

$$= \frac{2x^2 + 2x + 2 + 7x^2 - 8x + 1}{6(x^3 - 1} - \frac{2x^2 - 2x + 2 + x^2 - 1}{2(x^3+1)}$$

$$= \frac{3x^2 - 2x + 1}{2(x^3 - 1)} - \frac{3x^2 - 2x + 1}{2(x^3+1)}$$

$$= \frac{3x^2 - 2x + 1}{x^6 - 1}.$$

(58). $\frac{8x+9}{(x+1)^2} - \frac{8x+1}{x^2+x+1}$

$$= \frac{8x+1}{(x+1)^2(x^2+x+1)}\{x^2 + x + 1 - x^2 - 2x - 1\}$$

$$+ \frac{8(x^2+x+1)}{(x+1)^2(x^2+x+1)}$$

$$= \frac{7x+8}{(x+1)^2(x^2+x+1)}.$$

(59). $$\frac{1}{2a(x-a)^2} + \frac{1}{4a^2(x-a)} + \frac{1}{12a^2(x+a)} - \frac{x+a}{3a^2(x^2-ax+a^2)}$$

$$= \frac{x+a}{4a^2(x-a)^2} + \frac{x^2-ax+a^2-4(x^2+2ax+a^2)}{12a^2(x^3+a^3)}$$

$$= \frac{x^4+a^3x+ax^3+a^4-(x^2+3ax+a^2)(x-a)^2}{4a^2(x-a)^2(x^3+a^3)}$$

$$= \frac{a^3x+ax^3-(ax^3-4a^2x^2+a^3x)}{4a^2(x-a)^2(x^3+a^3)} = \frac{x^2}{(x-a)^2(x^3+a^3)}.$$

(60). $$\frac{a^2-(b-c)^2}{(a+c)^2-b^2} + \frac{b^2-(c-a)^2}{\overline{a+b}|^2-c^2} + \frac{c^2-(a-b)^2}{(b+c)^2-a^2}$$

$$= \frac{(a-b+c)(a+b-c)}{(a+c-b)(a+b+c)} + \frac{(b-c+a)(b+c-a)}{(a+b-c)(a+b+c)} + \frac{(c-a+b)(c+a-b)}{(b+c-a)(a+b+c)}$$

$$= \frac{a+b-c+b+c-a+c+a-b}{a+b+c} = 1.$$

(61). $$\frac{bc(a+d)}{(a-b)(a-c)} - \frac{ac(b+d)}{(a-b)(b-c)} + \frac{ab(c+d)}{(a-c)(b-c)}$$

$$= \frac{ab^2c+b^2cd-abc^2-bc^2d-(a^2bc+a^2cd-ac^2b-ac^2d)}{(a-b)(a-c)(b-c)} + \frac{ab(c+d)}{(a-c)(b-c)}$$

$$= \frac{-abc(a-b)-cd(a^2-b^2)+c^2d(a-b)}{(a-b)(a-c)(b-c)} + \frac{ab(c+d)}{(a-c)(b-c)}$$

$$= \frac{c^2d-acd-bcd-abc+abc+abd}{(a-c)(b-c)} = \frac{bd(a-c)-cd(a-c)}{(a-c)(b-c)}$$

$$= \frac{d(b-c)}{(b-c)} = d.$$

(62). $$\left(\frac{1}{a}+\frac{1}{b}\right)(a^2+b^2-c^2) + \left(\frac{1}{b}+\frac{1}{c}\right)(b^2+c^2-a^2) + \left(\frac{1}{a}+\frac{1}{c}\right)(a^2+c^2-b^2)$$

$$= \frac{1}{a}(a^2+b^2-c^2+a^2+c^2-b^2)$$

$$+ \frac{1}{b}(a^2+b^2-c^2+b^2+c^2-a^2) + \frac{1}{c}(a^2+c^2-b^2+b^2+c^2-a^2)$$

$$= 2(a+b+c).$$

(63). $$\frac{a}{b} - \frac{(a^2 - b^2)x}{b^2} + \frac{ax^2(a^2 - b^2)}{b^2(b + ax)} - \frac{a + bx}{b + ax}$$

$$= \frac{ab + a^2x - b(a + bx)}{b(b + ax)} + \frac{(a^2 - b^2)(ax^2 - bx - ax^2)}{b^2(b + ax)}$$

$$= \frac{(a^2x - b^2x)b - (a^2 - b^2)bx}{b^2(b + ax)} = 0.$$

(64). $$\frac{(b^3 + 1)}{ab^2(b^2 - b + 1)} \div \frac{4b^2 + 4b + 1}{4(2b + 1)}$$

$$= \frac{b + 1}{ab^2} \times \frac{4}{2b + 1} = \frac{4}{ab^2} \times \frac{b + 1}{2b + 1}.$$

(65). $$\frac{a + ba^2 + b - a}{1 + ba - ab + a^2} \times \frac{a^2}{b^2} \times \frac{(a^2 + ab + b^2)b}{a(ab + b^2 + a^2)}$$

$$= \frac{b(1 + a^2)}{1 + a^2} \times \frac{a}{b} = a.$$

(66). $$\frac{1}{b + \frac{1}{c}} = \frac{c}{bc + 1};$$

$$\therefore \frac{1}{a + \frac{c}{bc + 1}} = \frac{bc + 1}{abc + a + c}:$$

here $$\frac{1}{1 + \frac{x}{4 - x}} = \frac{4 - x}{4};$$

$$\therefore \frac{1}{x - 1 + \frac{4 - x}{4}} = \frac{4}{4x - 4 + 4 - x} = \frac{4}{3x}.$$

(67). $$\frac{a + c}{(a - b)(x - a)} - \frac{b + c}{(a - b)(x - b)} - \frac{x + c}{(x - a)(x - b)}$$

$$= \frac{ax + cx - ab - bc - (bx + cx - ab - ac)}{(a - b)(x - a)(x - b)} - \frac{x + c}{(x - a)(x - b)}$$

$$= \frac{(a - b)x + c(a - b)}{(a - b)(x - a)(x - b)} - \frac{x + c}{(x - a)(x - b)} = 0.$$

(68). $$\frac{1}{(a-b)(a-c)(x+a)}-\frac{1}{(a-b)(b-c)(x+b)}+\frac{1}{(a-c)(b-c)(x+c)}$$

$$=\frac{xb+b^2-cx-bc-(ax+a^2-cx-ac)}{(a-b)(a-c)(b-c)(x+a)(x+b)}+\frac{1}{(a-c)(b-c)(x+c)}$$

$$=\frac{c(a-b)-(a^2-b^2)-x(a-b)}{(a-b)(a-c)(b-c)(x+a)(x+b)}+\&c.$$

$$=\frac{c-x-(a+b)}{(a-c)(b-c)(x+a)(x+b)}+\&c.$$

$$=\frac{c^2-x^2-ax-ac-bx-bc+x^2+ax+bx+ab}{(a-c)(b-c)(x+a)(x+b)(x+c)}$$

$$=\frac{b(a-c)-c(a-c)}{(a-c)(b-c)(x+a)(x+b)(x+c)}=\frac{1}{(x+a)(x+b)(x+c)}.$$

(69). $$\frac{a(b-c)(x-b)-b(a-c)(x-a)}{(a-b)(a-c)(x-a)(b-c)(x-b)}+\frac{c}{(a-c)(b-c)(x-c)},$$

whence by dividing out $(a-b)$, we have

$$\frac{(ab-cx)(x-c)+c(x-a)(x-b)}{(a-c)(b-c)(x-a)(x-b)(x-c)}$$

$$=\frac{(a-c)(b-c)x}{(\quad)}=\frac{x}{(x-a)(x-b)(x-c)}.$$

(70). $$\frac{a^2(b-c)(x-b)-b^2(a-c)(x-a)}{(a-b)(a-c)(b-c)(x-a)(x-b)}+\frac{c^2}{(a-c)(b-c)(x-c)},$$

whence by dividing out $(a-b)$, we have

$$\frac{\{ab(c+x)-cx(a+b)\}(x-c)+c^2(x-a)(x-b)}{(a-c)(b-c)(x-a)(x-b)(x-c)}$$

$$=\frac{x^2(b-c)(a-c)}{(\quad)}=\frac{x^2}{(x-a)(x-b)(x-c)}.$$

(71). The sum of the latter part of the fraction

$$\frac{1}{(a-b)(a-c)(x-a)}+\frac{1}{(b-a)(b-c)(x-b)}+\frac{1}{(c-a)(c-b)(x-c)}$$

$$=\frac{1}{(x-a)(x-b)(x-c)},$$

add therefore the sums of 70 and 69, and we have the result

$$\frac{x^2}{(x-a)(x-b)(x-c)} + \frac{x}{(x-a)(x-b)(x-c)} + \frac{1}{(x-a)(x-b)(x-c)}$$

$$= \frac{x^2+x+1}{(x-a)(x-b)(x-c)}.$$

(72). This is the same as 71, except that the numerators of two fractions to be added are multiplied respectively by m and n, or thus, let

$$\frac{1}{(a-b)(a-c)(x-a)} = p, \quad \frac{1}{(a-b)(b-c)(x-b)} = q,$$

$$\frac{1}{(a-c)(b-c)(x-c)} = r,$$

then $a^2p + b^2q + c^2r + m(ap + bq + cr) + n(p + q + r)$

(from the preceding fractions)

$$= \frac{x^2 + mx + n}{(x-a)(x-b)(x-c)}.$$

Reduce to their lowest terms:—

(73). $\frac{a^3-x^3}{a^3+x^3} \times \frac{a^2+x^2}{a^2-x^2} \times \frac{a+x}{a-x} \times \frac{a^2-ax+x^2}{a^2+ax+x^2}$

$$= \frac{a^2+ax+x^2}{a^2-ax+x^2} \times \frac{a^2+x^2}{a^2-x^2} \times \frac{a^2-ax+x^2}{a^2+ax+x^2} = \frac{a^2+x^2}{a^2-x^2}.$$

(74). $\frac{x^2+3x+2}{x^2+2x+1} \times \frac{x^2+5x+4}{x^2+7x+12} = \frac{(x+1)(x+2)(x+4)(x+1)}{(x+1)^2(x+4)(x+3)} = \frac{x+2}{x+3}.$

(75). $\frac{x \times (x-1)(x-2)}{x^2-3x+2+x^2-2x+x^2-x} \times \frac{3x^2-6x+2}{x}$

$$= \frac{x^2-3x+2}{3x^2-6x+2} \times \frac{3x^2-6x+2}{1} = x^2-3x+2.$$

(76). $\frac{(x+3)(x-5)(x+7)}{x^2+2x-35+x^2+10x+21+x^2-2x-15} \times \frac{3x^2+10x-29}{(x+7)(x+3)} = x-5.$

(77). $\frac{x^2-4x+3}{x^2-2x-3}\times\frac{x+1}{x-1}\times\frac{x^2-x+1}{x^3+1}$

$$=\frac{(x-3)(x-1)}{(x-3)(x+1)}\times\frac{x+1}{x-1}\times\frac{1}{x+1}=\frac{1}{x+1}.$$

(78). $\frac{x^2+(a-b)x-ab}{x^2+(a+b)x+ab}\times\frac{x+b}{x-b}=\frac{x(x-b)+a(x-b)}{x(x+b)+a(x+b)}\times\frac{x+b}{x-b}=1.$

(79). $\frac{x^2+11x+30}{9x^3+53x^2-9x-18}\times\frac{9x^2-x-3}{x+5}$

$$=\frac{(x+5)(x+6)}{(9x^2-x-3)(x+6)}\times\frac{9x^2-x-3}{x+5}=1.$$

(80). $\frac{b(c+d)+a(c+d)}{a(f+2x)+b(f+2x)}\times\frac{f+2x}{c+d}=\frac{a+b}{a+b}=1.$

(81). $\frac{a(a^2+b^2)-b(a^2+b^2)}{2a^2(2a^2-b^2)-2ab(2a^2-b^2)}\times\frac{2a(2a^2-b^2)}{a^2+b^2}=\frac{a-b}{a-b}=1.$

(82). $\frac{x^m(a-bx)}{bx(a^2-b^2x^2)}\times\frac{b(a+bx)}{x^{m-1}}=1.$

(83). $\frac{x^4+x^2y^2+y^4}{x^4+x^2y^2+y^4+2xy(x^2+xy+y^2)}\times\frac{x^2+xy+y^2}{x^2-xy+y^2}$

$$=\frac{x^2+xy+y^2}{x^2-xy+y^2+2xy}=1.$$

(84). $\frac{x^4-y^4-2xy(x^2-y^2)}{x^4+2x^2y^2+y^4-2xy(x^2+y^2)}\times\frac{x^2+y^2}{x^2-y^2}$

$$=\frac{x^2+y^2-2xy}{x^2+y^2-2xy}=1.$$

(85). $\frac{3x^4(2x+5y)-2x^2z^2(2x+5y)}{9x^2y(x-3z)-6yz^2(x-3z)}\times\frac{x-3z}{2x+5y}$

$$=\frac{3x^4-2x^2z^2}{3y(3x^2-2z^2)}=\frac{x^2}{3y}.$$

(86). $\frac{(a+b+c)^2}{a^2-(b+c)^2}\times\frac{a-(b+c)}{a+b+c}=\frac{a+b+c}{a+b+c}=1.$

(87).

$$\begin{array}{r|l}
\frac{a}{c} & \frac{a^2}{bc} - \frac{2a}{d} + \frac{ac}{be} + \frac{bc}{d^2} - \frac{c^2}{de}. \\
\hline
-\frac{b}{d} & -\frac{a^2}{bc} + \frac{a}{d} \\
+\frac{c}{e} & \qquad + \frac{a}{d} - \frac{bc}{d^2} \\
 & \qquad\qquad - \frac{ac}{be} + \frac{c^2}{de} \\
\hline
\text{quotient} & \frac{a}{b} - \frac{c}{d}
\end{array}$$

Write down under like quantities the products of the divisor and the quotient, with their signs changed, and then their sum + the dividend = 0.

(88).

$$\begin{array}{r|l}
\frac{a^2}{x^2} & \frac{3a^4}{x^4} - \frac{19a^3b}{10x^3y} + \frac{21a^2b^2}{5x^2y^2} - \frac{9ab^3}{10xy^3} + \frac{b^4}{y^4} \\
\hline
-\frac{ab}{2xy} & -\frac{3a^4}{x^4} + \frac{2a^3b}{5x^3y} - \frac{a^2b^2}{x^2y^2} \\
+\frac{b^2}{y^2} & \qquad + \frac{3a^3b}{2x^3y} - \frac{a^2b^2}{5x^2y^2} + \frac{ab^3}{2xy^3} \\
 & \qquad\qquad - \frac{3a^2b^2}{x^2y^2} + \frac{2ab^3}{5xy^3} - \frac{b^4}{y^4} \\
\hline
\text{quotient} & \frac{3a^2}{x^2} - \frac{2ab}{5xy} + \frac{b^2}{y^2}
\end{array}$$

(89).

$$\begin{array}{r|l}
a^3 & a^6 + a^4 + a^2 + 2 + \frac{1}{a^2} + \frac{1}{a^4} + \frac{1}{a^6} \\
\hline
+\frac{1}{a^3} & -a^6 - a^4 - a^2 - 1 \\
 & \qquad\qquad -1 - \frac{1}{a^2} - \frac{1}{a^4} - \frac{1}{a^6} \\
\hline
\text{quotient} & a^3 + a + \frac{1}{a} + \frac{1}{a^3}
\end{array}$$

SURDS.

***V.** (1). $a^{\frac{1}{3}}b^{\frac{1}{2}}c^{\frac{1}{4}} \times a^{\frac{1}{3}}b^{\frac{1}{2}}c^{\frac{1}{4}} = a^{\frac{2}{3}}bc^{\frac{1}{2}}$;

$$a^{\frac{1}{3}}b^{\frac{1}{2}}c^{\frac{1}{4}} \times a^{\frac{1}{4}}b^{\frac{1}{3}}c^{\frac{1}{2}} = a^{\frac{7}{12}}b^{\frac{5}{6}}c^{\frac{3}{4}};$$

$$(a^{\frac{1}{2}}b^{\frac{1}{2}} - c^{\frac{1}{2}}x^{\frac{1}{2}})(a^{\frac{1}{2}}b^{\frac{1}{2}} + c^{\frac{1}{2}}x^{\frac{1}{2}}) = ab - cx.$$

(2). This is of the form $(a+b)(a-b) = a^2 - b^2$;

$$\therefore (3a^{\frac{1}{3}}b^{\frac{2}{3}} - 5a^{\frac{2}{3}}b^{\frac{4}{3}})(3a^{\frac{1}{3}}b^{\frac{2}{3}} + 5a^{\frac{2}{3}}b^{\frac{4}{3}}) = 9a^{\frac{2}{3}}b^{\frac{4}{3}} - 25a^{\frac{4}{3}}b^{\frac{8}{3}}.$$

(3). $(a^{\frac{4}{5}} + a^{\frac{3}{5}}y^{\frac{1}{5}} + a^{\frac{2}{5}}y^{\frac{2}{5}} + a^{\frac{1}{5}}y^{\frac{2}{5}} + y^{\frac{4}{5}}) \times (a^{\frac{1}{5}} - y^{\frac{1}{5}}) = a - y.$

(4). This is of the form $(a^2 - ab + b^2)(a+b) = a^3 + b^3$;

$$\therefore (a^{\frac{2}{3}} - a^{\frac{1}{3}}b^{\frac{1}{3}} + b^{\frac{2}{3}})(a^{\frac{1}{3}} + b^{\frac{1}{3}}) = a + b.$$

(5). This is of the form $(a^2 + ab + b^2)(a^2 - ab + b^2) = a^4 + a^2b^2 + b^4$;

$$\therefore (x^{\frac{3}{4}} - x^{\frac{1}{2}} + x^{\frac{1}{4}})(x^{\frac{3}{4}} + x^{\frac{1}{2}} + x^{\frac{1}{4}}) = x^{\frac{3}{2}} + x + x^{\frac{1}{2}}.$$

(6). $(a^{\frac{3}{8}} + a^{\frac{1}{4}}b^{\frac{1}{8}} + a^{\frac{1}{8}}b^{\frac{1}{4}} + b^{\frac{3}{8}})(a^{\frac{1}{8}} - b^{\frac{1}{8}}) = a^{\frac{1}{2}} - b^{\frac{1}{2}}.$

(7). (See Ex. 5)

$$(3x^{\frac{1}{2}} + 4x^{\frac{1}{3}} + 5x^{\frac{1}{6}})(3x^{\frac{1}{2}} - 4x^{\frac{1}{3}} + 5x^{\frac{1}{6}}) = 9x + 14x^{\frac{2}{3}} + 25x^{\frac{1}{3}}.$$

(8). (See Ex. 5)

$$3(x^{\frac{2}{5}} + x^{\frac{1}{5}}y^{\frac{1}{5}} + y^{\frac{2}{5}})(x^{\frac{2}{5}} - x^{\frac{1}{5}}y^{\frac{1}{5}} + y^{\frac{2}{5}}) = 3(x^{\frac{4}{5}} + x^{\frac{2}{5}}y^{\frac{2}{5}} + y^{\frac{4}{5}}).$$

(9). $(a^{\frac{3}{4}} + a^{\frac{1}{2}}b^{\frac{1}{4}} + a^{\frac{1}{4}}b^{\frac{1}{2}} + b^{\frac{3}{4}}) \times (a^{\frac{1}{4}} - b^{\frac{1}{4}}) = a - b.$

(10). $(x^{\frac{3}{4}} - m^{\frac{1}{4}}p^{\frac{1}{4}}x^{\frac{1}{2}} + m^{\frac{1}{2}}p^{\frac{1}{2}}x^{\frac{1}{4}} - m^{\frac{3}{4}}p^{\frac{3}{4}})(x^{\frac{1}{4}} + p^{\frac{1}{4}}m^{\frac{1}{4}} = x - mp.$

(11). $\dfrac{a^{\frac{3}{4}} - 3a^{\frac{1}{2}}b^{\frac{1}{4}} + 3a^{\frac{1}{4}}b^{\frac{1}{2}} - b^{\frac{3}{4}}}{a^{\frac{1}{4}} - b^{\frac{1}{4}}} = \dfrac{(a^{\frac{1}{4}} - b^{\frac{1}{4}})^3}{a^{\frac{1}{4}} - b^{\frac{1}{4}}} = a^{\frac{1}{2}} - 2a^{\frac{1}{4}}b^{\frac{1}{4}} + b^{\frac{1}{2}}.$

(12).

$$\begin{array}{r|l} 1 & 1 - y^{\frac{3}{4}} \\ -y^{\frac{1}{12}} & -1 - y^{\frac{1}{12}} - y^{\frac{1}{6}} - y^{\frac{1}{4}} - y^{\frac{1}{3}} - y^{\frac{5}{12}} - y^{\frac{1}{2}} - y^{\frac{7}{12}} - y^{\frac{2}{3}} \\ & \quad + y^{\frac{1}{12}} + y^{\frac{1}{6}} + y^{\frac{1}{4}} + y^{\frac{1}{3}} + y^{\frac{5}{12}} + y^{\frac{1}{2}} + y^{\frac{7}{12}} + y^{\frac{2}{3}} + y^{\frac{3}{4}} \\ \hline \end{array}$$

the quotient $1 + y^{\frac{1}{12}} + y^{\frac{1}{6}} + y^{\frac{1}{4}} + y^{\frac{1}{3}} + y^{\frac{5}{12}} + y^{\frac{1}{2}} + y^{\frac{7}{12}} + y^{\frac{2}{3}}$

(13). This is of the form $\dfrac{a^2 - b^2}{a + b} = a - b$;

$$(9ab^{\frac{4}{3}} - 25a^{\frac{2}{3}}b^{\frac{4}{3}}) \div (3a^{\frac{1}{2}}b^{\frac{2}{3}} + 5a^{\frac{1}{3}}b^{\frac{2}{3}}) = 3a^{\frac{1}{2}}b^{\frac{2}{3}} - 5a^{\frac{1}{3}}b^{\frac{2}{3}}.$$

(14). $(x^{\frac{1}{2}} - 2 + x^{-\frac{1}{2}}) \div (x^{\frac{1}{8}} - x^{-\frac{1}{8}})$

$$= \frac{(x - 2x^{\frac{1}{2}} + 1)}{(x^{\frac{1}{4}} - 1)\, x^{\frac{3}{8}}} = \frac{(x^{\frac{1}{2}} - 1)\,(x^{\frac{1}{2}} - 1)}{(x^{\frac{1}{4}} - 1)\, x^{\frac{3}{8}}} = \frac{(x^{\frac{1}{4}} + 1)\,(x^{\frac{1}{2}} - 1)}{x^{\frac{3}{8}}}$$

$$= x^{\frac{3}{8}} + x^{\frac{1}{8}} - x^{-\frac{1}{8}} - x^{-\frac{3}{8}}.$$

(15). $(a^{\frac{1}{2}} - b^{\frac{1}{2}}) \div (a^{\frac{1}{8}} - b^{\frac{1}{8}}) = (a^{\frac{1}{4}} - b^{\frac{1}{4}})\,(a^{\frac{1}{4}} + b^{\frac{1}{4}}) \div (a^{\frac{1}{8}} - b^{\frac{1}{8}})$

$$= (a^{\frac{1}{8}} - b^{\frac{1}{8}})\,(a^{\frac{1}{8}} + b^{\frac{1}{8}})(a^{\frac{1}{4}} + b^{\frac{1}{4}}) \div (a^{\frac{1}{8}} - b^{\frac{1}{8}}) = a^{\frac{3}{8}} + a^{\frac{1}{4}}b^{\frac{1}{8}} + b^{\frac{1}{4}}a^{\frac{1}{8}} + b^{\frac{3}{8}}.$$

(16). $\dfrac{(x^{12} - 3x + 3x^4 - 1)\, x^2}{x^6 \times (x^4 - 1)} = \dfrac{(x^4 - 1)^3}{x^4\,(x^4 - 1)} = \dfrac{x^8 - 2x^4 + 1}{x^4} = x^4 - 2 + \dfrac{1}{x^4}.$

(17). $(x^{\frac{1}{2}} + x^{-\frac{1}{2}} - x^{\frac{1}{4}} - x^{-\frac{1}{4}}) \div (x^{\frac{1}{8}} - x^{-\frac{1}{8}})$

$$= \frac{(x - x^{\frac{3}{4}} + 1 - x^{\frac{1}{4}})\, x^{\frac{1}{8}}}{x^{\frac{1}{2}}\,(x^{\frac{1}{4}} - 1)}$$

$$= \frac{(x^{\frac{1}{4}} - 1)\, x^{\frac{3}{4}} - (x^{\frac{1}{4}} - 1)}{x^{\frac{3}{8}}\,(x^{\frac{1}{4}} - 1)} = \frac{x^{\frac{3}{4}} - 1}{x^{\frac{3}{8}}} = x^{\frac{3}{8}} - x^{-\frac{3}{8}}.$$

(18). This and 19 are of the form $\dfrac{a^4 + a^2b^2 + b^4}{a^2 + ab + b^2} = a^2 - ab + b^2$;

$$\therefore (a^{\frac{1}{3}} + a^{\frac{1}{6}}x^{\frac{1}{6}} + x^{\frac{1}{3}}) \div (a^{\frac{1}{6}} + a^{\frac{1}{12}}x^{\frac{1}{12}} + x^{\frac{1}{6}}) = a^{\frac{1}{6}} - a^{\frac{1}{12}}x^{\frac{1}{12}} + x^{\frac{1}{6}}.$$

(19). $\left(x^{\frac{1}{3}} + \frac{x^{\frac{1}{6}}}{4} + \frac{1}{16}\right) \div \left(x^{\frac{1}{6}} - \frac{x^{\frac{1}{12}}}{2} + \frac{1}{4}\right) = x^{\frac{1}{6}} + \frac{x^{\frac{1}{12}}}{2} + \frac{1}{4}.$

(20). $\{x^{\frac{1}{3}} + x^{-\frac{1}{3}} + 2(x^{\frac{1}{6}} - x^{-\frac{1}{6}}) - 1\} \div (x^{\frac{1}{6}} - x^{-\frac{1}{6}} + 1)$

$$= \frac{x^{\frac{2}{3}} + 1 + 2x^{\frac{1}{6}}(x^{\frac{1}{3}} - 1) - x^{\frac{1}{3}}}{x^{\frac{1}{6}}(x^{\frac{1}{3}} - 1 + x^{\frac{1}{6}})} = x^{\frac{1}{6}} - x^{-\frac{1}{6}} + 1,$$

or thus

$$\begin{array}{r|l} x^{\frac{1}{6}} & x^{\frac{1}{3}} + 2x^{\frac{1}{6}} - 2x^{-\frac{1}{6}} + x^{-\frac{1}{3}} + 1 \\ \hline +1 & -x^{\frac{1}{3}} - x^{\frac{1}{6}} + 1 \\ -x^{-\frac{1}{6}} & \quad - x^{\frac{1}{6}} - 1 + x^{-\frac{1}{6}} \\ & \qquad + 1 + x^{-\frac{1}{6}} - x^{-\frac{1}{3}} \\ \hline \text{the quotient} & x^{\frac{1}{6}} + 1 - x^{-\frac{1}{6}} \end{array}$$

VI. (1). Find the value of $\sqrt{(32)} + \sqrt{(50)} - \sqrt{(18)} + \sqrt{(98)}$.

Here $32 = 2 \times 16$, $50 = 2 \times 25$, $18 = 2 \times 9$, and $98 = 2 \times 49$;

$$\therefore \sqrt{(32)} + \sqrt{(50)} - \sqrt{(18)} + \sqrt{(98)}$$
$$= \sqrt{(2 \times 16)} + \sqrt{(2 \times 25)} - \sqrt{(2 \times 9)} + \sqrt{(2 \times 49)}$$
$$= 4\sqrt{(2)} + 5\sqrt{(2)} + 7\sqrt{(2)} - 3\sqrt{(2)}$$
$$= (4 + 5 + 7 - 3)\sqrt{(2)} = 13\sqrt{(2)}.$$

(2). $3\sqrt{(8)} \times \sqrt{(32)} \times 3\sqrt{(48)} \times \sqrt{(75)}$

$$= 3\sqrt{(4 \times 2)} \times \sqrt{(2 \times 16)} \times 3\sqrt{(16 \times 3)} \times \sqrt{(25 \times 3)}$$
$$= 3 \times 2 \times 16 \times 3 \times 5\sqrt{(2 \times 2 \times 3 \times 3)}$$
$$= 1440 \times 6 = 8640.$$

(3). $3\sqrt{(8)} \times 2\sqrt{(6)} \times \sqrt{(15)} \times \sqrt{(20)}$

$$= 6 \times 2 \times 5 \times \sqrt{(2 \times 6 \times 3 \times 4)} = 720,$$

for $\sqrt{(8)} = 2\sqrt{(2)}$ and $\sqrt{(15)} \times \sqrt{20}$

$$= \sqrt{(3 \times 5)} \times \sqrt{(4 \times 5)} = 5\sqrt{(3 \times 4)}.$$

(4). $\sqrt[3]{(81)} \times \sqrt[3]{(64)} \times \sqrt[3]{(375)} \times \sqrt[3]{(-24)}$

$$= 3 \times 4 \times 5 \times 2 \times \sqrt[3]{\{3 \times 3 \times (-3) = -360\}},$$

for $\sqrt[3]{(81)} = \sqrt[3]{(27 \times 3)} = 3\sqrt[3]{(3)}$, $\sqrt[3]{(64)} = 4$, $\sqrt[3]{(375)} = 5\sqrt[3]{(3)}$,

and $\sqrt[3]{\{3 \times 3 \times (-3)\}} = \sqrt[3]{(-27)} = -3$.

(5). $4\sqrt{(147)} - 3\sqrt{(75)} - 2\sqrt{(3)} = (28 - 15 - 2)\sqrt{3} = 11\sqrt{(3)}$.

(6). $3\sqrt[3]{(8 \times 5)} + \sqrt[3]{(27 \times 5)} + \sqrt[3]{(64 \times 5)} = (6 + 3 + 4)\sqrt[3]{(5)} = 13\sqrt[3]{(5)}$.

(7). $2a^2b\sqrt[3]{(b)} + 4a^2b\sqrt[3]{(b)} - 5a^2b\sqrt[3]{(b)} = a^2b^{\frac{4}{3}}$.

(8). $16\sqrt{(2)} - 10\sqrt{(2)} + 12\sqrt{(2)} - 3\sqrt{(2)} = 15\sqrt{(2)}$.

(9). $5\sqrt{3} + 7\sqrt{3} + 6\sqrt{2} + 8\sqrt{2} = 12\sqrt{3} + 14\sqrt{2}$.

(10). $\sqrt[3]{(125 \times 4)} - \sqrt[3]{(27 \times 4)} + \sqrt{(49 \times 7)} + \sqrt{(9 \times 3)} + \sqrt{(25 \times 3)}$

$$= (5 - 3)\sqrt[3]{(4)} + 7\sqrt{(7)} + 8\sqrt{(3)} = 2\sqrt[3]{4} + 7\sqrt{7} + 8\sqrt{(3)}.$$

(11). $8\sqrt{(2)} + 8\sqrt{(3)} + 13\sqrt{(2)} + 13\sqrt{3} = 21(\sqrt{2} + \sqrt{3})$.

(12). $4\sqrt{2} - 21\sqrt{2} + 30\sqrt{2} - 5\sqrt{2} = 8\sqrt{(2)}$.

(13). $2\sqrt{3} - \sqrt{3} + 12\sqrt{3} - \frac{1}{2}\sqrt{3} = \frac{25}{2}\sqrt{3}$.

(14). $4\sqrt{2} - 21\sqrt{(2)} + 30\sqrt{2} + 5\sqrt{2} = 18\sqrt{2}$.

(15). $2\sqrt{3} + 18\sqrt{(3)} + 15\sqrt{(3)} + 13\sqrt{3} = 48\sqrt{3}$.

(16). $7 \times \sqrt[3]{(4 \times 6 \times 5 \times 100)} = 7 \times 2 \times 5 \times \sqrt[3]{(6 \times 2)} = 70\sqrt[3]{(12)}$.

(17). $\sqrt{\left(\frac{2}{3} \times \frac{5}{7} \times \frac{3}{8} \times \frac{20}{3} \times \frac{245}{44} \times \frac{44}{42}\right)} = \frac{5}{3}\sqrt{\frac{5}{2}}$.

(18). $\sqrt[3]{\left(\frac{1}{2} \times \frac{1}{3} \times \frac{1}{4} \times \frac{3}{5} \times \frac{24}{25} \times \frac{250}{27} \times \frac{189}{70}\right)} = \sqrt[3]{\left(\frac{3}{5}\right)}$.

(19). $7\frac{3}{11}\sqrt{\left(\frac{2}{7}\right)} \times \frac{18}{15}\sqrt{\frac{14}{15}} \times \frac{77}{5\sqrt{(15)}}$

$$= \frac{80}{11} \times \frac{6}{5} \times \frac{77}{5} \times \sqrt{\left(\frac{2}{7} \times \frac{14}{15} \times \frac{1}{15}\right)} = \frac{80 \times 9 \times 7}{25} \times \frac{2}{15} = 17\frac{23}{25}.$$

(20). $\frac{3}{7}\sqrt{\frac{5}{11}}+\sqrt{\left(\frac{11}{5}\right)}+\frac{1}{2}\sqrt{\left(\frac{11}{245}\right)}$

$$=\left\{\frac{3}{7}+\frac{11}{5}+\frac{1}{2}\sqrt{\left(\frac{11}{245}\times\frac{11}{5}\right)}\right\}\sqrt{\frac{5}{11}}$$

$$=\left(\frac{92}{35}+\frac{11}{70}\right)\sqrt{\frac{5}{11}}=\frac{39}{14}\sqrt{\frac{5}{11}}.$$

(21). $\sqrt{\frac{a}{x}}\times\sqrt{\frac{x}{b}}\times\sqrt{\frac{b}{y}}=\sqrt{\left(\frac{a}{x}\times\frac{x}{b}\times\frac{b}{y}\right)}=\sqrt{(ay^{-1})}.$

(22). $\left(\frac{3a}{4}\times 2a-\frac{3a^2b}{6b}+a^2\right)\sqrt[3]{(b)}=2a^2\sqrt[3]{(b)}.$

(23). $\sqrt{\left(\frac{ay}{x}\right)}\sqrt[3]{\left(\frac{bx}{y^2}\right)}\times\sqrt[6]{\left(\frac{y^2}{b^2a^3}\times\frac{y}{x}\right)}$

$$=\left(\frac{a^3y^3}{x^3}\times\frac{b^2x^2}{y^4}\times\frac{y^3}{a^3b^2}\right)^{\frac{1}{6}}=\sqrt[6]{(x^{-1}y^2)}.$$

(24). $\{4\sqrt[3]{(2)}+3\sqrt[3]{2}+5\sqrt[3]{(2)}\}\{3\sqrt[3]{(2)}-4\sqrt[3]{(2)}+5\sqrt[3]{2}\}$

$$=12\times 4\times\sqrt[3]{(4)}=48\sqrt[3]{(4)}.$$

(25). $18a\sqrt{(b+x)}-12a\sqrt{(b+x)}=6a\sqrt{(b+x)}.$

(26). $2a\sqrt{(b)}+24x\sqrt{(b)}=(2a+24x)\sqrt{(b)}=2(a+12x)\sqrt{(b)},$

and $4a^2\sqrt{(5x)}-2ax\sqrt{(5x)}=(4a^2-2ax)\sqrt{(5x)}.$

(27). $\sqrt[3]{\frac{2}{3}}-\sqrt[3]{\left(\frac{9}{32}\right)}=\frac{4-3}{\sqrt[3]{(96)}}=\frac{1}{2\sqrt[3]{(12)}}=\frac{1}{12}\sqrt[3]{(18)},$

and $\sqrt{\left(\frac{8}{27}\right)}-\sqrt{\left(\frac{1}{6}\right)}=\sqrt{\left(\frac{16}{54}\right)}-\sqrt{\left(\frac{9}{54}\right)}=\frac{4-3}{3\sqrt{(6)}}=\frac{1}{3\sqrt{(6)}}.$

(28). $10a\sqrt{(b)}+12a\sqrt{(b)}-17a\sqrt{(b)}=5a\sqrt{(b)}.$

(29). $(a^{-\frac{1}{6}}+a^{\frac{1}{12}}b^{\frac{1}{6}})(a^{-\frac{1}{6}}-a^{\frac{1}{12}}b^{\frac{1}{6}})=a^{-\frac{1}{3}}-a^{\frac{1}{6}}b^{\frac{1}{3}}.$

(30). $\{5-2\sqrt{(-3)}\}\{5+2\sqrt{(-3)}\}=25-\{2\sqrt{(-3)}\}^2=25+12=37,$

$$\{3-\sqrt{(-5)}\}\{6+2\sqrt{(-5)}\}$$

$$=18+6\sqrt{(-5)}-6\sqrt{(-5)}+10=18+10=28.$$

(31). $\{2 - 2\sqrt{(-3)}\}\{5 - 5\sqrt{(-3)}\}$

$$= 10 - 30 - 20\sqrt{(-5)} = -20\{1 + \sqrt{(-5)}\},$$

also $(5\sqrt{6} - 3\sqrt{5})(5\sqrt{6} + 3\sqrt{5}) = 150 - 45 = 105.$

(32). $\sqrt{(-a)} \times \sqrt{(-b)} = a^{\frac{1}{2}} \times b^{\frac{1}{2}}(-1)^{\frac{1}{2}} \times (-1)^{\frac{1}{2}} = a^{\frac{1}{2}}b^{\frac{1}{2}}(-1) = -a^{\frac{1}{2}}b^{\frac{1}{2}},$

$$(-a)^{\frac{1}{4}}(-b)^{\frac{1}{4}} = a^{\frac{1}{4}}b^{\frac{1}{4}}(-1)^{\frac{1}{2}} = a^{\frac{1}{4}}b^{\frac{1}{4}}\sqrt{(-1)}.$$

(33). $\{(x-1)^2 - 2\}\{(x+2)^2 - 3\}$

$$= (x^2 - 2x - 1)(x^2 + 4x + 1) = x^4 + 2x^3 - 8x^2 - 6x - 1.$$

(34). $(x+y)^{\frac{1}{m}}(x-y)^{\frac{1}{n}}(x-y)^{\frac{1}{m}}(x-y)^{\frac{1}{n}}$

$$= (x^2 - y^2)^{\frac{1}{m}}(x^2 - y^2)^{\frac{1}{n}} = (x^2 - y^2)^{\frac{m+n}{mn}}.$$

(35). $(1 - \sqrt{3})(\sqrt{2} - \sqrt{3}) \times (1 + \sqrt{3})(\sqrt{2} + \sqrt{3}) = (2 - 3)(1 - 3) = 2.$

(36). $2ax\sqrt{(3x)} + 18ax\sqrt{(3x)} + 12ax\sqrt{(3x)} - 20ax\sqrt{(3x)} = 12ax\sqrt{(3x)}.$

(37). $2a^2b\sqrt[3]{(b)} + 4a^2b\sqrt[3]{(b)} - 5a^2b\sqrt[3]{(b)} = a^2b^{\frac{4}{3}}.$

(38). $\frac{b}{c}\sqrt{(ab)} + \frac{\sqrt{(ab)}}{2c} \times \sqrt{(a^2 - 4ab + 4b^2)}$

$$= \frac{\sqrt{(ab)}}{2c}(2b + a - 2b) = \frac{a\sqrt{(ab)}}{2c}.$$

(39). $\dfrac{\frac{1}{2}\sqrt{(\frac{1}{6})}}{\sqrt{(\frac{2}{3})} + 3\sqrt{(\frac{1}{6})}} = \dfrac{\frac{1}{2}}{\sqrt{(4)} + 3} = \dfrac{1}{10},$

$$\frac{\frac{1}{8}\{\sqrt{(5)} \mp 1\}}{\frac{1}{4\sqrt{2}}\sqrt{(5 \pm \sqrt{5})}} = \frac{1}{\sqrt{2}}\sqrt{\left(\frac{6 \mp 2\sqrt{5}}{5 \pm \sqrt{5}}\right)} = \sqrt{\left\{\frac{(3 \mp \sqrt{5})(5 \mp \sqrt{5})}{(5 \pm \sqrt{5})(5 \mp \sqrt{5})}\right\}}$$

$$= \sqrt{\left(\frac{20 \mp 8\sqrt{5}}{20}\right)} = \sqrt{\left(1 \mp \frac{2}{\sqrt{5}}\right)}.$$

(40). $\dfrac{\sqrt{2}}{1}\sqrt{\left(\dfrac{5 \mp \sqrt{5}}{6 \pm 2\sqrt{5}}\right)} = \sqrt{\left(\dfrac{5 \mp \sqrt{5}}{3 \pm \sqrt{5}} \times \dfrac{3 \mp \sqrt{5}}{3 \mp \sqrt{5}}\right)}$

$$= \sqrt{\left(\frac{20 \mp 8\sqrt{5}}{4}\right)} = \sqrt{(5 \mp 2\sqrt{5})},$$

$$\frac{\sqrt[3]{20}}{2} \times \frac{2^{\frac{4}{3}} - 2^{\frac{2}{3}}}{2^{\frac{2}{3}} - 2^{\frac{1}{3}}} = \frac{(2^{\frac{2}{3}} + 2^{\frac{1}{3}}) \times 5^{\frac{1}{3}}}{2^{\frac{1}{3}}} = \sqrt[3]{(10)} + \sqrt[3]{(5)}.$$

(41). $(a^2 - b^2)^{\frac{1}{m}} (a^2 - b^2)^{\frac{1}{n}} = (a^2 - b^2)^{\frac{m+n}{mn}}.$

(42). $\sqrt[3]{\left\{\frac{1}{x}(1 - 3x^2 + 3x - x^3)\right\}} = \frac{1-x}{\sqrt[3]{x}} = x^{-\frac{1}{3}} - x^{\frac{2}{3}}.$

(43). $\frac{(x+y)\,x^{\frac{1}{3}}}{x^{\frac{1}{3}}\{x^{\frac{2}{3}} - \sqrt[3]{(xy)} + y^{\frac{2}{3}}\}} = x^{\frac{1}{3}} + y^{\frac{1}{3}} \left(\text{by the form } \frac{a^3+b^3}{a^2-ab+b^2} = a+b\right).$

(44). $\left(\frac{x}{y}\right)^{\frac{1}{4}} \left(\frac{y}{x}\right)^{\frac{1}{2}} \times \left(\frac{1}{x^3y^2}\right)^{\frac{1}{6}} \times x^{\frac{2}{5}} y^{\frac{1}{3}} = \left(\frac{y}{x}\right)^{\frac{1}{4}} \times \left(\frac{1}{x^3y^2}\right)^{\frac{1}{6}} \times x^{\frac{2}{5}} y^{\frac{1}{3}}$

$$= \left(\frac{y^3}{x^3} \times \frac{y^4}{x^6y^4}\right)^{\frac{1}{12}} \times x^{\frac{2}{5}} = \frac{y^{\frac{1}{4}} \times x^{\frac{2}{5}}}{x^{\frac{3}{4}}} = y^{\frac{1}{4}} x^{-\frac{7}{20}}.$$

(45). $a^{\frac{3p}{m}} \times x^{\frac{2m-n}{m}} \times y^{\frac{5m+1}{m}} \times a^{\frac{m-3p}{m}} \times x^{\frac{n}{m}} \times y^{\frac{m-1}{m}} = ax^2y^6;$

for the index of $a = \frac{3p}{m} + \frac{m-3p}{m} = 1$, &c. &c. = &c.

(46). $\frac{ab}{b-c} \pm \frac{1}{b-c} \sqrt{(a^2b^2 - a^2b^2 + a^2bc)} = \frac{ab \pm a\sqrt{(bc)}}{b-c}$

$$= \frac{a\sqrt{b}(\sqrt{b} \pm \sqrt{c})}{b-c} = \frac{a\sqrt{b}}{\sqrt{b} \mp \sqrt{c}}.$$

(47). $(x^m - x^{\frac{m}{2}} \times x^{\frac{2}{p}} \times b^{\frac{1}{n}} + \frac{1}{4} b^{\frac{2}{n}} x^{\frac{4}{p}})^{\frac{1}{2}} = x^{\frac{m}{2}} - \frac{1}{2} b^{\frac{1}{n}} x^{\frac{2}{p}}$

(48). $\frac{2x^2 - 2y^2 - \sqrt{2}xy}{\sqrt{2}\{(\sqrt{2}x - y)(\sqrt{2}x + y)\}}$

$$= \frac{(2x^2 - y^2) - y(\sqrt{2}x + y)}{\sqrt{2}(\sqrt{2}x - y)(\sqrt{2}x + y)} = \frac{\sqrt{2}x - y - y}{\sqrt{2}(\sqrt{2}x - y)} = \frac{x - \sqrt{2}y}{\sqrt{2}x - y}.$$

(49). $\frac{3a^2b\sqrt{(2ab)} + 5ab^2\sqrt{(2ab)}}{\sqrt{(2ab)}\{2b + \sqrt{(a^2 - 4ab + 4b^2)}\}} = \frac{3a^2b + 5ab^2}{2b + a - 2b} = b(3a + 5b).$

(50). $\dfrac{1}{2(1-x)}+\dfrac{1}{2(1+x)}=\dfrac{1}{1-x^2}$.

(51). $\dfrac{2x^2+1-x^2}{(1-x^2)^{\frac{3}{2}}}=\dfrac{1+x^2}{(1-x^2)^{\frac{3}{2}}}$.

(52). $\dfrac{x^2}{a-\surd(a^2-x^2)}+\dfrac{x^2}{a+\surd(a^2+a^2)}$

$$=\frac{x^2\{a+\surd(a^2-x^2)\}}{x^2}+\frac{x^2.\{a-\surd(a^2+x^2)\}}{-x^2}$$

$$=a+\surd(a^2-x^2)-a+\surd(a^2+x^2)=\surd(a^2-x^2)+\surd(a^2+x^2).$$

(53). $\dfrac{\{\surd(x^2+1)+\surd(x^2-1)\}^2}{2}+\dfrac{\{\surd(x^2+1)-\surd(x^2-1)\}^2}{2}$

$$=\frac{2(x^2+1)+2(x^2-1)}{2}=2x^2.$$

(54). $\dfrac{1}{x+\surd(x^2-1)}+\dfrac{1}{x-\surd(x^2-1)}=\dfrac{x-\surd(x^2-1)+x+\surd(x^2-1)}{1}=2x.$

(55). $\left[\sqrt{\left\{\dfrac{a+\surd(a^2-b)}{2}\right\}}+\sqrt{\left\{\dfrac{a-\surd(a^2-b)}{2}\right\}}\right]^2$

$$=a+\surd\{a^2-(a^2-b)\}=a+\surd(b).$$

(56). $\left[\sqrt{\left\{\dfrac{x^5y}{(x\surd3-3y)^2}\right\}}+\dfrac{y^2\surd(3xy)}{x-y\surd3}\right]\times\dfrac{\surd3}{\surd(xy)\times\surd(x+y\surd3)}$

$$=\frac{\surd(xy)(x^2+3y^2)\times\surd3}{\surd3(x-y\surd3)\surd(xy)\times\surd(x+y\surd3)}=\frac{x^2+3y^2}{x^2-3y^2}.$$

(57). $\left(a^{-\frac{2}{xy}}-2b^{\frac{1}{2r}}a^{-\frac{1}{xy}}a^{-\frac{y}{2r}}+a^{-\frac{y}{r}}b^{\frac{1}{r}}\right)^{\frac{1}{2}}=a^{-\frac{1}{xy}}-a^{-\frac{y}{2r}}b^{\frac{1}{2r}}.$

(58). $\dfrac{8-5\surd(2)}{3-2\surd(2)}$;

$$\because\ \{3-2\surd(2)\}\{3+2\surd(2)\}=9-8,$$

$$\therefore\ \frac{8-5\surd(2)}{3-2\surd(2)}\times\frac{3+2\surd(2)}{3+2\surd(2)}=\frac{24+\surd(2)-20}{9-8}=4+\surd(2).$$

(59). $\dfrac{3+\surd5}{4-\surd5}\times\dfrac{4+\surd5}{4+\surd5}=\dfrac{17+7\surd5}{11}$;

$$\frac{\surd3+7\surd5}{\surd3+\surd5}\times\frac{\surd5-\surd3}{\surd5-\surd3}=\frac{35-3-6\surd15}{2}=16-3\surd(15).$$

(60). $\frac{6+10\sqrt{(6)}}{2\sqrt{3}+3\sqrt{2}}=\frac{10+\sqrt{(6)}}{\sqrt{2}+\sqrt{3}}$

$\times\frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}-\sqrt{2}}=\frac{10\sqrt{(3)}-10\sqrt{2}+\sqrt{(18)}-\sqrt{(12)}}{+1}=8\sqrt{3}-7\sqrt{2};$

$\frac{2\{\sqrt{(3)}+1\}}{\sqrt{3}+1-\sqrt{2}}=\frac{2(\sqrt{3}+1)(\sqrt{3}+1+\sqrt{2})}{(\sqrt{3}+1)^2-2}$

$=\frac{2(\sqrt{3}+1)^2+\sqrt{2}(\sqrt{3}+1)}{2+2\sqrt{3}}=\sqrt{3}+1+\sqrt{2}.$

(61). $\frac{\sqrt{2}(2+\sqrt{3})}{2+\sqrt{\{4+2\sqrt{(3)}\}}}+\frac{\sqrt{2}(2-\sqrt{3})}{2-\sqrt{(4-2\sqrt{3})}}=\frac{\sqrt{2}(2+\sqrt{3})}{\sqrt{3}(\sqrt{3}+1)}+\frac{\sqrt{2}.(2-\sqrt{3})}{\sqrt{3}(\sqrt{3}-1)}$

$=\frac{4+2\sqrt{3}}{\sqrt{6}(\sqrt{3}+1)}+\frac{4-2\sqrt{3}}{\sqrt{6}(\sqrt{3}-1)}=\frac{\sqrt{3}+1+\sqrt{3}-1}{\sqrt{6}}=\sqrt{2}.$

(62). $\frac{5+2\sqrt{-3}}{2-\sqrt{(-3)}}+\frac{2}{2+\sqrt{(-3)}}-\frac{4}{1-\sqrt{(-3)}}$

$=\frac{4+9\sqrt{(-3)}+4-2\sqrt{(-3)}}{4+3}-\frac{4}{1-\sqrt{(-3)}}$

$=\frac{8+7\sqrt{(-3)}}{7}-\frac{4+4\sqrt{(-3)}}{4}=\frac{1}{7}.$

(63). $\frac{1+\sqrt{3}}{1-\sqrt{3}}+\frac{2+\sqrt{3}}{2-\sqrt{3}}-\frac{2\sqrt{3}+1}{2+\sqrt{3}}=\frac{4+2\sqrt{3}}{-2}+\frac{7+4\sqrt{3}}{1}-\frac{3\sqrt{3}-4}{1}$

$=-2-\sqrt{3}+7+4\sqrt{3}+4-3\sqrt{3}=9.$

(64). $\frac{\{a+\sqrt{(-b)}\}^2+\{a-\sqrt{(-b)}\}^2}{a^2+b}=\frac{2(a^2-b)}{a^2+b}.$

(65.) $\frac{x^{\frac{1}{3}}+y^{\frac{1}{2}}}{x^{\frac{1}{3}}-y^{\frac{1}{2}}}-\frac{2x^{\frac{1}{3}}y^{\frac{1}{2}}}{x^{\frac{2}{3}}-y}=\frac{x^{\frac{2}{3}}+2x^{\frac{1}{3}}y^{\frac{1}{2}}+y-2x^{\frac{1}{3}}y^{\frac{1}{2}}}{x^{\frac{2}{3}}-y}=\frac{x^{\frac{2}{3}}+y}{x^{\frac{2}{3}}-y}.$

(66). $\frac{a^2+ab+b^2}{(a^3-b^3)(x^{\frac{1}{2}}-a^{\frac{1}{2}})}-\frac{x^{\frac{1}{2}}}{(a-b)(x-a)}$

$=\frac{x^{\frac{1}{2}}+a^{\frac{1}{2}}-x^{\frac{1}{2}}}{(a-b)(x-a)}=\frac{a^{\frac{1}{2}}}{(a-b)(x-a)}.$

(67). $\sqrt{\left\{\frac{a+b-c}{(a+c-b)(b+c-a)}\right\}}+\sqrt{\left\{\frac{a+c-b}{(b+c-a)(a+b-c)}\right\}}$

$$+\sqrt{\left\{\frac{b+c-a}{(a+c-b)(a+b-c)}\right\}}$$

$$=\frac{a+b-c+a+c-b+b+c-a}{\sqrt{\{(a+c-b)(b+c-a)(a+b-c)\}}}$$

$$=\frac{a+b+c}{\sqrt{\{(a+c-b)(a+b-c)(b+c-a)\}}}.$$

(68). $\frac{a^{\frac{3}{2}}+b^{\frac{3}{2}}}{(a^{\frac{1}{2}}+b^{\frac{1}{2}})^3}+\frac{a^{\frac{1}{2}}b^{\frac{1}{2}}}{(a^{\frac{1}{2}}+b^{\frac{1}{2}})^2}+\frac{a^{\frac{1}{2}}-b^{\frac{1}{2}}}{a^{\frac{1}{2}}+b^{\frac{1}{2}}}$

$$=\frac{a-a^{\frac{1}{2}}b^{\frac{1}{2}}+b+a^{\frac{1}{2}}b^{\frac{1}{2}}+a-b}{(a^{\frac{1}{2}}+b^{\frac{1}{2}})^2}=\frac{2a}{(a^{\frac{1}{2}}+b^{\frac{1}{2}})^2}$$

(69). $\frac{1}{x-1}+\frac{2}{2x+1-\sqrt{(-3)}}+\frac{2}{2x+1+\sqrt{(-3)}}=\frac{1}{x-1}+\frac{4(2x+1)}{4x^2+4x+4}$

$$=\frac{x^2+x+1+2x^2-x-1}{x^3-1}=\frac{3x^2}{x^3-1}.$$

(70). $\frac{4\sqrt{3}+\sqrt{2}}{\sqrt{3}+\sqrt{2}-\sqrt{5}}\times\frac{\sqrt{3}+\sqrt{2}+\sqrt{5}}{\sqrt{3}+\sqrt{2}+\sqrt{5}}=\frac{14+5\sqrt{6}+4\sqrt{(15)}+\sqrt{10}}{2\sqrt{6}}.$

$$\frac{1+\sqrt{2}}{1+\sqrt{2}-\sqrt{3}}=\frac{1+\sqrt{2}}{1+\sqrt{2}-\sqrt{3}}\times\frac{1+\sqrt{2}+\sqrt{3}}{1+\sqrt{2}+\sqrt{3}}$$

$$=\frac{3+2\sqrt{2}+\sqrt{3}(\sqrt{2}+1)}{2\sqrt{(2)}}=\frac{3+2\sqrt{2}+\sqrt{6}+\sqrt{3}}{2\sqrt{2}}.$$

(71). $\frac{\sqrt{2}(2+\sqrt{3})}{\sqrt{3}(1+\sqrt{3})}-\frac{\sqrt{2}(2-\sqrt{3})}{\sqrt{3}(\sqrt{3}-1)}$

$$=\frac{1}{\sqrt{6}}\frac{4+2\sqrt{3}}{(\sqrt{3}+1)}-\frac{4-2\sqrt{3}}{\sqrt{6}(\sqrt{3}-1)}=\frac{1}{\sqrt{6}}(1+\sqrt{3}-\sqrt{3}+1)=\frac{\sqrt{6}}{3}.$$

(72). Find the root of $5+2\sqrt{(6)}$.

Let $\sqrt{\{5+2\sqrt{(6)}\}}=\sqrt{(x)}+\sqrt{(y)}$.

Squaring both sides, $5+2\sqrt{(6)}=x+y+2\sqrt{(xy)}$;

$\therefore x+y=5$, and $\sqrt{(xy)}=\sqrt{(6)}$;

$$\begin{array}{rl} \text{and } x^2 + 2xy + y^2 &= 25 \\ 4xy \quad &= 24 \\ \hline \therefore\ x^2 - 2xy + y^2 &= 1; \end{array}$$

$$\text{and } x - y = 1, \text{ and } x + y = 5;$$

$$\therefore\ x = 3,\ y = 2, \text{ and } \sqrt{(5 + 2\sqrt{6})} = \sqrt{(3)} + \sqrt{(2)}.$$

(73). $\sqrt{(7 + 2\sqrt{10})} = \sqrt{5} + \sqrt{2} : \sqrt{(8 + 2\sqrt{7})} = 1 + \sqrt{7}.$

(74). Let $\sqrt{\{\cdot 03 + \cdot 04\sqrt{(-1)}\}} = \sqrt{x} + \sqrt{y};$

$$\therefore\ \cdot 03 + \cdot 04\sqrt{(-1)} = x + y + 2\sqrt{(xy)};$$

$$\begin{array}{rl} x^2 + 2xy + y^2 &= \cdot 0009 \\ 4xy \quad &= -\cdot 0016 \\ \hline x^2 - 2xy + y^2 &= \cdot 0025; \end{array}$$

$$\therefore\ x - y = \cdot 05,\ (x + y) = \cdot 03;$$

$$\therefore\ x = \cdot 04,\ y = -\cdot 01;$$

$$\therefore\ \sqrt{x} + \sqrt{y} = \cdot 2 + \cdot 1\sqrt{(-1)};$$

$$\sqrt{(37 - 20\sqrt{3})} = 5 - 2\sqrt{3}.$$

(75). $\sqrt{(28 - 10\sqrt{3})} = 5 - \sqrt{3}.$

$$\text{Let } \sqrt{\{-2 + 2\sqrt{(-3)}\}} = \sqrt{x} + \sqrt{y};\ \therefore\ x + y = -2;$$

$$\therefore\ x^2 + 2xy + y^2 = 4, \text{ and } 4xy = -12;$$

$$\therefore\ x - y = 4, \text{ or } x = 1, \text{ and } y = -3;$$

$$\therefore\ \sqrt{\{-2 + 2\sqrt{(-3)}\}} = 1 + \sqrt{(-3)}.$$

(76). Let $\sqrt{\{21 - \sqrt{-(400)}\}} = \sqrt{x} - \sqrt{y};$

$$\therefore\ x + y = 21, \text{ and } 2\sqrt{(xy)} = \sqrt{(-400)};$$

$$\therefore\ x^2 + 2xy + y^2 = 441, \text{ and } x^2 - 2xy + y^2 = 841;\ \therefore\ x - y = 29;$$

$$\therefore\ x = 25, \text{ and } y = -4;$$

$$\therefore\ \sqrt{x} - \sqrt{y} = 5 - 2\sqrt{(-1)}.$$

$$\text{Let } \sqrt{\{4\sqrt{(-5)} - 1)\}} = \sqrt{x} - \sqrt{y};$$

$$\therefore\ x + y = -1, \text{ and } 2\sqrt{(xy)} = 4\sqrt{(-5)};$$

$$\therefore\ x^2 + 2xy + y^2 = 1, \text{ and } x^2 - 2xy + y^2 = 81;\ \therefore\ x - y = 9;$$

$$\therefore\ x = 4, \text{ and } y = -5;$$

$$\therefore\ \sqrt{x} - \sqrt{y} = 2 - \sqrt{(-5)}.$$

(77). $\sqrt{(2+\sqrt{3})} = \sqrt{\left\{\frac{4+2\sqrt{(3)}}{2}\right\}} = \frac{1}{\sqrt{2}}(1+\sqrt{3})$.

$$\text{Let } \sqrt{\{-18\sqrt{(-1)}\}} = \sqrt{x} - \sqrt{y};$$

$$\therefore x + y = 0, \text{ and } 2\sqrt{(xy)} = 18\sqrt{(-1)};$$

$$\therefore x^2 - 2xy + y^2 = 324, \text{ and } x - y = 18; \therefore x = 9, \text{ and } y = -9;$$

$$\therefore \sqrt{x} - \sqrt{y} = 3 - 3\sqrt{(-1)}.$$

(78). $\sqrt{\{4-\sqrt{(7)}\}} = \frac{\sqrt{7}-1}{\sqrt{2}}$; $\sqrt{\{16 \pm 8\sqrt{(3)}\}} = 2 \pm 2\sqrt{(3)}$.

Find the fourth root of—

(79). $\sqrt[4]{\{97+28\sqrt{(12)}\}} = \sqrt{\{7+4\sqrt{(3)}\}} = 2+\sqrt{3}$;

$$\sqrt[4]{\left(\frac{17-12\sqrt{2}}{3}\right)} = \frac{\sqrt{(3-2\sqrt{2})}}{\sqrt[4]{(3)}} = \frac{1}{\sqrt[4]{3}}(1-\sqrt{2}).$$

(80). $\sqrt[4]{(14+8\sqrt{3})} = \sqrt{(2\sqrt{2}+\sqrt{6})} = \sqrt{\{\sqrt{2}\,(2+\sqrt{3})\}} = \frac{1+\sqrt{3}}{\sqrt[4]{(2)}}$,

$$\sqrt[2]{(-16a^4)} = 4a^2\sqrt{(-1)}, \text{ let } \sqrt{\{4a^2\sqrt{(-1)}\}} = \sqrt{x}+\sqrt{y},$$

$$\text{then } x + y = 0, \text{ and } 2\sqrt{(xy)} = 4a^2\sqrt{(-1)};$$

$$\therefore x^2 - 2xy + y^2 = +16a^4, \text{ and } x - y = 4a^2;$$

$$\therefore x = 2a^2, \text{ and } y = -2a^2;$$

$$\therefore \sqrt[4]{(-16a^2)} = a\,\{\sqrt{2}+\sqrt{(-2)}\}.$$

Find the square root of—

(81). Let $\sqrt{\{a^2+2x\sqrt{(a^2-x^2)}\}} = \sqrt{x}+\sqrt{y}$;

$$\therefore a^2 = x + y, \text{ and } 2x\sqrt{(a^2-x^2)} = 2\sqrt{(xy)},$$

$$\text{and } x^2 + 2xy + y^2 = a^4,$$

$$\text{also } 4xy = 4a^2x^2 - 4x^4;$$

$$\therefore x^2 - 2xy + y^2 = (a^2-2x^2)^2, \text{ and } x - y = a^2 - 2x^2,$$

$$\text{but } x + y = a^2;$$

$$\therefore x = a^2 - x^2, \text{ and } y = x^2;$$

$$\therefore \sqrt{x}+\sqrt{y} = \sqrt{(a^2-x^2)} + x.$$

$$\text{Let } \sqrt{\{2+2(1-x)\sqrt{(1+2x-x^2)}\}} = \sqrt{x}+\sqrt{y};$$

$$\therefore x + y = 2, \text{ and } 2\sqrt{(xy)} = 2(1-x)\sqrt{(1+2x-x^2)},$$

$$\text{and } x^2 + 2xy + y^2 = 4,$$

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$$\text{but} \quad 4xy = 4 - 16x^2 + 16x^3 - 4x^4;$$

$$\therefore x^2 - 2xy + y^2 = 16x^2 - 16x^3 + 4x^4,$$

$$\text{and } x - y = 4x - 2x^2,$$

$$\text{also } x + y = 2; \therefore x = 1 + 2x - x^2, \text{ and } y = 1 - 2x + x^2,$$

$$\sqrt{x} + \sqrt{y} = 1 - x + \sqrt{(1 + 2x - x^2)}.$$

(82). Let $\sqrt{\{1 + \sqrt{(1 - m^2)}\}} = \sqrt{x} + \sqrt{y}$;

$$\therefore x + y = 1, \text{ and } 2\sqrt{(xy)} = \sqrt{(1 - m^2)},$$

$$\text{and } x^2 + 2xy + y^2 = 1,$$

$$\text{also } 4xy = 1 - m^2;$$

$$\therefore x^2 - 2xy + y^2 = m^2, \text{ and } x - y = m,$$

$$\text{but } x + y = 1; \therefore x = \frac{1+m}{2}, \; y = \frac{1-m}{2},$$

$$\text{and } \sqrt{x} + \sqrt{y} = \sqrt{\left(\frac{1+m}{2}\right)} + \sqrt{\left(\frac{1-m}{2}\right)},$$

$$\text{so also } \sqrt{\{xy - 2x\sqrt{(xy - x^2)}\}} = x - \sqrt{(xy - x^2)}.$$

Find the cube root of—

(83). Let $\sqrt[3]{(16 + 8\sqrt{5})} = x + \sqrt{y}$; $\therefore \sqrt[3]{(16 - 8\sqrt{5})} = x - \sqrt{y}$;

$$\therefore x^2 - y = \sqrt[3]{(256 - 320)} = \sqrt[3]{(-64)} = -4,$$

$$\text{also } 16 + 8\sqrt{5} = x^3 + 3x^2\sqrt{y} + 3xy + y^{\frac{3}{2}},$$

$$\therefore x^3 + 3xy = 16, \text{ and } y = x^2 + 4;$$

$$\therefore x^3 + 3x^3 + 12x = 16, \text{ or } x^3 + 3x = 4, \text{ if } x = 1,$$

$$\text{then } y = 5;$$

$$\therefore \sqrt[3]{(16 + 8\sqrt{5})} = 1 + \sqrt{5},$$

$$\text{if } \sqrt[3]{(22 + 10\sqrt{7})} = x + \sqrt{y}; \therefore \sqrt[3]{(22 - 10\sqrt{7})} = x - \sqrt{y},$$

$$\text{then } x^2 - y = \sqrt[3]{(484 - 700)} = -6; \therefore y = x^2 + 6,$$

$$\text{and } x^3 + 3xy = 22, \text{ or } x^3 + 3x^3 + 18x = 22, \text{ let } x = 1;$$

$$\therefore y = 7, \text{ and } \sqrt[3]{(22 + 10\sqrt{7})} = 1 + \sqrt{7}.$$

(84). $\sqrt[3]{\{11\sqrt{(2)} + 9\sqrt{3}\}} = \sqrt{2} + \sqrt{3}$; $\sqrt[3]{(2\sqrt{7} + 3\sqrt{3})} = \frac{1}{2}(\sqrt{7} + \sqrt{3})$.

(85). Let $\sqrt[3]{\{2\sqrt{(-1)} - 11\}} = \sqrt{y} - x$; $\therefore \sqrt[3]{\{2\sqrt{(-1)} + 11\}} = \sqrt{(y)} + x$,

$$\text{and } y - x^2 = \sqrt[3]{(-4 - 121)} = -5; \therefore y = x^2 - 5,$$

$$\text{and } (\sqrt{y} - x)^3 = y\sqrt{(y)} - 3xy + 3x^2\sqrt{(y)} - x^3; \ \therefore\ x^3 + 3xy = 11.$$

$$\text{and } x^3 + 3x^3 - 15x = 11; \text{ let } x = -1, \text{ then } y = -4;$$

$$\therefore\ \sqrt[3]{\{2\sqrt{(-1)} - 11\}} = 1 - 2\sqrt{(-1)}.$$

$$\text{Let } \sqrt[3]{(25 + 21\sqrt{3} + 17\sqrt{5} + 6\sqrt{15})} = 1 + A\sqrt{3} + B\sqrt{5}$$

$$\therefore\ 25 + 21\sqrt{3} + 17\sqrt{5} + 6\sqrt{15}$$

$$= 1 + 3A\sqrt{3} + 3B\sqrt{5} + 3\{3A^2 + 5B^2 + 2AB\sqrt{(15)}\}$$

$$+ 3A^3\sqrt{3} + 9AB^2\sqrt{(15)} + 15AB^2\sqrt{3} + 5B^3\sqrt{5},$$

$$\text{whence } 9A^2 + 15B^2 + 1 = 25; \ \therefore\ 3A^2 = 8 - 5B^2,$$

$$\text{if } 6AB\sqrt{(15)} = 6\sqrt{(15)}; \ \therefore\ AB = 1; \ \therefore\ A = 1, B = 1,$$

$$\text{and } \sqrt[3]{(25 + 21\sqrt{3} + 17\sqrt{5} + 6\sqrt{15})} = 1 + \sqrt{3} + \sqrt{5}.$$

(86). $$\sqrt[3]{(x^{\frac{3}{2}} - 3x^{\frac{4}{3}} + 3x^{\frac{7}{6}} + 3x^{\frac{5}{4}} + 2x - 3x^{\frac{5}{6}} - 6x^{\frac{13}{12}} + 3x^{\frac{11}{12}} + x^{\frac{3}{4}})} = x^{\frac{1}{2}} - x^{\frac{1}{3}} + x^{\frac{1}{4}}$$

$$-x^{\frac{1}{3}} \times (3x - 3x^{\frac{5}{6}} + x^{\frac{2}{3}}) \ \Big|\ \underline{-3x^{\frac{4}{3}} + 3x^{\frac{7}{6}} - x}$$

$$x^{\frac{1}{4}} \times \{3(x^{\frac{1}{2}} - x^{\frac{1}{3}})^2 + 3(x^{\frac{1}{2}} - x^{\frac{1}{3}})x^{\frac{1}{4}} + x^{\frac{1}{2}}\} \ \Big|\ 3x^{\frac{5}{4}} + 3x$$

$$x^{\frac{1}{4}}(3x - 6x^{\frac{5}{6}} + 3x^{\frac{2}{3}} + 3x^{\frac{3}{4}} - 3x^{\frac{7}{12}} + x^{\frac{1}{2}} \ \Big|\ \underline{3x^{\frac{5}{4}} - 6x^{\frac{13}{12}} + 3x^{\frac{11}{12}} + 3x - 3x^{\frac{5}{6}} + x^{\frac{3}{4}}}$$

(87). (1) $2^{\frac{1}{5}} >< 3^{\frac{1}{8}}$, or $2^8 >< 3^5$; $\therefore$ but $256 > 243$;

$$\therefore\ 2^{\frac{1}{5}} > 3^{\frac{1}{8}}.$$

(2) $\sqrt{(10)} + \sqrt{7} >< \sqrt{19} + \sqrt{3}$; $\therefore\ 17 + 2\sqrt{(70)} >< 22 + 2\sqrt{(57)}$,

$$\text{but } 2\sqrt{(70)} < 5 + 2\sqrt{(57)};$$

$$\therefore\ \sqrt{(10)} + \sqrt{7} < \sqrt{19} + \sqrt{3}.$$

(3) $(\sqrt[3]{2} + \sqrt[3]{5})^3 >< 27$,

$$\text{or } 2 + 3\sqrt[3]{(10)}\,(\sqrt[3]{2} + \sqrt[3]{5}) + 5 >< 27,$$

$$3\sqrt[3]{(10)}\,\{\sqrt[3]{(2)} + \sqrt[3]{5}\} >< 20,$$

$$\text{let } \sqrt[3]{2} + \sqrt[3]{5} = 3, \text{ then } 729 \times 10 < 8000,$$

$$\text{much more then is } \sqrt[3]{2} + \sqrt[3]{5} < 3.$$

(4) $\left(\frac{1}{2}\right)^{\frac{1}{2}} >< \left(\frac{4}{9}\right)^{\frac{1}{3}}$; $\therefore\ \left(\frac{1}{2}\right)^3 >< \left(\frac{4}{9}\right)^2$,

$$\text{or } 81 < 128; \ \therefore\ \left(\frac{1}{2}\right)^{\frac{1}{2}} < \left(\frac{2}{3}\right)^{\frac{2}{3}}.$$

(88). $2(1+n^2+n^4) >< 3n(1+n^2)$;

$$\therefore\ 1-n+(1-n^3)-2n(1-n)-2n^3(1-n) >< 0,$$

$$\text{or } 2-n+n^2-2n^3 >< 0,$$

$$\therefore\ 1-n+(1-n^2)+2n^2(1-n) >< 0,$$

or $2+n+2n^2$ a positive quantity > 0;

$$\therefore\ 2(1+n^2+n^4) > 3(n+n^3),$$

if $n^{\frac{3}{2}}-1 >< (n-1)^{\frac{3}{2}}$, if $n=1$, the expression vanishes;

$$\therefore\ n^3-2n^{\frac{3}{2}}+1 >< n^3-3n^2+3n-1,$$

$$\text{or } 3n(n-1) >< 2(n^{\frac{3}{2}}-1),$$

$$\text{or } 3n^{\frac{3}{2}}+n-2n^{\frac{1}{2}}-2 >< 0,$$

$$\text{whence } 2n^{\frac{1}{2}}(n-1)+(n^{\frac{3}{2}}-1)+n-1 > 0,$$

or the positive quantity

$$3n+4n^{\frac{1}{2}}+2 > 0;\ \therefore\ n^{\frac{3}{2}}-1 > (n-1)^{\frac{3}{2}}.$$

(89). If $a+\sqrt{a} >< 1+a^{\frac{3}{2}}$,

$$\text{then } \sqrt{a}\,(1+a^{\frac{1}{2}}) >< 1+a^{\frac{3}{2}},$$

$$\text{or } \sqrt{a} >< 1-a^{\frac{1}{2}}+a,$$

$$\text{or } 0 < (1-a^{\frac{1}{2}})^2;$$

$$\therefore\ a+\sqrt{a} < 1+a^{\frac{3}{2}};$$

$$\text{if } a^{\frac{3}{2}}-1 >< a-1,$$

$$\text{then } a+a^{\frac{1}{2}}+1 >< a^{\frac{1}{2}}+1, \text{ or } a > 0,$$

$$\text{but } a > 0;\ \therefore\ a^{\frac{3}{2}}-1 > a-1.$$

(90). $2^{\frac{1}{2}} >< 3^{\frac{1}{3}}$; $\therefore\ 2^3 >< 3^2$, but $8 < 9$,

also $3^{\frac{1}{3}} >< 5^{\frac{1}{5}}$; $\therefore\ 3^5 >< 5^3$, but $243 > 125$,

also $2^{\frac{1}{2}} >< 5^{\frac{1}{5}}$; $\therefore\ 2^5 >< 5^2$, but $32 > 25$;

$\therefore\ \sqrt[3]{3}$ is the greatest, and $\sqrt[5]{5}$ is the least.

(91). Let $\sqrt[4]{(-1)} = \sqrt{x} - \sqrt{y}$; $\therefore (-1)^{\frac{1}{2}} = x + y - 2\sqrt{(xy)}$;

$$\therefore x^2 + 2xy + y^2 = 0,$$
$$4xy = -1,$$

and
$$x - y = 1,$$
$$x + y = 0;$$

$$\therefore \sqrt{x} = \frac{1}{\sqrt{2}}, \text{ and } \sqrt{y} = \sqrt{\frac{-1}{2}}; \quad \therefore \sqrt[4]{(-1)} = \frac{1+\sqrt{(-1)}}{\sqrt{2}};$$

$$\frac{a+b\sqrt{(-1)}}{c+d\sqrt{(-1)}} = \frac{a+b\sqrt{(-1)}}{c+d\sqrt{(-1)}} \times \frac{c-d\sqrt{(-1)}}{c-d\sqrt{(-1)}} = \frac{ac+bd+(bc-ad)\sqrt{(-1)}}{c^2+d^2}.$$

SIMPLE EQUATIONS.

VII. (1). $\frac{x}{6} + \frac{11}{2} = \frac{x+5}{2} + \frac{x-8}{4} + \frac{1}{3}$,

by multiplying the equation by 12, we have

$$2x + 66 = 6x + 30 + 3x - 24 + 4,$$
$$7x = 56; \therefore x = 8.$$

(2). $\frac{3x}{8} + 1 + \frac{1}{21} - \frac{2x}{7} = \frac{31}{28}$,

$$\text{(by 56) } 21x + 56 + \frac{8}{3} - 16x = 62,$$
$$15x = 18 - 8; \therefore x = \frac{2}{3}.$$

(3). $\frac{25x+5}{6} - \frac{8+2x}{5} = \frac{3x+9}{4}$,

$$\text{(by 60) } 250x + 50 - 96 - 24x = 45x + 135,$$
$$x = \frac{181}{181} = 1.$$

(4). $\frac{x-18}{4} + \frac{2x-24}{11} + \frac{11x-34}{22} = \frac{7}{44}$,

$$\text{(by 44) } 11x - 198 + 8x - 96 + 22x - 68 = 7,$$
$$41x = 369; \therefore x = 9.$$

(5). $\frac{2x+12}{6}+\frac{2x-15}{6}+\frac{x}{2}=3,$

$$(\text{by } 6)\ 2x+12+2x-15+3x=18,$$

$$x=\frac{21}{7}=3.$$

(6). $\frac{3x}{4}+\frac{7x}{15}+\frac{11x}{6}-\frac{183}{5}=0,$

$$(\text{by } 60)\ 45x+28x+110x=183\times 12;$$

$$\therefore x=\frac{183\times 12}{183}=12.$$

(7). $\frac{1}{x}+\frac{1}{2x}-\frac{1}{3x}=\frac{7}{24},$

$$(\text{by } 24x)\ 24+12-8=7x;\ \therefore x=\frac{28}{7}=4.$$

(8). $10\left(x+\frac{1}{2}\right)-6x\left(\frac{1}{x}-\frac{1}{3}\right)=23,$

by taking away the brackets by multiplication, we have

$$10x+5-6+2x=23;\ \therefore x=\frac{24}{12}=2.$$

(9). $x+\frac{11-x}{3}=\frac{19-x}{2}+\frac{7}{3}+\frac{2x-14}{7},$

$$(\text{by } 42)\ 42x+154-14x=399-21x+98+12x-84;$$

$$\therefore 37x=259;\ \therefore x=7.$$

(10). $3x+20=7-\frac{1}{2}\left[3-\frac{4}{3}(x-1)\right],$

$$18x+78=-9+4x-4;$$

$$\therefore 14x=-91;\ \therefore x=-\frac{13}{2}=-6\tfrac{1}{2}.$$

(11). $\frac{x}{2}+\frac{5x+4}{3}=\frac{4x+9}{3}+\frac{5}{12},$

$$(\text{by } 12)\ 6x+20x+16=16x+36+5,$$

$$10x=25;\ \therefore x=\frac{5}{2}=2\tfrac{1}{2}.$$

(12). $\frac{x}{12} - \frac{8-x}{8} - \frac{1}{4}(5+x) + \frac{11}{4} = 0,$

$$\text{(by 24) } 2x - 24 + 3x - 30 - 6x + 66 = 0,$$

$$x = 12.$$

(13). $\frac{x}{8} - \frac{x-1}{2\frac{1}{2}} = \frac{3x-4}{15} + \frac{x}{12},$

$$\text{(by 60) } \frac{15x}{2} - 24x + 24 = 12x - 16 + 5x,$$

$$15x - 82x = -80,$$

$$x = \frac{-80}{-67} = +1\tfrac{13}{67}.$$

(14). $\frac{1}{6}(x+3) - \frac{1}{7}(11-x) = \frac{2}{5}(x-4) - \frac{1}{21}(x-3),$

$$\text{(by 21) } \frac{7x+21}{2} - 33 + 3x = \frac{42}{5}(x-4) - x + 3,$$

$$35x + 105 - 360 + 40x = 84x - 336;$$

$$\therefore 9x = 81, \text{ and } x = 9.$$

(15). $\frac{x+1}{2} + \frac{x+2}{3} = 16 - \frac{x-6}{4},$

$$6x + 6 + 4x + 8 = 192 - 3x + 18,$$

$$13x = 196; \therefore x = 15\tfrac{1}{13}.$$

(16). $\frac{1}{7}\left(x - \frac{1}{2}\right) - \frac{1}{5}\left(\frac{2}{3} - x\right) = 1\tfrac{13}{30},$

$$\text{(by 30) then } \frac{30x-15}{7} - 4 + 6x = 43,$$

$$\text{thence } 30x - 15 + 42x = 329;$$

$$\therefore x = \frac{344}{72} = \frac{43}{9} = 4\tfrac{7}{9}.$$

(17). $\frac{x+2}{3} + \frac{x+3}{4} = 7 + \frac{x-4}{6},$

$$4x + 8 + 3x + 9 = 84 + 2x - 8;$$

$$\therefore x = \frac{59}{5} = 11\tfrac{4}{5}.$$

(18). $\frac{2}{3}(x-5)-\frac{3}{11}(x-13\frac{1}{3})=15-\frac{3}{5}(19-\frac{1}{3}x)$,

(by 15) then $10x-50-\frac{15}{11}(3x-40)=225-171+3x$;

$$\therefore 77x-45x+600=1144;$$

$$\therefore x=\frac{544}{32}=17.$$

(19). $\frac{5x-7}{3}-\frac{4x-9}{5}+2x=13\frac{4}{5}$,

(by 15) $25x-35-12x+27+30x=69\times 3=207$

$$\therefore x=\frac{215}{43}=5.$$

(20). $3x-4-\frac{4}{5}\times\frac{7x-9}{3}=\frac{4}{5}\times\frac{17+x}{3}$,

(by 15) $45x-60-28x+36=68+4x$;

$$x=\frac{92}{13}=7\tfrac{1}{13}.$$

(21). $\frac{5x}{9}-\frac{2x-1}{3}=\frac{4}{15}+\frac{5x-3}{5}$,

(by 45) $25x-30x+15=12+45x-27$;

$$\therefore x=\frac{30}{50}=\frac{3}{5}.$$

(22). $\frac{5x-7}{3}-\frac{3x-2}{7}=\frac{x-5}{4}+\frac{83x-67}{83}$,

(by 84) $140x-196-36x+24=21x-105+84\left(x-\frac{67}{83}\right)$;

$$\therefore 83x-67=\frac{84}{83}(83x-67);\quad \therefore 83x=67,\text{ or } x=\frac{67}{83}.$$

(23). $\frac{x}{2}-\frac{x-2}{3}=\frac{1}{4}\left[x-\frac{2}{3}(2\frac{1}{2}-x)\right]-\frac{1}{3}(x-5)$,

by multiplying by 12 and taking away a bracket, we have

$$6x-4x+8=3x-2(2\tfrac{1}{2}-x)-4x+20;$$

$$\therefore 3x=12-5+2x;\quad \therefore x=7.$$

(24). $\frac{5x-4}{9} - \frac{2x-13}{7} = \frac{x+1}{3}$,

$$\text{(by 63) } 35x - 28 - 18x + 117 = 21x + 21;$$

$$\therefore x = \frac{68}{4} = 17.$$

(25). $3\frac{1}{3}\left[28 - \left(\frac{x}{8} + 24\right)\right] = 3\frac{1}{2}\left(2\frac{1}{3} + \frac{x}{4}\right)$,

$$\frac{10}{3}\left(28 - \frac{x}{8} - 24\right) = \frac{7}{2}\left(\frac{7}{3} + \frac{x}{4}\right),$$

$$\text{then } \frac{40}{3} - \frac{5x}{12} = \frac{49}{6} + \frac{7x}{8},$$

$$\text{and } 320 - 10x = 196 + 21x;$$

$$\therefore x = \frac{124}{31} = 4.$$

(26). $\frac{2}{3}(x-5) - \frac{3}{11}(x-14) = 5 - \frac{1}{5}(9-x)$,

$$\text{(by 15) } 10x - 50 - \frac{15}{11}(3x - 42) = 75 - 27 + 3x,$$

by transposition and multiplying by 11,

$$\text{then } 77x - 45x + 630 = 98 \times 11 = 1078;$$

$$\therefore x = \frac{448}{32} = 14.$$

(27). $\frac{x-1}{7} + \frac{23-x}{5} = 7\frac{81}{140} - \frac{x+4}{4}$,

$$\text{(by 20) } \frac{20x-20}{7} + 92 - 4x = \frac{1061}{7} - 5x - 20,$$

$$\text{then } 20x - 20 + 784 = 1061 - 7x;$$

$$\therefore x = \frac{297}{27} = 11.$$

(28). $\frac{3x+4}{5} - \frac{7x-3}{2} - \frac{9x-16}{4} + \frac{183}{20} = 0$,

$$12x + 16 - 70x + 30 - 45x + 80 + 183 = 0;$$

$$x = \frac{309}{103} = 3.$$

D 5

(29). $8\frac{3}{4} + \frac{3x}{7} - \frac{5}{6} + 2x - \frac{12x}{5} + 13 + \frac{x}{4} = 22\frac{13}{15}$,

$$\text{(by 12) } 105 + \frac{36x}{7} - 10 + 24x - \frac{144x}{5} + 156 + 3x = \frac{343 \times 4}{5},$$

$$\text{or } 251 + \frac{36x}{7} + 27x - \frac{144x}{5} = \frac{1372}{5},$$

$$\text{then } 1255 + \frac{180x}{7} + 135x - 144x = 1372;$$

$$\therefore 180x - 63x = 117 \times 7;$$

$$\therefore x = \frac{117 \times 7}{117} = 7.$$

(30). $\frac{3}{2}(x + 2) - \frac{8}{7}[1\frac{1}{3} - (1\frac{1}{2} + x)] = 4\frac{20}{21}$,

$$\frac{3x + 6}{2} - \frac{8}{7}\left(\frac{4}{3} - \frac{3}{2} - x\right) = \frac{104}{21};$$

$$\text{(by 42) then } 63x + 126 + 8 + 48x = 208;$$

$$\therefore x = \frac{74}{111} = \frac{2}{3}.$$

(31). $\frac{x + 1}{2} + \frac{x + 3}{3} = 14 + \frac{5 - x}{4} + \frac{14x}{51}$,

$$6x + 6 + 4x + 12 = 168 + 15 - 3x + \frac{56x}{17},$$

$$13x \times 17 = 165 \times 17 + 56x,$$

$$x = \frac{165 \times 17}{165} = 17.$$

(32). $\frac{1}{14}\left(3x + \frac{2}{3}\right) - \frac{1}{7}(4x - 6\frac{2}{3}) = \frac{1}{2}(5x - 6)$,

$$3x + \frac{2}{3} - 8x + \frac{40}{3} = 35x - 42,$$

$$42 = 120x - 126;$$

$$\therefore x = \frac{168}{120} = \frac{21}{15} = \frac{7}{5} = 1\frac{2}{5}.$$

(33). $\frac{x}{21} - \frac{x-7}{3} + \frac{3x-1}{5} - \frac{2x}{7} = 2\frac{11}{15}$,

$$\text{(by 105) } 5x - 35x + 245 + 63x - 21 - 30x = 41 \times 7 = 287,$$

$$\therefore\ x = \frac{63}{3} = 21.$$

(34). $\frac{5x-1}{2} - \frac{7x-2}{10} = 29\frac{2}{5} - \frac{x}{2} + \frac{117}{10}$,

$$25x - 5 - 7x + 2 = 294 - 5x + 117.$$

$$\therefore\ x = \frac{414}{23} = 18.$$

(35). $\frac{4x-21}{7} + 7\frac{5}{6} + \frac{7x-28}{3} = x + 3\frac{3}{4} - \frac{9-7x}{8} + \frac{1}{12}$,

$$\frac{96x}{7} - 72 + 188 + 56x - 224 = 24x + 90 - 27 + 21x + 2,$$

$$\text{or } \frac{96x}{7} - 173 + 11x = 0,$$

$$96x + 77x = 173 \times 7;$$

$$\therefore\ x = \frac{173 \times 7}{173} = 7.$$

(36). $\frac{2x}{3} - \frac{1 - \frac{x}{2}}{4x} = \frac{x-1}{2} + \frac{x}{6} + \frac{7}{12}$,

$$\text{(by 12) } 8x - \frac{3 - \frac{3x}{2}}{x} = 6x - 6 + 2x + 7,$$

$$\text{then } -6 + 3x = 2x;\quad \therefore\ x = 6.$$

(37). $\frac{11x+12}{25} + \frac{19x+3}{7} - \frac{5x - 25\frac{1}{8}}{4} + \frac{17x-17}{21} = 28\frac{1}{7}$,

$$\frac{11x+12}{25} \times 84 + 228x + 36 - 105x + 532 + 68x - 68 = 2364,$$

by transposition, multiplying by 25,

$$\text{then } 924x + 1008 + 191x \times 25 = 1864 \times 25;$$

$$\therefore\ x = \frac{45592}{5699} = 8.$$

(38). $\frac{4x-17}{x-4}-\frac{8x-30}{2x-7}=\frac{5x-4}{x-1}-\frac{10x-13}{2x-3}$;

$$\therefore \frac{8x^2-62x+119-(8x^2-62x+120)}{(x-4)(2x-7)}$$

$$=\frac{10x^2-23x+12-(10x^2-23x+13)}{(x-1)(2x-3)},$$

or $(x-1)(2x-3)=(x-4)(2x-7)$;

$$\therefore 2x^2-5x+3=2x^2-15x+28;$$

$$\therefore x=\frac{25}{10}=\frac{5}{2}.$$

(39). $\frac{x+3}{x+1}-\frac{x+4}{x+2}+\frac{x-6}{x-4}=\frac{x^2-2x-15}{x^2-9}$,

$$\frac{x^2+5x+6-(x^2+5x+4)}{(x+1)(x+2)}=\frac{(x-5)(x+3)}{x^2-9}-\frac{x-6}{x-4}$$

$$=\frac{x^2-9x+20-(x^2-9x+18)}{(x-3)(x-4)};$$

$$\therefore x^2-7x+12=x^2+3x+2;$$

$$\therefore x=1.$$

(40). $\frac{7x+6}{28}-\frac{2x+4\frac{2}{7}}{23x-6}+\frac{x}{4}=\frac{11x}{21}-\frac{x^2-9}{42(x+3)}$,

by multiplying by 84 and observing that

$$(x^2-9)\div(x-3)=x+3,$$

$$21x+18-12\times\frac{14x+30}{23x-6}+21x=44x-2x+6,$$

$$\therefore 1-\frac{14x+30}{23x-6}=0,$$

or $23x-6-14x-30=0$; $\therefore x=\frac{36}{9}=4$.

(41). $\frac{3x-1}{x-4}+\frac{2x-1}{x+4}-5=\frac{96}{x^2-16}$,

by multiplying by x^2-16, then

$$3x^2+11x-4+2x^2-9x+4-5x^2+80=96;$$

$$\therefore x=\frac{16}{2}=8.$$

(42). $\frac{3x}{2} + \frac{9(9x^2-1)}{(1-3x)(x+3)} + \frac{6x^2-3}{2(x+3)} = \frac{9x-57}{2}$,

$$\frac{9x^2-1}{1-3x} = -\frac{9x^2-1}{3x-1} = -(3x+1),$$

and multiplying by $2(x+3)$,

then $3x^2 + 9x - 54x - 18 + 6x^2 - 3 = 9x^2 - 30x - 171$,

then $-15x = -150$; $\therefore x = 10$.

(43). $\frac{5x^2+x-3}{5x-4} - \frac{7x^2-3x-9}{7x-10} = \frac{x-3}{35x^2-78x+40}$,

observe that $(5x-4)(7x-10) = 35x^2 - 78x + 40$.

$35x^3 - 43x^2 - 31x + 30 - (35x^3 - 43x^2 - 33x + 36) = x - 3$.

then $2x - 6 = x - 3$; $\therefore x = 3$.

(44). $\frac{1}{1-x^2} + \frac{x+2}{4-x^2} - \frac{3+x}{9-x^2} = 0$,

$$\frac{1}{1-x^2} - \frac{1}{x-2} + \frac{1}{x-3} = 0;$$

$$\therefore \frac{1}{1-x^2} = \frac{x-3-x+2}{x^2-5x+6} = -\frac{1}{x^2-5x+6},$$

then $x^2 - 5x + 6 = x^2 - 1$; $\therefore x = 1\frac{2}{5}$.

(45). $\frac{12x+2}{3x-2} + \frac{3x-2}{3x+2} = \frac{15x+11}{3x+2}$,

$$\frac{12x+2}{3x-2} = \frac{15x+11-3x+2}{3x+2} = \frac{12x+13}{3x+2},$$

then $36x^2 + 30x + 4 = 36x^2 + 15x - 26$;

$$\therefore x = \frac{-30}{+15} = -2.$$

(46). $\frac{1}{3x-1} + \frac{2(x+1)}{x^2-1} - \frac{16x^2+x+3}{4(3x^2-4x+1)} = \frac{1}{x}$.

$$\frac{1}{3x-1} + \frac{2}{x-1} - \frac{1}{x} = \frac{16x^2+x+3}{4(3x^2-4x+1)x};$$

$\therefore 4\{x^2 - x + 6x^2 - 2x - (3x^2 - 4x + 1)\} = 16x^2 + x + 3$,

$3x = 7$; $\therefore x = 2\frac{1}{3}$.

(47). $\dfrac{x+\frac{4}{3}}{x+\frac{1}{2}} - \dfrac{x+20}{x+12} - 1 = 0,$

$$\frac{6x+8}{6x+3} = \frac{x+20+x+12}{x+12} = \frac{2x+32}{x+12},$$

$$6x^2 + 80x + 96 = 12x^2 + 198x + 96,$$

$$6x^2 = -118x; \therefore x = -19\tfrac{2}{3}.$$

(48). $\dfrac{x+8}{x+12} - \dfrac{3x+14}{3x+8} + \dfrac{6x-24}{x^2-16} = 0,$

$$\frac{x+8}{x+12} - \frac{3x+14}{3x+8} + \frac{6}{x+4} = 0,$$

$$3x^2 + 32x + 64 - (3x^2 + 50x + 168) + \frac{6(3x^2 + 44x + 96)}{x+4} = 0,$$

$$(9x+52)(x+4) = 9x^2 + 132x + 288,$$

$$\text{whence } x = \frac{80}{44} = 1\tfrac{9}{11}.$$

(49). $\dfrac{25-\frac{x}{3}}{x+1} + \dfrac{16x+4\frac{1}{5}}{3x+2} = 5 + \dfrac{23}{x+1},$

$$\frac{75-x-69}{3(x+1)} + \frac{80x+21}{5(3x+2)} = 5,$$

whence $5(12+16x-3x^2)+3(80x^2+101x+21)=75(3x^2+5x+2),$

$$\text{or } x = \frac{27}{8} = 3\tfrac{3}{8}.$$

(50). $\left(\dfrac{8x-3}{4x-1}\right)^2 - \dfrac{4x-5}{x-1} = 0,$

$$(64x^2 - 48x + 9)(x-1) = (16x^2 - 8x + 1)(4x-5),$$

whence $-112x^2 + 57x - 9 = -112x^2 + 44x - 5,$

$$\text{or } x = \frac{4}{13}.$$

(51). $\dfrac{x}{x-2} + \dfrac{x-9}{x-7} = \dfrac{x+1}{x-1} + \dfrac{x-8}{x-6},$

$$\frac{x^2-6x-(x^2-10x+16)}{(x-2)(x-6)} = \frac{x^2-6x-7-(x^2-10x+9)}{(x-1)(x-7)},$$

$$\text{whence } 4x - 16 = 0, \text{ and } x = 4,$$

$$\text{also } x^2 - 8x + 12 = x^2 - 8x + 7, \text{ or } x = 0.$$

(52). $\frac{3}{1-3x} + \frac{5}{1-5x} + \frac{4}{2x-1} = 0,$

$$(3 - 15x + 5 - 15x)(2x - 1) + 4(1 - 8x + 15x^2) = 0,$$

$$\text{whence } 46x - 8 + 4 - 32x = 0,$$

$$\text{or } x = \frac{4}{14} = \frac{2}{7}.$$

(53). $\frac{x-3}{x^2-9} - \frac{12-2x}{x^2-36} = \frac{3x-27}{x^2-81},$

$$\frac{1}{x+3} + \frac{2}{x+6} = \frac{3}{x+9},$$

$$\text{or } (x + 6 + 2x + 6)(x + 9) = 3(x^2 + 9x + 18),$$

$$\text{whence } x^2 + 13x + 36 = x^2 + 9x + 18,$$

$$\text{or } x = -\frac{18}{4} = -4\tfrac{1}{2}.$$

(54). $\frac{17-4x}{2} : \frac{15+2x}{3} - 2x :: 5 : 2,$

by multiplying the extremes and means together, we have

$$3(17 - 4x) = 5(15 + 2x - 6x),$$

$$x = \frac{24}{8} = 3.$$

(55). $\frac{2x+7}{9x+31} : \frac{16x+5}{2} :: 1 : 36x + 10,$

$$\frac{2x+7}{(9x+31)} \times (36x + 10) = \frac{16x+5}{2},$$

$$\text{or } 144x^2 + 544x + 140 = 144x^2 + 541x + 155;$$

$$\therefore x = \frac{15}{3} = 5.$$

56). $4x + 3 : 6x - 43 :: 2x + 19 : 3x - 19,$

$$(4x + 3)(3x - 19) = (6x - 43)(2x + 19),$$

$$12x^2 - 67x - 57 = 12x^2 + 28x - 817,$$

$$\text{or } x = \frac{760}{95} = 8.$$

(57). $10 + x : 4x - 9 :: 2 : 1$,

$$10 + x = 8x - 18; \therefore x = \frac{28}{7} = 4.$$

(58). $\cdot 3x + 3\cdot 15 - 1\cdot 75x = \cdot 125x$,

by multiplying the equation by 100, we have

$$30x + 315 - 175x = 12\cdot 5x;$$

$$\therefore x = \frac{315}{157\cdot 5} = 2.$$

(59). $\cdot 6x + \cdot 2 - \cdot 7x + \cdot 75x - \cdot 875x + \cdot 1 = 0$,

by multiplying by 10, then

$$6x + 2 - 7x + 7\cdot 5x - 8\cdot 75x + 1 = 0;$$

$$\therefore x = \frac{3}{2\cdot 25} = \frac{4}{3} = 1\tfrac{1}{3}.$$

(60). $\cdot 375x + \cdot 05 = \cdot 225x + \cdot 8$,

by multiplying by 100, then

$$37\cdot 5x + 5 = 22\cdot 5x + 80;$$

$$\therefore x = \frac{75}{15} = 5.$$

(61). $\cdot 05x - \cdot 25 = \cdot 075x - \cdot 45$,

by multiplying by 100, then

$$5x - 25 = 7\cdot 5x - 45; \therefore x = \frac{20}{2\cdot 5} = 8.$$

(62). $2\cdot 4x - \cdot 072x + \cdot 1 = \cdot 8x + 9\cdot 268$,

$$240x - 7\cdot 2x + 10 = 80x + 926\cdot 8; \therefore x = \frac{916\cdot 8}{152\cdot 8} = 6.$$

(63). $\cdot 6x + \cdot 8 - 3\cdot 5x + 1\cdot 5 + 4 = \cdot 25x$,

$$6x + 8 - 35x + 15 + 40 = 2\cdot 5x; \because x = \frac{63}{31\cdot 5} = 2.$$

(64). $\frac{\cdot 06x - \cdot 02}{4} = \frac{\cdot 036x - \cdot 005}{5} + \cdot 23$,

$$\frac{6x - 2}{4} = \frac{3\cdot 6x - \cdot 5}{5} + 23,$$

$$\text{or } 30x - 10 = 14\cdot 4x - 2 + 460; \therefore x = \frac{468}{15\cdot 6} = 30.$$

(65). $\cdot 15x - \cdot 875x + 1\cdot 575 = \cdot 0625x$,

$$15x - 87\cdot 5x + 157\cdot 5 = 6\cdot 25x; \therefore x = \frac{157\cdot 5}{78\cdot 75} = 2.$$

(66). $\frac{5ab}{6} + \frac{4ac}{5} - \frac{2cx}{3} = \frac{3ac}{4} + 2ab - 6cx$,

$$10ab + \frac{48ac}{5} - 8cx = 9ac + 24ab - 72cx,$$

$$\text{whence } x = \frac{70ab - 3ac}{320c}.$$

(67). $ax - \frac{a^2 - 3bx}{a} - ab^2 = bx + \frac{6bx - 5a^2}{2a} - \frac{bx + 4a}{4}$,

$$4ax - 4a + \frac{12bx}{a} - 4ab^2 = 4bx + \frac{12bx}{a} - 10a - bx - 4a,$$

$$\text{or } 4ax - 4ab^2 = 3bx - 10a;$$

$$\therefore x = \frac{4ab^2 - 10a}{4a - 3b}.$$

(68). $(a + b)(b - x) + (a - b)(a + x) = c^2$,

$$ab + b^2 - ax - bx + a^2 - ab + ax - bx = c^2;$$

$$\therefore x = \frac{a^2 + b^2 - c^2}{2b}.$$

(69). $\frac{x - a}{b} - \frac{x - b}{a} = \frac{b}{a}$,

$$(x - a)a - (x - b)b = b^2,$$

$$ax - a^2 - bx + b^2 = b^2; \therefore x = \frac{a^2}{a - b}.$$

(70). $\frac{c + x}{c - x} - \frac{c - x}{c + x} = \frac{5b^2}{4(c^2 - x^2)}$,

$$4(c + x)^2 - 4(c - x)^2 = 5b^2; \therefore x = \frac{5b^2}{16c}.$$

(71). $\frac{a}{x - a} - \frac{a}{7(x - a)} = \frac{2a}{x + 7a}$,

$$\frac{7a - a}{7(x - a)} = \frac{2a}{x + 7a},$$

$$\text{whence } 3x + 21a = 7x - 7a, \text{ and } x = \frac{28a}{4} = 7a.$$

(72). $\frac{3a}{x} - \frac{2a}{x+a} = \frac{5a}{4(x+a)}$,

$$\frac{3a}{x} = \frac{5a + 8a}{4(x+a)},$$

whence $12ax + 12a^2 = 13ax$, and $x = 12a$.

(73). $\frac{x^2 + a^2}{4x^2 - a^2} - \frac{x}{2x + a} + \frac{1}{4} = 0$,

$4(x^2 + a^2) - 4x(2x - a) + 4x^2 - a^2 = 0$, and $x = -\frac{3a^2}{4a} = -\frac{3a}{4}$.

(74). $\frac{2a}{x+a} + \frac{5a}{2x + 2a} - \frac{21}{8} = \frac{(a - x)}{x^2 - a^2}$,

$$\frac{4a + 5a}{2(x+a)} + \frac{6}{x+a} = \frac{21}{8},$$

$$36a + 48 = 21x + 21a, \text{ and } x = \frac{5a + 16}{7}.$$

(75). $\frac{7(a+x)^2}{5} - a(a+x) = \frac{6(a^2 - x^2)}{7} - \frac{16a^2}{35}$,

$$49(a+x)^2 - 35a(a+x) = 30(a^2 - x^2) - 16a^2,$$

whence $79x^2 + 63ax = 0$, and $x = -\frac{63a}{79}$.

(76). $\frac{20x + 11a}{25a} + \frac{5x + 20a}{9x - 16a} = \frac{4x}{5a} + \frac{61}{25}$,

by multiplying by $25a$,

$$20x + 11a + \frac{25a(5x + 20a)}{9x - 16a} = 20x + 61a,$$

by cancelling and multiplying by $(9x - 16a)$ and dividing by $25a$, then

$$5x + 20a = 2(9x - 16a);$$

$$\therefore x = \frac{52a}{13} = 4a.$$

(77). $\frac{9x - 16a}{36a} = \frac{4x - 12a}{5x - 4x} + \frac{x - 4a}{4a}$,

by multiplying the equation by $36a$,

$$9x - 16a = \frac{36a(4x - 12a)}{5x - 4a} + 9x - 36a,$$

by cancelling, dividing by $4a$, and multiplying by $5x - 4a$,

$$\text{or } 5(5x - 4a) = 9(4x - 12a);$$

$$\therefore x = \frac{88a}{11} = 8a.$$

(78). $\dfrac{a+x}{a^2+ax+x^2} + \dfrac{a-x}{a^2-ax+x^2} = \dfrac{3a}{x(a^4+a^2x^2+x^4)}$,

observe that

$$(x^2+ax+a^2)(x^2-ax+a^2) = x^4+a^2x^2+a^4;$$

$$\therefore x(a^3+x^3) + x(a^3-x^3) = 3a; \quad \therefore x = \frac{3}{2a^2}.$$

(79). $\dfrac{ad-bc}{d(c+dx)} + \dfrac{b}{d} = \dfrac{2a-bx}{c+dx}$,

by multiplying by $d(c+dx)$,

$$ad - bc + b(c+dx) = 2ad - bdx,$$

$$\text{whence } x = \frac{ad}{2bd} = \frac{a}{2b}.$$

(80). $\dfrac{1}{(x-a)(x-c)} + \dfrac{2}{(a-c)(a-x)} = \dfrac{1}{(c-a)(c-x)}$,

by changing the signs, to have the expressions uniform, and multiplying by $(a-c)(x-a)(x-c)$,

$$\frac{1}{(x-a)(x-c)} - \frac{2}{(a-c)(x-a)} = \frac{1}{(a-c)(x-c)},$$

$$\therefore a - c - 2(x-c) = (x-a),$$

$$\text{whence } x = \frac{2a+c}{3}.$$

(81). $\dfrac{c}{a-b}\left(1+\dfrac{1}{x}\right) - \dfrac{b}{a-c}\left(1+\dfrac{1}{x}\right) = \dfrac{a+c}{(a-c)x} + 1$,

$$\left(\frac{c}{a-b} - \frac{b}{a-c} - \frac{a+c}{a-c}\right)\frac{1}{x} = 1 + \frac{b}{a-c} - \frac{c}{a-b};$$

$$\therefore \frac{c(a-c)-(a+b+c)(a-b)}{x} = (a-c)(a-b) + b(a-b) - c(a-c),$$

$$\text{or } x = \frac{a^2-b^2-c(b-c)}{b(b-c)-(a-c)^2}.$$

(82). $\frac{1}{ab - ax} + \frac{1}{bc - bx} - \frac{1}{ac - ax} = 0,$

$$\frac{1}{a(b - x)} + \frac{1}{b(c - x)} - \frac{1}{a(c - x)} = 0,$$

by multiplying by $ab(b - x)(c - x)$, we have

$$bc - bx + ab - ax - b^2 + bx = 0;$$

$$\therefore\ x = \frac{b(a + c - b)}{a}.$$

(83). $\frac{x + c}{a + b} - \frac{ax}{(a + b)^2} = \frac{ac}{a^2 - b^2} - \frac{b^2x}{a^3 - ab^2 + a^2b - b^3},$

$$\frac{(x + c)(a + b) - ax}{(a + b)^2} = \frac{ac}{a^2 - b^2} - \frac{b^2x}{(a + b)^2(a - b)},$$

by multiplying by $(a + b)^2(a - b)$, we have

$$\{bx + (a + b)c\}(a - b) = ac(a + b) - b^2x;$$

$$\therefore\ abx - b^2x + c(a^2 - b^2) = ac(a + b) - b^2x,$$

$$\text{and } x = \frac{ac + bc}{a}.$$

(84). $\frac{a}{1 + a^2} + \frac{x}{1 + x^2} - \frac{(a + b)(1 + ax)}{1 + x^2} = 0,$

by multiplying by $(1 + a^2)(1 + x^2)$, we have

$$a + ax^2 + x + a^2x = (a + b)(1 + ax)(1 + a^2);$$

$$\therefore\ (1 + ax)(a + x) = (a + b)(1 + ax)(1 + a^2);\ \therefore\ x = -\frac{1}{a},$$

$$\text{and } x = a^3 + a^2b + b.$$

(85). $\frac{x + a + b + c}{x^2 + a^2 + b^2 + c^2} = \frac{1}{x + a + b + c},$

$$x^2 + 2(a + b + c)x + (a + b + c)^2 = x^2 + a^2 + b^2 + c^2;$$

$$\therefore\ x = -\frac{ab + ac + bc}{a + b + c}.$$

(86). $x + a : x - b :: (2x + a)^2 : (2x - b)^2,$

1st − 2nd : 1st :: 3rd − 4th : 3rd,

then $a + b : x + a :: 4ax + 4bx + a^2 - b^2 : (2x + a)^2,$

by dividing 1st and 3rd terms by $a + b$;

$$\therefore\ 1 : x + a :: 4x + a - b : 4x^2 + 4ax + a^2,$$

by multiplying extremes and means together, we have

$$4x^2 + 4ax + a^2 = 4x^2 + 4ax + (a - b)\,x + a^2 - ab;$$

$$\therefore\ x = \frac{ab}{a - b}.$$

(87). $4x + a : 4x - b :: (2x + a)^{\frac{1}{2}} : (2x - b)^{\frac{1}{2}}$,

by squaring every term, we have

$$\therefore\ (4x + a)^2 : (4x - b)^2 :: 2x + a : 2x - b,$$

then 1st – 2nd : 1st :: 3rd – 4th : 3rd term,

and $8ax + 8bx + a^2 - b^2 : (4x + a)^2 :: a + b : (2x + a)$,

by dividing 1st and 3rd terms by $(a + b)$, we have

$$\therefore\ (8x + a - b)\,(2x + a) = (4x + a)^2, \text{ and } x = \frac{ab}{2\,(a - b)}.$$

(88). $a : b :: (x + 2a + b)^2 : (x + 2b + a)^2$,

by taking the square roots of each and then

1st : 1st – 2nd :: 3rd : 3rd – 4th;

$$\therefore\ a^{\frac{1}{2}} : a^{\frac{1}{2}} - b^{\frac{1}{2}} :: x + 2a + b : a - b,$$

by dividing the 2nd and 4th terms by $a^{\frac{1}{2}} - b^{\frac{1}{2}}$;

$$\therefore\ x + 2a + b = (a^{\frac{1}{2}} + b^{\frac{1}{2}})\,a^{\frac{1}{2}} = a + a^{\frac{1}{2}}b^{\frac{1}{2}},$$

$$\text{and } x = a^{\frac{1}{2}}b^{\frac{1}{2}} - (a + b).$$

(89). $2\,(x + 12)^{\frac{1}{2}} = 1$,

by squaring both sides, we have

$$4x + 48 = 1;\ \therefore\ x = -\,11\tfrac{3}{4}.$$

(90). $(10x + 35)^{\frac{1}{3}} = 5$,

by squaring both sides, we have

$$10x + 35 = 125;\ \therefore\ x = \frac{90}{10} = 9.$$

(91). $(9x - 4)^{\frac{1}{5}} = 2$,

$$\text{whence } 9x - 4 = 32; \therefore x = \frac{36}{9} = 4.$$

(92). $\sqrt{(x + 16)} = 2 + \sqrt{(x)}$;

$$\therefore x + 16 = 4 + 4\sqrt{(x)} + x,$$

$$4\sqrt{(x)} = 12; \therefore \sqrt{(x)} = 3, \text{ and } x = 9.$$

(93). $\sqrt{(4x + 21)} = 1 + 2\sqrt{(x)}$,

$$4x + 21 = 1 + 4\sqrt{(x)} + 4x;$$

$$\therefore \sqrt{(x)} = 5, \text{ and } x = 25.$$

(94). $\sqrt{(16 + x)} = 2\sqrt{(6 + x)}$;

$$\therefore 16 + x = 24 + 4x; \therefore x = -\frac{8}{3} = -2\tfrac{2}{3}.$$

(95). $8\sqrt[3]{(7x - 6)} = 16$,

$$\therefore \sqrt[3]{(7x - 6)} = 2; \therefore 7x - 6 = 8, \text{ and } x = 2.$$

(96). $\sqrt{(8 + x)} + \sqrt{(x)} = 2\sqrt{(1 + x)}$,

$$8 + x + x + 2\sqrt{(8x + x^2)} = 4 + 4x,$$

$$2 - x = \sqrt{(8x + x^2)},$$

$$\text{and } 4 - 1x + x^2 = 8x + x^2; \therefore x = \frac{4}{12} = \frac{1}{3}.$$

(97). $\sqrt{(x + 9)} = 1 + \sqrt{(x)}$;

$$\therefore x + 9 = 1 + 2\sqrt{(x)} + x, \ 2\sqrt{(x)} = 8; \therefore x = 16.$$

(98). $\dfrac{\sqrt{(x)} + 28}{\sqrt{(x)} + 38} = \dfrac{\sqrt{(x)} + 4}{\sqrt{(x)} + 6}$,

$$x + 34\sqrt{(x)} + 168 = x + 42\sqrt{(x)} + 152;$$

$$\therefore 8\sqrt{(x)} = 16, \text{ and } x = 4.$$

(99). $\dfrac{5x - 9}{\sqrt{(5x)} - 3} = \dfrac{\sqrt{(5x)} + 3}{2} + \dfrac{\sqrt{(5x)} + 3}{\sqrt{(5x)} - 3}$,

$2(5x - 9) = 5x - 9 + \{\sqrt{(5x)} + 3\}2$, and $\sqrt{(5x)} + 3$ is a factor,

$$\text{whence } \sqrt{(5x)} + 3 = 0, \ 5x = 9, \text{ and } x = 1\tfrac{4}{5},$$

$$\text{also } \sqrt{5x} - 3 = 2; \therefore x = 5.$$

(100). $x - \sqrt{(a)} = \sqrt{(ax + x^2)}$,

$$x^2 - 2x\sqrt{(a)} + a = ax + x^2,$$

$$\text{or } (a + 2\sqrt{a})\,x = a; \;\therefore\; x = \frac{\sqrt{a}}{\sqrt{a} + 2}.$$

(101). $a + x + \sqrt{(2ax + x^2)} = b$,

$$(a - b) + x = -\sqrt{(2ax + x^2)},$$

$$(a - b)^2 + 2x(a - b) + x^2 = 2ax + x^2;$$

$$\therefore\; x = \frac{(a - b)^2}{2b}.$$

(102). $\sqrt{(4a + x)} + \sqrt{(x)} = 2\sqrt{(a + x)}$,

$$4a + x + 2\sqrt{(4ax + x^2)} + x = 4a + 4x,$$

$$\text{or } \sqrt{(4ax + x^2)} = x,$$

$$4ax + x^2 = x^2; \;\therefore\; x = 0.$$

(103). $\sqrt{(1 + x + x^2)} + \sqrt{(1 - x + x^2)} = a$,

$$1 + x + x^2 + 1 - x + x^2 + 2\sqrt{(1 + x^2 + x^4)} = a^2,$$

$$\text{and } 2\sqrt{(1 + x^2 + x^4)} = a^2 - 2(1 + x^2);$$

$$\therefore\; 4 + 4x^2 + 4x^4 = a^4 - 4a^2(1 + x^2) + 4(1 + 2x^2 + x^4),$$

$$\text{and } 4(a^2 - 1)\,x^2 = a^4 - 4a^2;$$

$$\therefore\; x = \pm\frac{a}{2}\sqrt{\left(\frac{a^2 - 4}{a^2 - 1}\right)}.$$

(104). $bx\sqrt{(a + x)} + ab\sqrt{(a + x)} = ax^{\frac{3}{2}}$,

$$b(a + x)(a + x)^{\frac{1}{2}} = ax^{\frac{3}{2}};$$

$$\therefore\; \left(\frac{x}{a + x}\right)^{\frac{3}{2}} = \frac{b}{a}, \text{ and } \frac{x}{a + x} = \left(\frac{b}{a}\right)^{\frac{2}{3}};$$

$$\therefore\; \frac{x}{-a} = \frac{b^{\frac{2}{3}}}{(b^{\frac{2}{3}} - a^{\frac{2}{3}})}; \;\therefore\; x = \frac{b^{\frac{2}{3}}a}{a^{\frac{2}{3}} - b^{\frac{2}{3}}}.$$

(105). $x\sqrt{(a^2 + x^2)} + x^2 = (n^2 - 1)\,a^2$,

$$x\sqrt{(a^2 + x^2)} = (n^2 - 1)\,a^2 - x^2;$$

$$\therefore\; a^2x^2 + x^4 = a^4(n^2 - 1)^2 - 2a^2x^2(x^2 - 1) + x^4,$$

and $x^2(2a^2n^2 - a^2) = a^4(n^2-1)^2$;

$$\therefore\ x = \pm\frac{(n^2-1)\,a}{\sqrt{(2n^2-1)}}.$$

(106). $x - \sqrt{(x^2 - x)} = (a-1)\sqrt{(x)}$,

by dividing by $\sqrt{(x)}$, we have

$$\sqrt{(x-1)} = \sqrt{(x)} - (a-1);$$

$$\therefore\ x - 1 = x - 2(a-1)\sqrt{(x)} + (a-1)^2;$$

$$\therefore\ 2(a-1)\sqrt{(x)} = a^2 - 2a + 2,$$

$$\text{and } x = \frac{(a^2-2a+2)^2}{4(a-1)^2}.$$

(107). $ax + \sqrt{(a^2x^2 + b^2)} = \sqrt{[b^2 + \sqrt{\{a^2x^2(4b^2+x^2)\}}]}$,

$$a^2x^2 + 2ax\sqrt{(a^2x^2+b^2)} + a^2x^2 + b^2 = b^2 + \sqrt{\{a^2x^2(4b^2+x^2)\}},$$

$$\text{whence } 2ax + 2\sqrt{(a^2x^2+b^2)} = \sqrt{(4b^2+x^2)};$$

$$\therefore\ 4a^2x^2 + 8ax\sqrt{(a^2x^2+b^2)} + 4(a^2x^2+b^2) = 4b^2 + x^2,$$

$$\text{and } (8a^2-1)\,x = -8a\sqrt{(a^2x^2+b^2)};$$

$$\therefore\ (64a^4 - 16a^2 + 1)\,x^2 = 64a^2(a^2x^2+b^2);$$

$$\therefore\ x^2 = \frac{64a^2b^2}{1-16a^2},\text{ and } x = \pm\frac{8ab}{\sqrt{(1-16a^2)}}.$$

(108). $a^2\sqrt{(1-x)} - \sqrt{(a^2-x)} = \sqrt{\{(a^2-1)\,x\}}$;

$$\therefore\ a^4(1-x) - 2a^2\sqrt{\{(a^2-x)(1-x)\}} + a^2 - x = a^2x - x,$$

$$\text{and } a^2(1-x) + (1-x) = 2\sqrt{\{(a^2-x)(1-x)\}};$$

$$\therefore\ (a^2+1)\sqrt{(1-x)} = 2\sqrt{(a^2-x)},\text{ and } x = 1),$$

$$\text{also } (a^2+1)^2(1-x) = 4(a^2-x);$$

$$\therefore\ x(a^4 + 2a^2 - 3) = a^4 - 2a^2 + 1;$$

$$\therefore\ x = \frac{(a^2-1)^2}{a^4+2a^2-3}.$$

(109). $\sqrt{(b)}\{\sqrt{(x^2+3a^2)} - \sqrt{(x^2-3a^2)}\} = 2x\sqrt{(a)}$,

$$b\{x^2 + 3a^2 - 2\sqrt{(x^4-9a^4)} + x^2 - 3x^2\} = 4ax^2,$$

$$(b-2a)\,x^2 = b\sqrt{(x^4-9a^4)};$$

$$\therefore\ (b^2 - 4ab + 4a^2)\,x^4 = b^2(x^4 - 9a^4);$$

$$\therefore\ x^4 = \frac{9a^4b^2}{4a^2-4ab},\text{ and } x = \frac{a\sqrt{(3b)}}{\sqrt{[2\sqrt{\{(a^2-ab)\}}]}}.$$

(110). $\sqrt{(x+a)} - \sqrt{(x-a)} = \sqrt{(a)}$,

$$x + a - 2\sqrt{(x^2 - a^2)} + x - a = a;$$

$$\therefore\ 2x - a = 2\sqrt{(x^2 - a^2)},$$

$$\text{and } 4x^2 - 4ax + a^2 = 4x^2 - 4a^2;$$

$$\therefore\ x = \frac{5a}{4}.$$

(111). $\sqrt{(x)} - \sqrt{\{n - \sqrt{(nx + x^2)}\}} = \sqrt{(n)}$,

$$\sqrt{(x)} - \sqrt{(n)} = \sqrt{\{n - \sqrt{(nx + x^2)}\}};$$

$$\therefore\ x + n - 2\sqrt{(nx)} = n - \sqrt{(nx + x^2)},$$

$$\text{and } \sqrt{(x)} - 2\sqrt{(n)} = \sqrt{(n + x)};$$

$$\therefore\ x - 4\sqrt{(nx)} + 4n = n + x,$$

$$\text{and } 4\sqrt{(nx)} = 3n;\ \therefore\ x = \frac{9n}{16}.$$

(112). $\sqrt{(2x - 45)} = 3\sqrt{(15)} - \sqrt{(2x)}$,

$$2x - 45 = 135 - 6\sqrt{(30x)} + 2x;$$

$$\therefore\ 6\sqrt{(30x)} = 180;\ \therefore\ x = 30.$$

(113). $\dfrac{1 - ax}{1 + ax}\sqrt{\left(\dfrac{1 + bx}{1 - bx}\right)} = 1$,

$$\left(\frac{1 - ax}{1 + ax}\right)^2 = \frac{1 - bx}{1 + bx};$$

$$\therefore\ \frac{1 + a^2x^2}{2ax} = \frac{1}{bx};\ \therefore\ 1 + a^2x^2 = \frac{2a}{b};$$

$$\therefore\ x^2 = \frac{2a - b}{a^2b},\ \text{and } x = \pm\frac{1}{a}\sqrt{\left(\frac{2a - b}{b}\right)}.$$

(114). $\{\sqrt{(1 - x)} - 1\}\{\sqrt{(1 + x)} + 1\} = \sqrt{(1 - x^2)}$,

$$\sqrt{(1 - x^2)} + \sqrt{(1 - x)} - \sqrt{(1 + x)} - 1 = \sqrt{(1 - x^2)};$$

$$\therefore\ \sqrt{(1 - x)} - \sqrt{(1 + x)} = 1,$$

$$\text{and } 1 - x + 1 + x - 2\sqrt{(1 - x^2)} = 1;$$

$$\therefore\ 1 = 4(1 - x^2),\ \text{and } x = \pm\frac{\sqrt{3}}{2}.$$

(115). $\sqrt{(x + 4a + 4b)} + \sqrt{(x)} = 2\sqrt{(b + x)}$,

$$x + 4a + 4b = 4b + 4x - 4\sqrt{(bx + x^2)} + x,$$

$$\text{whence } (a - x) = \sqrt{(bx + x^2)};$$

$$\therefore\ a^2 - 2ax + x^2 = bx + x^2, \text{ or } x = \frac{a^2}{2a + b}.$$

(116). $\left(\frac{a + x}{a - x}\right)^2 - 1 = \frac{cx}{ab}$,

$$\left(\frac{a + x}{a - x}\right)^2 = \frac{ab + cx}{ab}; \quad \therefore\ \frac{4ax}{(a - x)^2} = \frac{cx}{ab},$$

$$\text{or } (a - x)^2 = \frac{4a^2b}{c}; \quad \therefore\ a - x = \pm\, 2a\sqrt{\left(\frac{b}{c}\right)};$$

$$\therefore\ x = \frac{a(\sqrt{c} \mp 2\sqrt{b})}{\sqrt{c}}.$$

(117). $(a + b)\, x = (a - b)\sqrt{(1 + x^2)}$,

$$\left[\frac{a + b}{a - b}\right]^2 = \frac{1 + x^2}{x^2};$$

$$\therefore\ \frac{4ab}{(a - b)^2} = \frac{1}{x^2}; \quad \therefore\ x = \pm\, \frac{a - b}{2\sqrt{(ab)}}.$$

(118). $a + x = \sqrt{\{a^2 + x\sqrt{(4c^2 + x^2)}\}}$,

$$a^2 + 2ax + x^2 = a^2 + x\sqrt{(4c^2 + x^2)},$$

$$\text{whence } 2a + x = \sqrt{(4c^2 + x^2)},$$

$$\text{and } 4a^2 + 4ax + x^2 = 4c^2 + x^2; \quad \therefore\ x = \frac{c^2 - a^2}{a}.$$

(119). $\{m + n\sqrt{(x)}\}\{q + p\sqrt{(x)}\} = \{n + m\sqrt{(x)}\}\{p + q\sqrt{(x)}\}$,

$$mq + (mp + nq)\sqrt{(x)} + npx = np + (mp + nq)\sqrt{(x)} + mqx;$$

$$\therefore\ x = \frac{mq - np}{mq - np} = 1.$$

(120). $x + \sqrt{(x^2 - 2ax + b^2)} = a + b$,

$$x - (a + b) = -\sqrt{(x^2 - 2ax + b^2)};$$

$$\therefore\ x^2 - 2(a + b)\, x + (a + b)^2 = x^2 - 2ax + b^2,$$

$$\text{whence } -2bx + a^2 + 2ab = 0, \text{ and } x = \frac{a^2 + 2ab}{2b}.$$

(121). $(a-1)(1+x+x^2)^2 = (a+1)(1+x^2+x^4)$,

$$\text{since } \frac{1+x^2+x^4}{1+x+x^2} = 1 - x + x^2;$$

$$\therefore \frac{1+x+x^2}{1-x+x^2} = \frac{a+1}{a-1},$$

$$\text{and } \frac{1+x^2}{2x} = \frac{a}{2},$$

$$\text{whence } x^2 - ax + \frac{a^2}{4} = \frac{a^2}{4} - 1; \;\therefore\; x = \frac{a}{2} \pm \sqrt{\left(\frac{a^2}{4} - 1\right)}.$$

(122). $x^2 + \sqrt{\{4x^2 + x + \sqrt{(9x^2+12x)}\}} = (1+x)^2$,

$$\{4x^2 + x + \sqrt{(9x^2+12x)}\}^{\frac{1}{2}} = 1 + 2x;$$

$$\therefore\; 4x^2 + x + \sqrt{(9x^2+12x)} = 1 + 4x + 4x^2,$$

$$\text{and } 9x^2 + 12x = 1 + 6x + 9x^2; \;\therefore\; x = \frac{1}{6}.$$

(123). $a + x - \sqrt{(ax - x^2)} = \sqrt{(2a^2 - ax - x^2)} - \sqrt{(2ax + x^2)}$,

$$\text{since } 2a^2 - ax - x^2 = a^2 - x^2 + a(a-x)$$

$$= (a + x + a)(a - x) = (2a + x)(a - x);$$

$$\therefore\; a + x - \sqrt{\{x(a-x)\}} = \sqrt{(2a+x)}\,\{\sqrt{(a-x)} - \sqrt{(x)}\},$$

$$\text{and } a^2 + 2ax + x^2 - 2(a+x)\sqrt{\{x(a-x)\}} + ax - x^2$$

$$= (2a + x)\,[a - 2\sqrt{\{x(a-x)\}}],$$

$$\text{whence } a^2 - 2ax = 2a\sqrt{\{x(a-x)\}};$$

$$\therefore\; a^2 - 4ax + 4x^2 = 4ax - 4x^2,$$

$$\text{and } x^2 - ax + \frac{a^2}{4} = -\frac{a^2}{8} + \frac{a^2}{4} = \frac{a^2}{8};$$

$$\therefore\; x = \frac{a}{2}\left(1 \pm \frac{1}{\sqrt{2}}\right).$$

(124). $\sqrt{\{x^2 + 1 + x\sqrt{(x - a - 4)}\}} = 1 + x$,

$$x^2 + 1 + x\sqrt{(x-a-4)} = 1 + 2x + x^2;$$

$$\therefore\; \sqrt{(x - a - 4)} = 2,$$

$$\text{and } x - a - 4 = 4; \;\therefore\; x = a + 8.$$

(125). $4x\left(x+\frac{a}{4}\right)^{\frac{1}{2}} = a\{\sqrt{(a)}+\sqrt{(a+4x)}\}$,

$$2x\sqrt{(a+4x)} = a\sqrt{(a)} + a\sqrt{(a+4x)};$$

$$\therefore (2x-a)\sqrt{(a+4x)} = a\sqrt{(a)},$$

$$\text{or } 16x^3 - 12ax^2 + a^3 = a^3; \;\therefore x = \frac{3a}{4}.$$

(126). $\sqrt{\{x+\sqrt{(x)}\}} - \frac{a}{b}\sqrt{\left\{\frac{x}{x+\sqrt{(x)}}\right\}} = \sqrt{\{x-\sqrt{(x)}\}}$,

$$b\sqrt{\{x+\sqrt{(x)}\}}\times\sqrt{\{x+\sqrt{(x)}\}} - a\sqrt{(x)} = b\sqrt{\{x-\sqrt{(x)}\}}\sqrt{\{x+\sqrt{(x)}\}},$$

$$\text{or } bx + b\sqrt{(x)} - a\sqrt{(x)} = b\sqrt{(x^2-x)};$$

$$\therefore b\sqrt{(x)} - (a-b) = b\sqrt{(x-1)},$$

$$\text{and } b^2x - 2(a-b)\,b\sqrt{(x)} + (a-b)^2 = b^2x - b^2;$$

$$\therefore \sqrt{x} = \frac{a^2-2ab+2b^2}{2b(a-b)}, \text{ and } x = \frac{(a^2-2ab+2b^2)^2}{4b^2(a-b)^2}.$$

(127). $x+a : x-a :: \sqrt{(x^2-ax+2a^2)} : \sqrt{(x^2-5ax+14a^2)}$,

$$\frac{(x+a)^2}{(x-a)^2} = \frac{x^2-ax+2a^2}{x^2-5ax+14a^2}; \;\therefore \frac{(x+a)^2}{x} = \frac{x^2-ax+2x^2}{x-3a};$$

$$\therefore \frac{(x+a)^2}{x^2-ax+2a^2} = \frac{x}{x-3a},$$

$$\text{or } \frac{(x+a)^2}{3ax-a^2} = \frac{x}{3a};$$

$$\therefore 3x^2 + 6ax + 3a^2 = 3x^2 - ax,$$

$$\text{and } x = -\frac{3a}{7}.$$

(128). $\frac{ax-b^2}{\sqrt{(ax)}+b} = \frac{\sqrt{(ax)}-b}{n} - c$,

$$\text{since } \frac{ax-b^2}{\sqrt{(ax)}+b} = \sqrt{(ax)} - b;$$

$$\therefore n\{\sqrt{(ax)}-b\} = \sqrt{(ax)} - b - nc,$$

$$\text{and } (n-1)\sqrt{(ax)} = (n-1)\,b - nc;$$

$$\therefore ax = \left(b - \frac{nc}{n-1}\right)^2, \text{ or } x = \frac{1}{a}\left(b - \frac{nc}{n-1}\right)^2.$$

(129). $\frac{\sqrt{(1-x^3)}}{\sqrt{(1-x)}} + \frac{\sqrt{(1+x^3)}}{\sqrt{(1+x)}} = mx,$

$$\sqrt{(1+x+x^2)} + \sqrt{(1-x+x^2)} = mx;$$

$$\therefore\ 1+x+x^2+1-x+x^2+2\sqrt{(1+x^2+x^4)} = m^2x^2,$$

$$\text{whence } 2\sqrt{(1+x^2+x^4)} = (m^2-2)\,x^2-2,$$

$$\text{and } 4+4x^2+4x^4 = (m^4-4m^2+4)\,x^4-4\,(m^2-2)\,x^2+4,$$

$$x^2 = \frac{4\,(m^2-1)}{m^2\,(m^2-4)};\ \therefore\ x = \pm\frac{2}{m}\sqrt{\left(\frac{m^2-1}{m^2-4}\right)}.$$

(130). $(1+x)^{\frac{1}{3}} + (1-x)^{\frac{1}{3}} = 2^{\frac{1}{3}},$

$$1+x+3\sqrt[3]{(1-x^2)}\,\{(1+x)^{\frac{1}{3}}+(1-x)^{\frac{1}{3}}\}+1-x = 2;$$

$$\therefore\ 3\sqrt[3]{(1-x^2)}\times 2^{\frac{1}{3}} = 0,$$

$$\text{or } \sqrt[3]{(1-x^2)} = 0;\ \therefore\ x = 1.$$

(131). $\sqrt[3]{(a+x)} + \sqrt[3]{(a-x)} = b,$

$$a+x+a-x+3\sqrt[3]{(a^2-x^2)}\,\{\sqrt[3]{(a+x)}+\sqrt[3]{(a-x)}\} = b^3,$$

$$\text{whence } \sqrt[3]{(a^2-x^2)} = \frac{b^2}{3}-\frac{2a}{3b};$$

$$\therefore\ x = \pm\sqrt{\left\{a^2-\left(\frac{b^2}{3}-\frac{2a}{3b}\right)^3\right\}}.$$

(132). $\sqrt{(x^2+2ax)} + \sqrt{(x^2-2ax)} = \frac{ax}{\sqrt{(x^2+2ax)}},$

$$x^2+2ax+\sqrt{(x^4-4a^2x^2)} = ax;$$

$$\therefore\ x+a = -\sqrt{(x^2-4a^2)},$$

$$\text{and } x^2+2ax+a^2 = x^2-4a^2;\ \therefore\ x = -\frac{5a}{2}.$$

(133). $\frac{1+x^3}{(1+x)^2} + \frac{1-x^3}{(1-x)^2} = a,$

$$\text{since } \frac{1+x^3}{(1+x)^2} = \frac{1-x+x^2}{1+x},\ \text{and } \frac{1-x^3}{(1-x)^2} = \frac{1+x+x^2}{1-x};$$

$$\therefore\ \frac{1-x+x^2}{1+x} + \frac{1+x+x^2}{1-x} = a;$$

$$\therefore\ (1-x)^2+x^2\,(1-x)+(1+x)^2+x^2\,(1+x) = a\,(1-x^2);$$

$$\therefore\ 2 + 4x^2 = a(1 - x^2),$$

$$\text{and } x = \pm\sqrt{\left(\frac{a-2}{a+4}\right)}.$$

(134). $\dfrac{x + \sqrt{(1 + x^2)}}{2a\sqrt{(1 + x^2)}} = \dfrac{1}{a + b},$

$$(a + b)x = (a - b)\sqrt{(1 + x^2)};$$

$$\therefore\ \frac{1 + x^2}{x^2} = \left(\frac{a + b}{a - b}\right)^2, \text{ or } x^2 = \frac{(a - b)^2}{4ab};$$

$$\therefore\ x = \pm\frac{a - b}{2\sqrt{(ab)}}.$$

(135). $\dfrac{\sqrt{\{a + \sqrt{(a^2 - x^2)}\}}}{\sqrt{(a + x)}} + \dfrac{\sqrt{\{a - \sqrt{(a^2 - x^2)}\}}}{\sqrt{(a + x)}} = n\{a + \sqrt{(a^2 - x^2)}\}^{-\frac{1}{2}},$

$$a + \sqrt{(a^2 - x^2)} + \sqrt{\{a^2 - (a^2 - x^2)\}} = n\sqrt{(a + x)},$$

$$\text{whence } a + x + \sqrt{(a^2 - x^2)} = n\sqrt{(a + x)};$$

$$\therefore\ \sqrt{(a + x)} + \sqrt{(a - x)} = n,$$

$$\text{and } 2a + 2\sqrt{(a^2 - x^2)} = n^2;$$

$$\therefore\ 4(a^2 - x^2) = n^4 - 4an^2 + 4a^2;$$

$$\therefore\ x = \pm n\sqrt{\left(a - \frac{n^2}{4}\right)}, \text{ and } -a.$$

(136). $\dfrac{x}{\sqrt{(1 - x)} + 1} + \dfrac{x}{\sqrt{(1 + x)} - 1} = 1,$

$$\frac{x\{\sqrt{(1 - x)} - 1\}}{-x} + \frac{x\{\sqrt{(1 + x)} + 1\}}{x} = 1;$$

$$\therefore\ \sqrt{(1 + x)} + 1 - \sqrt{(1 - x)} + 1 = 1,$$

$$\text{and } 1 + x + 1 - x - 2\sqrt{(1 - x^2)} = 1;$$

$$\therefore\ 4 - 4x^2 = 1, \text{ and } x = \pm\frac{\sqrt{(3)}}{2}.$$

(137). $\dfrac{a(1 - x)}{\sqrt{(1 + a^2)}} + 1 = \dfrac{a\sqrt{(1 + x^2)}}{\sqrt{(1 + a^2)}},$

$$a(1 - x) + \sqrt{(1 + a^2)} = a\sqrt{(1 + x^2)};$$

$$\therefore\ a^2(1 - x)^2 + 2a(1 - x)\sqrt{(1 + a^2)} + 1 + a^2 = a^2(1 + x^2);$$

$$\therefore\ 2a^2x + 2ax\sqrt{(1 + a^2)} = 1 + a^2 + 2a\sqrt{(1 + a^2)},$$

$$\text{or } x = \frac{\sqrt{(1 + a^2)}\{2a + \sqrt{(1 + a^2)}\}}{2a\{a + \sqrt{(1 + a^2)}\}} = \frac{\sqrt{(1 + a^2)}}{2a}\{a\sqrt{(1 + a^2)} + 1 - a^2\}.$$

(138). $\sqrt{\left(\frac{3a}{4} - x\right)} + \sqrt{(3ax - x)} = \frac{3a}{2}\sqrt{(1 - 4x)}$,

$$\sqrt{(3a - 4x)} - 3a\sqrt{(1 - 4x)} = -2\sqrt{(3ax - x)};$$

$$\therefore 3a - 4x + 9a^2(1 - 4x) - 6a\sqrt{(1 - 4x)} \times \sqrt{(3a - 4x)} = 12ax - 4x,$$

whence $3a(1 - 4x) + 9a^2(1 - 4x) - 6a\sqrt{(1 - 4x)} \times \sqrt{(3a - 4x)} = 0$,

$$\text{and } x = \frac{1}{4};$$

$$\therefore (1 + 3a)\sqrt{(1 - 4x)} = 2\sqrt{(3a - 4x)},$$

whence $16x - 4(1 + 6a + 9a^2)x = 12a - (1 + 3a)^2$;

$$\therefore x = \frac{1 - 6a + 9a^2}{4(9a^2 + 6a - 3)} = \frac{3a - 1}{12(a + 1)}$$

(139). $\frac{1 + x}{1 + x + \sqrt{(1 + x^2)}} + \frac{1 - x}{1 - x + \sqrt{(1 + x^2)}} = a$,

$$\frac{(1 + x)^2 - (1 + x)\sqrt{(1 + x^2)}}{2x} - \frac{(1 - x)^2 - (1 - x)\sqrt{(1 + x^2)}}{2x} = a,$$

$$\therefore 2 - a = \sqrt{(1 + x^2)},$$

and $1 + x^2 = 4 - 4a + a^2$; $\therefore x = \pm\sqrt{(a^2 - 4a + 3)}$.

(140). $\frac{1 + x + \sqrt{(2x + x^2)}}{1 - x + \sqrt{(2x + x^2)}} = 1 - ax$,

$$\frac{1 + x + \sqrt{(2x + x^2)}}{2x} = \frac{1 - ax}{-ax};$$

$$\therefore a + ax + a\sqrt{(2x + x^2)} = 2ax - 2,$$

$$a^2(2x + x^2) = a^2(x - 1)^2 - 4a(x - 1) + 4;$$

$$\therefore x = \frac{(a + 2)^2}{4a(a + 1)}.$$

(141). $\left(\frac{1 + x}{1 - x}\right)^{\frac{1}{4}} + \sqrt{\left(\frac{1 - a}{1 + a}\right)}\sqrt[4]{\left(\frac{1 - x}{1 + x}\right)} = 2 \times \sqrt[4]{\left\{\frac{1 - a^2}{(1 + a)}\right\}^2}$,

$$\sqrt{\left(\frac{1 + x}{1 - x}\right)} - 2\sqrt[4]{\left\{\frac{1 - a^2}{(1 + a)^2}\right\}} \times \sqrt[4]{\left(\frac{1 + x}{1 - x}\right)} + \sqrt{\left\{\frac{1 - a^2}{(1 + a)^2}\right\}} = 0;$$

$$\therefore \sqrt[4]{\left(\frac{1 + x}{1 - x}\right)} = \sqrt[4]{\left\{\frac{1 - a^2}{(1 + a)^2}\right\}}, \text{ or } \frac{1 + x}{1 - x} = \frac{1 - a}{1 + a},$$

whence $x = -a$.

(142). $\dfrac{1+x-\sqrt{(2x+x^2)}}{1+x+\sqrt{(2x+x^2)}} = a \times \dfrac{\sqrt{(2+x)}+\sqrt{(x)}}{\sqrt{(2+x)}-\sqrt{(x)}}$,

$$\left[\frac{\{\sqrt{(2+x)}-\sqrt{(x)}\}}{\sqrt{(2+x)}+\sqrt{(x)}}\right]^2 = a \times \frac{\sqrt{(2+x)}+\sqrt{(x)}}{\sqrt{(2+x)}-\sqrt{(x)}};$$

$$\therefore \left\{\frac{\sqrt{(2+x)}-\sqrt{(x)}}{\sqrt{(2+x)}+\sqrt{(x)}}\right\}^3 = a, \text{ or } \frac{\sqrt{(2+x)}-\sqrt{(x)}}{\sqrt{(2+x)}+\sqrt{(x)}} = a^{\frac{1}{3}},$$

$$\text{whence } \frac{2+x}{x} = \left(\frac{1+a^{\frac{1}{3}}}{1-a^{\frac{1}{3}}}\right)^2, \text{ or } x = \frac{(1-a^{\frac{1}{3}})^2}{2a^{\frac{1}{3}}}.$$

(143). $\dfrac{a\sqrt{(x)}+\sqrt{(1+a^2x)}}{2a\sqrt{(1+a^2x)}} = \dfrac{1}{a+x}$,

$$(a^2+ax)\sqrt{(x)} = (a-x)\sqrt{(1+a^2x)},$$

$$\text{or } \left(\frac{a+x}{a-x}\right)^2 = \frac{1+a^2x}{a^2x}; \ \therefore \frac{4ax}{(a-x)^2} = \frac{1}{a^2x},$$

$$\text{whence } \frac{x}{(a-x)} = \frac{1}{2a^{\frac{3}{2}}}; \ \therefore x = \frac{a}{1+2a^{\frac{3}{2}}}.$$

(144). $\dfrac{a-\sqrt{(2ax-x^2)}}{a+\sqrt{(2ax-x^2)}} = b$,

$$\frac{\sqrt{(2ax-x^2)}}{a} = \frac{1-b}{1+b},$$

$$\frac{2ax-x^2}{a^2} = \overline{\frac{1-b}{1+b}}\Big]^2; \ \therefore 1 - \frac{2ax-x^2}{a^2} = 1 - \overline{\frac{1-b}{1+b}}\Big]^2,$$

$$\text{or } \frac{x^2-2ax+a^2}{a^2} = \frac{4b}{(1+b)^2}; \ \therefore x = a \pm \frac{2a\sqrt{(b)}}{(1+b)},$$

$$\text{and } x = a\,\frac{(1\pm\sqrt{b})^2}{1+b}.$$

(145). $\dfrac{1}{x^{\frac{1}{2}}} + \dfrac{1}{a^{\frac{1}{2}}} - \left\{\dfrac{1}{a} + \sqrt{\left(\dfrac{4}{ax}+\dfrac{9}{x^2}\right)}\right\}^{\frac{1}{2}} = 0$,

$$\frac{1}{x} + \frac{2}{a^{\frac{1}{2}}x^{\frac{1}{2}}} + \frac{1}{a} = \frac{1}{a} + \sqrt{\left(\frac{4}{ax}+\frac{9}{x^2}\right)};$$

$$\therefore \frac{1}{x^{\frac{1}{2}}} + \frac{2}{a^{\frac{1}{2}}} = \sqrt{\left(\frac{4}{a}+\frac{9}{x}\right)},$$

$$\text{and } \frac{1}{x} + \frac{4}{a^{\frac{1}{2}}x^{\frac{1}{2}}} + \frac{4}{a} = \frac{4}{a} + \frac{9}{x}; \ \therefore x = 4a.$$

(146). $\dfrac{a+x}{a+\sqrt{(a^2+ax)}}+\dfrac{a-x}{a+\sqrt{(a^2-ax)}}=1,$

$$(a+x)\,a+(a+x)\sqrt{(a^2-ax)}+(a-x)\,a+(a-x)\sqrt{(a^2+ax)}$$
$$=a^2+a\sqrt{(a^2-ax)}+a\sqrt{(a^2+ax)}+\sqrt{(a^4-a^2x^2)};$$
$$\therefore\ x\sqrt{(a^2-ax)}-x\sqrt{(a^2+ax)}=\sqrt{(a^4-a^2x^2)}-a^2,$$
$$\text{or } x^2\{2a^2-2\sqrt{(a^4-a^2x^2)}\}=2a^4-a^2x^2-2a^2\sqrt{(a^4-a^2x^2)},$$
$$3ax^2-2a^3=2(x^2-a^2)\sqrt{(a^2-x^2)},$$
$$9a^2x^4-12a^4x^2+4a^6=4(a^6-3a^4x^2+3a^2x^4-x^6),$$
$$4x^2=3a^2;\quad \therefore\ x=\pm\frac{a\sqrt{3}}{2}.$$

(147). $\dfrac{\sqrt{(1+x)}-1}{\sqrt{(1-x)}+1}+\dfrac{\sqrt{(1-x)}+1}{\sqrt{(1+x)}-1}=a,$

$$\frac{\sqrt{(1-x^2)}-\sqrt{(1+x)}-\sqrt{(1-x)}+1}{-x}+\frac{\sqrt{(1-x^2)}+\sqrt{(1+x)}+\sqrt{(1-x)}+1}{x}=a;$$
$$\therefore\ 2\sqrt{(1+x)}+2\sqrt{(1-x)}=ax,$$
$$\text{and } 8\sqrt{(1-x^2)}=a^2x^2-8;$$
$$\therefore\ a^4x^2=16a^2-64 \text{ and } x=\pm\frac{4}{a^2}\sqrt{(a^2-4)}.$$

(148). $\sqrt[4]{(x^4-1)}+x\sqrt{(x^4-1)}=x^3,$

$$\sqrt{(x^4-1)}=x^6-2x^4\sqrt{(x^4-1)}+x^6-x^2,$$
$$\text{whence } \frac{1+2x^4}{1-2x^4}=\frac{-x^2}{\sqrt{(x^4-1)}} \text{ and } \left.\overline{\frac{1+2x^4}{1-2x^4}}\right]^2=\frac{x^4}{x^4-1};$$
$$\therefore\ \frac{8x^4}{(1+2x^4)^2}=\frac{1}{x^4};\ \therefore\ x^4 2\sqrt{(2)}=1+2x^4,$$
$$\text{and } x=\sqrt[4]{\left\{\frac{1}{\sqrt{(2)}}+\frac{1}{2}\right\}}.$$

(149). $\dfrac{\sqrt{(a)}-\sqrt{\{a-\sqrt{(a^2-ax)}\}}}{\sqrt{(a)}+\sqrt{\{a-\sqrt{(a^2-ax)}\}}}=b,$

$$\frac{\sqrt{\{a-\sqrt{(a^2-ax)}\}}}{\sqrt{a}}=\frac{1-b}{1+b} \text{ and } \frac{a-\sqrt{(a^2-ax)}}{a}=\left.\overline{\frac{1-b}{1+b}}\right]^2;$$
$$\therefore\ \frac{\sqrt{(a^2-ax)}}{a}=\frac{(1+b)^2-(1-b)^2}{(1+b)^2}=\frac{4b}{(1+b)^2};$$
$$\therefore\ x=a\left\{1-\left(\frac{2\sqrt{b}}{1+b^2}\right)^4\right\}.$$

(150). $\frac{\sqrt{(a^2+x^2)}-a}{\sqrt{(a^2-x^2)}+a}=b,$

$$\sqrt{(a^2+x^2)}-b\sqrt{(a^2-x^2)}=a+ab;$$

$$\therefore\ a^2+x^2+b^2(a^2-x^2)-2b\sqrt{(a^4-x^4)}=(a+ab)^2,$$

$$\text{whence } 2a^2b-(1-b)^2x^2=-2b\sqrt{(a^4-x^4)};$$

$$\therefore\ 4a^4b^2-4a^2b(1-b)^2x^2+(1-b^2)^2x^4=4a^4b^2-4b^2x^4,$$

$$\text{whence } x=\frac{2a\sqrt{b}}{1+2b}\times\sqrt{(1-b^2)}.$$

(151). $\frac{1+\sqrt{(x^2-1)}}{1+2a\sqrt{(x^2-1)}}=\frac{\sqrt{(x^2-1)}-1}{x^2-2},$

$$\frac{1+\sqrt{(x^2-1)}}{1+2a\sqrt{(x^2-1)}}=\frac{\sqrt{(x^2-1)}-1}{(x^2-1)-1}=\frac{1}{\sqrt{(x^2-1)}+1},$$

$$\text{wherefore } \sqrt{(x^2-1)}-1=0,\ \text{and } x=\pm\sqrt{2},$$

$$\text{and } 1+x^2-1+2\sqrt{(x^2-1)}=1+2a\sqrt{(x^2-1)};$$

$$\therefore\ x^2-1=2(a-1)\sqrt{(x^2-1)}\ \ \therefore\ \sqrt{(x^2-1)}=2(a-1),$$

$$\text{whence } x^2-1=0,\ \text{and } x=\pm1,\ \text{also } x=\pm\sqrt{\{4(a-1)^2+1\}}.$$

(152). $\frac{\sqrt{(a+bx^n)}+\sqrt{(a-bx^n)}}{\sqrt{(a+bx^n)}-\sqrt{(a-bx^n)}}=c,$

$$\frac{\sqrt{(a+bx^n)}}{\sqrt{(a-bx^n)}}=\frac{c+1}{c-1};\ \ \therefore\ \frac{a+bx^n}{a-bx^n}=\left(\frac{c+1}{c-1}\right)^2,$$

$$\frac{bx^n}{a}=\frac{4c}{2(c^2+1)};\ \ \therefore\ x=\sqrt[n]{\left\{\frac{2ac}{b(c^2+1)}\right\}}.$$

(153). $\sqrt{(a+x)}+\sqrt{(a-x)}=\sqrt[4]{(a^2+x^2)}+\sqrt[4]{(a^2-x^2)},$

$$a+x+a-x+2\sqrt{(a^2-x^2)}=\sqrt{(a^2+x^2)}+\sqrt{(a^2-x^2)}+2\sqrt[4]{(a^4-x^4)},$$

$$\text{whence } 2\{a-\sqrt[4]{(a^4-x^4)}\}=\sqrt{(a^2+x^2)}-\sqrt{(a^2-x^2)};$$

$$\therefore\ 4\{a^2-2a\sqrt[4]{(a^4-x^4)}+\sqrt{(a^4-x^4)}\}=2a^2-2\sqrt{(a^4-x^4)},$$

$$\text{or } a^2+3\sqrt{(a^4-x^4)}=4a\sqrt[4]{(a^4-x^4)};$$

$$\therefore\ a^4+9(a^4-x^4)+6a^2\sqrt{(a^4-x^4)}=16a^2\sqrt{(a^4-x^4)};$$

$$\therefore\ 9(a^4-x^4)-10a^2\sqrt{(a^4-x^4)}+\frac{25a^4}{9}=\frac{16a^4}{9},$$

$$3\sqrt{(a^4 - x^4)} = 3a^2, \text{ or } \frac{a^2}{3}; \ \therefore x = 0,$$

$$\text{or } x = \frac{2a}{3}\sqrt[4]{(5)}.$$

(154). $\dfrac{\sqrt{(4x+1)} + \sqrt{(4x)}}{\sqrt{(4x+1)} - \sqrt{(4x)}} = 9,$

$$\frac{\sqrt{(4x+1)}}{\sqrt{(4x)}} = \frac{5}{4}, \text{ or } \frac{4x+1}{4x} = \frac{25}{16};$$

$$\therefore 16x + 4 = 25x; \ \therefore x = \frac{4}{9}.$$

(155). $\sqrt{\{(1+x)^2 - ax\}} + \sqrt{\{(1-x)^2 + ax\}} = x,$

$$\overline{1+x}|^2 - ax + x^2 - 2x\sqrt{\{(1+x)^2 - ax\}} = (1-x)^2 + ax,$$

$$\text{whence } 2(2-a) + x = 2\sqrt{\{(1+x)^2 - ax\}},$$

$$\text{and } 4(2-a)^2 + 4x(2-a) + x^2 = 4(1+x)^2 - 4ax;$$

$$\therefore x^2 = 4 - \frac{16a}{3} + \frac{4a^2}{3}; \ \therefore x = \pm 2\sqrt{\left(1 - \frac{4a}{3} \pm \frac{a^2}{3}\right)}.$$

(156). $\dfrac{1-ax}{1+ax}\sqrt{\left(\dfrac{1+bx}{1-bx}\right)} = 1,$

$$\overline{\frac{1-ax}{1+ax}}\Big|^2 = \frac{1-bx}{1+bx}; \ \therefore \frac{1+a^2x^2}{2ax} = \frac{1}{bx};$$

$$\therefore (1 + a^2x^2)\, b = 2a; \ \therefore x = \frac{1}{a}\sqrt{\left(\frac{2a}{b} - 1\right)}.$$

(157). $(1+x)\sqrt{(1+a)} + (1-x)\sqrt{(1-a)} = 2\sqrt{(1+x^2)},$

$$(1+x)^2(1+a) + (1-x)^2(1-a) + 2(1-x^2)\sqrt{(1-a^2)} = 4 + 4x^2,$$

$$(1-x^2)\sqrt{(1-a^2)} = (1+x^2) - 2ax;$$

$$\therefore 1 - 2x^2 + x^4 - a^2(1 - 2x^2 + x^4) = 1 + 2x^2 + x^4 - 4ax(1+x^2) + 4a^2x^2;$$

$$\therefore a^2(1+x^2)^2 - 4ax(1+x^2) + 4x^2 = 0;$$

$$\therefore 1 + x^2 = \pm\frac{2x}{a}, \text{ and } x^2 \mp \frac{2x}{a} + \frac{1}{a^2} = -1 + \frac{1}{a^2};$$

$$\therefore x = \pm\frac{1}{a} \pm \sqrt{\left(\frac{1}{a^2} - 1\right)}.$$

(158). $(x-a)\sqrt{(x)}-(x+a)\sqrt{(b)}=b\{\sqrt{(x)}-\sqrt{(b)},$

$$x^{\frac{3}{2}}+b^{\frac{3}{2}}-\sqrt{(bx)}\,(\sqrt{x}+\sqrt{b})-a\,(\sqrt{x}+\sqrt{b})=0;$$

$$\therefore\ \sqrt{x}+\sqrt{b}=0,\ \text{and}\ x=b.$$

$$\text{also}\ x-\sqrt{(bx)}+b-\sqrt{(bx)}-a=0;$$

$$\therefore\ x-2\sqrt{(bx)}+b=a;\ \therefore\ x^{\frac{1}{2}}=a^{\frac{1}{2}}+b^{\frac{1}{2}},$$

$$\text{and}\ x=\{\sqrt{(a)}+\sqrt{(b)}\}^2.$$

(159). $2x^2+1+x\sqrt{(4x^2+3)}=a\{2x^2+3+x\sqrt{(4x^2+3)}\},$

$$\frac{2x^2+3+x\sqrt{(4x^2+3)}}{2x^2+1+x\sqrt{(4x^2+3)}}=\frac{1}{a};$$

$$\therefore\ 2x^2+3+x\sqrt{(4x^2+3)}=\frac{2}{1-a},$$

$$\text{whence}\ x\sqrt{(4x^2+3)}=\frac{3a-1}{1-a}-2x^2,$$

$$\text{and}\ 4x^4+3x^2=\left(\frac{3a-1}{1-a}\right)^2-4x^2\left(\frac{3a-1}{1-a}\right)+4x^4,$$

$$\text{wherefore}\ x^2\left(\frac{12a-4}{1-a}+3\right)=\left(\frac{3a-1}{1-a}\right)^2;$$

$$\therefore\ x^2=\frac{(3a-1)^2}{(1-a)}\ \frac{(1-a)}{9a-1};$$

$$\therefore\ x=\pm\frac{3a-1}{\sqrt{\{(1-a)\,(9a-1)\}}}.$$

(160). $\dfrac{a+x+\sqrt{(2ax+x^2)}}{a+x-\sqrt{(2ax+x^2)}}=b^2,$

$$\frac{\sqrt{(2ax+x^2)}}{a+x}=\frac{b^2-1}{b^2+1},\ \text{and}\ \frac{2ax+x^2}{(a+x)^2}=\left(\frac{b^2-1}{b^2+1}\right)^2;$$

$$\therefore\ \frac{(a+x)^2}{a^2}=\frac{(b^2+1)^2}{4b^2};\ \therefore\ a+x=\frac{a}{2b}(b^2+1);$$

$$\therefore\ x=\frac{a}{2b}(b-1)^2.$$

161). $\dfrac{243+324\sqrt{(3x)}}{\{4\sqrt{(x)}-\sqrt{(3)}\}^2}=16x-3,$

$$\frac{81\sqrt{3}\,(4\sqrt{x}+\sqrt{3})}{(4\sqrt{x}-\sqrt{3})^2}=(4\sqrt{x}-\sqrt{3})\,(4\sqrt{x}+\sqrt{3});$$

$$\therefore 81\sqrt{3}=(4\sqrt{x}-\sqrt{3})^3;$$

$$\text{or } 3\sqrt{3}=4\sqrt{x}-\sqrt{3};\ \therefore 4\sqrt{x}=\sqrt{3}+3\sqrt{3},$$

$$\text{and } x=\frac{30+18}{16}=3.$$

(162). $\dfrac{(a-x)+\sqrt{(2ax-x^2)}}{a-x}=b,$

$$\text{or } \frac{\sqrt{(2ax-x^2)}}{a-x}=b-1;\ \therefore \frac{(a-x)^2}{2ax-x^2}=\frac{1}{(b-1)^2},$$

$$\text{and } \frac{(a-x)^2}{a^2}=\frac{1}{(b-1)^2+1};\ \therefore a-x=\frac{a}{\sqrt{\{(b^2-1)^2+1\}}}$$

$$\therefore x=a-\frac{a}{\sqrt{\{(b-1)^2+1\}}}.$$

(163). $\sqrt[4]{(x^4-1)}+x\sqrt{(x^4-1)}=2x^{-1},$

$$\sqrt{(x^4-1)}+\frac{1}{x}\sqrt[4]{(x^4-1)}+\frac{1}{4x^2}=\frac{2}{x^2}+\frac{1}{4x^2}=\frac{9}{4x^2};$$

$$\therefore \sqrt[4]{(x^4-1)}=\frac{-1\pm 3}{2x}=\frac{1}{x},\text{ or } -2x;$$

$$\therefore x^4-1=\frac{1}{x^4}\text{ or } 16x^4;$$

$$\therefore x^8-x^4+\frac{1}{4}=1+\frac{1}{4};$$

$$\therefore x=\left(\frac{1\pm\sqrt{5}}{2}\right)^{\frac{1}{4}},\text{ or } \left(-\frac{1}{15}\right)^{\frac{1}{4}}.$$

(164). $ax+1=\dfrac{2ax\sqrt{(x+a^2)}}{a+\sqrt{(x+a^2)}},$

$$(ax+1)\,a=(ax-1)\sqrt{(x+a^2)};$$

$$\therefore \overline{\frac{ax+1}{ax-1}}\Big|^2=\frac{(x+a^2)}{a^2};\ \therefore \frac{(ax-1)^2}{4ax}=\frac{a^2}{x};$$

$$\therefore ax-1=\pm 2a\sqrt{(a)},\text{ and } x=\pm 2\sqrt{a}+\frac{1}{a}.$$

(165). $\dfrac{2a\sqrt{(1+x^2)}}{a+b} = 1 - x + \sqrt{(1+x^2)},$

$$(a-b)\sqrt{(1+x^2)} = (1-x)(a+b);$$

$$\therefore \frac{1+x^2}{(1-x)^2} = \left(\frac{a+b}{a-b}\right)^2,$$

$$\frac{(1-x)^2}{2x} = \frac{(a-b)^2}{4ab};$$

$$\therefore 1 - x = \frac{(a-b)\times\sqrt{(x)}}{\sqrt{(2ab)}};$$

$$\therefore x + \frac{(a-b)\sqrt{(x)}}{\sqrt{(2ab)}} + (\quad)^2 = 1 + \frac{(a-b)^2}{8ab};$$

$$\therefore \sqrt{x} = -\frac{a-b}{2\sqrt{(2ab)}} \pm \frac{\sqrt{\{(a-b)^2+8ab\}}}{2\sqrt{(2ab)}};$$

$$\therefore x = \frac{(a-b)^2}{4ab}\left[1 + \frac{8ab}{(a-b)^2} \mp \sqrt{\left\{1 + \frac{8ab}{(a-b)^2}\right\}}\right].$$

(166). $\sqrt{\{(1+a)^2 + (1-a)x\}} + \sqrt{\{(1-a)^2 + (1+a)x\}} = 2a,$

$$\therefore (1+a)^2 + (1-a)x + (1-a)^2 + (1+a)x$$
$$+ 2\sqrt{\{(1-a^2)^2 + (1-a^2)x^2 + 2(1+3a^2)x\}} = 4a^2,$$

whence $\sqrt{\{(1-a^2)^2 + (1-a^2)x^2 + 2(1+3a^2)x\}} = -(1-a^2) - x,$

or $(1-a^2)^2 + (1-a^2)x^2 + 2(1+3a^2)x = (1-a^2)^2 + 2x(1-a)^2 + x^2$

and $a^2x^2 = 8a^2x$; $\therefore x = 8.$

(167). $\dfrac{a^2(a+x) - a - x\sqrt{(2a^2-1)}}{a^2(a+x) - a + x\sqrt{(2a^2-1)}} = a^2 + x^2 - a^2(a+x)^2,$

$$\frac{a(a^2-1) + a^2x - x\sqrt{(2a^2-1)}}{a(a^2-1) + a^2x + x\sqrt{(2a^2-1)}} = a^2 + x^2 - a^2(a+x)^2,$$

by multiplying numerator and denominator by denominator, we have

$$\frac{a^2(a^2-1)^2 + 2a^3x(a^2-1) + x^2(a^2-1)^2}{(\text{denom.})^2} = a^2 + x^2 - a^2(a+x)^2,$$

but $(a^2-1)(a^4 - a^2 + 2a^3x + a^2x^2 - x^2)$

$$= (a^2-1)\{a^2(a+x)^2 - (a^2+x^2)\}$$
$$= (1-a^2)\{a^2 + x^2 - a^2(a+x)^2\};$$

$$\therefore \frac{1-a^2}{\overline{\text{denom.}}|^2} = 1,$$

$$\text{or } a(a^2-1) + a^2x + x\sqrt{(2a^2-1)} = \pm\sqrt{(1-a^2)};$$

$$\therefore x = \pm\frac{\sqrt{(1-a^2)} - a(a^2-1)}{a^2+\sqrt{(2a^2-1)}} = \pm\frac{\sqrt{(1-a^2)}\{1 \mp a\sqrt{(1-a^2)}\}}{a^2+\sqrt{(2a^2-1)}},$$

this equation may be put under the form

$$\frac{\{1-\sqrt{(2a^2-1)}\}}{\{1+\sqrt{(2a^2-1)}\}}\frac{\{(x-a)-(x+a)\sqrt{(2a^2-1)}\}}{\{(x-a)+(x+a)\sqrt{(2a^2-1)}\}} = \frac{1}{2}\{(x-a)^2-(x+a)^2(2a^2-1)\},$$

whence $x - a - (x+a)\sqrt{(2a^2-1)} = 0$, and $x = \dfrac{a^3 + a\sqrt{(2a^2-1)}}{1-a^2}$,

or $\{(x-a)+(x+a)\sqrt{(2a^2-1)}\}^2 = 2\left\{\dfrac{1-\sqrt{(2a^2-1)}}{1+\sqrt{(2a^2-1)}}\right\} = \dfrac{\{1-\sqrt{(2a^2-1)}^2}{1-a^2}$,

$$\text{whence } x = \frac{a(1-a^2) \pm \sqrt{(1-a^2)}}{a^2+\sqrt{(2a^2-1)}}.$$

QUADRATIC EQUATIONS.

VIII. (1). $12x^2 - 44 = 6x^2 + 10$; $\therefore$ $6x^2 = 54$, and $x = \pm 3$.

(2). $4x^2 - 4 = 28 + 2x^2$; $\therefore$ $2x^2 = 32$, and $x = \pm 4$.

(3). $(x+2)^2 - 5 = 4x$,

$$x^2 + 4x + 4 - 5 = 4x; \therefore x = \pm 1.$$

(4). $3x^2 + 63 = 10x^2$; $\therefore$ $x^2 = 9$, and $x = \pm 3$.

(5). $\dfrac{3}{x+1} - 8 = \dfrac{3}{x-1}$,

or $3x - 3 - 8(x^2-1) = 3x + 3$; $\therefore$ $8x^2 = 2$, and $x = \pm\dfrac{1}{2}$.

(6). $\dfrac{2x^2+10}{15} + \dfrac{50+x^2}{25} = 7$,

$10x^2 + 50 + 150 + 3x^2 = 525$; $\therefore$ $x^2 = \dfrac{325}{13} = 25$, and $x = \pm 5$

(7). $x^2 - 12x + 20 = 0$,

$$x^2 - 12x = -20 \text{ by completing the square,}$$

$$x^2 - 12x + 36 = -20 + 36 = 16;$$

$$\therefore x = 6 \pm 4 = 10 \text{ or } 2.$$

(8). $x^2 - 10x + 16 = 0$,

$$x^2 - 10x = -16; \therefore x^2 - 10x + 25 = -16 + 25 = 9;$$

$$\therefore x = 5 \pm 3 = 8 \text{ or } 2.$$

(9). $x^2 + 32x = 320$; $\therefore x^2 + 32x + \overline{16}|^2 = 320 + 256 = 576$;

$$\therefore x = -16 \pm 24 = 8 \text{ or } -40.$$

(10). $2x^2 - 4x = 6$; $\therefore x^2 - 2x + 1 = 3 + 1 = 4$,

$$\text{and } x = 1 \pm 2 = 3 \text{ or } -1.$$

(11). $x^2 + 13x + 12 = 0$,

$$x^2 + 13x + \frac{169}{4} = -12 + \frac{169}{4} = \frac{121}{4};$$

$$\therefore x = -\frac{13}{2} \pm \frac{11}{2} = -1 \text{ or } -12.$$

(12). $x^2 - 13x = 68$,

$$x^2 - 13x + \frac{169}{4} = 68 + \frac{169}{4} = \frac{441}{4};$$

$$\therefore x = \frac{13}{2} \pm \frac{21}{2} = 17 \text{ or } -4.$$

(13). $x^2 + 25x + 100 = 0$,

$$x^2 + 25x + \overline{\frac{25}{2}}\Big|^2 = -100 + \frac{625}{4} = \frac{225}{4};$$

$$\therefore x = -\frac{25}{2} \pm \frac{15}{2} = -5 \text{ or } -20.$$

(14). $x^2 + 19x = 20$,

$$x^2 + 19x + \overline{\frac{19}{2}}\Big|^2 = 20 + \frac{361}{4} = \frac{441}{4};$$

$$\therefore x = -\frac{19}{2} \pm \frac{21}{2} = 1 \text{ or } -20.$$

(15). $3x^2 - x = 102$,

$$\left. x^2 - \frac{x}{3} + \overline{\frac{1}{6}} \right]^2 = \frac{102}{3} + \frac{1}{36} = \frac{1225}{36}; \therefore x = \frac{1}{6} \pm \frac{35}{6} = 6 \text{ or } -\frac{17}{3}.$$

(16). $20x^2 + 9 = 36x$,

$$20x^2 - 36x = -9; \therefore \left. x^2 - \frac{9x}{5} + \overline{\frac{9}{10}} \right]^2 = -\frac{9}{20} + \frac{81}{100} = \frac{36}{100};$$

$$\therefore x = \frac{9}{10} \pm \frac{6}{10} = \frac{3}{2} \text{ or } \frac{3}{10}.$$

(17). $\frac{x^2}{3} + \frac{3x}{2} = 21$,

$$x^2 + \frac{9x}{2} = 63; \therefore \left. x^2 + \frac{9x}{2} + \overline{\frac{9}{4}} \right]^2 = 63 + \frac{81}{16} = \frac{1089}{16};$$

$$\therefore x = -\frac{9}{4} \pm \frac{33}{4} = 6 \text{ or } -\frac{21}{2}.$$

(18). $\frac{2x^2}{3} - \frac{x}{2} = 4\frac{1}{2}$,

$$2x^2 - \frac{3x}{2} = \frac{27}{2}; \therefore \left. x^2 - \frac{3x}{4} + \overline{\frac{3}{8}} \right]^2 = \frac{27}{4} + \frac{9}{64} = \frac{441}{64};$$

$$\therefore x = \frac{3}{8} \pm \frac{21}{8} = 3 \text{ or } -\frac{9}{4}.$$

(19). $6x^6 - 12x^3 = 288$,

$$x^6 - 2x^2 = 48; \therefore x^6 - 2x^3 + 1 = 49, \text{ and } x^3 = 1 \pm 7 = 8 \text{ or } -6$$

$$\text{and } x = 2 \text{ or } \sqrt[3]{(-6)}.$$

(20). $5x^4 - 11x^2 = 306$,

$$\left. x^4 - \frac{11}{5}x^2 + \overline{\frac{11}{10}} \right]^2 = \frac{306}{5} + \frac{121}{100} = \frac{6241}{100};$$

$$\therefore x^2 = \frac{11}{10} \pm \frac{79}{10} = 9 \text{ or } -\frac{34}{5}; \therefore x = \pm 3 \text{ or } \pm \sqrt{\left(\frac{-34}{5}\right)}.$$

(21). $x^6 - 4x^3 = 32$,

$$x^6 - 4x^3 + 4 = 36; \therefore x^3 = 2 \pm 6 = 8 \text{ or } -4; \therefore x = 2 \text{ or } \sqrt[3]{(-4)}.$$

(22). $3x^6 + 42x^3 = 3321$,

$$x^6 + 14x^3 = 1107; \therefore x^6 + 14x^3 + 49 = 1156,$$

$$\text{and } x^3 = -7 \pm 34 = 27 \text{ or } -41; \therefore x = 3 \text{ or } \sqrt[3]{(-41)}.$$

(23). $x^2(x^2 - 18) = 4(x^4 - 12)$,

$$x^4 - 18x^2 = 4x^4 - 48; \therefore 3x^4 + 18x^2 = 48,$$

and $x^4 + 6x^2 + 9 = 16 + 9 = 25$; $\therefore x^2 = 3 \pm 5 = 2$ or -8;

$$\therefore x = \pm\sqrt{2} \text{ or } \pm 2\sqrt{(-2)}.$$

(24). $x^4 - 5x^2 + 6 = 0$,

$$\overline{x^4 - 5x^2 + \frac{5}{2}}\Big]^2 = -6 + \frac{25}{4} = \frac{1}{4};$$

$$\therefore x^2 = \frac{5}{2} \pm \frac{1}{2} = 3 \text{ or } 2; \therefore x = \pm\sqrt{3} \text{ or } \pm\sqrt{2}.$$

(25). $\dfrac{4}{x-3} - \dfrac{3}{x+5} = \dfrac{11}{3}$,

$$12(x+5) - 9(x-3) = 11(x-3)(x+5);$$

$$\therefore 3x + 87 = 11(x^2 + 2x - 15);$$

$$\therefore \overline{x^2 + \frac{19x}{11} + \frac{19}{22}}\Big]^2 = \frac{252}{11} + \frac{361}{484} = \frac{11449}{484};$$

$$\therefore x = -\frac{19}{22} \pm \frac{107}{22} = 4 \text{ or } -\frac{63}{11}.$$

(26). $\dfrac{2x}{x-4} + \dfrac{2x-5}{x-3} = 8\frac{1}{3}$,

$$6x(x-3) + 3(2x-5)(x-4) = 25(x-4)(x-3);$$

$$\therefore 6x^2 - 18x + 3(2x^2 - 13x + 20) = 25(x^2 - 7x + 12),$$

whence $13x^2 - 118x = -240$;

$$\therefore \overline{x^2 - \frac{118x}{13} + \frac{59}{13}}\Big]^2 = -\frac{240}{13} + \frac{3481}{169} = \frac{361}{169};$$

$$\therefore x = \frac{59}{13} \pm \frac{19}{13} = 6 \text{ or } 3\tfrac{1}{13}.$$

(27). $\dfrac{3x}{x+1} + \dfrac{2x-5}{3x-1} = \dfrac{217}{69}$,

$$3x(3x-1) + (2x-5)(x+1) = \frac{217}{69}(3x-1)(x+1),$$

and $69(11x^2 - 6x - 5) = 217(3x^2 + 2x - 1)$;

$$\therefore 108x^2 - 848x = 128;$$

$$\therefore\ x^2 - \frac{212x}{27} + \left(\frac{106}{27}\right)^2 = \frac{32}{27} + \frac{11236}{729} = \frac{12100}{729};$$

$$\therefore\ x = \frac{106}{27} \pm \frac{110}{27} = 8 \text{ or } -\frac{4}{27}.$$

(28). $\dfrac{2x+3}{10-x} = \dfrac{2x}{25-3x} - 6\tfrac{1}{2}$,

$$2(2x+3)(25-3x) = 4x(10-x) - 13(25-3x)(10-x);$$

$$\therefore\ 2(75 - 6x^2 + 41x) = 40x - 4x^2 - 13(250 + 3x^2 - 55x);$$

$$\therefore\ 31x^2 - 673x = -3400;$$

$$\therefore\ x^2 - \frac{673x}{31} + \left(\frac{673}{62}\right)^2 = -\frac{3400}{31} + \frac{452929}{(62)^2} = \frac{31329}{(62)^2};$$

$$\therefore\ x = \frac{673}{62} \pm \frac{177}{62} = 13\tfrac{22}{31} \text{ or } 8.$$

(29). $\dfrac{7}{x^2+4x} + \dfrac{21}{3x^2-8x} = \dfrac{22}{x}$,

$$7(3x-8) + 21(x+4) = 22(x+4)(3x-8);$$

$$\therefore\ 21x - 56 + 21x + 84 = 22(3x^2 + 4x - 32);$$

$$\therefore\ x^2 + \frac{23x}{33} + \left(\frac{23}{66}\right)^2 = \frac{732}{66} + \frac{529}{4356} = \frac{48441}{4356};$$

$$\therefore\ x = -\frac{23}{66} \pm \frac{221}{66} = 3 \text{ or } -3\tfrac{23}{33}.$$

(30). $\dfrac{4x-35}{5} + \dfrac{36-5x}{5x} = 0$,

$$4x^2 + 35x + 36 - 5x = 0;$$

$$\therefore\ x^2 - 10x + 25 = -9 + 25 = 16;$$

$$\therefore\ x = 5 \pm 4 = 9 \text{ or } 1.$$

(31). $\dfrac{x+3}{2} + \dfrac{16-2x}{2x-5} = 5\tfrac{1}{5}$,

$$5(x+3)(2x-5) + 10(16-2x) = 26(4x-10);$$

$$\therefore\ 5(2x^2 + x - 15) + 160 - 20x = 104x - 260;$$

$$\therefore x^2 - \frac{119x}{10} + \overline{\frac{119}{20}}\Big|^2 = -\frac{345}{10} + \frac{14161}{400} = \frac{361}{400};$$

$$\therefore x = \frac{119}{30} \pm \frac{19}{20} = 6\tfrac{9}{10} \text{ or } 5.$$

(32). $\dfrac{3x-7}{x} + \dfrac{4x-10}{x+5} = 3\tfrac{1}{2}$,

$$2(3x-7)(x+5) + 2x(4x-10) = 7x(x+5);$$

$$\therefore 2(3x^2 + 8x - 35) + 8x^2 - 20x = 7x^2 + 35x,$$

$$\text{and } x^2 - \frac{39x}{7} + \overline{\frac{39}{14}}\Big|^2 = \frac{70}{7} + \frac{1521}{196} = \frac{3481}{196};$$

$$\therefore x = \frac{39}{14} \pm \frac{59}{14} = 7 \text{ or } -1\tfrac{3}{7}.$$

(33). $\dfrac{x+2}{x-1} - \dfrac{7-2x}{2x} = \dfrac{7}{3}$,

$$6x(x+2) + 3(2x-7)(x-1) = 14x(x-1);$$

$$\therefore 6x^2 + 12x + 3(2x^2 - 9x + 7) = 14x^2 - 14x,$$

$$\text{and } x^2 + \frac{x}{2} + \left(\frac{1}{4}\right)^2 = \frac{21}{2} + \frac{1}{16} = \frac{169}{16};$$

$$\therefore x = -\frac{1}{4} \pm \frac{13}{4} = 3 \text{ or } -3\tfrac{1}{2}.$$

(34). $\dfrac{x^2}{4} = \dfrac{x^4-12}{x^2-18} + 126$,

$$x^2(x^2-18) = 4(x^4-12) + 504(x^2-18);$$

$$\therefore x^4 + 174x^2 + (87)^2 = 3040 + 7569 = 10609;$$

$$\therefore x^2 = -87 \pm 103 = 16 \text{ or } -190;$$

$$\therefore x = \pm 4 \text{ or } \pm\sqrt{(-190)}.$$

(35). $\dfrac{4x}{5-x} - \dfrac{20-4x}{x} = 15$,

$$4x^2 - (20-4x)(5-x) = 15x(5-x);$$

$$\therefore 4x^2 - (100 - 40x + 4x^2) = 75x - 15x^2;$$

$$\therefore x^2 - \frac{7x}{3} + \left(\frac{7}{6}\right)^2 = \frac{20}{3} + \frac{49}{36} = \frac{289}{36};$$

$$\therefore x = \frac{7}{6} \pm \frac{17}{6} = 4 \text{ or } -1\tfrac{2}{3}.$$

(36). $\dfrac{x^2-4}{x+2}=\dfrac{x^3-8}{2(x^2-10)}$,

$$x^3-8=(x^2+2x+4)(x-2),$$

$$\text{also } x^2-4=(x-2)\times(x+2); \therefore x=-2, \text{ and } x=2,$$

$$\text{and } 1=\frac{x^2+2x+4}{2(x^2-10)}; \therefore 2x^2-20=x^2+2x+4,$$

$$\text{and } x^2-2x+1=25; \therefore x=1\pm 5=6 \text{ or } -4.$$

(37). $\dfrac{x+2}{x-1}+\dfrac{x-4}{2x}=2\frac{1}{3}$,

$$6x(x+2)+3(x-4)(x-1)=14x(x-1),$$

$$\text{and } 6x^2+12x+3(x^2-5x+4)=14x^2-14x;$$

$$\therefore x^2-\frac{11x}{5}+\overline{\frac{11}{10}}\Big]^2=\frac{12}{5}+\frac{121}{100}=\frac{361}{100};$$

$$\therefore x=\frac{11}{10}\pm\frac{19}{10}=3 \text{ or } -\frac{4}{5}.$$

(38). $\dfrac{4(x^2-1)}{x+1}-\dfrac{x-1}{2x}=\dfrac{15}{4}$,

$$4(x-1)-\frac{x-1}{2x}=\frac{15}{4};$$

$$\therefore 16x^2-16x-2x+2=15x,$$

$$\text{and } x^2-\frac{33x}{16}+\overline{\frac{33}{32}}\Big]^2=-\frac{2}{16}+\frac{1089}{(32)^2}=\frac{961}{1024};$$

$$\therefore x=\frac{33}{32}\pm\frac{31}{32}=2 \text{ or } +\frac{1}{16}.$$

(39). $\dfrac{x-1}{x^2-1}+\dfrac{x^2-1}{x-1}=\dfrac{13}{6}$,

$$\text{or } \frac{1}{x+1}+\frac{x+1}{1}=\frac{13}{6}; \therefore 6+6(x^2+2x+1)=13(x+1)$$

$$\therefore x^2-\frac{x}{6}+\overline{\frac{1}{12}}\Big]^2=\frac{1}{6}+\frac{1}{144}=\frac{25}{144};$$

$$\therefore x=\frac{1}{12}\pm\frac{5}{12}=\frac{1}{2} \text{ or } -\frac{1}{3}.$$

(40). $\frac{x^2-4}{x-2}-1=\frac{6(x-2)}{x^2-4}$,

$$x+2-1=\frac{6}{x+2};\ \therefore\ x^2+3x+2=6,$$

$$\therefore\ x=2,\ \text{and}\ x^2+3x+\frac{9}{4}=4+\frac{9}{4}=\frac{25}{4};$$

$$\therefore\ x=-\frac{3}{2}\pm\frac{5}{2}=1,\ \text{or}\ -4.$$

(41). $\frac{7+x}{7-x}+\frac{7-x}{7+x}=\frac{29}{10}$,

$$\overline{\frac{7+x}{7-x}}\Big]^2-\frac{29}{10}\times\frac{7+x}{7-x}+\overline{\frac{29}{20}}\Big]^2=-1+\frac{841}{400}=\frac{441}{400};$$

$$\therefore\ \frac{7+x}{7-x}=\frac{29}{20}\pm\frac{21}{20}=\frac{5}{2},\ \text{or}\ \frac{2}{5},\ \text{by the formula, if}\ \frac{a}{b}=\frac{c}{d},$$

$$\text{then}\ \frac{a\pm b}{a\mp b}=\frac{c\pm d}{c\mp d},$$

$$\frac{x}{7}=\frac{3}{7}\ \text{or}\ -\frac{3}{7};\ \therefore\ x=3,\ \text{or}\ -3.$$

(42). $\frac{3x+5}{3x-5}-\frac{135}{176}=\frac{3x-5}{3x+5}$,

$$\overline{\frac{3x+5}{3x-5}}\Big]^2-\frac{135}{176}\ \frac{3x+5}{3x-5}+\overline{\frac{135}{352}}\Big]^2=1+\overline{\frac{135}{352}}\Big]^2=\frac{142129}{(352)^2},$$

$$\therefore\ \frac{3x+5}{3x-5}=\frac{135}{352}\pm\frac{377}{352}=\frac{16}{11},\ \text{or}\ -\frac{11}{16},$$

$$\text{whence}\ x=9\ \text{or}\ -\frac{25}{81}.$$

(43). $\frac{x^2}{(x^2-4)^2}=\frac{351}{25x^2}-\frac{6}{x^2-4}$,

$$\frac{x^2}{(x^2-4)^2}+\frac{6}{x^2-4}+\frac{9}{x^2}=\frac{351}{25x^2}+\frac{9}{x^2}=\frac{576}{25x^2},$$

$$\text{whence}\ \frac{x}{x^2-4}=-\frac{3}{x}\pm\frac{24}{5x}=\frac{9}{5x},\ \text{or}\ -\frac{39}{5x};$$

$$\therefore\ 5x^2=9x^2-36;\ \therefore\ x=\pm 3,$$

$$\text{or}\ x=\pm\sqrt{\left(\frac{39}{11}\right)}.$$

(44). $\frac{x+7}{x-7} - \frac{7x}{x^2-73} = \frac{x-7}{x+7}$,

$$x^2 + 14x + 49 - \frac{7x(x^2-49)}{x^2-73} = x^2 - 14x + 49,$$

$$\text{whence } 4(x^2 - 73) = x^2 - 49;$$

$$\therefore\ x^2 = 81, \text{ and } x = \pm 9.$$

(45). $x^2 + \frac{1}{x^2} + x + \frac{1}{x} = 4$,

$$\therefore\ x^2 + 2 + \frac{1}{x^2} + x + \frac{1}{x} + \frac{1}{4} = 6 + \frac{1}{4} = \frac{25}{4};$$

$$\therefore\ x + \frac{1}{x} = -\frac{1}{2} \pm \frac{5}{2} = 2, \text{ or } -3;$$

$$\therefore\ x^2 - 2x + 1 = 0, \text{ and } x = 1, \text{ or } x^2 + 3x + \frac{9}{4} = -1 + \frac{9}{4} = \frac{5}{4};$$

$$\therefore\ x = -\frac{3}{2} \pm \frac{\sqrt{5}}{2}.$$

(46). $x^2 + \frac{4}{x^2} + 6\left(x + \frac{2}{x}\right) = 23$,

$$x^2 + 4 + \frac{4}{x^2} + 6\left(x + \frac{2}{x}\right) + 9 = 27 + 9 = 36;$$

$$\therefore\ x + \frac{2}{x} = -3 \pm 6 = 3, \text{ or } -9,$$

$$\text{and } x^2 - 3x + \frac{9}{4} = -2 + \frac{9}{4} = \frac{1}{4};\ \therefore\ x = 2, \text{ or } 1,$$

$$\text{also } x^2 + 9x + \left(\frac{81}{4}\right) = -2 + \frac{81}{4};\ \therefore\ x = -\frac{9}{2} \pm \frac{\sqrt{(73)}}{2}.$$

(47). $(3x^2+1)^2 - 9(3x^2+1) = 630$,

$$(3x^2+1)^2 - 9(3x^2+1) + \frac{81}{4} = 630 + \frac{81}{4} = \frac{2601}{4};$$

$$\therefore\ 3x^2 + 1 = \frac{9}{2} \pm \frac{51}{2} = 30 \text{ or } -21;$$

$$\therefore\ x^2 = \frac{29}{3} \text{ or } -\frac{22}{3};\ \therefore\ x = \pm\sqrt{\left(\frac{29}{3}\right)} \text{ or } \pm\sqrt{\left(-\frac{22}{3}\right)}.$$

(48). $(x + 8x^{-1})^2 + x = 42 - 8x^{-1}$,

$$\left(x + \frac{8}{x}\right)^2 + \left(x + \frac{8}{x}\right) + \frac{1}{4} = 42 + \frac{1}{4} = \frac{169}{4};$$

$$\therefore\ x + \frac{8}{x} = -\frac{1}{2} \pm \frac{13}{2} = 6 \text{ or } -7,$$

$$\text{whence } x^2 - 6x + 9 = 1, \text{ or } x^2 + 7x + \frac{49}{4} = \frac{17}{4};$$

$$\therefore\ x = 4 \text{ or } 2, \text{ or } x = \frac{1}{2}\{-7 \pm \sqrt{(17)}\}.$$

(49). $(x^2 + 5)^2 - 4x^2 = 160$,

$$(x^2 + 5)^2 - 4(x^2 + 5) + 4 = 144;$$

$$\therefore\ x^2 + 5 = 2 \pm 12 = 14 \text{ or } -10,$$

$$\text{and } x = \pm 3 \text{ or } \pm \sqrt{(-15)}.$$

(50). $(x - 5)^3 - 3(x - 5)^{\frac{3}{2}} = 40$,

$$(x - 5)^3 - 3(x - 5)^{\frac{3}{2}} + \frac{9}{4} = 40 + \frac{9}{4} = \frac{169}{4};$$

$$\therefore\ (x - 5)^{\frac{3}{2}} = \frac{3}{2} \pm \frac{13}{2} = 8 \text{ or } -5,$$

$$\text{whence } x - 5 = 4 \text{ or } (-5)^{\frac{2}{3}};$$

$$\therefore\ x = 9, \text{ or } 5 \pm (-5)^{\frac{2}{3}}.$$

(51). $x^2 - (a + b)x + ab = 0$,

$$x^2 - (a + b)x + \overline{\frac{a + b}{2}}\Big|^2 = -ab + \frac{a^2 + 2ab + b^2}{4} = \frac{a^2 - 2ab + b^2}{4};$$

$$\therefore\ x = \frac{a + b}{2} \pm \frac{a - b}{2} = a \text{ or } b.$$

(52). $2b^2x^2 + ab(x - 1) = 2b^2x$,

$$b^2x^2 + \frac{b(a - 2b)x}{2} + \overline{\frac{a - 2b}{4}}\Big|^2 = \frac{ab}{2} + \frac{a^2 - 4ab + 4b^2}{16} = \overline{\frac{a + 2b}{4}}\Big|^2;$$

$$\therefore\ bx = -\frac{a - 2b}{4} \pm \frac{a + 2b}{4} = b \text{ or } -\frac{a}{2}, \text{ and } x = 1 \text{ or } -\frac{a}{2b}.$$

(53). $adx - acx^2 = bcx - bd$,

$$x^2 - \frac{ad - bc}{ac}x + \left(\frac{ad - bc}{2ac}\right)^2 = \frac{bd}{ac} + \left(\frac{ad - bc}{2ac}\right)^2 = \left(\frac{ad + bc}{2ac}\right)^2;$$

$$\therefore x = \frac{ad - bc}{2ac} \pm \frac{ad + bc}{2ac} = \frac{d}{c} \text{ or } -\frac{b}{a}.$$

(54). $bc(x - a)^2 - bc = (b^2 - c^2)(x - a)$,

$$(x - a)^2 - \frac{(b^2 - c^2)(x - a)}{bc} + \left(\frac{b^2 - c^2}{2bc}\right)^2 = 1 + \frac{(b^2 - c^2)^2}{4b^2c^2} = \frac{(b^2 + c^2)^2}{4b^2c^2};$$

$$\therefore x - a = \frac{b^2 - c^2}{2bc} \pm \frac{b^2 + c^2}{2bc} = \frac{b}{c} \text{ or } -\frac{c}{b};$$

$$\text{and } \therefore x = \frac{ac + b}{c} \text{ or } \frac{ab - c}{b}.$$

(55). $1 + b^2x^2 = \frac{b}{a^2} \times (2a^2x + b)$,

$$b^2x^2 - 2bx + 1 = \frac{b^2}{a^2};$$

$$\therefore bx = 1 \pm \frac{b}{a}, \text{ and } x = \frac{a \pm b}{ab}.$$

(56). $x^2 - (a + b - c)x = (a + b)c$,

$$x^2 - (a + b - c)x + \left(\frac{a + b - c}{2}\right)^2 = ac + bc + \frac{(a + b - c)^2}{4} = \frac{(a + b + c)^2}{4};$$

$$\therefore x = \frac{a + b - c}{2} \pm \frac{(a + b + c)}{2} = (a + b) \text{ or } -c.$$

(57). $a(x^2 - ab) = b(x^2 - 2ax)$,

$$x^2 + \frac{2abx}{a - b} + \frac{a^2b^2}{(a - b)^2} = \frac{a^2b}{a - b} + \frac{a^2b^2}{(a - b)^2} = \frac{a^3b}{(a - b)^2};$$

$$\therefore x = -\frac{ab}{a - b} \pm \frac{a\sqrt{(ab)}}{a - b} = \pm\frac{a\sqrt{b}}{a - b}(\sqrt{a} \mp \sqrt{b}) = \pm\frac{a\sqrt{b}}{\sqrt{(a)} \pm \sqrt{b}}.$$

(58). $a^2bx + b^2x^2 = 2a^4$,

$$b^2x^2 + a^2bx + \frac{a^4}{4} = 2a^4 + \frac{a^4}{4} = 9\frac{a^4}{4};$$

$$\therefore bx = -\frac{a^2}{2} \pm \frac{3a^2}{2} = a^2 \text{ or } -2a^2;$$

$$\therefore x = \frac{a^2}{b} \text{ or } -\frac{2a^2}{b}.$$

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(59). $9a^4b^4x^2 - 6a^3b^2x - b^2 = 0,$

$$9a^4b^4x^2 - 6a^3b^2x + a^2 = a^2 + b^2;$$

$$\therefore\ x = \frac{a \pm \sqrt{(a^2+b^2)}}{3a^2b^2}.$$

(60). $8x(a - x) = a(3a - 2x),$

$$8ax - 8x^2 = 3a^2 - 2ax;$$

$$\therefore\ \overline{x^2 - \frac{5ax}{4} + \frac{5a}{8}}\Big]^2 = -\frac{3a^2}{8} + \frac{25a^2}{64} = \frac{a^2}{64};$$

$$\therefore\ x = \frac{5a}{8} \pm \frac{a}{8} = \frac{3a}{4} \text{ or } \frac{a}{2},$$

$$a + 2ax - a + 2ax = a^2 - 4x^2.$$

(61). $a(1 + 2x) - a(1 - 2x) = a^2 - 4x^2,$

$$\therefore\ 4x^2 + 4ax + a^2 = 2a^2;$$

$$\therefore\ x = \frac{a(-1 \pm \sqrt{2})}{2}.$$

(62). $mqx^2 - mnx + pqx = np,$

$$\overline{x^2 - \frac{(mn-pq)x}{mq} + \frac{mn-pq}{2mq}}\Big]^2 = \frac{np}{mq} + \frac{(mn-pq)^2}{4m^2q^2} = \frac{(mn+pq)^2}{4m^2q^2};$$

$$\therefore\ x = \frac{mn-pq}{2mq} \pm \frac{mn+pq}{2mq} = \frac{n}{q} \text{ or } -\frac{p}{m}.$$

(63). $8x^2 + 6(a - 2b)x = 9ab,$

$$x^2 + \frac{3}{4}(a-2b)x + \frac{9}{64}(a-2b)^2 = \frac{9ab}{8} + \frac{9}{64}(a-2b)^2 = \frac{9}{64}(a+2b)^2;$$

$$\therefore\ x = -\frac{3}{8}(a-2b) \pm \frac{3}{8}(a+2b) = -\frac{3a}{4} \text{ or } +\frac{3b}{2}.$$

(64). $6x^2 + 9(a - 8b)x = 108ab,$

$$x^2 + \frac{3}{2}(a-8b)x + \frac{9}{16}(a-8b)^2 = 18ab + \frac{9}{16}(a-8b)^2 = \frac{9}{16}(a+8b)^2;$$

$$\therefore\ x = -\frac{3}{4}(a-8b) \pm \frac{3}{4}(a+8b) = 12b \text{ or } -\frac{3a}{2}.$$

(65). $9x^2 - 6bx = a^2 - b^2,$

$$9x^2 - 6bx + b^2 = a^2;\ \therefore\ x = \frac{b \pm a}{3}.$$

(66). $25x^2 - 10bx = a^2 - b^2$,

$$25x^2 - 10bx + b^2 = a^2; \quad \therefore\ x = \frac{b \pm a}{5}.$$

(67). $4x^2 - 4bx - a^2 + b^2 = 0$,

$$4x^2 - 4bx + b^2 = a^2; \quad \therefore\ x = \frac{b \pm a}{2}.$$

(68). $6a^2x^2 - 5abx = b^2$,

$$a^2x^2 - \frac{5abx}{6} + \left.\overline{\frac{5b}{12}}\right]^2 = \frac{b^2}{6} + \frac{25b^2}{144} = \frac{49b^2}{144};$$

$$\therefore\ x = \frac{1}{a}\left(\frac{5b}{12} \pm \frac{7b}{12}\right) = \frac{b}{a} \text{ or } -\frac{b}{6a}.$$

(69). $4x^2 - 4(3a + 2b)x + 24ab = 0$,

$$4x^2 - 4(3a + 2b)x + (3a + 2b)^2$$

$$= -24ab + 9a^2 + 12ab + 4b^2 = (3a - 2b)^2;$$

$$\therefore\ x = \frac{3a + 2b}{2} \pm \frac{3a - 2b}{2} = 3a \text{ or } 2b.$$

(70). $mnx^2 - (n^2 - m^2)x - mn = 0$,

$$x^2 - \frac{n^2 - m^2}{mn}x + \left.\overline{\frac{n^2 - m^2}{2mn}}\right]^2 = 1 + \frac{(n^2 - m^2)^2}{4m^2n^2} = \frac{(n^2 + m^2)^2}{4m^2n^2};$$

$$\therefore\ x = \frac{n^2 - m^2}{2mn} \pm \frac{n^2 + m^2}{2mn} = \frac{n}{m} \text{ or } -\frac{m}{n}.$$

(71). $4x^2 - 4mx = n^2 - m^2$,

$$4x^2 - 4mn + m^2 = n^2; \quad \therefore\ x = \frac{m \pm n}{2}.$$

(72). $4x^2 - 12ax + 9a^2 - 4b^2 = 0$,

$$4x^2 - 12ax + 9a^2 = 4b^2; \quad \therefore\ x = \frac{3a \pm 2b}{2}.$$

(73). $px^2 + qx + 1 = 0$,

$$x^2 + \frac{qn}{p} + \left.\overline{\frac{q}{2p}}\right]^2 = -1 + \frac{q^2}{4p^2}; \quad \therefore\ x = -\frac{q}{2p} \pm \frac{\sqrt{(q^2 - 4p^2)}}{2p}.$$

(74). $x^2 \pm px + q = 0$,

$$x^2 \pm px + \frac{p^2}{4} = -q; \quad \therefore\ x = \mp\frac{p}{2} \pm \frac{1}{2}\sqrt{(p^2 - 4q)}.$$

(75). $(a - b)x^2 - (a + b)x + 2b = 0$,

$$\overline{x^2 - \frac{(a+b)x}{a-b} + \frac{a+b}{2(a-b)}}\Big|^2 = -\frac{2b}{a-b} + \frac{(a+b)^2}{4(a-b)^2} = \frac{(a-3b)^2}{4(a-b)^2}.$$

$$\therefore\ x = \frac{a+b}{2(a-b)} \pm \frac{a-3b}{2(a-b)} = 1 \text{ or } \frac{2b}{a-b}.$$

(76). $anx^2 + ab = na^2x + bx$,

$$\overline{x^2 - \frac{(na^2+b)x}{an} + \frac{na^2+b}{2an}}\Big|^2 = -\frac{ab}{an} + \frac{(na^2+b^2)^2}{4a^2n^2} = \frac{(na^2-b^2)^2}{4a^2x^2};$$

$$\therefore\ x = \frac{na^2+b}{2an} \pm \frac{na^2-b}{2an} = a \text{ or } \frac{b}{an}.$$

(77). $x^2 - \frac{cx}{a+b} - \frac{2c^2}{(a+b)^2} = 0$,

$$x^2 - \frac{cx}{a+b} + \frac{c^2}{4(a+b)^2} = \frac{2c^2}{(a+b)^2} + \frac{c^2}{4(a+b)^2} = \frac{9c^2}{4(a+b)^2};$$

$$\therefore\ x = \frac{c}{4(a+b)} \pm \frac{3c}{2(a+b)} = \frac{2c}{a+b} \text{ or } -\frac{c}{a+b}.$$

(78). $x^2 - ax + \frac{a^2-b^2}{4} = 0$,

$$x^2 - ax + \frac{a^2}{4} = \frac{b^2}{4};\ \therefore\ x = \frac{a \pm b}{2}.$$

(79). $\frac{a^2}{b+x} + \frac{a^2}{b-x} = \frac{c}{x}$,

$$a^2(b-x)x + a^2(b+x)x = c(b^2 - x^2);$$

$$\therefore\ x^2 + \frac{2a^2bx}{c} + \frac{a^4b^2}{c^2} = \frac{b^2c}{c} + \frac{a^4b^2}{c^2};\ \therefore\ x = \frac{b}{c}\{-a^2 \pm \sqrt{(a^4 + c^2)}\}.$$

(80). $\frac{1}{a} + \frac{1}{a+x} + \frac{1}{a+2x} = 0$,

$$(a+x)(a+2x) + a(a+2x) + a(a+x) = 0,$$

$$\text{whence } x^2 + 3ax + \frac{9a^2}{4} = -\frac{3a^2}{2} + \frac{9a^2}{4} = \frac{3a^2}{4};$$

$$\therefore\ x = \frac{a}{2}\{-3 \pm \sqrt{(3)}\}.$$

(81). $bx^2 - \frac{ba}{b+c} = dx - cx^2$,

$$x^2 - \frac{dx}{(b+c)} + \frac{d^2}{4(b+c)^2} = \frac{ba}{(b+c)^2} + \frac{d^2}{4(b+c)^2};$$

$$\therefore\ x = \frac{1}{2(b+c)}\{d + \sqrt{(4ab + d^2)}\}.$$

(82). $\frac{1}{a} + \frac{1}{a-x} - \frac{5}{a-2x} = 0$,

$$(a-x)(a-2x) + a(a-2x) - 5a(a-x) = 0,$$

whence $2x^2 - 3a^2 = 0$; $\therefore\ x = \pm a\sqrt{\left(\frac{3}{2}\right)}$.

(83). $\left(\frac{x}{x-1}\right)^2 + \left(\frac{x}{x+1}\right)^2 = n(n-1)$,

$$\frac{x^2(x+1)^2 + x^2(x-1)^2}{(x^2-1)^2} = n^2 - n,$$

and $\frac{2x^2(x^2+1)}{(x^2-1)^2} + \frac{1}{4} = n^2 - n + \frac{1}{4}$;

$$\therefore\ \frac{9x^4 + 6x^2 + 1}{4(x^2-1)^2} = \left(n - \frac{1}{2}\right)^2,$$

or $\frac{3x^2+1}{(x^2-1)} = 2\left(n - \frac{1}{2}\right) = \pm(2n-1)$,

and $\frac{4x^2}{x^2-1} = \frac{2n}{1}$, whence $4x^2 = 2nx^2 - 2n$;

$$\therefore\ x = \pm\sqrt{\left(\frac{n}{n-2}\right)} \text{ or } \pm\sqrt{\left(\frac{n-1}{n+1}\right)}.$$

(84). $\frac{(x-a)(x-c)}{a+b-c} = b$,

$$x^2 - (a+c)x + ac = ab + b^2 - bc,$$

and $x^2 - (a+c)x + \overline{\frac{(a+c)}{2}}\Big|^2$

$$= ab + b^2 - bc + \frac{(a+c)^2}{4} - ac = \frac{(a+2b-c)^2}{4};$$

$$\therefore\ x = \frac{a+c}{2} \pm \frac{a+2b-c}{2} = a + b \text{ or } c - b.$$

(85). $\frac{(x+a)^3}{(x-a)^2} - \frac{(x-a)^3}{(x+a)^2} = 10a,$

$$(x+a)^5 - (x-a)^5 = 10a\,(x^2-a^2)^2,$$

whence $10ax^4 + 20a^3x^2 + 2a^5 = 10a\,(x^4 - 2a^2x^2 + a^4),$

and $40a^3x^2 = 8a^5$; $\therefore\ x = \pm\frac{a}{\sqrt{(5)}}$.

(86). $\frac{x^2+1}{a^2+3ab+b^2} = \frac{2x}{a^2+ab+b^2},$

$$x^2 - \frac{2x\,(a^2+3ab+b^2)}{a^2+ab+b^2} + \overline{\frac{a^2+3ab+b^2}{a^2+ab+b^2}}\Bigg|^2$$

$$= -1 + \overline{\frac{a^2+3ab+b^2}{a^2+ab+b^2}}\Bigg|^2 = \frac{4ab\,(a+b)^2}{(a^2+ab+b^2)^2};$$

$$\therefore\ x = \frac{a^2+3ab+b^2}{a^2+ab+b^2} \pm \frac{(a+b)\,2\sqrt{(ab)}}{(a^2+ab+b^2)} = \frac{a+\sqrt{(ab)}+b}{a-\sqrt{(ab)}+b} \text{ or } \frac{a-\sqrt{(ab)}+b}{a+\sqrt{(ab)}+b},$$

for $\frac{a^2+3ab+b^2 \pm 2\sqrt{(ab)}(a+b)}{a^2+ab+b^2} = \frac{a^2+ab+b^2 \pm 2\sqrt{(ab)}\{a+b\pm\sqrt{(ab)}\}}{\{a+\sqrt{(ab)}+b\}\{a-\sqrt{(ab)}+b\}}$

$$= \frac{\{a\pm\sqrt{(ab)}+b\}\{a\pm\sqrt{(ab)}+b\}}{\{a+\sqrt{(ab)}+b\}\{a-\sqrt{(ab)}+b\}}.$$

(87). $\frac{6\,(a^2-b^2)}{x^2-1} = \frac{12\,(a^2+b^2)\,x}{x^4-1} + \frac{5ab}{x^2+1},$

$$6\,(a^2-b^2)\,(x^2+1) = 12\,(a^2+b^2)\,x + 5ab\,(x^2-1),$$

whence $x^2 - \frac{12\,(a^2+b^2)\,x}{6a^2-6b^2-5ab} + (\quad)^2$

$$= \frac{6b^2-6a^2-5ab}{(2a-3b)(3a+2b)} + \frac{36\,(a^2+b^2)^2}{(2a-3b)^2(3a+2b)^2} = \frac{169a^2b^2}{(2a-3b)^2(3a+2b)^2}$$

$$\therefore\ x = \frac{(6a^2+6b^2\pm 13ab)}{(2a-3b)\,(3a+2b)} = \frac{2a+3b}{2a-3b} \text{ or } \frac{3a-2b}{3a+2b}.$$

(88). $(a-b)\,x^2 + 2b = (a+b)\,x,$

$$x^2 - \frac{a+b}{a-b}x + \overline{\frac{a+b}{2\,(a-b)}}\Bigg|^2 = -\frac{2b}{a-b} + \frac{(a+b)^2}{4\,(a-b)^2} = \frac{(a-3b)^2}{4\,(a-b)^2};$$

$$\therefore\ x = \frac{a+b}{a-b} \pm \frac{a-3b}{2\,(a-b)} = 1 \text{ or } \frac{2b}{a-b}.$$

(89). $\sqrt{(a)}\,(nx+b)=\sqrt{(x)}\,(na+b)$,

$$x-\frac{(na+b)}{n\sqrt{(a)}}\sqrt{(x)}+\left\{\frac{na+b}{2n\sqrt{(a)}}\right\}^2$$

$$=-\frac{b\sqrt{(a)}}{n\sqrt{(a)}}+\left\{\frac{na+b}{2n\sqrt{(a)}}\right\}^2=\overline{\frac{na-b}{2n\sqrt{(a)}}}\Big]^2;$$

$$\therefore \sqrt{x}=\frac{na+b}{2n\sqrt{(a)}}\pm\frac{na-b}{2n\sqrt{(a)}}=\sqrt{a}\text{ or }\frac{b}{n\sqrt{(a)}};\ \therefore x=a\text{ or }\frac{b^2}{n^2a}.$$

(90). $(x-c)\sqrt{(ab)}=(a-b)\sqrt{(cx)}$,

$$x-\frac{(a-b)\sqrt{(cx)}}{\sqrt{(ab)}}+\overline{\frac{(a-b)\sqrt{(c)}}{2\sqrt{(ab)}}}\Big]^2=c+\overline{\frac{(a-b)\sqrt{(c)}}{2\sqrt{(ab)}}}\Big]^2=\frac{(a+b)^2c}{4ab};$$

$$\therefore \sqrt{x}=\frac{\sqrt{c}}{2\sqrt{(ab)}}\{(a-b)\pm(a+b)\}=\frac{\sqrt{(ac)}}{\sqrt{(b)}}\text{ or }-\frac{\sqrt{(bc)}}{\sqrt{(a)}};$$

$$\therefore x=\frac{ac}{b}\text{ or }\frac{bc}{a}.$$

(91). $abx^2+2\sqrt{(ab)}\,(a+b)\,x+(a-b)^2=0$,

$$x^2+\frac{2(a+b)x}{\sqrt{(ab)}}+\frac{(a+b)^2}{ab}=-\frac{(a-b)^2}{ab}+\frac{(a+b)^2}{ab}=\frac{4ab}{ab};$$

$$\therefore x=-\frac{a+b}{\sqrt{(ab)}}\pm 2=-\frac{(\sqrt{a}\mp\sqrt{b})^2}{\sqrt{(ab)}}.$$

(92). $x-15=\sqrt{(x)}+5$,

$$x-\sqrt{(x)}+\frac{1}{4}=20+\frac{1}{4}=\frac{81}{84};\ \therefore \sqrt{(x)}=\frac{1}{2}\pm\frac{9}{2},\text{ and } x=25\text{ or }16.$$

(93). $\sqrt{(x)}-2=x-8$,

$$x-\sqrt{(x)}+\frac{1}{4}=6+\frac{1}{4}=\frac{25}{4};\ \therefore \sqrt{x}=\frac{1}{2}\pm\frac{5}{2}=3\text{ or }-2,\text{ and } x=9\text{ or }4.$$

(94). $\sqrt{(28x^2+39x+5)}=30$,

$$28x^2+39x+5=900;$$

$$\therefore x^2+\frac{39x}{28}+\overline{\frac{39}{56}}\Big]^2=\frac{895}{28}+\frac{1521}{(56)^2}=\frac{101761}{(56)^2};$$

$$\therefore x=-\frac{39}{56}\pm\frac{319}{56}=5\text{ or }-6\tfrac{11}{28}.$$

(95). $x + 2 = \sqrt{\{4 + x\sqrt{(8 - x)}\}}$,

$$x^2 + 4x + 4 = 4 + x\sqrt{(8 - x)},$$

$$\text{or } x + 4 = \sqrt{(8 - x)}; \ \therefore \ x^2 + 8x + 16 = 8 - x,$$

$$\text{and } x^2 + 9x + \overline{\frac{9}{2}}\Big|^2 = -8 + \frac{81}{4} = \frac{49}{4};$$

$$\therefore \ x = -\frac{9}{2} \pm \frac{7}{2} = -1 \text{ or } -8.$$

(96). $\sqrt{(x^4 + 3x^2)} + x^4 = 6 - 3x^2$,

$$x^4 + 3x^2 + \sqrt{(x^4 + 3x^2)} + \frac{1}{4} = \frac{25}{4};$$

$$\therefore \ \sqrt{(x^4 + 3x^2)} = -\frac{1}{2} \pm \frac{5}{2} = 2 \text{ or } -3,$$

$$x^4 + 3x^2 + \frac{9}{4} = 4 + \frac{9}{4} \text{ or } 9 + \frac{9}{4};$$

$$\therefore \ x^2 = -\frac{3}{2} \pm \frac{5}{2} \text{ or } -\frac{3}{2} \pm \frac{\sqrt{(45)}}{2};$$

$$\therefore \ x = \pm 1 \text{ or } \pm 2\sqrt{(-1)} \text{ or } \frac{3}{2}\{-1 \pm \sqrt{(5)}\}.$$

(97). $2x^2 - 11x + 14\sqrt{(11x - 2x^2 + 2)} = 42$,

$$11x - 2x^2 + 2 - 14\sqrt{(11x - 2x^2 + 2)} + 49 = 9;$$

$$\therefore \ \sqrt{(11x - 2x^2 + 2)} = -7 \pm 3 = -4 \text{ or } -10,$$

$$\text{and } 11x - 2x^2 + 2 = 16 \text{ or } 100;$$

$$\therefore \ x^2 - \frac{11x}{2} + \frac{121}{16} = \frac{9}{16} \text{ or } \frac{663}{16};$$

$$\therefore \ x = 3\tfrac{1}{2} \text{ or } 2, \text{ or } \frac{11}{4} \pm \frac{1}{4}\sqrt{-663}.$$

(98). $x^3 + 11 + \sqrt{(x^2 + 11)} = 42$,

$$x^2 + 11 + \sqrt{(x^2 + 11)} + \frac{1}{4} = 42 + \frac{1}{4} = \frac{169}{4};$$

$$\therefore \ \sqrt{(x^2 + 11)} = -\frac{1}{2} \pm \frac{13}{2} = 6 \text{ or } -7,$$

$$\text{and } x^2 + 11 = 36 \text{ or } 49; \ \therefore \ x = \pm 5 \text{ or } \pm\sqrt{(38)}.$$

(99). $9x + 4 = 15x^2 + 2x\sqrt{(9x+4)}$,

$$9x + 4 - 2x\sqrt{(9x+4)} + x^2 = 16x^2;$$

$$\therefore \sqrt{(9x+4)} = x \pm 4x = 5x \text{ or } -3x;$$

$$\therefore 9x + 4 = 25x^2 \text{ or } 9x^2,$$

whence $\overline{25x^2 - 9x + \frac{9}{10}}\Big|^2 = \frac{481}{100}$; $\therefore x = \frac{9}{50} \pm \frac{\sqrt{(481)}}{50}$,

or $\overline{9x^2 - 9x + \frac{3}{2}}\Big|^2 = 4 + \frac{9}{4} = \frac{25}{4}$; $\therefore x = \frac{4}{3}$ or $-\frac{1}{3}$.

(100). $3x = 8\sqrt{(x+1)} - 7$,

$$9x^2 + 42x + 49 = 64x + 64;$$

$$\therefore \overline{x^2 - \frac{22x}{9} + \frac{11}{9}}\Big|^2 = \frac{15}{9} + \frac{121}{81} = \frac{256}{81}$$

$$\therefore x = \frac{11}{9} \pm \frac{16}{9} = 3 \text{ or } -\frac{5}{9}.$$

(101). $2x^2 - 2x + 6\sqrt{(2x^2 - 3x + 2)} = x + 14$,

$$2x^2 - 3x + 2 + 6\sqrt{(2x^2 - 3x + 2)} + 9 = 25;$$

$$\therefore \sqrt{(2x^2 - 3x + 2)} = 2 \text{ or } -8,$$

and $2x^2 - 3x + 2 = 4$ or 64,

whence $x^2 - \frac{3x}{2} + \frac{9}{16} = \frac{25}{16}$; $\therefore x = 2$ or $-\frac{1}{2}$,

also $x^2 - \frac{3x}{2} + \frac{9}{16} = \frac{505}{16}$; $\therefore x = \frac{1}{4}\{3 \pm \sqrt{(505)}\}$.

(102). $(a^2 - x^2) + a\sqrt{(a^2 - x^2)} = 12a^2$,

$$a^2 - x^2 + a\sqrt{(a^2 - x^2)} + \frac{a^2}{4} = \frac{49a^2}{4};$$

$$\therefore \sqrt{(a^2 - x^2)} = 3a \text{ or } -4a,$$

whence $x = 2a\sqrt{(-2)}$ or $a\sqrt{(-15)}$.

(103). $ax + a\sqrt{(x^2 - ax + b^2)} = x^2 + ab$,

$$x^2 - ax + b^2 - a\sqrt{(x^2 - ax + b^2)} + \frac{a^2}{4} = \frac{a^2}{4} - ab + b^2;$$

$$\therefore \sqrt{(x^2 - ax + b^2)} = a - b \text{ or } b;$$

$$\therefore\ x^2 - ax + \frac{a^2}{4} = \frac{a^2}{4} + a^2 - 2ab \text{ or } \frac{a^2}{4};$$

$$\therefore\ x = \frac{a}{2} \pm \frac{1}{2}\sqrt{(5a^2 - 8ab)} \text{ or } a.$$

(104). $x - 15 = \frac{x-9}{\sqrt{x}+3}$, $\because \frac{x-9}{\sqrt{x}+3} = \sqrt{x} - 3$;

$$\therefore x - 15 = \sqrt{x} - 3;\ \therefore\ x - \sqrt{x} + \frac{1}{4} = 12 + \frac{1}{4} = \frac{49}{4};$$

$$\therefore\ \sqrt{x} = 4 \text{ or } -3;\ \therefore\ x = 16 \text{ or } 9.$$

(105). $\frac{x-4}{\sqrt{(x)}+2} = x - 8$, $\because \frac{x-4}{\sqrt{x}+2} = \sqrt{x} - 2$;

$$\therefore\ \sqrt{x} - 2 = x - 8;\ \therefore\ x - \sqrt{x} + \frac{1}{4} = 6 + \frac{1}{4} = \frac{25}{4};$$

$$\therefore\ \sqrt{x} = 3 \text{ or } -2;\ \therefore\ x = 9 \text{ or } 4.$$

(106). $(a - 3n)x + n + 2\sqrt{(n^2x + nax^2)} = 0$,

$$(a - 3n)x + n = -2\sqrt{(n^2x + nax^2)};$$

$$\therefore\ (a - 3n)^2x^2 + 2nx(a - 3n) + n^2 = 4n^2x + 4nax^2,$$

whence $(a^2 - 10an + 9n^2)x^2 + 2nx(a - 5n) + n^2 = 0$;

$$\therefore x^2 + \frac{2nx(a-5n)}{(a-9n)(a-n)} + \frac{n^2(a-5n)^2}{(a^2-10an+9n^2)^2} = \frac{16n^4}{(a^2-10an+9n^2)^2}$$

$$\therefore\ x = \frac{n\{-(a-5)n \pm 4n\}}{(a-9n)(a-n)} = \frac{n}{9n-a} \text{ or } \frac{n}{n-a}.$$

(107). $x^2 - 3x = \sqrt{(x^2 - 3x + 5)} + 1$,

$$x^2 - 3x + 5 - \sqrt{(x^2 - 3x + 5)} + \frac{1}{4} = \frac{25}{4};$$

$$\therefore\ \sqrt{(x^2 - 3x + 5)} = \frac{1}{2} \pm \frac{5}{2} = 3 \text{ or } -2,$$

$$x^2 - 3x + \frac{9}{4} = +4 + \frac{9}{4}, \text{ or } -1 + \frac{9}{4} = \frac{25}{4} \text{ or } \frac{5}{4};$$

$$\therefore\ x = 4 \text{ or } -1, \text{ or } \frac{3 \pm \sqrt{(5)}}{2}.$$

(108). $3x^2 + 2\sqrt{(3x^2 + 3x)} = 48 - 3x$,

$$3x^2 + 3x + 2\sqrt{(3x^2 + 3x)} + 1 = 49;$$

$$\therefore\ \sqrt{(3x^2 + 3x)} = -1 \pm 7 = 6 \text{ or } -8;$$

$$\therefore x^2 + x + \frac{1}{4} = \frac{64}{3} + \frac{1}{4}, \text{ or } \frac{36}{3} + \frac{1}{4} = \frac{259}{12} \text{ or } \frac{49}{4};$$

$$\therefore x = -\frac{1}{2} \pm \frac{7}{2} = 3 \text{ or } -4, \text{ or } \frac{1}{2}\left\{-1 \pm \sqrt{\left(\frac{259}{3}\right)}\right\}.$$

(109). $x - 1 = 2\sqrt{(x - x^2)}$,

$$2\sqrt{\{x(1-x)\}} - (1-x) = 0; \therefore \sqrt{(1-x)} = 0, \text{ and } x = 1,$$

$$\text{and } 2\sqrt{(x)} = \sqrt{(1-x)}, \text{ or } 4x = 1 - x; \therefore x = \frac{1}{5}.$$

(110). $9x - 4x^2 + \sqrt{(4x^2 - 9x + 11)} = 5$,

$$4x^2 - 9x + 11 - \sqrt{(4x^2 - 9x + 11)} + \frac{1}{4} = 6 + \frac{1}{4} = \frac{25}{4};$$

$$\therefore \sqrt{(4x^2 - 9x + 11)} = \frac{1}{2} \pm \frac{5}{2} = 3 \text{ or } -2;$$

$$\therefore 4x^2 - 9x + 11 = 9 \text{ or } 4,$$

$$\text{whence } \overline{x^2 - \frac{9x}{4} + \frac{9}{8}}\Big|^2 = \frac{49}{64} \text{ or } -\frac{31}{64};$$

$$\therefore x = \frac{9}{8} \pm \frac{7}{8} = 2 \text{ or } \frac{1}{4}, \text{ or } \frac{1}{8}\{9 \pm \sqrt{(-31)}\}.$$

(111). $2x^2 + \sqrt{(x^2 - 9)} = x^2 + 21$,

$$x^2 - 9 + \sqrt{(x^2 - 9)} + \frac{1}{4} = 12 + \frac{1}{4} = \frac{49}{4};$$

$$\therefore \sqrt{(x^2 - 9)} = -\frac{1}{2} \pm \frac{7}{2} = 3 \text{ or } -4;$$

$$\therefore x^2 = 18 \text{ or } 25, \text{ and } x = \pm 5 \text{ or } \pm 3\sqrt{(2)}.$$

(112). $x\sqrt{(6 - x^2)} = x^2 - 6$,

$$x^2(6 - x^2) = x^4 - 12x^2 + 36;$$

$$\therefore x^4 - 9x^2 + \frac{81}{4} = -18 + \frac{81}{4} = \frac{9}{4};$$

$$\therefore x^2 = \frac{9}{2} \pm \frac{3}{2} = 6 \text{ or } 3;$$

$$\therefore x = \pm\sqrt{(6)} \text{ or } \pm\sqrt{(3)}.$$

(113). $25x^2 - 31 = 25\sqrt{(1-x^2)}$,

$$31 - 25x^2 + 25\sqrt{(1-x^2)} = 0;$$

$$\therefore 1 - x^2 + \sqrt{(1-x^2)} + \frac{1}{4} = -\frac{6}{25} + \frac{1}{4} = \frac{1}{100};$$

$$\therefore \sqrt{(1-x^2)} = -\frac{1}{2} \pm \frac{1}{10} = -\frac{2}{5} \text{ or } -\frac{3}{5};$$

$$\therefore 1 - x^2 = \frac{4}{25} \text{ or } \frac{9}{25}; \ \therefore x = \pm\frac{1}{5}\sqrt{(21)} \text{ or } \pm\frac{4}{5}.$$

(114). $x\sqrt{(6+x^2)} = 1 + x^2$,

$$x^2(6+x^2) = 1 + 2x^2 + x^4;$$

$$\therefore 4x^2 = 1, \text{ and } x = \pm\frac{1}{2}.$$

(115). $\dfrac{x + \sqrt{(x^2-9)}}{x - \sqrt{(x^2-9)}} = (x-2)^2$,

$$\{x + \sqrt{(x^2-9)}\}^2 = 9(x-2)^2; \ \therefore x + \sqrt{(x^2-9)} = \pm 3(x-2),$$

$$\text{if } \sqrt{(x^2-9)} = 2x - 6, \text{ then } x^2 - 9 = 4x^2 - 24x + 36;$$

$$\therefore x^2 - 8x + 16 = 1; \ \therefore x = 5 \text{ or } 3,$$

$$\text{if } \sqrt{(x^2-9)} = -4x + 6, \text{ then } x^2 - 9 = 16x^2 - 48x + 36,$$

$$\text{and } x = \frac{1}{5}\{8 \pm \sqrt{(-11)}\}.$$

(116). $ab + b\sqrt{(a^2-x^2)} = x^2$,

$$a^2 - x^2 + b\sqrt{(a^2-x^2)} + \frac{b^2}{4} = a^2 - ab + \frac{b^2}{4};$$

$$\therefore \sqrt{(a^2-x^2)} = -\frac{b}{2} \pm \left(a - \frac{b}{2}\right) = a - b \text{ or } a;$$

$$\therefore a^2 - x^2 = (a-b)^2 \text{ or } a^2,$$

$$\text{and } x = \pm\sqrt{(2ab - b^2)} \text{ or } 0.$$

(117). $\dfrac{x^2}{a - \sqrt{(a^2-x^2)}} - \dfrac{x^2}{a + \sqrt{(a^2-x^2)}} = a$,

$$x^2\{a + \sqrt{(a^2-x^2)}\} - x^2\{a - \sqrt{(a^2-x^2)}\} = ax^2;$$

$$\therefore 2\sqrt{(a^2-x^2)} = a, \text{ and } a^2 - x^2 = \frac{a^2}{4}; \ \therefore x = \pm\frac{a}{2}\sqrt{(3)}.$$

(118). $\dfrac{x+\sqrt{(a^2+x^2)}}{\sqrt{(a^2-x^2)}}=\dfrac{2a^2}{\sqrt{(a^4-x^4)}}$,

$$\frac{x+\sqrt{(a^2+x^2)}}{1}=\frac{2a^2}{\sqrt{(a^2+x^2)}},$$

$$\therefore\ x\sqrt{(a^2+x^2)}+a^2+x^2=2a^2,\ \text{or } x\sqrt{(a^2+x^2)}=a^2-x^2,$$

$$\text{whence } x^4-\frac{3a^2x^2}{2}+\frac{9a^4}{16}=-\frac{a^4}{2}+\frac{9a^2}{16}=\frac{a^2}{16};$$

$$\therefore\ x^2=\frac{3a^2}{4}\pm\frac{a^2}{4}=a^2 \text{ or } \frac{a^2}{2};\ \therefore\ x=\pm a \text{ or } \pm a\sqrt{\left(\frac{1}{2}\right)}.$$

(119). $x+a+2\sqrt{(ax)}=bx$,

$$(1-b)\,x+2\sqrt{(ax)}+a=0,$$

$$\therefore\ x+\frac{2\sqrt{(ax)}}{1-b}+\frac{a}{(1-b)^2}=\frac{-a}{1-b}+\frac{a}{(1-b)^2}=\frac{ab}{(1-b)^2};$$

$$\therefore\ \sqrt{x}=-\frac{\sqrt{a}\mp\sqrt{(ab)}}{1-b}=\frac{-\sqrt{a}}{1\pm\sqrt{b}};\ \therefore\ x=\frac{a}{(1\pm\sqrt{b})^2}.$$

(120). $\sqrt{(1+x+x^2)}+\sqrt{(1-x+x^2)}=mx$,

$$1+x+x^2+2\sqrt{(1+x^2+x^4)}+1-x+x^2=m^2x^2;$$

$$\therefore\ 2\sqrt{(1+x^2+x^4)}=(m^2-2)\,x^2-2,$$

$$\text{and } 4+4x^2+4x^4=(m^4-4m^2+4)\,x^4-4\,(m^2-2)\,x^2+4,$$

$$\text{whence } m^2\,(m^2-4)\,x^4-4\,(m^2-1)\,x^2=0;$$

$$\therefore\ x=\pm\frac{2}{m}\sqrt{\left(\frac{m^2-1}{m^2-4}\right)}.$$

(121). $nx=\{\sqrt{(1+x)}-1\}\,\{\sqrt{(1-x)}+1\}$,

$$nx\,\{\sqrt{(1+x)}+1\}=x\,\{\sqrt{(1-x)}+1\};$$

$$\therefore\ (n-1)=\sqrt{(1-x)}-n\sqrt{(1+x)},$$

$$\text{and } n^2-2n+1=1-x+n^2\,(1+x)-2n\sqrt{(1-x^2)},$$

$$\text{whence } (n^2-1)\,x+2n=2n\sqrt{(1-x^2)};$$

$$\therefore\ (n^2-1)^2\,x^2+4nx\,(n^2-1)+4n^2=4n^2\,(1-x^2);$$

$$\therefore\ (n^4-2n^2+1+4n^2)\,x=4n\,(1-n^2),$$

$$\text{or } x=\frac{4n\,(1-n^2)}{(n^2+1)^2}.$$

(122). $(x+a)^{\frac{3}{2}} = a^{\frac{1}{2}}(3x-a)$,

$$(x+a)^3 = a(3x-a)^2,$$

$$\text{or } x^3 + 3ax^2 + 3a^2x + a^3 = 3ax^2 - 6a^2x + a^3;$$

$$\therefore\ x^2 + 9a^2 = 0;\ \therefore\ x = \pm 3a\sqrt{(-1)}.$$

(123). $\dfrac{x - \sqrt{(x+1)}}{x + \sqrt{(x+1)}} = \dfrac{5}{11}$,

$$11x - 11\sqrt{(x+1)} = 5x + 5\sqrt{(x+1)};$$

$$\therefore\ 3x = 8\sqrt{(x+1)},\ \text{and } 9x^2 = 64x + 64,$$

$$\text{and } \overline{x^2 - \frac{64x}{9} + \frac{32}{9}}\Big|^2 = \frac{64}{9} + \frac{1024}{81} = \frac{1600}{81};$$

$$\therefore\ x = \frac{32}{9} \pm \frac{40}{9} = 8 \text{ or } -\frac{8}{9}.$$

(124). $\dfrac{1 + \sqrt{(1-x^2)}}{1 - \sqrt{(1-x^2)}} = \dfrac{1}{x^2}$,

$$\frac{\{1 + \sqrt{(1-x^2)}\}^2}{x^2} = \frac{1}{x^2};\ \therefore\ 1 + \sqrt{(1-x^2)} = 1,$$

$$\text{and } \sqrt{(1-x^2)} = 0;\ \therefore\ x = \pm 1,$$

$$\text{also } 1 + x^2 = \sqrt{(1-x^2)};\ \therefore\ x = \pm\sqrt{(-3)}.$$

(125). $\dfrac{\sqrt{(1+x)}}{\sqrt{(1-x)}} = \dfrac{1 + \sqrt{(1+x)}}{1 - \sqrt{(1-x)}}$,

$$\sqrt{(1+x)} - \sqrt{(1-x^2)} = \sqrt{(1-x)} + \sqrt{(1-x^2)}$$

$$\therefore\ \sqrt{(1+x)} - \sqrt{(1-x)} = 2\sqrt{(1-x^2)},$$

$$\text{whence } 2 - 2\sqrt{(1-x^2)} = 4(1-x^2);$$

$$\therefore\ 2x^2 - 1 = \sqrt{(1-x^2)};$$

$$\therefore\ 4x^4 - 4x^2 + 1 = 1 - x^2;$$

$$\therefore\ x = \pm\frac{1}{2}\sqrt{(3)}.$$

(126). $\dfrac{ax + 1 + \sqrt{(a^2x^2-1)}}{ax + 1 - \sqrt{(a^2x^2-1)}} = \dfrac{b}{x}$;

$$\therefore\ \frac{ax+1}{\sqrt{(a^2x^2-1)}} = \frac{b+x}{b-x},\ \text{but } \frac{ax+1}{\sqrt{(a^2x^2-1)}} = \frac{\sqrt{(ax+1)}}{\sqrt{(ax-1)}};$$

$$\therefore \frac{ax+1}{ax-1} = \overline{\frac{b+x}{b-x}}\Big|^2, \text{ and } x = \frac{b^2+x^2}{2bx},$$

$$\text{or } 2abx^2 = b^2 + x^2; \ \therefore \ x = \frac{\pm b}{\sqrt{(2ab-1)}}.$$

(127). $\dfrac{x+\sqrt{(x^2-a^2)}}{x-\sqrt{(x^2-a^2)}} = b^2,$

$$\frac{x}{\sqrt{(x^2-a^2)}} = \frac{b^2+1}{b^2-1}, \text{ or } \frac{x^2}{x^2-a^2} = \overline{\frac{b^2+1}{b^2-1}}\Big|^2;$$

$$\therefore \frac{x^2}{a^2} + \frac{(b^2+1)^2}{4b^2}; \ \therefore \ x = \pm \frac{a}{2b}(b^2+1).$$

(128). $\dfrac{\sqrt{(a+bx^n)}+\sqrt{(a-bx^n)}}{\sqrt{(a+bx^n)}-\sqrt{(a-bx^n)}} = m,$

$$\frac{\sqrt{(a+bx^n)}}{\sqrt{(a-bx^n)}} = \frac{m+1}{m-1}; \ \therefore \ \frac{a+bx^n}{a-bx^n} = \overline{\frac{m+1}{m-1}}\Big|^2,$$

$$\text{and } \frac{bx^n}{a} = \frac{2m}{m^2+1}; \ \therefore \ x = \left\{\frac{2am}{b(m^2+1)}\right\}^{\frac{1}{n}}.$$

(129). $\dfrac{a+x+\sqrt{(a^2-x^2)}}{a+x-\sqrt{(a^2-x^2)}} = \dfrac{b}{x},$

$$\frac{a+x}{\sqrt{(a^2-x^2)}} = \frac{b+x}{b-x}, \text{ but } \frac{a+x}{\sqrt{(a^2-x^2)}} = \sqrt{\left(\frac{a+x}{a-x}\right)};$$

$$\therefore \frac{a+x}{a-x} = \overline{\frac{b+x}{b-x}}\Big|^2, \text{ and } \frac{x}{a} = \frac{2bx}{b^2+x^2}; \ \therefore \ b^2+x^2 = 2ab,$$

$$\text{and } x = \pm\sqrt{(2ab-b^2)}.$$

(130). $\dfrac{x}{x+4} + \dfrac{4}{\sqrt{(x+4)}} = \dfrac{21}{x},$

$$\frac{x^2}{x+4} + \frac{4x}{\sqrt{(x+4)}} + 4 = 25; \ \therefore \ \frac{x}{\sqrt{(x+4)}} = -2 \pm 5 = 3 \text{ or } -7;$$

$$\therefore \ x^2 = 9x + 36, \text{ or } 49x + 196,$$

$$\text{whence } x^2 - 9x + \frac{81}{4} = 36 + \frac{81}{4} = \frac{225}{4}; \ \therefore x = \frac{9}{2} \pm \frac{15}{2} = 12 \text{ or } -3,$$

$$\text{or } x^2 - 49x + \overline{\frac{49}{2}}\Big|^2 = 196 + \frac{1401}{4}; \ \therefore x = \frac{49}{2} \pm \frac{1}{2}\sqrt{(2185)}.$$

(131). $\dfrac{x-18}{x^{\frac{1}{2}}-18^{\frac{1}{2}}}+\dfrac{(x-18)^{\frac{1}{2}}}{x^{\frac{1}{2}}}+\dfrac{x^{\frac{1}{2}}}{(x-18)^{\frac{1}{2}}}=x^{\frac{1}{2}}$,

$$\text{since } (x-18)\div(x^{\frac{1}{2}}-18)^{\frac{1}{2}}=x^{\frac{1}{2}}+18^{\frac{1}{2}};$$

$$\therefore\ \frac{x-18}{x}+18^{\frac{1}{2}}\frac{\sqrt{(x-18)}}{x^{\frac{1}{2}}}+\frac{9}{2}=-1+\frac{9}{2}=\frac{7}{2},$$

$$\text{and } \sqrt{(x-18)}=x^{\frac{1}{2}}\left\{-\frac{3}{\sqrt{2}}\pm\frac{\sqrt{(7)}}{\sqrt{2}}\right\};$$

$$\therefore\ x-18=x(8\mp 3\sqrt{7});$$

$$\therefore\ x=\frac{-18}{7\mp 3\sqrt{7}}=9\pm\frac{27}{\sqrt{(7)}}.$$

(132). $\dfrac{250}{7x\sqrt{(x^2-9)}}+\dfrac{\sqrt{(x^2-9)}}{7x}-\dfrac{19}{2x}=0$,

$$x^2-9-\frac{133}{2}\sqrt{(x^2-9)}+\overline{\frac{133}{4}}\Big|^2=-250+\frac{17689}{16}=\frac{13689}{16};$$

$$\therefore\ \sqrt{(x^2-9)}=\frac{133}{4}\pm\frac{117}{4}=\frac{125}{2}\text{ or }4;$$

$$\therefore\ x^2-9=16\text{ or }\overline{\frac{125}{2}}\Big|^2;$$

$$\therefore\ x=\pm 5\text{ or }\pm\frac{1}{2}\sqrt{(15161)}.$$

(133). $\dfrac{x+\sqrt{(x)}}{x^2-x}=\dfrac{x-\sqrt{(x)}}{4}$,

$$\text{whence } \frac{1}{x-\sqrt{x}}=\frac{x-\sqrt{x}}{4};\ \therefore\ x-\sqrt{x}=\pm 2,$$

$$\text{and } x-\sqrt{x}+\frac{1}{4}=\pm 2+\frac{1}{4}=\frac{9}{4}\text{ or }-\frac{7}{4};$$

$$\therefore\ \sqrt{x}=2\text{ or }-1,\text{ or }\frac{1}{2}\{1\pm\sqrt{(-7)}\};$$

$$\therefore\ x=4,\ 1\text{ or }\frac{1}{2}\{-3\pm\sqrt{(-7)}\}.$$

(134). $x + 4 + \sqrt{\left(\frac{x+4}{x-4}\right)} = \frac{12}{x-4}$,

$$x^2 - 16 + \sqrt{(x^2 - 16)} + \frac{1}{4} = 12 + \frac{1}{4} = \frac{49}{4};$$

$$\therefore\ \sqrt{(x^2 - 16)} = 3 \text{ or } -4,$$

$$\text{and } x^2 - 16 = 9 \text{ or } 16;\ \therefore\ x = \pm 5 \text{ or } \pm 4\sqrt{(2)}.$$

(135). $x^3 - 3x^{\frac{3}{2}} = 40$,

$$x^3 - 3x^{\frac{3}{2}} + \frac{9}{4} = 40 + \frac{9}{4} = \frac{169}{4};$$

$$\therefore\ x^{\frac{3}{2}} = \frac{3}{2} \pm \frac{13}{2} = 8 \text{ or } -5,$$

$$\text{whence } x = 4 \text{ or } (-5)^{\frac{2}{3}}.$$

(136). $\frac{5(x-4)}{\sqrt{(x)} + 2} = (x^{\frac{3}{2}} - 8)$,

$$\therefore\ 5(x^{\frac{1}{2}} - 2) = x^{\frac{3}{2}} - 8 \text{ or } 5 = x + x^{\frac{1}{2}} + 4,$$

$$\text{and } x^{\frac{1}{2}} = 2, \text{ and } x = 4,$$

$$\text{also } x + x^{\frac{1}{2}} + \frac{1}{4} = 1 + \frac{1}{4} = \frac{5}{4},$$

$$\text{and } \sqrt{(x)} = \frac{1}{2}(-1 \pm \sqrt{5});\ \therefore\ x = \frac{1}{2}\{3 \mp 2\sqrt{(5)}\}.$$

(137). $x^3 - 1 = 2x^2 - 2x$,

$$x^3 - 1 = 2(x-1)x;\ \therefore\ x^2 + x + 1 = 2x, \text{ and } x = 1,$$

$$\text{also } x^2 - x + \frac{1}{4} = -1 + \frac{1}{4} = -\frac{3}{4};$$

$$\therefore\ x = \frac{1}{2}\{1 \pm \sqrt{(-3)}\}.$$

(138). $x^3 + 1 = 2x^2$,

$$x^2(x-1) - (x^2 - 1) = 0;\ \therefore\ x = 1,$$

$$\text{and } x^2 - x - 1 = 0, \text{ or } x^2 - x + \frac{1}{4} = \frac{5}{4};\ \therefore\ x = \frac{1}{2}(1 \pm \sqrt{5}).$$

(139). $x^3 - 6x^2 + 10x = 5$;

$$\therefore\ x^2(x-1) - 5x(x-1) + 5(x-1) = 0;\ \therefore\ x = 1,$$

$$\text{and } x^2 - 5x + \frac{25}{4} = -5 + \frac{25}{4} = \frac{5}{4};$$

$$\therefore\ x = \frac{1}{2}(5 \pm \sqrt{5}).$$

(140). $x^2 + \dfrac{64}{x^2} - \dfrac{5x-30}{2} = \dfrac{25x^2}{16}$,

$$x^2 + \frac{64}{x^2} - \frac{5x}{2} - \frac{25x^2}{16} + 15 = 0,$$

$$\text{whence } x^2 + 16 + \frac{64}{x^2} = \frac{25x^2}{16} + \frac{5x}{2} + 1,$$

$$\text{or } x + \frac{8}{x} = \pm\left(\frac{5x}{4} + 1\right),$$

$$\text{whence } x^2 + 4x + 4 = 36, \text{ or } x = 4 \text{ or } -8,$$

$$\text{also } x = \frac{2}{9}\{-1 \pm \sqrt{(71)}\}.$$

(141). $8x^4 + 8x = 16x^3 + 2$,

$$x^4 - 2x^3 + x^2 = x^2 - x + \frac{1}{4};$$

$$\therefore\ x^2 = x \pm \left(x - \frac{1}{2}\right) = 2x - \frac{1}{2} \text{ or } \frac{1}{2};$$

$$\therefore\ x^2 - 2x + 1 = \frac{1}{2};\ \therefore\ x = 1 \pm \sqrt{\frac{1}{2}} \text{ or } \pm\sqrt{\left(\frac{1}{2}\right)}.$$

(142). $4x^4 - 4x^3 + 4 = 2x - 5x^2$,

$$4x^4 - 4x^3 + x^2 + 2(2x^2 - x) + 4 = 0;$$

$$\therefore\ 2x^2 - x = -2,$$

$$\text{and } x^2 - \frac{x}{2} + \frac{1}{16} = -1 + \frac{1}{16};$$

$$\therefore\ x = \frac{1}{4}\{1 \pm \sqrt{(-15)}\}.$$

(143). $x^4 - 2x^3 + x - 1 = 0,$

$$x^4 - 2x^3 + x^2 - (x^2 - x) + \frac{1}{4} = 1 + \frac{1}{4} = \frac{5}{4};$$

$$\therefore\ x^2 - x = \frac{1}{2} \pm \frac{\sqrt{5}}{2},$$

$$\text{and } x^2 - x + \frac{1}{4} = \frac{3}{4} \pm \frac{\sqrt{5}}{2};$$

$$\therefore\ x = \frac{1}{2} \pm \frac{1}{2}\{\sqrt{(3 \pm 2\sqrt{5})}\}.$$

(144). $x^4 - 2x^3 + x - 132 = 0,$

$$x^4 - 2x^3 + x^2 - (x^2 - x) + \frac{1}{4} = 132 + \frac{1}{4} = \frac{529}{4};$$

$$\therefore\ x^2 - x = +\frac{1}{2} \pm \frac{23}{2} = 12 \text{ or } -11,$$

$$x^2 - x + \frac{1}{4} = \frac{49}{4} \text{ or } -\frac{43}{4};\ \therefore\ x = 4 \text{ or } -3, \text{ or } \frac{1 \pm \sqrt{(-43)}}{2}.$$

(145). $x^{12} - 1 = 0,$

$$\because\ x^{12} - 1 = 0;\ \therefore\ (x^6 - 1)(x^6 + 1) = 0,$$

$$\text{whence } (x^3 + 1)(x^3 - 1) = 0, \text{ or } (x^2 - x + 1)(x + 1) = 0,$$

$$\text{also } (x^2 + x + 1)(x - 1) = 0, \text{ whence } x = -1 \text{ or } 1.$$

$$\text{and from } x^2 + x + 1 = 0,\ x = \frac{1}{2}\{-1 \pm \sqrt{(-3)}\},$$

$$\text{from } x^2 - x + 1 = 0,\ x = \frac{1}{2}\{1 \pm \sqrt{(-3)}\},$$

$$\text{also from } x^6 + 1 = 0,\ (x^4 - x^2 + 1)(x^2 + 1) = 0;$$

$$\therefore\ x = \pm\sqrt{(-1)},$$

$$\text{also from } x^4 - x^2 + 1 = 0,\ x^2 = \frac{1}{2}\{1 \pm \sqrt{(-3)}\};\ \therefore\ x = \pm\sqrt{\left\{\frac{1 \pm \sqrt{(-3)}}{2}\right\}}.$$

(146). $2x^3 - x^2 = 1,$

$$\therefore\ x^3 - 1 + x^2(x - 1) = 0;$$

$$\therefore\ x^2 + x + 1 + x^2 = 0, \text{ and } x = 1,$$

$$\text{also } \overline{x^2 + \frac{x}{2} + \frac{1}{4}}\Big|^2 = -\frac{1}{2} + \frac{1}{16} = -\frac{7}{16};$$

$$\therefore\ x = \frac{1}{4}\{-1 \pm \sqrt{(-7)}\}.$$

(147). $x^3 + px^2 + px + 1 = 0$,

$$x^3 + 1 + px(x+1) = 0; \quad \therefore x = -1,$$

$$\text{and } x^2 - x + 1 + px = 0,$$

$$\text{or } \overline{x^2 + (p-1)x + \frac{p-1}{2}}\Big]^2 = -1 + \frac{(p-1)^2}{4} = \frac{(p-1)^2 - 4}{4};$$

$$\therefore x = -\frac{1}{2}\left[(p-1) \pm \sqrt{\{(p-1)^2 - 4\}}\right].$$

(148). $x^3 - px - p + 1 = 0$,

$$x^3 + 1 - p(x+1) = 0; \quad \therefore x + 1 = 0, \text{ and } x = -1,$$

$$\text{also } x^2 - x + 1 - p = 0;$$

$$\therefore \text{ whence } x = \frac{1}{2}\{1 \pm \sqrt{(4p-3)}\}.$$

(149). $x^4 - 8x^3 + 10x^2 + 24x + 5 = 0$,

$$x^4 - 8x^3 + 16x^2 - 6(x^2 - 4x) + 9 = 4;$$

$$\therefore x^2 - 4x = 3 \pm 2 = 5 \text{ or } 1,$$

$$\text{whence } x^2 - 4x + 4 = 9, \text{ and } x = 5 \text{ or } -1,$$

$$\text{also } x = 2 \pm \sqrt{(5)}.$$

(150). $x^4 + 5x^3 + 2x^2 + 5x + 1 = 0$,

$$x^2 + 5x + 2 + \frac{5}{x} + \frac{1}{x^2} = 0,$$

$$\therefore x^2 + 2 + \frac{1}{x^2} + 5\left(x + \frac{1}{x}\right) = 0;$$

$$\therefore x + \frac{1}{x} = 0, \text{ and } x = \pm\sqrt{(-1)},$$

$$\text{also } x + \frac{1}{x} + 5 = 0, \text{ whence } x = \frac{1}{2}\{-5 \pm \sqrt{(21)}\}.$$

(151). $x^4 - \frac{5x^3}{2} + 2x^2 - \frac{5x}{2} + 1 = 0$,

$$x^2 - \frac{5x}{2} + 2 - \frac{5}{2x} + \frac{1}{x^2} = 0;$$

$$\therefore x^2 + 2 + \frac{1}{x^2} - \frac{5}{2}\left(x + \frac{1}{x}\right) = 0;$$

$$\therefore\ x + \frac{1}{x} = 0, \text{ and } x = \pm\sqrt{(-1)},$$

$$\text{also } x + \frac{1}{x} = \frac{5}{2};\ \therefore x = 2 \text{ or } \frac{1}{2}.$$

(152). $x^4 - \frac{5x^3}{2} + 3x^2 - \frac{5x}{2} + 1 = 0,$

$$x^2 - \frac{5x}{2} + 3 - \frac{5}{2x} + \frac{1}{x^2} = 0,$$

$$x^2 + 2 + \frac{1}{x^2} - \frac{5}{2}\left(x + \frac{1}{x}\right) + \frac{25}{16} = -1 + \frac{25}{16} = \frac{9}{16};$$

$$\therefore\ x + \frac{1}{x} = \frac{5}{4} \pm \frac{3}{4} = 2 \text{ or } \frac{1}{2},$$

$$\text{whence } x = 1, \text{ or } \frac{1}{4}\{1 \pm \sqrt{(-15)}\}.$$

(153). $x^4 + x^3 + x + 1 = 4x^2,$

$$x^2 + x + \frac{1}{x} + \frac{1}{x^2} = 4;$$

$$\therefore x^2 + 2 + \frac{1}{x^2} + \left(x + \frac{1}{x}\right) + \frac{1}{4} = 6 + \frac{1}{4} = \frac{25}{4};$$

$$\therefore\ x + \frac{1}{x} = -\frac{1}{2} \pm \frac{5}{2} = 2 \text{ or } -3,$$

$$\text{whence } x = 1, \text{ or } x = \frac{3}{2}(1 \pm \sqrt{5}).$$

(154). $a + x + \sqrt{(2ax + x^2)} = \frac{15x^2 + 6ax}{4\sqrt{(2ax + x^2)}},$

$$4(a + x)\sqrt{(2ax + x^2)} + 4(2ax + x^2) = 15x^2 + 6ax,$$

$$\text{whence } 16(a + x)^2(2ax + x^2) = 121x^4 - 44ax^3 + 4a^2x^2,$$

$$\text{and } 105x^4 - 108ax^3 - 76a^2x^2 - 32a^3x = 0,$$

$$\text{whence } 21x^2(5x - 8a) + 12ax(5x - 8a) + 4a^2(5x - 8a) = 0,$$

$$\therefore\ x = \frac{8a}{5},$$

$$\text{and } x^2 + \frac{4ax}{7} + \frac{4a^2}{49} = -\frac{4a^2}{21} + \frac{4a^2}{49} = -\frac{16a^2}{63};$$

$$\therefore x = -\frac{2a}{7} \pm \frac{4a\sqrt{(-1)}}{7\sqrt{3}} = \frac{2a}{7}\left(-1 \pm \frac{2\sqrt{(-1)}}{3}\right).$$

(155). $\frac{x}{\sqrt{(x)} + \sqrt{(a-x)}} + \frac{x}{\sqrt{(x)} - \sqrt{(a-x)}} = \frac{b}{\sqrt{(x)}}$,

$$x^{\frac{3}{2}} \{\sqrt{(x)} - \sqrt{(a-x)} + \sqrt{(x)} + \sqrt{(a-x)}\} = b\,(2x-a),$$

$$\text{or } x^2 - bx + \frac{b^2}{4} = -\frac{ab}{2} + \frac{b^2}{4};$$

$$\therefore\ x = \frac{1}{2}\{b \pm \sqrt{(b^2 - 2ab)}\}.$$

(156). $\sqrt{(1+a)}\sqrt{(1-x)} - \sqrt{(1-a)}\sqrt{(1+x)} = 2a$,

$$(1+a)(1-x) - 2\sqrt{(1-a^2)}\sqrt{(1-x^2)} + (1-a)(1+x) = 4a^2,$$

$$\text{whence } (1-2a^2) - ax = \sqrt{(1-a^2)}\sqrt{(1-x^2)},$$

$$\text{and } \overline{1-2a^2}]^2 - 2ax(1-2a^2) + a^2x^2 = 1 - a^2 - (1-a^2)x^2,$$

$$x^2 - 2ax(1-2a^2) + a^2(1-2a^2)^2 = 1 - a^2 - (1-2a^2)^2 + a^2(1-2a^2)^2$$

$$= 4a^2(1-a^2)(1-a^2);$$

$$\therefore x = a(1-2a^2) \pm 2a(1-a^2) = a(3-4a^2) \text{ or } -a.$$

(157). $\sqrt{(x^4-1)} + \sqrt{(x^2-1)} = x^3$,

$$\sqrt{(x^2-1)}\{\sqrt{(x^2+1)} + 1\} = x^3,$$

$$\text{whence } \sqrt{(x^2-1)}(x^2) = x^3\{\sqrt{(x^2+1)} - 1\};$$

$$\therefore\ \sqrt{(x^2-1)} - x\sqrt{(x^2+1)} = -x,$$

$$\text{or } x^2 - 1 + x^2(x^2+1) - 2x\sqrt{(x^4-1)} = x^2,$$

$$\text{and } x^4 - 1 - 2x\sqrt{(x^4-1)} + x^2 = 0;$$

$$\therefore\ \sqrt{(x^4-1)} = x,$$

$$\text{when } x^4 - x^2 + \frac{1}{4} = \frac{5}{4}, \text{ and } x^2 = \frac{1 \pm \sqrt{5}}{2};$$

$$\therefore x = \pm\sqrt{\left(\frac{1 \pm \sqrt{5}}{2}\right)}.$$

(158). $\sqrt[5]{\{(a+x)^2\}} + \sqrt[5]{\{(a-x)^2\}} - 3\sqrt[5]{(a^2-x^2)} = 0$,

$$(a+x)^{\frac{2}{5}} - 3(a^2-x^2)^{\frac{1}{5}} + \frac{9}{4}(a-x)^{\frac{2}{5}} = \left(-1 + \frac{9}{4}\right)(a-x)^{\frac{2}{5}};$$

$$\therefore (a+x)^{\frac{1}{5}} = \frac{(a-x)^{\frac{1}{5}}}{2}\{3 \pm \sqrt{(5)}\};$$

$$\therefore \frac{a+x}{a-x} = \left(\frac{3 \pm \sqrt{5}}{2}\right)^5,$$

$$x = \frac{(3 \pm \sqrt{5})^5 - 2^5}{(3 \pm \sqrt{5})^5 + 2^5}\, a.$$

(159). $2x^{\frac{3}{2}}(a^3 + x^3)^{\frac{1}{2}} - 2x^2(x + 2a) = a^2(x - a),$

$$a^3 + x^3 + 2x^{\frac{3}{2}}(a^3 + x^3)^{\frac{1}{2}} + x^3 = 4x^3 + 4ax^2 + a^2x,$$

$$\text{whence } \sqrt{(a^3 + x^3)} = -x^{\frac{3}{2}} \pm x^{\frac{1}{2}}(2x + a)$$

$$= x^{\frac{3}{2}} + ax^{\frac{1}{2}}, \text{ or } -(3x^{\frac{3}{2}} + ax^{\frac{1}{2}});$$

$$\therefore a^3 + x^3 = x^3 + 2ax^2 + a^2x, \text{ or } 9x^3 + 6ax^2 + a^2x,$$

$$\text{and } x^2 + \frac{ax}{2} + \frac{a^2}{16} = \frac{9a^2}{16}; \ \therefore x = \frac{a}{2} \text{ or } -a,$$

and the second part produces a cubic.

(160). $\frac{1}{x} + \frac{1}{a} = \sqrt{\left\{\frac{1}{a^2} + \sqrt{\left(\frac{1}{a^2x^2} + \frac{5}{x^4}\right)}\right\}};$

$$\therefore \frac{1}{x^2} + \frac{1}{a^2} + \frac{2}{ax} = \frac{1}{a^2} + \frac{1}{x}\sqrt{\left(\frac{1}{a^2} + \frac{5}{x^2}\right)},$$

$$\text{or } \frac{1}{x} + \frac{2}{a} = \sqrt{\left(\frac{1}{a^2} + \frac{5}{x^2}\right)}; \ \therefore \frac{1}{x^2} + \frac{4}{a^2} + \frac{4}{ax} = \frac{1}{a^2} + \frac{5}{x^2};$$

$$\therefore \frac{4}{x^2} - \frac{4}{ax} + \frac{1}{a^2} = \frac{4}{a^2};$$

$$\therefore \frac{2}{x} = \frac{1}{a} \pm \frac{2}{a} = \frac{3}{a} \text{ or } -\frac{1}{a}; \ \therefore x = \frac{2a}{3} \text{ or } -2a.$$

(161). $\frac{x^2}{8a} - \frac{2a}{3} = \sqrt{\left(\frac{x^3}{3a} + \frac{x^2}{4}\right)} - \frac{4\sqrt{ax}}{3} - \frac{3x}{32},$

$$\frac{x^2}{8a} - \frac{2a}{3} = \sqrt{\left(\frac{x^3}{3a} + \frac{x^2}{4}\right)} - \frac{4\sqrt{(ax)}}{3} - \frac{3x}{32},$$

$$\text{whence } \frac{x}{3a} + \frac{1}{4} - \frac{8}{3}\sqrt{\left(\frac{x}{3a} + \frac{1}{4}\right)} + \frac{16}{9} = \frac{16}{9} - \frac{32\sqrt{a}}{9\sqrt{x}} + \frac{16a}{9x};$$

$$\therefore \sqrt{\left(\frac{x}{3a} + \frac{1}{4}\right)} = \frac{4}{3} \pm \frac{4}{3}\left(\sqrt{\frac{a}{x}} - 1\right) = \frac{4}{3}\sqrt{\frac{a}{2}} \text{ or } \frac{8}{3} - \frac{4}{3}\sqrt{\frac{a}{x}},$$

$$\text{from the first } x = \frac{a}{8}\left\{-3 \pm \sqrt{\left(\frac{1051}{3}\right)}\right\},$$

or let $x = az$, then the equation reduces to

$$z\,\{(12z+9)^{\frac{1}{2}} - 8\}^2 = 64\,(1 - z^{\frac{1}{2}})^2,$$

whence $z^{\frac{1}{2}}\,(12z+9)^{\frac{1}{2}} = 8$, and $x = az = \frac{a}{8}\left\{-3 \pm \sqrt{\left(\frac{1051}{3}\right)}\right\}$,

and the other values may be found from the equation

$$12z^2 - 247z + 256z^{\frac{1}{2}} - 64 = 0.$$

(162). $\sqrt{(3+\sqrt{x})} + \sqrt{(4-\sqrt{x})} = \sqrt{(7+2\sqrt{x})}$,

$$3 + \sqrt{x} + 4 - \sqrt{x} + 2\sqrt{(12 + \sqrt{x} - x)} = 7 + 2\sqrt{x}\,;$$

$$\therefore\ 4\,(12 + \sqrt{x} - x) = 4x,$$

$$\text{or } x - \frac{\sqrt{x}}{2} + \overline{\frac{1}{4}}\Big]^2 = 6 + \frac{1}{16} = \frac{97}{16}\,;$$

$$\therefore\ \sqrt{x} = \frac{1}{4}\{1 \pm \sqrt{(97)}\},$$

$$\text{and } x = \frac{1}{8}\{49 \pm \sqrt{(97)}\}.$$

(163). $(a+x)\sqrt{(a^2+x^2)} = 6\,(a-x)^2$,

$$(a+x)^2\,(a^2+x^2) = 36\,(a^2+x^2)^2 - 144ax\,(a^2+x^2) + 144a^2x^2,$$

$$35\,(a^2+x^2)^2 - 146ax\,(a^2+x^2) + 144a^2x^2 = 0\,;$$

$$\therefore\ (a^2+x^2)^2 - \frac{146ax}{35}(a^2+x^2) + \overline{\frac{73ax}{35}}\Big]^2$$

$$= -\frac{144a^2x^2}{35} + \frac{5329a^2x^2}{(35)^2} = \frac{289a^2x^2}{(35)^2}\,;$$

$$\therefore\ a^2 + x^2 = \frac{73ax}{35} \pm \frac{17ax}{35} = \frac{18ax}{7} \text{ or } \frac{8ax}{5}\,;$$

$$\therefore\ x^2 - \frac{18ax}{7} + \overline{\frac{9a}{7}}\Big]^2 = -a^2 + \frac{81a^2}{49} = \frac{32a^2}{49}\,;$$

$$\therefore\ x = \frac{a}{7}\{9 \pm 4\sqrt{(2)}\},$$

$$\text{also } x^2 - \frac{8ax}{5} + \frac{16a^2}{25} = -a^2 + \frac{16a^2}{25} = -\frac{9a^2}{25}\,;$$

$$\therefore\ x = \frac{a}{5}\{4 \pm 3\sqrt{(-1)}\}.$$

(164). $\sqrt{(x^2+1)} - \sqrt{(x^2-1)} = x(x^4-1)^{-\frac{1}{2}}$,

$$\sqrt{(x^2+1)} - \sqrt{(x^2-1)} = \frac{x}{\sqrt{(x^4-1)}};$$

$$\therefore\ (x^2+1)\sqrt{(x^2-1)} - (x^2-1)\sqrt{(x^2+1)} = x,$$

$$\text{or } (x^4-1)\{x^2+1+x^2-1-2\sqrt{(x^4-1)}\} = x^2;$$

$$\therefore\ 2(x^4-1)\{x^2-\sqrt{(x^4-1)}\} = x^2;$$

$$\therefore\ 2(x^4-1) = x^2\{x^2+\sqrt{(x^4-1)}\},$$

$$\text{or } x^4-2 = x^2\sqrt{(x^4-1)},$$

$$x^8-4x^4+4 = x^8-x^4;$$

$$\therefore\ x^4 = \frac{4}{3};\ \therefore\ x^2 = \pm\frac{2}{\sqrt{3}},\ \text{and } x = \pm\sqrt{\left(\pm\frac{2}{\sqrt{3}}\right)}.$$

(165). $\dfrac{16-4\sqrt{(x)}}{8-3\sqrt{(x)}} = \dfrac{88+33\sqrt{(x)}}{4+\sqrt{(x)}} + \dfrac{x^2-5x+11}{\{8-3\sqrt{(x)}\}\{4+\sqrt{(x)}\}}$,

$$\frac{4(4-\sqrt{x})}{8-3\sqrt{x}} = \frac{11(8+3\sqrt{x})}{4+\sqrt{x}} + \frac{x^2-5x+11}{(8-3\sqrt{x})(4+\sqrt{x})},$$

$$\text{whence } 4(16-x) = 11(64-9x) + x^2 - 5x + 11;$$

$$\therefore\ x^2 - 100x + \overline{50}|^2 = -651 + 2500 = 1849;$$

$$\therefore\ x = 50 \pm 43 = 93 \text{ or } 7.$$

(166). $\dfrac{123+41\sqrt{(x)}}{\{5\sqrt{(x)}-x\}} + \dfrac{2x^2}{\{5\sqrt{(x)}-x\}\{3-\sqrt{(x)}\}} = \dfrac{20\sqrt{(x)}+4x}{3-\sqrt{(x)}}$,

$$\frac{41(3+\sqrt{x})}{(5\sqrt{x}-x)} + \frac{2x^2}{(5\sqrt{x}-x)(3-\sqrt{x})} = \frac{4(5\sqrt{x}+x)}{3-\sqrt{x}};$$

$$\therefore\ 41(9-x) + 2x^2 = 4(25x-x^2),$$

$$\text{and } x^2 - \frac{47x}{2} + \overline{\frac{47}{4}}\Big|^2 = -\frac{123}{2} + \frac{2209}{16} = \frac{1225}{16};$$

$$\therefore\ x = \frac{47}{4} \pm \frac{35}{4} = 20\tfrac{1}{2} \text{ or } 3.$$

(167). $\dfrac{54-9\sqrt{(x)}}{x+2\sqrt{(x)}} - \dfrac{7x^2-3x+4}{\{x+2\sqrt{(x)}\}\times\{6+\sqrt{(x)}\}} = \dfrac{23x-46\sqrt{(x)}}{6+\sqrt{(x)}}$,

$$\frac{9(6-\sqrt{x})}{x+2\sqrt{x}} - \frac{7x^2-3x+4}{(x+2\sqrt{x})(6+\sqrt{x})} = \frac{23(x-2\sqrt{x})}{(6+\sqrt{x})};$$

$$\therefore\ 9(36-x)-7x^2+3x-4=23(x^2-4x);$$

$$\therefore\ 30x^2-86x=320,$$

$$\text{whence } \overline{x^2-\frac{43x}{15}+\frac{43}{30}}\Big|^2=-\frac{320}{30}+\frac{1848}{900}=\frac{11449}{900};$$

$$\therefore\ x=\frac{43}{30}\pm\frac{107}{30}=5 \text{ or } -2\tfrac{2}{15}.$$

(168). $$\sqrt{\{x+\sqrt{(2x-1)}\}}-\sqrt{\{x-\sqrt{(2x-1)}\}}=\frac{3\sqrt{(10x)}}{5\sqrt{\{x+\sqrt{(2x-1)}\}}},$$

$$\therefore\ x+\sqrt{(2x-1)}-\sqrt{(x^2-2x+1)}=\frac{3\sqrt{(10x)}}{5},$$

$$\text{or } x+\sqrt{(2x-1)}-x+1=\frac{3\sqrt{(10x)}}{5};$$

$$\therefore\ 2x-1=\frac{18x}{5}-\frac{6}{5}\sqrt{(10x)}+1,$$

$$\text{whence } x-\frac{3}{4}\sqrt{(10x)}+\frac{90}{64}=-\frac{5}{4}+\frac{90}{64}=\frac{10}{64};$$

$$\therefore\ \sqrt{x}=\frac{1}{8}(3\sqrt{10}\pm\sqrt{10})=\frac{\sqrt{10}}{2} \text{ or } \frac{\sqrt{10}}{4};$$

$$\therefore\ x=\frac{5}{2} \text{ or } \frac{5}{8}.$$

(169). $$\frac{x}{a+x}+\frac{a}{\sqrt{(a+x)}}=\frac{6a^2}{x},$$

$$\frac{x^2}{a+x}+\frac{ax}{\sqrt{(a+x)}}+\frac{a^2}{4}=6a^2+\frac{a^2}{4}=\frac{25a^2}{4};$$

$$\therefore\ \frac{x}{\sqrt{(a+x)}}=-\frac{a}{2}\pm\frac{5a}{2}=2a \text{ or } -3a,$$

$$\text{whence } x^2=4a^2(a+x) \text{ or } 9a^2(a+x),$$

$$\text{or } x^2-4a^2x+4a^4=4a^3(a+1);\ \therefore\ x=2a^2\pm 2a\sqrt{(a^2+a)},$$

$$\text{or } x^2-9a^2x+\frac{81a^4}{4}=\frac{9a^2}{4}(9a^2+4a);\ \therefore\ x=\frac{9a}{2}\{a\pm\sqrt{(9a^2+4a)}\}.$$

(170). $$\sqrt{\left(12-\frac{12}{x^2}\right)}-x^2+\sqrt{\left(x^2-\frac{12}{x^2}\right)}=0,$$

$$\sqrt{\left(12-\frac{12}{x^2}\right)}=x^2-\sqrt{\left(x^2-\frac{12}{x^2}\right)};$$

$$\therefore\ 12 - \frac{12}{x^2} = x^4 - 2x^2\sqrt{\left(x^2 - \frac{12}{x^2}\right)} + x^2 - \frac{12}{x^2},$$

whence $x^4 - 12 - 2x\sqrt{(x^4 - 12)} + x^2 = 0$;

$\therefore\ \sqrt{(x^4 - 12)} = x$; $\therefore\ x = \pm 2$ or $\pm\sqrt{(-3)}$.

(171). $\dfrac{2x + \sqrt{x}}{2x - \sqrt{x}} = \dfrac{52}{15} - 3\,\dfrac{2x - \sqrt{x}}{2x + \sqrt{(x)}}$,

$$\overline{\frac{2\sqrt{x}+1}{2\sqrt{x}-1}}\Big|^2 - \frac{52}{15}\,\frac{2\sqrt{x}+1}{2\sqrt{x}-1} + \overline{\frac{26}{15}}\Big|^2 = -3 + \frac{676}{225} = \frac{1}{225};$$

$$\therefore\ \frac{2\sqrt{x}+1}{2\sqrt{x}-1} = \frac{26}{15} \pm \frac{1}{15} = \frac{9}{5} \text{ or } \frac{5}{3};$$

$$\therefore\ \sqrt{x} = \frac{7}{4} \text{ or } 2, \text{ and } x = \frac{49}{16} \text{ or } 4.$$

(172). $\sqrt[2pq]{(x^{p+q})} - \dfrac{1}{2}\,\dfrac{a^2 - b^2}{a^2 + b^2}\,(x^{\frac{1}{p}} + x^{\frac{1}{q}}) = 0$,

$$x^{\frac{1}{p}} - \frac{2(a^2+b^2)}{a^2-b^2}\,x^{\frac{p+q}{2pq}} + \overline{\frac{a^2+b^2}{a^2-b^2}}\Big|^2 x^{\frac{1}{q}} = -x^{\frac{1}{q}} + \overline{\frac{a^2+b^2}{a^2-b^2}}\Big|^2 x^{\frac{1}{q}} = \frac{4a^2b^2x^{\frac{1}{q}}}{(a^2-b^2)^2};$$

$$\therefore\ x^{\frac{1}{2p} - \frac{1}{2q}} = \frac{a^2 + b^2 \pm 2ab}{a^2 - b^2} = \frac{(a \pm b)^2}{a^2 - b^2} = \frac{a \pm b}{a \mp b},$$

$$\text{and } x = \left(\frac{a \pm b}{a \mp b}\right)^{\frac{2qp}{q-p}}.$$

(173). $\dfrac{x^{(m-n)^2} + x^{-4mn}}{x^{(m-n)^2} - x^{-4mn}} = a^{\frac{r}{s}}$,

$$\frac{x^{(m-n)^2}}{x^{-4mn}} = \frac{a^{\frac{r}{s}} + 1}{a^{\frac{r}{s}} - 1} \text{ or } x^{(m+n)^2} = \frac{a^{\frac{r}{s}} + 1}{a^{\frac{r}{s}} - 1};$$

$$\therefore\ x = \overline{\frac{a^{\frac{r}{s}} + 1}{a^{\frac{r}{s}} - 1}}\Big|^{\frac{1}{(m+n)^2}},$$

observe that $x^{m^2 - 2mn + n^2 + 4mn} = x^{(m+n)^2}$.

(174). $a^2b^2x^{\frac{1}{n}} - 4 \times (ab)^{\frac{3}{2}} \times x^{\frac{m+n}{2mn}} = (a - b)^2 x^{\frac{1}{m}}$,

$$a^2b^2x^{\frac{1}{n}} - 4ab\sqrt{(ab)}\,x^{\frac{1}{2n} + \frac{1}{2m}} + 4abx^{\frac{1}{m}} = (a + b)^2 x^{\frac{1}{m}};$$

$$\therefore\ abx^{\frac{1}{2n}-\frac{1}{2m}} = 2\sqrt{(ab)} \pm (a+b) = \pm(\sqrt{a} \pm \sqrt{b})^2;$$

$$\therefore\ x^{\frac{m-n}{2mn}} = \pm\frac{(\sqrt{a} \pm \sqrt{b})^2}{ab}, \text{ and } x = \pm\left(\frac{1}{\sqrt{b}} \pm \frac{1}{\sqrt{a}}\right)^{\frac{4mn}{m-n}}.$$

(175). $(a^{4m}+1)(x^{\frac{1}{2}}-1)^2 = 2(x+1)$,

$$\therefore (a^{4m}-1)x - 2x^{\frac{1}{2}}(a^{4m}+1) = -a^{4m} - 1 + 2 = -(a^{4m}-1),$$

$$\text{whence } x - 2x^{\frac{1}{2}}\frac{a^{4m}+1}{a^{4m}-1} + \left(\frac{a^{4m}+1}{a^{4m}-1}\right)^2 = \frac{4a^{4m}}{(a^{4m}-1)^2};$$

$$\therefore\ x^{\frac{1}{2}} = \frac{a^{4m}+1 \pm 2a^{2m}}{a^{4m}-1} = \frac{(a^{2m} \pm 1)^2}{a^{4m}-1} = \frac{a^{2m} \pm 1}{a^{2m} \mp 1};$$

$$\therefore\ x = \left(\frac{a^{2m} \pm 1}{a^{2m} \mp 1}\right)^2.$$

(176). $x^{3m} - a = x^m(a+1)$,

$$\text{and } x^{3m} - ax^m - a - x^m = 0;$$

$$\therefore\ x^m(x^{2m}-1) - a(x^m+1) = 0; \ \therefore\ x = (-1)^{\frac{1}{m}},$$

$$\text{and } x^{2m} - x^m + \frac{1}{4} = a + \frac{1}{4}; \ \therefore\ x = \left[\frac{1}{2}\{1 \pm \sqrt{(1+4a)}\}\right]^{\frac{1}{m}}.$$

(177). $(1+x)^{\frac{2}{m}} - (1-x)^{\frac{2}{m}} = (1-x^2)^{\frac{1}{m}}$,

$$(1+x)^{\frac{2}{m}} - (1-x^2)^{\frac{1}{m}} + \frac{1}{4}(1-x)^{\frac{2}{m}} = \frac{5}{4}(1-x)^{\frac{2}{m}};$$

$$\therefore \overline{\frac{1+x}{1-x}}\Big] = \overline{\frac{1 \pm \sqrt{5}}{2}}\Big]^m, \text{ and } x = \frac{(1 \pm \sqrt{5})^m - 2^m}{(1 \pm \sqrt{5})^m + 2^m}.$$

(178). $a[(1-x)\sqrt{(x^2+x^3)} - (1+x)\sqrt{(x^2-x^3)}] = \sqrt{\{2+2\sqrt{(1-x^2)}\}}$,

$$ax\sqrt{(1-x^2)}\{\sqrt{(1-x)} - \sqrt{(1+x)}\} = \sqrt{\{2+2\sqrt{(1-x^2)}\}},$$

$$\text{whence } a^2x^2(1-x^2)\{2 - 2\sqrt{(1-x^2)}\} = 2 + 2\sqrt{(1-x^2)},$$

$$\text{or } a^2x^2(1-x^2)\{1 - \sqrt{(1-x^2)}\} = 1 + \sqrt{(1-x^2)},$$

$$\text{and } a^2x^2(1-x^2)x^2 = \{1 + \sqrt{(1-x^2)}\}^2;$$

$$\therefore\ ax^2\sqrt{(1-x^2)} = 1 + \sqrt{(1-x^2)},$$

$$\text{or } (ax^2-1)^2(1-x^2) = 1;$$

$$\therefore\ a^2x^4 - 2ax^2 + 1 - x^2(ax^2 - 1)^2 = 1,$$

$$\text{or } a^2x^2 - 2a = a^2x^4 - 2ax^2 + 1,$$

$$\text{or } a^2x^4 - (2+a)\,ax^2 + \overline{\frac{2+a}{2}}\Big|^2 = -(2a+1) + \frac{(2+a)^2}{4} = \frac{a^2-4a}{4};$$

$$\therefore\ ax^2 = \frac{2+a}{2} \pm \sqrt{\left(\frac{a^2-4a}{4}\right)};$$

$$\therefore\ x = \pm\sqrt{\left\{\frac{1}{a} + \frac{1}{2} \pm \sqrt{\left(\frac{1}{4} - \frac{1}{a}\right)}\right\}},$$

or putting the equation under the form

$$ax\sqrt{(1-x^2)}\,\{\sqrt{(1-x)} - \sqrt{(1+x)}\} = \pm\sqrt{(1+x)} + \sqrt{(1-x)},$$

and multiplying both sides by $\sqrt{(1+x)} - \sqrt{(1-x)}$, we have

$$\sqrt{\{(1-x^2)\}} - (1-x^2) = \pm\frac{1}{a},$$

$$\text{whence } 1 - x^2 - \sqrt{(1-x^2)} + \frac{1}{4} = \frac{1}{4} \mp \frac{1}{a};$$

$$\therefore\ \sqrt{(1-x^2)} = \frac{1}{2} \pm \sqrt{\left(\frac{1}{4} \mp \frac{1}{a}\right)},$$

$$\text{and } x = \left\{\frac{1}{2} \mp \frac{1}{a} \pm \sqrt{\left(\frac{1}{4} \pm \frac{1}{a}\right)}\right\}^{\frac{1}{2}}.$$

(179). $\sqrt[3]{\{\sqrt{(a)} + \sqrt{(x)}\}} + \sqrt[3]{\{\sqrt{(a)} - \sqrt{(x)}\}} = \sqrt[6]{(a)}$,

by cubing both sides and reducing

$$2\sqrt{a} + 3(a-x)^{\frac{1}{3}}\,(\overline{\sqrt{a}+\sqrt{x}}\,\big|^{\frac{1}{3}} + \overline{\sqrt{a}-\sqrt{x}}\,\big|^{\frac{1}{3}}) = a^{\frac{1}{2}},$$

$$\text{or } a^{\frac{1}{2}} + 3(a-x)^{\frac{1}{3}}\,a^{\frac{1}{6}} = 0;$$

$$\therefore\ 3(a-x)^{\frac{1}{3}} = -a^{\frac{1}{3}},$$

$$\text{whence } 27(a-x) = -a, \text{ or } x = \frac{28a}{27}.$$

(180). $\dfrac{x}{2} - 2 = \dfrac{x^2}{2\{1+\sqrt{(1+x)}\}^2}$,

$$\therefore\ \sqrt{(x-4)} = \frac{x\{1-\sqrt{(1+x)}\}}{-x} = -1 + \sqrt{(1+x)};$$

$$\therefore\ x - 4 + 1 + 2\sqrt{(x-4)} = 1 + x,$$

$$4(x-4) = 16;\ \therefore\ x = 8.$$

(181). $4x^2 + 12x\sqrt{(1+x)} = 27(1+x)$,

$$1 + x - \frac{4x}{9}\sqrt{(1+x)} + \frac{4x^2}{81} = \frac{4x^2}{27} + \frac{4x^2}{81} = \frac{16x^2}{81};$$

$$\therefore \sqrt{(1+x)} = \frac{2x}{9} \pm \frac{4x}{9} = \frac{2x}{3} \text{ or } -\frac{2x}{9};$$

$$\therefore 9 + 9x = 4x^2, \text{ or } 81 + 81x = 4x^2,$$

whence $x = 3$ or $-\frac{3}{4}$, also $x = \frac{9}{8}\{9 \pm \sqrt{(97)}\}$.

(182). $(a+x)^{\frac{2}{3}} - 5 \times (a^2 - x^2)^{\frac{1}{3}} + 4(a-x)^{\frac{2}{3}} = 0$,

$$(a+x)^{\frac{2}{3}} - 5(a^2 - x^2)^{\frac{1}{3}} + \frac{25}{4}(a-x)^{\frac{2}{3}}$$

$$= -4(a-x)^{\frac{2}{3}} + \frac{25}{4}(a-x)^{\frac{2}{3}} = \frac{9}{4}(a-x)^{\frac{2}{3}};$$

$$\therefore \left(\frac{a+x}{a-x}\right)^{\frac{1}{3}} = \frac{5}{2} \pm \frac{3}{2} = 4 \text{ or } 1;$$

$$\therefore \frac{a+x}{a-x} = 64 \text{ or } 1, \text{ and } x = \frac{63a}{65} \text{ or } 0.$$

(183). $\dfrac{\sqrt{(a^{\frac{1}{n}} - x^{\frac{1}{n}})}}{x^{\frac{1}{n}}} - \dfrac{\sqrt{(a^{\frac{1}{n}} - x^{\frac{1}{n}})}}{a^{\frac{1}{n}}} = \dfrac{x^{\frac{1}{2n}}}{b^{\frac{1}{n}}}$,

$$(a^{\frac{1}{n}} - x^{\frac{1}{n}})\sqrt{(a^{\frac{1}{n}} - x^{\frac{1}{n}})} = a^{\frac{1}{n}} x^{\frac{3}{2n}} b^{-\frac{1}{n}},$$

$$\text{or } (a^{\frac{1}{n}} - x^{\frac{1}{n}})^{\frac{3}{2}} = a^{\frac{1}{n}} x^{\frac{3}{2n}} b^{-\frac{1}{n}}; \quad \therefore a^{\frac{1}{n}} - x^{\frac{1}{n}} = a^{\frac{2}{3n}} b^{-\frac{2}{3n}} x^{\frac{1}{n}};$$

$$\therefore x = \left(\frac{a^{\frac{1}{n}} b^{\frac{2}{3n}}}{a^{\frac{2}{3n}} + b^{\frac{2}{3n}}}\right)^n = \frac{ab^{\frac{2}{3}}}{(a^{\frac{2}{3n}} + b^{\frac{2}{3n}})^n}.$$

(184). $x - 2\sqrt{(x+2)} = 1 + \sqrt[4]{(x^3 - 3x + 2)}$,

$$(x-1) - 2\sqrt{(x+2)} = \sqrt[4]{(x^3 - 3x + 2)} = \sqrt{(x-1)}\sqrt[4]{(x+2)},$$

whence $x - 1 - \sqrt{(x+2)} = \sqrt[4]{(x+2)}\{\sqrt{(x-1)} + \sqrt[4]{(x+2)}\}$,

and dividing by $\sqrt{(x-1)} + \sqrt[4]{(x+2)} = 0$, $(x-1)^2 = x + 2$.

$$\text{whence } x = \frac{3 \pm \sqrt{(13)}}{2},$$

$$\text{also } \sqrt{(x-1)} - \sqrt[4]{(x+2)} = \sqrt[4]{(x+2)};$$

$$\therefore (x-1)^2 = 16(x+2),$$

$$\text{whence } x = 9 \pm 4\sqrt{7}.$$

(185). $\sqrt{(x)}\sqrt{(x^3-a^3)} - 2a^2 = x^2 - 3ax,$

$$\sqrt{x}\sqrt{(x^3-a^3)} = x^2 - 3ax + 2a^2 = (x-a)(x-2a),$$

$$\text{or } x(x^3-a^3) = (x-a)^2(x-2a)^2; \ \therefore x = a;$$

$$\therefore x(x^2+ax+a^2) = (x-a)(x-2a)^2 = x^3 - 5ax^2 + 8a^2x - 4a^3;$$

$$\therefore \overline{x^2 - \frac{7ax}{6} + \frac{7a}{12}}\Big]^2 = -\frac{4a^2}{6} + \frac{49a^2}{144} = -\frac{47a^2}{144};$$

$$\therefore x = \frac{a}{12}\{7 \pm \sqrt{(-47)}\}.$$

(186). $x^4 - x^3 + 1 = \dfrac{4x-5x^2}{4},$

$$x^4 - x^3 + \frac{5x^2}{4} - x + 1 = 0;$$

$$\therefore x^2 + 2 + \frac{1}{x^2} - \left(x + \frac{1}{x}\right) + \frac{1}{4} = 2 - \frac{5}{4} + \frac{1}{4} = 1;$$

$$\therefore x + \frac{1}{x} = \frac{1}{2} \pm 1 = \frac{3}{2} \text{ or } -\frac{1}{2},$$

$$\text{whence } x = \frac{1}{4}\{3 \pm \sqrt{(-7)}\}, \text{ or } x = \frac{1}{4}\{-1 \pm \sqrt{(-15)}\}.$$

(187). $\dfrac{(1+x)^3}{1+x^3} = \dfrac{1}{a},$

$$\frac{(1+x)^2}{1-x+x^2} = \frac{1}{a}, \text{ or } \frac{(1+x)^2}{3x} = \frac{1}{1-a},$$

$$\text{whence } (1-a)(1+2x+x^2) = 3x,$$

$$\text{and } \overline{x^2 - \frac{(1+2a)x}{1-a} + \frac{1+2a}{2(1-a)}}\Big]^2 = -1 + \frac{(1+2a)^2}{4(1-a)^2};$$

$$\therefore x = \frac{1}{2(1-a)}[1 + 2a \pm \sqrt{\{3(4a-1)\}}].$$

(188). $\frac{(1+x)^2}{1+x^3} + \frac{(1-x)^2}{1-x^3} = a,$

$$\frac{1+x}{1-x+x^2} + \frac{1-x}{1+x+x^2} = a;$$

$$\therefore (1+x)^2 + \overline{1-x}|^2 + 2x^2 = a(1+x^2+x^4),$$

$$\text{or } 2(1+2x^2) = a(1+x^2+x^4);$$

$$\therefore x^4 + \frac{(a-4)x^2}{a} + \overline{\frac{a-4}{2a}}\Big]^2 = -\frac{a-2}{a} + \frac{(a-4)^2}{4a^2} = \frac{16-3a^2}{4a^2};$$

$$\therefore x = \pm\sqrt{\left[\frac{1}{2a}\{(a-4) \pm \sqrt{(16-3a^2)}\}\right]}.$$

(189). $(x+a)\left(1+\frac{1}{x^2+a^2}\right) + \sqrt{(2ax)}\left(1-\frac{1}{x^2+a^2}\right) = 4a,$

$$x + \sqrt{(2ax)} + a + \frac{x - \sqrt{(2ax)} + a}{\{(x+a) - \sqrt{(2ax)}\}\{x+a+\sqrt{(2ax)}\}} = 4a;$$

$$\therefore x - \sqrt{(2ax)} + a = 0, \text{ and } \sqrt{x} = \sqrt{\frac{a}{2}} \pm \sqrt{\left(-\frac{a}{2}\right)};$$

$$\therefore x = a\sqrt{(-1)},$$

$$\text{also } \{x + \sqrt{(2ax)} + a\}^2 - 4a\{x + \sqrt{(2ax)} + a\} + 4a^2 = 4a^2 - 1;$$

$$\therefore x + \sqrt{(2ax)} + a = 2a \pm \sqrt{(4a^2-1)} = m,$$

$$\text{and } x + \sqrt{(2ax)} + \frac{a}{2} = \frac{a+2m}{2};$$

$$\therefore \sqrt{x} = -\sqrt{\left(\frac{a}{2}\right)} \pm \sqrt{\left(\frac{a+2m}{2}\right)};$$

$$\therefore x = a + m + \sqrt{(a^2+2am)}.$$

(190). $\frac{1+x^3}{(1+x)^3} + \frac{1-x^3}{(1-x)^3} = a,$

$$\frac{1-x+x^2}{(1+x)^2} + \frac{1+x+x^2}{(1-x)^2} = a,$$

$$\text{whence } (1-x)^3 + (1+x)^3 + 2x^2(1+x^2) = a(1-x^2)^2;$$

$$\therefore 2 + 8x^2 + 2x^4 = a(1-2x^2+x^4),$$

$$\text{and } x^4 - \frac{2(a+4)}{a-2}x^2 + \overline{\frac{a+4}{a-2}}\Big|^2 = -\frac{a-2}{a-2} + \overline{\frac{a+4}{a-2}}\Big|^2 = \frac{12(a+1)}{a-2};$$

$$\therefore x = \left[\frac{a+4 \pm 2\sqrt{\{3(a+1)\}}}{a-2}\right]^{\frac{1}{2}}.$$

(191). $\dfrac{1+x^4}{(1+x)^4} = \dfrac{1}{2}$,

$$2 + 2x^4 = 1 + 4x + 6x^2 + 4x^3 + x^4,$$

$$\text{whence } x^2 - 4x - 6 - \frac{4}{x} + \frac{1}{x^2} = 0,$$

$$\text{or } x^2 + 2 + \frac{1}{x^2} - 4\left(x + \frac{1}{x}\right) + 4 = 12;$$

$$\therefore x + \frac{1}{x} = 2 \pm 2\sqrt{(3)},$$

$$\text{whence } x^2 - 2\{1 \pm \sqrt{(3)}\}x + (1 \pm \sqrt{3})^2 = 3 \pm 2\sqrt{(3)};$$

$$\therefore x = 1 \pm \sqrt{3} \pm \sqrt{(3 \pm 2\sqrt{3})}.$$

(192). $\dfrac{1+x^4}{(1+x)^4} = a$,

$$1 + x^4 = a(1 + 4x + 6x^2 + 4x^3 + x^4),$$

$$\text{whence } (a-1)x^4 + 4a(x + x^3) + 6ax^2 + a - 1 = 0;$$

$$\therefore x^2 + \frac{4a\left(x + \frac{1}{x}\right)}{a-1} + \frac{6a}{a-1} + \frac{1}{x^2} = 0,$$

$$\text{or } \overline{x + \frac{1}{x}}\Big|^2 + \frac{4a}{a-1}\left(x + \frac{1}{x}\right) + \frac{4a^2}{(a-1)^2}$$

$$= -\frac{6a}{a-1} + \frac{4a^2}{(a-1)^2} + 2 = \frac{2(a+1)}{(a-1)^2};$$

$$\therefore x + \frac{1}{x} = \frac{2a}{a-1} \pm \frac{\sqrt{\{2(a+1)\}}}{a-1} = 2m,$$

$$\text{whence } x = m \pm \sqrt{(m^2 - 1)}.$$

(193). $\dfrac{(1+x)^5}{1+x^5} = \dfrac{1}{a}$,

$$\frac{(1+x)^5}{1+x^5} = \frac{(1+x)^4}{1 - x + x^2 - x^3 + x^4} = \frac{1}{a};$$

$$\therefore\ a\,(1 + 4x + 6x^3 + 4x^3 + x^4) = 1 - x + x^2 - x^3 + x^4;$$

$$\therefore\ (a-1)\,x^4 + (4a+1)\,x^3 + (6a-1)\,x^2 + (4a+1)\,x + a - 1 = 0$$

$$\text{or } x^2 + \frac{(4a+1)}{a-1}\,x + \frac{6a-1}{a-1} + \frac{4a+1}{a-1}\,\frac{1}{x} + \frac{1}{x^2} = 0;$$

$$\therefore\ \overline{x + \frac{1}{x}}\Big|^2 + \frac{(4a+1)}{a-1}\left(x + \frac{1}{x}\right) + (\quad)^2 = -\,\frac{6a-1}{a-1} + \frac{(4a-1)^2}{4(a-1)^2} + 2$$

$$\therefore\ x + \frac{1}{x} = \frac{(4a+1)}{2\,(a-1)} \pm \frac{\sqrt{(4a+5)}}{2\,(a-1)} = 2m;$$

$$\therefore\ x = m \pm \sqrt{(m^2 - 1)}.$$

(194). $\dfrac{1 + x^4}{\sqrt{(1 - x^4)}} = \dfrac{2x}{a}$,

$$a^2\,(1 + 2x^4 + x^8) = 4x^2\,(1 - x^4),$$

$$\text{whence } a^2x^8 + 4x^6 + 2a^2x^4 - 4x^2 + a^2 = 0,$$

$$\text{and } x^4 + \frac{1}{x^4} + \frac{4}{a^2}\left(x^2 - \frac{1}{x^2}\right) + 2 = 0;$$

$$\therefore\ \overline{x^2 - \frac{1}{x^2}}\Big|^2 + \frac{4}{a^2}\,\overline{x^2 - \frac{1}{x^2}}\Big| + \frac{4}{a^4} = -\,4 + \frac{4}{a^4} = 4 \times \frac{1 - a^4}{a^4};$$

$$\therefore\ x^2 - \frac{1}{x^2} = -\,\frac{2}{a^2} \pm \frac{2}{a^2}\sqrt{(1 - a^4)} = 2m;$$

$$\therefore\ \text{whence } x = \sqrt{\{m \pm \sqrt{(m^2 - 1)}\}}.$$

(195). $\dfrac{\sqrt{(1 + a^2)} - a\sqrt{(1 + x^2)}}{\sqrt{(1 + x^2)} - x\sqrt{(1 + a^2)}} = a$,

$$\therefore\ \sqrt{(1 + a^2)} - a\sqrt{(1 + x^2)} = a\sqrt{(1 + x^2)} - ax\sqrt{(1 + a^2)},$$

$$\text{or } (1 + ax)^2\,(1 + a^2) = 4a^2\,(1 + x^2),$$

$$a^2\,(a^2 - 3)\,x^2 + 2ax\,(1 + a^2) + (1 + a^2) = 4a^2;$$

$$\therefore\ a^2x^2 + \frac{2ax\,(1 + a^2)}{a^2 - 3} + \frac{(1 + a^2)^2}{(a^2 - 3)^2}$$

$$= \frac{3a^2 - 1}{a^2 - 3} + \frac{(1 + a^2)^2}{(a^2 - 3)^2} = \frac{4\,(a^4 - 2a^2 + 1)}{(a^2 - 3)^2},$$

$$\text{whence } x = \frac{1}{a} \text{ or } \frac{1 - 3a^2}{a\,(a^2 - 3)}.$$

(196). $(1+x)^{\frac{2}{5}} + (1-x)^{\frac{2}{5}} = (1-x^2)^{\frac{1}{5}}$,

$$(1+x)^{\frac{2}{5}} - (1-x^2)^{\frac{1}{5}} + \frac{1}{4}(1-x)^{\frac{2}{5}}$$

$$= -(1-x)^{\frac{2}{5}} + \frac{1}{4}(1-x)^{\frac{2}{5}} = -\frac{3}{4}(1-x)^{\frac{2}{5}};$$

$$\therefore \left(\frac{1+x}{1-x}\right)^{\frac{1}{5}} = \frac{1}{2} \pm \frac{\sqrt{(-3)}}{2} = \frac{1 \pm \sqrt{(-3)}}{2},$$

$$x = \frac{\{1 \pm \sqrt{(-3)}\}^5 - 2^5}{\{1 \pm \sqrt{(-3)}\}^5 + 2^5}.$$

(197). $(a^{\frac{1}{2}} + x^{\frac{1}{2}})^{\frac{2}{m}} - (a-x)^{\frac{1}{m}} = 6\,(a^{\frac{1}{2}} - x^{\frac{1}{2}})^{\frac{2}{m}}$,

$$(a^{\frac{1}{2}} + x^{\frac{1}{2}})^{\frac{2}{m}} - (a-x)^{\frac{1}{m}} + \left(\frac{a^{\frac{1}{2}} - x^{\frac{1}{2}}}{4}\right)^{\frac{2}{m}} = \frac{25}{4}\,(a^{\frac{1}{2}} - x^{\frac{1}{2}})^{\frac{2}{m}};$$

$$\therefore \left(\frac{a^{\frac{1}{2}} + x^{\frac{1}{2}}}{a^{\frac{1}{2}} - x^{\frac{1}{2}}}\right)^{\frac{1}{m}} = \frac{1}{2} \pm \frac{5}{2} = 3 \text{ or } -2;$$

$$\therefore \frac{a^{\frac{1}{2}} + x^{\frac{1}{2}}}{a^{\frac{1}{2}} - x^{\frac{1}{2}}} = 3^m \text{ or} -2^m, \text{ whence } x = a\left(\frac{3^m-1}{3^m+1}\right)^2 \text{ or } a\,\frac{(-2)^m-1}{(-2)^m+1}.$$

(198). $16 \times (x^2+2)^{\frac{3}{2}} + \dfrac{3}{\sqrt{(x^2+2)}} = 32x^2 + 48$,

$$16\,(x^2+2)^2 + 3 = 32\,(x^2+2)^{\frac{3}{2}} - 16\,(x^2+2)^{\frac{1}{2}},$$

$$\text{whence } 16\,(x^2+2)^2 - 32\,(x^2+2)^{\frac{3}{2}} + 16\,(x^2+2)$$

$$-16\,\{(x^2+2) - (x^2+2)^{\frac{1}{2}}\} + 3 = 0;$$

$$\therefore \{(x^2+2) - (x^2+2)^{\frac{1}{2}}\}^2 - \{x^2+2 - (x^2+2)^{\frac{1}{2}}\} + \frac{1}{4} = -\frac{3}{16} + \frac{1}{4} = \frac{1}{16};$$

$$\therefore x^2 + 2 - \sqrt{(x^2+2)} = \frac{3}{4} \text{ or } \frac{1}{4},$$

$$\text{whence } x^2 + 2 - \sqrt{(x^2+2)} + \frac{1}{4} = 1 \text{ or } \frac{1}{2};$$

$$\therefore\ \sqrt{(x^2+2)} = \frac{3}{2} \text{ or } -\frac{1}{2} \text{ or } \frac{1 \pm \sqrt{2}}{2},$$

$$\text{whence } x = \pm\frac{1}{2} \pm \frac{1}{2}\sqrt{(-7)} \text{ or } \frac{1}{2}\sqrt{\{\pm 2\sqrt{(2)} - 5\}},$$

$$\text{or let } x^2 = z^2 + 2, \text{ then } 16\overline{z^2 - z}|^2 = 16(z^2 - z) - 3;$$

$$\therefore\ z^2 - z = \frac{3}{4} \text{ or } \frac{1}{4}, \text{ and } z = \frac{3}{2}, -\frac{1}{2}, \text{ or } \frac{1}{2}\{1 \pm \sqrt{(2)}\};$$

$$\therefore\ x = \pm\frac{1}{2}, \pm\frac{1}{2}\sqrt{(-7)}, \pm\frac{1}{2}(-5 \pm 2\sqrt{2})^{\frac{1}{2}}.$$

(199). $$\frac{a^4 - \frac{3}{16}b^4}{a^2 + a\sqrt{(b^2+bx)} + \frac{1}{2}bx - \frac{1}{4}b^2 - \frac{1}{2}b\sqrt{(x^2 - bx + b^2)}}$$

$$= a^2 - a\sqrt{(b^2+bx)} + \frac{1}{2}bx - \frac{1}{4}b^2 + \frac{1}{2}b\sqrt{(x^2 - bx + b^2)},$$

$$\text{whence } \left\{\left(a^4 - \frac{3}{16}b^4\right)\right\}$$

$$= \left[\left(a^2 + \frac{1}{2}bx - \frac{b^2}{4}\right) + \left\{a\sqrt{(b^2+bx)} - \frac{1}{2}b\sqrt{(x^2 - bx + b^2)}\right\}\right]$$

$$\times \left[\left(a^2 + \frac{1}{2}bx - \frac{b^2}{4}\right) - \left\{a\sqrt{(b^2+bx)} - \frac{1}{2}b\sqrt{(x^2 - bx + b^2)}\right\}\right],$$

$$\text{or } a^4 - \frac{3b^4}{16} = a^4 + a^2bx + \frac{b^2x^2}{2} - \frac{a^2b^2}{2} - \frac{b^3x}{4} + \frac{b^4}{16}$$

$$-\left[\left\{a^2b^2 + a^2bx + \frac{1}{4}(b^2x^2 - b^3x + b^4) - ab\sqrt{\{(b^2+bx)(x^2 - bx + b^2)\}}\right\}\right],$$

$$\text{whence } 9a^2b^2 = bx^3 + b^4;\ \therefore\ x = \sqrt[3]{\left(\frac{9a^2b - 4b^3}{4}\right)}.$$

200). $(1+x)\sqrt{(1+a)} + (1-x)\sqrt{(1-a)} = 2\sqrt{(1+x^2)}$,

$$(1+x)^2(1+a) + (1-x)^2(1-a) + 2\sqrt{(1-a^2)(1-x^2)} = 4(1+x^2);$$

$$\sqrt{(1-a^2)}(1-x^2) = 1 - 2ax + x^2;$$

$$\therefore\ x^2\{1 + \sqrt{(1-a^2)}\} - 2ax = -1 + \sqrt{(1-a^2)},$$

$$\text{or } x^2 - \frac{2x}{a}\frac{\{1-\sqrt{(1-a^2)}\}}{1} + \frac{\{1-\sqrt{(1-a^2)}\}^2}{a^2} = 0,$$

$$\therefore\ x = \frac{1}{a}\{1-\sqrt{(1-a^2)}\}.$$

(201). $\dfrac{x}{a+x} + \dfrac{a}{\sqrt{(a+x)}} = \dfrac{b}{x}$,

$$\frac{x^2}{a+x} + \frac{ax}{\sqrt{(a+x)}} + \frac{a^2}{4} = b + \frac{a^2}{4};$$

$$\therefore\ \frac{x}{\sqrt{(a+x)}} = \frac{1}{2}\{-a \pm \sqrt{(a^2+4b)}\} = 2m,$$

whence $x^2 - 4m^2x + 4m^4 = 4m^2a$; $\therefore\ x = 2m(m \pm \sqrt{a})$.

(202). $n\sqrt{(a^2+x^2)} - (n-1)\sqrt{\{x^2-2(n-1)a^2\}} = 2n-1$,

$$n\sqrt{(a^2+x^2)} - (2n-1) = (n-1)\sqrt{\{x^2-2(n-1)a^2\}},$$

$$n^2(a^2+x^2) - 2(2n-1)n\sqrt{(a^2+x^2)} + (2n-1)^2$$
$$= (n-1)^2\{x^2-2(n-1)a^2\},$$

whence $(2n-1)(a^2+x^2) - 2n(2n-1)\sqrt{(a^2+x^2)} + (2n-1)^2$
$$= -(2n-1)a^2(n-1)^2;$$

$$\therefore\ a^2+x^2-2n\sqrt{(a^2+x^2)}+n^2 = (n-1)^2(1-a^2);$$

$$\therefore\ \sqrt{(a^2+x^2)} = n \pm (n-1)\sqrt{(1-a^2)},$$

whence $x = \pm\sqrt{[\{n \pm (n-1)\sqrt{(1-a^2)}\}^2 - a^2]}$.

(203). $\dfrac{x}{2} + 63x^{-\frac{1}{4}} = 220\tfrac{1}{2} \times x^{-\frac{1}{2}} + 49\sqrt{(x)} - 1196 = 0$,

$$x + \frac{126}{x^{\frac{1}{4}}} = \frac{441}{x^{\frac{1}{2}}} + 98\sqrt{x} - 2392,$$

whence $x - 98\sqrt{x} + 2401 = \dfrac{441}{x^{\frac{1}{2}}} - \dfrac{126}{x^{\frac{1}{4}}} + 9$;

$$\therefore\ x^{\frac{1}{2}} - 49 = \pm\left(\frac{21}{x^{\frac{1}{4}}} - 3\right) = \mp\frac{3}{x^{\frac{1}{4}}}(x^{\frac{1}{4}} \mp 7);$$

$$\therefore\ x^{\frac{1}{4}} \mp 7 = 0, \text{ and } x = 2401,$$

$$\text{also } \overline{x^{\frac{1}{2}} \pm 49x^{\frac{1}{4}} + \frac{49}{2}}\Big]^2 = \mp 3, \text{ whence } x = \frac{1}{16}\{\mp 7 \mp \sqrt{(37)}\}^4,$$

$$\text{or } \frac{1}{16}\{7 \pm \sqrt{(61)}\}^4,$$

$$\text{or let } x = z^4, \text{ then } z^2(z^2 - 49)^2 = 9(z-7)^2;$$

$$\therefore z = 7 \text{ or } -\frac{1}{2}\{7 \pm \sqrt{(37)}\} \text{ or } -\frac{1}{2}\{7 \pm \sqrt{(61)}\},$$

$$\text{and } x = z^4 = 2401, \text{ or } \frac{1}{16}\{7 \pm \sqrt{(37)}\}^4, \text{ or } \frac{1}{16}\{7 \pm \sqrt{(61)}\}^4.$$

(204). $$\frac{x^2}{\sqrt{(a)} + \sqrt{(b)}} - \{\sqrt{(a)} - \sqrt{(b)}\}\,x = \{(ab^2)^{-\frac{1}{2}} + (a^2b)^{-\frac{1}{2}}\}^{-1},$$

$$\overline{x^2 - (a-b)\,x + \frac{a-b}{2}}\Big]^2 = ab + \frac{(a-b)^2}{4} = \frac{(a+b)^2}{4};$$

$$\therefore x = \frac{a-b}{2} \pm \frac{a+b}{2} = a \text{ or } -b.$$

(205). $$(x+2)^2 + 2\sqrt{(x)}\,(x+2) - 3\sqrt{(x)} = 46 + 2x,$$

$$(x+2)^2 + 2\sqrt{x}\,(x+2) + x - 3\{(x+2) + \sqrt{x}\} + \frac{9}{4} = \frac{169}{4};$$

$$\therefore x + 2 + \sqrt{x} = \frac{3}{2} \pm \frac{13}{2} = 8 \text{ or } -5,$$

$$\text{whence } x + \sqrt{x} + \frac{1}{4} = \frac{25}{4}, \text{ or } x + \sqrt{x} + \frac{1}{4} = -7 + \frac{1}{4},$$

$$\text{and } x = 4 \text{ or } 9, \text{ or } x = \left\{-\frac{1}{2} \pm \frac{1}{2}\sqrt{-27}\right\}^2.$$

(206). $$\sqrt[4]{(a+x)} + \sqrt[4]{(a-x)} = b,$$

$$\sqrt{(a+x)} + \sqrt{(a-x)} + 2\sqrt[4]{(a^2-x^2)} = b^2,$$

$$4\sqrt{(a^2-x^2)} = b^4 - 2b^2\{\sqrt{(a+x)} + \sqrt{(a-x)}\} + 2a + 2\sqrt{(a^2-x^2)},$$

$$\text{whence } \sqrt{(a^2-x^2)} - 2b^2\sqrt[4]{(a^2-x^2)} + b^4 = a + \frac{b^4}{2};$$

$$\therefore \sqrt[4]{(a^2-x^2)} = b^2 \pm \sqrt{\left(a + \frac{b^4}{2}\right)};$$

$$\therefore x = \pm\sqrt{\left[a^2 - \left\{\sqrt{\left(b^2 \pm \sqrt{\left(a + \frac{b^4}{2}\right)}\right)}\right\}^4\right]}.$$

(207). $\sqrt{(2a^2 - ax - x^2)} - (a + x) - \sqrt{(ax - x^2)} = \sqrt{(2ax + x^2)}$,

$$\sqrt{(2a^2 - ax - x^2)} = \sqrt{\{(a - x)(2a + x)\}};$$

$$\therefore (a + x) + \sqrt{(ax - x^2)} = \sqrt{(2a + x)}\{\sqrt{(a - x)} - \sqrt{x}\};$$

$$\therefore (a + x)^2 + ax - x^2 + 2(a + x)\sqrt{(ax - x^2)}$$

$$= (2a + x)\{a - 2\sqrt{(ax - x^2)}\},$$

or $a^2 + 3ax - 2(a + x)\sqrt{(ax - x^2)} = 2a^2 + ax - 2(2a + x)\sqrt{(ax - x^2)}$,

$$\text{or } 2a\sqrt{(ax - x^2)} = a^2 - 2ax;$$

$$\therefore 4ax - 4x^2 = a^2 - 4ax + 4x^2;$$

$$\therefore x^2 - ax + \frac{a^2}{4} = -\frac{a^2}{8} + \frac{a^2}{4} = \frac{a^2}{8};$$

$$\therefore x = \frac{a}{2}\left(1 \pm \frac{1}{\sqrt{2}}\right),$$

or let $x = az$, then $2(1 - z)^{\frac{1}{2}}\{\sqrt{(2 + z)} + \sqrt{(z)}\} = \{\sqrt{(2 + z)} + \sqrt{(z)}\}^2$;

$$\therefore \sqrt{(2 + z)} + z^{\frac{1}{2}} = 0, \text{ and } z = 0,$$

$$\text{or } \sqrt{(2 + z)} + z^{\frac{1}{2}} = 2\sqrt{(1 - z)},$$

$$\text{whence } x = az = \frac{a}{2}\left(1 \pm \frac{1}{\sqrt{2}}\right).$$

(208). $x^2 - \dfrac{27x}{4} + 25 = 7x^{\frac{1}{2}}(5 - x)$,

$$x^2 - 10x + 25 + 7x^{\frac{1}{2}}(x - 5) + \frac{49x}{4} = \frac{36x}{4};$$

$$\therefore x - 5 = -\frac{7x^{\frac{1}{2}}}{2} \pm \frac{6x^{\frac{1}{2}}}{2} = \frac{x^{\frac{1}{2}}}{2} \text{ or } -\frac{13x^{\frac{1}{2}}}{2},$$

whence $x^{\frac{1}{2}} = \dfrac{1}{4} \pm \dfrac{9}{4} = \dfrac{5}{2}$ or -2, or $x^{\frac{1}{2}} = \dfrac{1}{4}\{13 \pm \sqrt{(249)}\}$;

$$\therefore x = \frac{25}{4} \text{ or } 4, \text{ or } x = \frac{1}{8}\{209 \pm 13\sqrt{(249)}\}.$$

(209). $4x^2 - 27x + 12x\sqrt{(1 + x)} = 27$,

$$27(1 + x) - 12x\sqrt{(1 + x)} = 4x^2,$$

$$1 + x - \frac{4x}{9}\sqrt{(1 + x)} + \frac{4x^2}{81} = \frac{4x^2}{27} + \frac{4x^2}{81} = \frac{16x^2}{81},$$

$$\sqrt{(1 + x)} = \frac{2x}{9} \pm \frac{4x}{9} = \frac{2x}{3} \text{ or } -\frac{2x}{9},$$

from the first,

$$\frac{4x^2}{9} = x + 1;$$

$$\therefore\ 4x^2 - 9x + \frac{81}{16} = 9 + \frac{81}{16} = \frac{225}{16},$$

$$x = \frac{1}{2}\left(\frac{9}{4} \pm \frac{15}{4}\right) = 3 \text{ or } -\frac{3}{4},$$

$$\text{also } \frac{4x^2}{81} = x + 1;$$

$$\therefore\ \overline{x^2 - \frac{81x}{4} + \frac{81}{8}}\Big]^2 = \frac{81}{4} + \left(\frac{81}{8}\right)^2 = \frac{6885}{64};$$

$$\therefore\ x = \frac{81}{8} \pm \frac{9}{8}\sqrt{(97)} = \frac{9}{8}\{9 \pm \sqrt{(97)}\}.$$

(210). $$\frac{\sqrt{(x+1)} + \sqrt{(3x-1)}}{(1-x)} = \sqrt{\left\{a\left(1 + \frac{1}{x}\right) - 2\right\}},$$

$$(1-x)\sqrt{\left\{a\left(1+\frac{1}{x}\right) - 2\right\}} - \sqrt{(1+x)} = \sqrt{(3x-1)};$$

$$\therefore\ (1-x)^2\left\{a\left(1+\frac{1}{x}\right) - 2\right\}$$

$$-2(1-x)\sqrt{(1+x)}\sqrt{\left\{a\left(1+\frac{1}{x}\right) - 2\right\}} + 1 + x = 3x - 1;$$

$$\text{or } (1-x)^2\left\{a\left(1+\frac{1}{x}\right) - 2\right\}$$

$$-2(1-x)\sqrt{(1+x)}\sqrt{\left\{a\left(1+\frac{1}{x}\right) - 2\right\}} + 2(1-x) = 0;$$

$$\therefore\ a - x = 0, \text{ and } x = 1,$$

$$\text{and } \{a \times (1 - x^2)\} - 2x(1-x)$$

$$-2\sqrt{(1+x)}\sqrt{\{ax + (a-2)x^2\}} + 2x = 0;$$

$$\therefore\ a - (a-2)x^2 - 2x - 2\sqrt{(1+x)}\sqrt{\{ax + (a-2)x^2\}} + 2x = 0,$$

$$\text{whence } ax + (a-2)x^2$$

$$+ 2\sqrt{(1+x)}\sqrt{\{ax + (a-2)x^2\}} + (1+x) = (a+1)(1+x),$$

$$\therefore\ \sqrt{\{ax + (a-2)x^2\}} = \sqrt{(1+x)}\{-1 \pm \sqrt{(1+a)}\};$$

$$\therefore\ ax + (a-2)x^2 = \{-1 \pm \sqrt{(1+a)}\}^2 + x\{2 + a \mp 2\sqrt{(1+a)}\};$$

$$\therefore\ x^2 + 2\,\frac{\pm\sqrt{(1+a)}-1}{a-2}\,x + (\ \ )^2$$

$$= \frac{\{\pm\sqrt{(1+a)}-1\}^2}{a-2} + \frac{\{\pm\sqrt{(1+a)}-1\}^2}{(a-2)^2} = \frac{\{\pm\sqrt{(1+a)}-1\}^2(a-1)}{(a-2)^2};$$

$$\therefore\ x = \frac{\pm\sqrt{(1+a)}-1}{(a-1)-1}\{\pm\sqrt{(a-1)}-1\}$$

$$= \frac{\{\pm\sqrt{(1+a)}-1\}\{\pm\sqrt{(a-1)}-1\}}{(a-1)-1} = \frac{\pm\sqrt{(1+a)}-1}{\pm\sqrt{(a-1)}+1}.$$

(211). $a + b\sqrt{(x)} + x = \{b - \sqrt{(x)}\}\sqrt{(2a+x)}$,

$$a + b\sqrt{x} + x = b\sqrt{(2a+x)} - \sqrt{(2ax+x^2)},$$

$$\text{or } \{a + x + \sqrt{(2ax+x^2)}\}^2 = [b\{\sqrt{(2a+x)} - \sqrt{x}\}]^2$$

$$= 2b^2\{a + x - \sqrt{(2ax+x^2)}\};$$

$$\therefore\ \{a+x+\sqrt{(2ax+x^2)}\}^3 = 2b^2\{(a+x)^2-(2ax+x^2)\} = 2a^2b^2 = m^3;$$

$$\therefore\ a + x + \sqrt{(2ax+x^2)} = m,$$

$$\text{and } 2ax + x^2 = m^2 - 2m(a+x) + a^2 + 2ax + x^2;$$

$$\therefore\ x = \frac{m^2+a^2-2am}{2m} = \frac{(m-a)^2}{2m} = \frac{a^{\frac{4}{3}}}{2\sqrt[3]{(2b^2)}} - a + \frac{\sqrt[3]{(a^2b^2)}}{\sqrt[3]{(4)}},$$

or by this method, let $(x+a) = z$; then the equation becomes

$$b\{(z+a)^{\frac{1}{2}} - (z-a)^{\frac{1}{2}}\} = z + (z^2-a^2)^{\frac{1}{2}} = \frac{1}{2}\{(z+a)^{\frac{1}{2}} + (z-a)^{\frac{1}{2}}\}^2;$$

$$\therefore\ \{(z+a)^{\frac{1}{2}} + (z-a)^{\frac{1}{2}}\}^3 = 4ab,$$

$$\text{and hence } (z+a)^{\frac{1}{2}} + (z-a)^{\frac{1}{2}} = (4ab)^{\frac{1}{3}} \ \ldots\ldots(\alpha);$$

$$\text{and } \therefore\ (z+a)^{\frac{1}{2}} - (z-a)^{\frac{1}{2}} = \frac{2a}{(4ab)^{\frac{1}{3}}} \ \ldots\ldots\ (\beta),$$

$$(\alpha)^2 + (\beta)^2 \text{ gives } 4z = 2(2a^2b^2)^{\frac{1}{3}} + a(4ab^{-2})^{\frac{1}{3}};$$

$$\therefore\ x = z - a = \frac{a}{2}\cdot\sqrt[3]{\left(\frac{a}{2b^2}\right)} + \frac{ab}{2}\cdot\sqrt[3]{\left(\frac{2}{ab}\right)} - a.$$

(212). $\{x + 2\sqrt{(x)}\}^{\frac{1}{2}} - \{x - 2\sqrt{(x)}\}^{\frac{1}{2}} = 2(x^2 - 4x)^{\frac{1}{8}}$,

$$x + 2\sqrt{(x)} + x - 2\sqrt{(x)} - 2\sqrt{(x^2 - 4x)} = 4(x^2 - 4x)^{\frac{1}{4}},$$

$$\text{whence } x^2 + (x^2 - 4x) - 2x\sqrt{(x^2 - 4x)} = 4(x^2 - 4x)^{\frac{1}{2}},$$

$$\text{or } x(x - 2) = (x + 2)\sqrt{(x^2 - 4x)};$$

$$\therefore \frac{(x-2)^2}{(x+2)^2} = \frac{x^2 - 4x}{x^2}; \ \therefore (x + 2)^2 = 2x^2,$$

$$\text{and } x + 2 = \pm\sqrt{2}x; \ \therefore x = \frac{2}{\pm\sqrt{2} - 1} = 2 \pm 2\sqrt{2}.$$

(213). $(x - 1)^2 + (a - 1)^2 - 2(ax + 1) = \sqrt{\{3(x + a)^2 + 4ax\}}$,

$$\text{or } (x - a)^2 - 2(x + a) = \sqrt{\{3(x + a)^2 + 4ax\}};$$

$$\therefore (x - a)^4 - 4(x - a)^2(x + a) + 4(x + a)^2 = 3(x + a^2) + 4ax,$$

$$\text{or } (x - a)^4 - 4(x - a)^2(x + a) + x^2 - 2ax + a^2 = 0;$$

$$\therefore (x - a)^2 = 0, \text{ and } x = a,$$

$$\text{whence } x^2 - 2(a + 2)x + (a + 2)^2 = 8a + 3;$$

$$\therefore x = a + 2 \pm \sqrt{(8a + 3)}.$$

(214). $\left(x^{\frac{1}{3}} + \dfrac{1}{x^{\frac{1}{3}}}\right)\sqrt{\left(x^{\frac{1}{2}} + \dfrac{1}{x^{\frac{1}{2}}}\right)} = \sqrt{\left\{\dfrac{1}{3}\left(x^{\frac{3}{2}} + \dfrac{1}{x^{\frac{3}{2}}}\right)\right\}}$,

$$\text{or } x^{\frac{2}{3}} + 2 + \frac{1}{x^{\frac{2}{3}}} = \frac{1}{3}\left(x^{\frac{3}{2}} + \frac{1}{x^{\frac{3}{2}}}\right) \div (x^{\frac{1}{2}} + x^{-\frac{1}{2}});$$

$$\therefore 3\left(x^{\frac{2}{3}} + \frac{1}{x^{\frac{2}{3}}}\right) = x - 1 + \frac{1}{x} - 6,$$

$$\therefore x = -1,$$

$$\text{and } 27\left(x^{\frac{2}{3}} + \frac{1}{x^{\frac{2}{3}}}\right)^3 = \overline{\left(x + \frac{1}{x}\right) - 7}\Big]^3;$$

$$\therefore 27\left\{x^2 + 3\left(x^{\frac{2}{3}} + \frac{1}{x^{\frac{2}{3}}}\right) + \frac{1}{x^2}\right\} = x^3 + 3x + \frac{3}{x} + \frac{1}{x^3}$$

$$- 21\left(x^2 + 2 + \frac{1}{x^2}\right) + 147\left(x + \frac{1}{x}\right) - 343,$$

and by substituting $x - 7 + \frac{1}{x}$ for $3\left(x^{\frac{2}{3}} + \frac{1}{x^{\frac{2}{3}}}\right)$, we have

$$x^3 - 48x^2 + 123x - 196 + \frac{123}{x} - \frac{48}{x^2} + \frac{1}{x^3} = 0,$$

$$\text{then } x^3 + \frac{1}{x^3} - 48\left(x + \frac{1}{x}\right)^2 + 123\left(x + \frac{1}{x}\right) = 0;$$

$$\therefore\ x^2 - 1 + \frac{1}{x^2} - 48\left(x + \frac{1}{x}\right) + 123 = 0;\ \therefore\ x = \pm\sqrt{(-1)},$$

$$\text{also } \left(x + \frac{1}{x}\right)^2 - 48\left(x + \frac{1}{x}\right) + (24)^2 = -120 + 576 = 456,$$

$$\text{and } x + \frac{1}{x} = 24 \pm 2\sqrt{(114)} = 2m;$$

$$\therefore\ x^2 - 2mx + m^2 = -1 + m^2;$$

$$\therefore\ x = m \pm \sqrt{(m^2 - 1)}$$

$$= 12 \pm \sqrt{114} \pm 2\sqrt{\{257 \pm 24\sqrt{(114)}\}},$$

or thus, by squaring, &c., we have

$$x^{\frac{1}{2}} + x^{-\frac{1}{2}} = 0,\ \{\therefore\ x = -1\},$$

$$\text{or } 3(x^{\frac{1}{3}} + x^{-\frac{1}{3}})^2 = (x - 1 + x^{-1})$$

$$= (x^{\frac{1}{3}} + x^{-\frac{1}{3}})\{(x^{\frac{1}{3}} + x^{-\frac{1}{3}})^2 - 3\} - 1 \ldots\ldots\ldots (\alpha).$$

Let $x^{\frac{1}{3}} + x^{-\frac{1}{3}} = z$; then, from (α), $(z+1)^3 = 2z^3$;

$$\therefore\ z = \frac{1}{\sqrt[3]{(2)} - 1} = 2^{\frac{2}{3}} + 2^{\frac{1}{3}} + 1 = x^{\frac{1}{3}} + x^{-\frac{1}{3}} \ldots. (\beta).$$

$$\text{From } (\beta) \begin{cases} (x^{\frac{1}{6}} + x^{-\frac{1}{6}})^2 = (2^{\frac{1}{3}} - 1)^{-1} + 2 \\ (x^{\frac{1}{6}} - x^{-\frac{1}{6}})^2 = (2^{\frac{1}{3}} - 1)^{-1} - 2 \end{cases};$$

$$\therefore\ 2x^{\frac{1}{6}} = \{(2^{\frac{1}{3}} - 1)^{-1} + 2\}^{\frac{1}{2}} \pm \{(2^{\frac{1}{3}} - 1)^{-1} - 2\}^{\frac{1}{2}};$$

$$\text{and } x = \frac{1}{32}[(2^{\frac{1}{3}} - 1)^{-1} \pm \sqrt{\{(2^{\frac{1}{3}} - 1)^{-2} - 4\}}]^3\ 1.$$

EQUATIONS INVOLVING TWO OR MORE UNKNOWN QUANTITIES.

IX. (1). $2x + 3y = 18$, and $3x - 2y = 1$,

$$\text{if } 2x + 3y = 18; \therefore x = \frac{18 - 3y}{2}, \text{ also } x = \frac{2y + 1}{3};$$

$$\therefore \frac{18 - 3y}{2} = \frac{2y + 1}{3}, \text{ and } 54 - 9y = 4y + 2,$$

$$\text{whence } 13y = 52, \text{ and } y = 4,$$

$$\text{also } x = \frac{8 + 1}{3} = 3.$$

(2). $7(x - 5) = y - 2$, and $4y - 3 = \frac{1}{3}(x + 10)$,

$$x = \frac{y + 33}{7}, \text{ also } x = 12y - 19;$$

$$\therefore y + 33 = 84y - 133; \therefore y = \frac{166}{83} = 2,$$

$$\text{and } x = 24 - 19 = 5.$$

(3). $\frac{1}{7}(2x - y) = 2y - 3(x + 2)$, and $\frac{1}{5}(y + 3) + \frac{1}{6}(y - x) = 2(x - 4)$,

$$2x - y = 14y - 21x - 42; \therefore x = \frac{15y - 42}{23},$$

$$\text{also } 6y + 18 + 5y - 5x = 60x - 240; \therefore x = \frac{11y + 258}{65};$$

$$\therefore \frac{15y - 42}{23} = \frac{11y + 258}{65}, \text{ or } 975y - 2730 = 253y + 5934;$$

$$\therefore y = \frac{8664}{722} = 12, \text{ and } x = \frac{180 - 42}{23} = \frac{138}{23} = 60.$$

(4). $\frac{x}{2} - \frac{y}{3} = 16$, and $\frac{x}{5} - \frac{y}{8} = 12$,

$$x = \frac{2y}{3} + 32, \text{ also } x = \frac{5y}{8} + 60;$$

$$\therefore \frac{2y}{3} + 32 = \frac{5y}{8} + 60, \text{ or } 16y - 15y = 24 \times 28, \text{ or } y = 672;$$

$$\text{and } x = 5 \times 84 + 60 = 480.$$

(5). $\frac{7x+6}{11} + y - 16 = \frac{5x-13}{2} - \frac{8y-x}{5}$, and $3(3x+4) = 10y - 15$,

$$70x + 60 + 110y - 1760 = 275x - 715 - 176y + 22x;$$

$$\therefore\ x = \frac{286y - 985}{227},\ \text{also}\ x = \frac{10y-27}{9};$$

$$\therefore\ \frac{10y-27}{9} = \frac{286y-985}{227},\ \text{or}\ y = \frac{2736}{304} = 9,$$

$$\text{and}\ x = \frac{90-27}{9} = \frac{63}{9} = 7.$$

(6). $9x - 4y = 8$, and $x + \frac{7y}{13} = 7\frac{10}{13}$,

$$x = \frac{8+4y}{9},\ \text{also}\ x = \frac{101}{13} - \frac{7y}{13};$$

$$\therefore\ 104 + 52y = 909 - 63y;\ \therefore\ y = \frac{805}{115} = 7,$$

$$\text{also}\ x = \frac{8+28}{9} = 4.$$

(7). $\frac{4}{x} + \frac{3}{y} = \frac{8}{15}$, and $\frac{3}{x} + \frac{4}{y} = \frac{31}{60}$,

$$(1)\ \frac{12}{x} + \frac{9}{y} = \frac{8}{5},\ \text{also}\ (2)\ \frac{12}{x} + \frac{16}{y} = \frac{31}{15},$$

and subtracting (1) from (2)

$$\frac{16-9}{y} = \frac{31-24}{15};\ \therefore\ y = \frac{7\times 15}{7} = 15,$$

$$\text{also}\ \frac{4}{x} = \frac{8}{15} - \frac{3}{15} = \frac{5}{15};\ \therefore\ x = 12.$$

(8). $(x+5)(y+7) = (x-1)(y-9) + 104$, and $2x + 10 = 3y + 1$,

$$xy + 7x + 5y + 35 = xy - 9x - y + 9 + 104;\ \therefore\ x = \frac{78-6y}{16};$$

$$\text{also}\ x = \frac{3y-9}{2} = \frac{78-6y}{16};\ \therefore\ 24y - 72 = 78 - 6y,\ \text{and}\ y = 5,$$

$$\text{also}\ x = \frac{15-9}{2} = 3.$$

(9). $\frac{6x^2 + 130 - 24y^2}{3x + 6y + 1} = 3 - 4y + 2x,$

$$\text{and } \frac{12xy - 19x + 137}{4y - 1} = \frac{9xy - 110}{3y - 4},$$

$$6x^2 + 130 - 24y^2 = 3 + 14y + 11x + 6x^2 - 24y^2;$$

$$\therefore x = \frac{127 - 14y}{11},$$

also $36xy^2 - 105xy + 411y + 76x - 548 = 36xy^2 - 440y - 9xy + 110;$

$$\therefore x = \frac{658 - 851y}{76 - 96y} = \frac{127 - 14y}{11},$$

$$\text{whence } 1344y^2 - 3895y + 2414 = 0,$$

$$\text{and } y^2 = \frac{3895y}{1344} + \left(\frac{3895}{2688}\right)^2 = \left(\frac{3895}{2688}\right)^2 - \frac{2414}{1344} = \frac{2193361}{(2688)^2};$$

$$\therefore y = \frac{3895}{2688} \pm \frac{1481}{2688} = 2, \text{ or } \frac{1207}{1344};$$

$$\therefore x = \frac{127 - 28}{11} = \frac{99}{11} = 9.$$

(10). $4x + 5y = 40(x - y)$, and $\frac{2x - y}{3} + 2y = \frac{1}{2}$,

$$x = \frac{45y}{36} = \frac{5y}{4}, \text{ also } x = \frac{3 - 10y}{4}; \therefore y = \frac{1}{5},$$

$$\text{and } x = \frac{1}{4}.$$

(11). $\frac{1}{3x} + \frac{1}{5y} = \frac{19}{90}$, and $\frac{1}{5x} + \frac{1}{3y} = \frac{7}{30}$,

$$\frac{1}{x} + \frac{3}{5y} = \frac{19}{30}, \text{ also } \frac{1}{x} + \frac{5}{3y} = \frac{7}{6},$$

and by subtraction,

$$\frac{5}{3y} - \frac{3}{5y} = \frac{35}{30} - \frac{19}{30} = \frac{8}{15}; \therefore y = 2, \text{ and } x = 3.$$

(12). $(x + 5)(y + 7) = (x + 1)(y - 9) + 152$, and $2x + 10 = 3y - 1$,

$$xy + 5y + 7x + 35 = xy + y - 9x - 9 + 152;$$

$$\therefore x = \frac{108 - 4y}{16} = \frac{27 - y}{4}, \text{ also } x = \frac{3y - 11}{2};$$

$$\therefore \frac{27 - y}{4} = \frac{3y - 11}{2}, \text{ and } 27 - y = 6y - 22; \therefore y = 7, \text{ and } x = 5.$$

(13). $\frac{7 + x}{5} - \frac{2x - y}{4} = 3y - 12$, and $\frac{5y - 7}{2} + \frac{4x - 3}{6} = \frac{308}{6} - 5x$,

$$28 + 4x - 10x + 5y = 60y - 240; \therefore x = \frac{268 - 55y}{6},$$

$$\text{also } 15y - 21 + 4x - 3 = 308 - 30x; \therefore x = \frac{332 - 15y}{34};$$

$$\therefore \frac{268 - 55y}{6} = \frac{332 - 15y}{34}, \text{ or } 4556 - 935y = 996 - 45y;$$

$$\therefore y = \frac{3560}{890} = 4, \text{ and } x = \frac{332 - 60}{34} = 8.$$

(14). $\frac{1}{3}(x + y) + \frac{1}{4}(x - y) = 59$, and $5x = 33y$,

$$4x + 4y + 3x - 3y = 708; \therefore x = \frac{708 - y}{7} = \frac{33y}{5},$$

$$\text{whence } 3540 - 5y = 231y, \text{ or } y = \frac{3540}{236} = 15, \text{ and } x = 99.$$

(15). $\dfrac{\frac{7x}{4} + 6y}{5} - \dfrac{\frac{3y + 6}{5} - \frac{3x - 2}{10}}{8} = \dfrac{80 - x}{16}$,

$$\text{and } \left(\frac{3x}{2} + \frac{2y}{3} + \frac{5}{2}\right) = 9\left(\frac{x}{2} - \frac{y}{3} + \frac{1}{6}\right),$$

$$\text{hence } \frac{7x + 24y}{20} - \frac{6y + 12 - 3x + 2}{80} = \frac{80 - x}{16};$$

$$\therefore 28x + 96y - 6y + 3x - 14 = 400 - 5x; \therefore x = \frac{414 - 90y}{36}$$

$$\text{also } 18x + 8y + 30 = 54x - 36y + 18; \therefore x = \frac{44y + 12}{36};$$

$$\therefore 44y + 12 = 414 - 90y, \text{ and } y = \frac{402}{134} = 3, \text{ and } x = 4.$$

(16). $\frac{7+x}{5} - \frac{2x-y}{4} = 3y - 5$, and $\frac{5y-7}{2} + \frac{4x-3}{6} = 18 - 5x$,

hence $28 + 4x - 10x + 5y = 60y - 100$; $\therefore y = \frac{128 - 6x}{55}$,

also $30y - 42 + 8x - 6 = 216 - 60x$; $\therefore y = \frac{264 - 68x}{30}$;

$$\therefore \frac{128 - 6x}{11} = \frac{264 - 68x}{6},$$

or $384 - 18x = 1452 - 374x$; $\therefore x = \frac{1068}{356} = 3$,

and $y = \frac{110}{55} = 2$.

(17). $x - \frac{2y - x}{23 - x} = 20 - \frac{59 - 2x}{2}$, and $y + \frac{y-3}{x-18} = 30 - \frac{73 - 3y}{3}$,

$$\frac{23x - x^2 - 2y + x}{23 - x} = \frac{40 - 59 + 2x}{2};$$

$$\therefore 48x - 2x^2 - 4y = 65x - 2x^2 - 437;$$

$\therefore x = \frac{437 - 4y}{17}$, also $\frac{xy - 18y + y - 3}{x - 18} = \frac{90 - 73 + 3y}{3}$;

$$\therefore 3xy - 51y - 9 = 3xy + 17x - 54y - 306;$$

$$\therefore x = \frac{297 + 3y}{17} = \frac{437 - 4y}{17};$$

$\therefore 297 + 3y = 437 - 4y$, and $y = \frac{140}{7} = 20$;

and $x = \frac{297 + 60}{17} = \frac{357}{17} = 21$.

(18). $\frac{3x - 5y}{3} - \frac{2x - 8y - 9}{12} = \frac{y}{2} + \frac{1}{3} + \frac{1}{4}$,

and $\frac{x}{7} + \frac{y}{4} + \frac{4}{3} : 4x - \frac{y}{8} - 24 :: 3\frac{1}{3} : 3\frac{1}{2}$,

$12x - 20y - 2x + 8y + 9 = 6y + 4 + 3$; $\therefore y = \frac{5x + 1}{9}$,

also $\frac{x}{7} + \frac{y}{4} + \frac{4}{3} : \frac{32x - y - 192}{8} :: 20 : 21$;

$$\therefore 2\left(3x + \frac{21y}{4} + 28\right) = 160x - 5y - 960;\ y = 2\frac{(154x - 1016)}{31};$$

$$\therefore \frac{5x+1}{9} = 2\frac{(154x - 1016)}{31};\ \therefore x = \frac{18319}{2617} = 7,$$

$$\text{and } y = \frac{36}{9} = 4.$$

(19). $2x + \cdot 4y = 1\cdot 2$, and $3\cdot 4x - \cdot 02y = \cdot 01$,

$$20x + 4y = 12;\ \therefore x = \frac{3-y}{5},$$

$$\text{also } 340x - 2y = 1;\ \therefore x = \frac{2y+1}{340};$$

$$\therefore 204 - 68y = 2y + 1, \text{ or } y = \frac{203}{70} = 2\cdot 9, \text{ and } x = \frac{6\cdot 8}{340} = \cdot 02.$$

(20). $2\cdot 4x + 0\cdot 32y - \dfrac{0\cdot 36x - 0\cdot 05}{0\cdot 5} = 0\cdot 8x + \dfrac{2\cdot 6 + 0\cdot 005y}{0\cdot 25}$,

$$\text{and } \frac{0\cdot 04y + 0\cdot 1}{0\cdot 3} = \frac{0\cdot 07x - 0\cdot 1}{0\cdot 6},$$

$$\text{from (1) } 1\cdot 6x + .32y - \cdot 72x + \cdot 1 = 10\cdot 4 + \cdot 02y;$$

$$\therefore x = \frac{10\cdot 3 - \cdot 3y}{\cdot 88} = \frac{1030 - 30y}{88};$$

$$\text{also } \cdot 08y + \cdot 2 = \cdot 07x - \cdot 1;\ \therefore x = \frac{\cdot 08y + \cdot 3}{\cdot 07} = \frac{8y+30}{7},$$

$$\text{whence } \frac{1030 - 30y}{88} = \frac{8y+30}{7}, \text{ and } y = \frac{4570}{914} = 5;$$

$$\therefore x = \frac{40+30}{7} = 10.$$

(21). $\dfrac{x}{a} + \dfrac{y}{b} = 1 = \dfrac{x-a}{b} + \dfrac{y-b}{a}$,

$$x + \frac{ay}{b} = a, \text{ also } x - a + \frac{b}{a}(y - b) = b;$$

therefore, by subtraction,

$$\frac{ay}{b} + a - \frac{b}{a}(y-b) = a - b, \text{ whence } y = \frac{-b^2}{a-b} = \frac{b^2}{b-a},$$

$$\text{and } x = a + \frac{ab}{a-b} = \frac{a^2}{a-b}.$$

(22). $\frac{y}{x} - \frac{x}{x+y} = \frac{x^2-y^2}{y}$, and $\frac{x}{y} - \frac{x+y}{x} = \frac{y}{x}$,

$$\text{from (2) } \frac{x^2-y^2}{xy} = \frac{x+y}{x}; \therefore x+y=0, \text{ and } x=-y,$$

$$\text{also } x-y-y=0; \therefore x=2y,$$

$$\text{from (1) } \frac{1}{2} - \frac{2}{3} = 3y; \therefore y = -\frac{1}{18}, \text{ and } x = -\frac{1}{9}.$$

(23). $3a^2 + ax = b(b+y)$, and $ax + 2by = a^2$,

$$x = \frac{b(b+y)-3a^2}{a} = \frac{a^2-2by}{a}; \therefore y = \frac{4a^2-b^2}{3b},$$

$$\text{and } x = a - \frac{2}{a} \cdot \frac{4a^2-b^2}{3} = \frac{2b^2-5a^2}{3a}.$$

(24). $\frac{m}{x} + \frac{n}{y} = a$, and $\frac{m}{y} + \frac{n}{x} = b$,

$$\frac{1}{x} + \frac{n}{my} = \frac{a}{m}, \text{ and } \frac{1}{x} + \frac{m}{ny} = \frac{b}{n};$$

$$\therefore \left(\frac{n}{m} - \frac{m}{n}\right)\frac{1}{y} = \frac{a}{m} - \frac{b}{n}; \therefore y = \frac{n^2-m^2}{an-bm},$$

$$\text{and } \frac{1}{x} = \frac{a}{m} - \frac{n}{m} \cdot \frac{an-bm}{n^2-m^2} = \frac{an^2-am^2-an^2+bmn}{m(n^2-m^2)};$$

$$\therefore x = \frac{n^2-m^2}{bn-am}.$$

(25). $x + y = a$, and $ax = by$,

$$\text{from (2) } x = \frac{by}{a}; \therefore \frac{by}{a} + y = a, \text{ or } y = \frac{a^2}{a+b},$$

$$\text{and } x = \frac{ab}{a+b}.$$

(26). $ax + by = m$, and $a'x + b'y = n$,

$$aa'x + a'by = a'm, \text{ also } aa'x + ab'y = an;$$

therefore, by subtraction,

$$(a'b - ab')\,y = a'm - an;$$

$$\therefore y = \frac{a'm-an}{a'b-ab'}, \text{ and } x = \frac{b'm-bn}{ab'-a'b}.$$

(27). $\frac{x}{a} + \frac{y}{b} = 1 - x$, and $\frac{y}{a} + \frac{x}{b} = 1 + y$,

$$\left(\frac{1}{a} + 1\right) x = 1 - \frac{y}{b}, \text{ also } x = b(1 + y) - \frac{by}{a};$$

$$\therefore \frac{(b - y)a}{b(a + 1)} = \frac{ab(1 + y) - by}{a}, \text{ whence } y = \frac{a^2b - a^2b^2 - ab^2}{a^2 + a^2b^2 - b^2},$$

$$\text{and } x = \frac{a}{a + 1} - \frac{a}{b(a + 1)} \times \frac{a^2b - a^2b^2 - ab^2}{a^2 + a^2b^2 - b^2} = \frac{ab(ab + a - b)}{a^2 + a^2b^2 - b^2}.$$

(28). $c(bx + ay) = axy$, and $c(ax - by) = bxy$,

$$x = \frac{acy}{ay - bc} = \frac{bcy}{ac - by}, \text{ or } \frac{a}{ay - bc} = \frac{b}{ac - by};$$

$$\therefore a^2c - aby = aby - b^2c; \therefore y = \frac{c(a^2 + b^2)}{2ab},$$

$$\text{and } bcx + acy = axy, \text{ whence } x = \frac{c(a^2 + b^2)}{a^2 - b^2}.$$

(29). $(a^2 - b^2)(3x + 5y) = 4\left(a - \frac{b}{4}\right) 2ab$,

$$\text{and } a^2x - \frac{ab^2c}{a + b} + (a + b + c) by = b^2x + (a + 2b) ab;$$

$$\therefore x = \frac{8ab\left(a - \frac{b}{4}\right) - 5y(a^2 - b^2)}{3(a^2 - b^2)},$$

$$\text{also } x = \frac{1}{a^2 - b^2}\left\{(a + 2b) ab + \frac{ab^2c}{a + b} - (a + b + c) by\right\};$$

$$\therefore 8ab\left(a - \frac{b}{4}\right) - 5y(a^2 - b^2) = 3ab(a + 2b) + \frac{3ab^2c}{a + b} - (a + b + c) 3by,$$

$$\text{whence } y\{(a + b + c) 3b - 5(a^2 - b^2)\} = \frac{ab\{3bc - (5a - 8b)(a + b)\}}{a + b};$$

$$\therefore y = \frac{ab}{a + b} \times \frac{3bc - 5a^2 + 3ab + 8b^2}{3bc - 5a^2 + 3ab + 8b^2} = \frac{ab}{a + b},$$

$$\text{and } x = \frac{ab}{a^2 - b^2}\left\{\frac{8a - 2b}{3} - \frac{5}{3} \times \frac{a - b}{1}\right\} = \frac{ab}{a - b}.$$

(30). $\frac{1}{mx} + \frac{1}{ny} = \frac{1}{a}$, and $\frac{1}{nx} + \frac{1}{my} = \frac{1}{b}$;

$$\therefore \frac{1}{x} + \frac{m}{ny} = \frac{m}{a}, \text{ also } \frac{1}{x} + \frac{n}{my} = \frac{n}{b},$$

whence, by subtraction,

$$\left(\frac{m}{n} - \frac{n}{m}\right)\frac{1}{y} = \frac{m}{a} - \frac{n}{b}; \quad \therefore y = \frac{m^2 - n^2}{mb - an} \times \frac{ab}{mn},$$

$$\text{also } \left(\frac{n}{m} - \frac{m}{n}\right)\frac{1}{x} = \frac{n}{a} - \frac{m}{b}; \quad \therefore x = \frac{n^2 - m^2}{bn - ma} \times \frac{ab}{nm}.$$

(31). $ax + by = m$, and $a^2x - b^2y = bm$,

$$x = \frac{m - by}{a} = \frac{bm + b^2y}{a^2}; \quad \therefore y = \frac{m(a-b)}{b(a+b)},$$

$$\text{and } x = \frac{m}{a} - \frac{b}{a} \times \frac{m(a-b)}{b(a+b)} = \frac{2mb}{a(a+b)}.$$

(32). $a(x^2 + y^2) - b(x^2 - y^2) = 2a$, and $(a^2 - b^2)(x^2 - y^2) = 4ab$,

$$(1)\ (a - b)x^2 + (a + b)y^2 = 2a,$$

$$\text{also } (2)\ (a^2 - b^2)x^2 - (a^2 - b^2)y^2 = 4ab,$$

multiplying (1) by $(a + b)$ and subtracting (2)

$$(a + b)^2 y^2 + (a^2 - b^2)y^2 = 2a(a + b) - 4ab,$$

$$\text{or } y = \pm\sqrt{\left(\frac{a-b}{a+b}\right)}, \text{ and } x = \pm\sqrt{\left(\frac{a+b}{a-b}\right)}.$$

(33). $(x + a)(y - b) + 2c = (x - a)(y + b)$,

$$\text{and } (x + b)(y - a) = (x + a)(y - b),$$

$$\text{from (1) } x(y + b - y + b) = 2c + a(y - b) + a(y + b);$$

$$\therefore x = \frac{2c + 2ay}{2b} = \frac{c + ay}{b},$$

$$\text{from (2) } x(y - a - y + b) = a(y - b) - b(y - a);$$

$$\therefore x = \frac{(a - b)y}{b - a} = -y;$$

$$\therefore c + ay = -by, \text{ or } y = \frac{-c}{a+b},\ x = \frac{c}{a+b}.$$

(34). $(x+y)a - b(x-y) = 2a^2$, and $(a^2-b^2)(x-y) = 4a^2b$,

$$\text{from (1) } x = \frac{2a^2-(a+b)y}{a-b}, \text{ from (2) } x = \frac{4a^2b+(a^2-b^2)y}{a^2-b^2};$$

$$\therefore 2a^2(a+b) - (a+b)^2y = 4a^2b + (a^2-b^2)y; \ \therefore y = \frac{a(a-b)}{a+b},$$

$$\text{and } x = \frac{2a^2 - a(a-b)}{a-b} = \frac{a(a+b)}{a-b}.$$

(35). $(x+y)b = a(x-y)$, and $(x+y)^2 = ax - b\left(\frac{1}{4}b + y\right)$,

$$\text{from (1) } (a-b)x = (a+b)y; \ \therefore x = \frac{(a+b)y}{a-b},$$

by substitution in (2)

$$\frac{4a^2y^2}{(a-b)^2} - \frac{(a^2+b^2)y}{a-b} + \frac{(a^2+b^2)^2}{16a^2} = \frac{(a^2+b^2)^2}{16a^2} - \frac{b^2}{4} = \frac{(a^2-b^2)^2}{16a^2};$$

$$\therefore y = \frac{a-b}{2a}\left(\frac{a^2+b^2}{4a} \pm \frac{a^2-b^2}{4a}\right) = \frac{a-b}{4}, \text{ and } x = \frac{a+b}{4}.$$

(36). $x^2 = ax + by$, and $y^2 = bx + ay$,

$$x^2 - y^2 = a(x-y) - b(x-y); \ \therefore x - y = 0, \text{ and } x = y,$$

$$\text{also } x + y = a - b; \ \therefore x = \frac{a-b}{2}, \ y = \frac{a-b}{2}.$$

(37). $\dfrac{10x^2-12y^2-14xy+2x}{5x+3y+3} = \dfrac{4x-8y+1}{2}$, and $2\sqrt{(6+x)} = 3\sqrt{(6-y)}$,

$$20x^2 - 24y^2 - 28xy + 4x = 20x^2 - 28xy - 24y^2 + 17x - 21y + 3;$$

$$\therefore y = \frac{13x+3}{21}, \text{ from (2) } y = \frac{30-4x}{9},$$

$$\text{whence } 39x + 9 = 210 - 28x, \text{ and } x = \frac{201}{67} = 3,$$

$$\text{and } y = 2.$$

(38). $x + y = 3y$, and $xy = 18$,

$$x = 2y = \frac{18}{y}; \ \therefore 2y^2 = 18, \ y^2 = 9; \ \therefore y = \pm 3, \ x = \pm 6.$$

(39). $x - y = a$, and $y(y + a) + bx = 0$,

$$x = (y + a);\ \therefore\ y(y + a) + b(y + a) = 0;$$

$$\therefore\ y + a = 0, \text{ and } y = -a,$$

$$\text{and } x = 0, \text{ also } y = -b;\ \therefore\ x = a - b.$$

(40). $x - 2y = 2$, and $3xy = 36$,

$$x = 2(y + 1);\ \therefore\ 6y(y + 1) = 36;$$

$$\therefore\ y^2 + y + \frac{1}{4} = 6 + \frac{1}{4} = \frac{25}{4}, \text{ and } y = -\frac{1}{2} \pm \frac{5}{2} = 2 \text{ or } -3;$$

$$\therefore\ x = 2(y + 1) = 6 \text{ or } -4.$$

(41). $x - y = 2$, and $\dfrac{x}{y} - \dfrac{y}{x} = \dfrac{16}{15}$,

from (1) $x = y + 2$, from (2) $15(x^2 - y^2) = 16xy$;
therefore, by substitution,

$$y^2 - \frac{7y}{4} + \left(\frac{7}{8}\right)^2 = \frac{15}{4} + \frac{49}{64} = \frac{289}{64};$$

$$\therefore\ y = \frac{7}{8} \pm \frac{17}{8} = 3 \text{ or } -\frac{5}{4}, \text{ and } x = 5 \text{ or } \frac{3}{4}.$$

(42). $5(x - y) = 4y$, and $x^2 + 4y^2 = 181$,

$$x = \frac{9y}{5};\ \therefore\ \frac{81y^2}{25} + 4y^2 = 181, \text{ and } y^2 = \frac{181 \times 25}{181};\ \therefore\ y = \pm 5,$$

$$\text{and } x = \pm \frac{9}{5} \times 5 = \pm 9.$$

(43). $x + 4y = 7$, and $x^2 + y^2 = 10$,

$$x = 7 - 4y;\ \therefore\ 49 - 56y + 16y^2 + y^2 = 10,$$

$$\text{whence } y^2 - \frac{56y}{17} + \left(\frac{28}{17}\right)^2 = -\frac{39}{17} + \left(\frac{28}{17}\right)^2 = \frac{121}{289};$$

$$\therefore\ y = \frac{28}{17} \pm \frac{11}{17} = 2\tfrac{5}{17} \text{ or } 1, \text{ and } x = 3 \text{ or } -1\tfrac{12}{17}.$$

(44). $\dfrac{x + y}{x^2 - y^2} = \dfrac{1}{4}$, and $xy = 21$,

$$\text{from (1) } \frac{1}{x - y} = \frac{1}{4};\ \therefore\ x - y = 4;\ \therefore\ x = y + 4,$$

$$\text{and } y^2 + 4y + 4 = 25;\ \therefore\ y = -2 \pm 5 = 3 \text{ or } -7,$$

$$\text{and } x = 7 \text{ or } -3.$$

(45). $x + y = 1$, and $x^2 - xy = 153$,

$$\text{from (1) } x = 1 - y; \therefore y^2 - 2y + 1 - y + y^2 = 153;$$

$$\therefore y^2 - \frac{3y}{2} + \left(\frac{3}{4}\right)^2 = \frac{152}{2} + \frac{9}{16} = \frac{1225}{16};$$

$$\therefore y = \frac{3}{4} \pm \frac{35}{4} = 9\tfrac{1}{2} \text{ or } -8, \text{ and } x = -8\tfrac{1}{2} \text{ or } 9.$$

(46). $\dfrac{x^3 + y^3}{x^3 - y^3} = \dfrac{559}{127}$, and $x^2y = 294$,

$$\text{from (1) } \frac{x^3}{y^3} = \frac{343}{216}, \text{ or } x = \frac{7y}{6},$$

$$\text{from (2) } \frac{49y^3}{36} = 294; \therefore y = 6, \text{ and } x = 7.$$

(47). $x^2 - xy = y^2 - \dfrac{5xy}{12}$, and $x - 2 = y$,

$$\text{from (2) } x = y + 2; \therefore (y + 2)^2 - \frac{7y}{12}(y + 2) = y^2,$$

$$\text{whence } (4y + 4)\,12 = 7y^2 + 14y,$$

$$\text{and } y^2 - \frac{34y}{7} + \left(\frac{17}{7}\right)^2 = \frac{48}{7} + \frac{289}{49} = \frac{625}{49};$$

$$\therefore y = \frac{17}{7} \pm \frac{25}{7} = 6 \text{ or } -1\tfrac{1}{7}, \text{ and } x = 8 \text{ or } +\frac{6}{7}.$$

(48). $x(x + y) = 66$, and $x^2 - y^2 = 11$,

$$\text{from (1) } y = \frac{66 - x^2}{x};$$

$$\therefore x^2 - y^2 = x^2 - \frac{(66)^2 - 132x^2 + x^4}{x^2} = 11;$$

$$\therefore x^4 - 66^2 + 132x^2 - x^4 = 11x^2; \therefore x = \pm 6,$$

$$\text{and } y = \frac{66 - 36}{\pm 6} = \pm 5.$$

(49). $x^2 + y^2 = 5$, and $x - y = 1$,

$$x^2 = 5 - y^2 = y^2 + 2y + 1;$$

$$\therefore y^2 + y + \frac{1}{4} = \frac{4}{2} + \frac{1}{4} = \frac{9}{4};$$

$$\therefore y = 1 \text{ or } -2, \text{ and } x = 2 \text{ or } -1.$$

(50). $x^2 - xy = 6$, and $x^2 + y^2 = 61$,

$$y = \frac{x^2 - 6}{x}; \ \therefore\ x^2 + \frac{x^4 - 12x^2 + 36}{x^2} = 61,$$

$$\text{whence } x^4 - \frac{73x^2}{2} + \left(\frac{73}{4}\right)^2 = -\frac{36}{2} + \frac{5329}{16} = \frac{5041}{16};$$

$$\therefore\ x^2 = 36 \text{ or } \frac{1}{2}; \ \therefore\ x = \pm 6 \text{ or } \pm\sqrt{\left(\frac{1}{2}\right)},$$

$$\text{and } y = \pm 5 \text{ or } \frac{-11}{\sqrt{(2)}}.$$

(51). $\dfrac{x^3 - y^3}{x^2y - xy^2} = \dfrac{7}{2}$, and $x + y = 6$,

$$\text{from (1) } \frac{x^2 + xy + y^2}{xy} = \frac{7}{2};$$

$$\therefore\ \frac{x^2 + 2xy + y^2}{xy} = \frac{9}{2} = \frac{(x+y)^2}{xy} = \frac{36}{xy} = \frac{9}{2}; \ \therefore\ xy = 8,$$

$$\text{and } x^2 - 2xy + y^2 = 36 - 32 = 4;$$

$$\therefore\ x - y = \pm 2, \text{ and } x + y = 6; \ \therefore\ x = 4 \text{ or } 2, \text{ and } y = 2 \text{ or } 4.$$

(52). $x^3 + y^3 = 35$, and $x + y = 5$,

$$\frac{x^3 + y^3}{x + y} = x^2 - xy + y^2 = 7, \text{ and } x^2 + 2xy + y^2 = 25,$$

whence by subtraction $xy = 6$,

$$\text{and } x^2 - 2xy + y^2 = 25 - 24; \ \therefore\ x - y = \pm 1, \text{ but } x + y = 5;$$

$$\therefore\ x = 3 \text{ or } 2, \text{ and } y = 2 \text{ or } 3.$$

(53). $\dfrac{1}{x} + \dfrac{1}{y} = \dfrac{1}{2}$, and $\dfrac{2}{xy} = \dfrac{1}{9}$,

$$\text{from (1) and (2) } \frac{1}{x^2} - \frac{2}{xy} + \frac{1}{y^2} = \frac{1}{4} - \frac{2}{9} = \frac{1}{36};$$

$$\therefore\ \frac{1}{x} - \frac{1}{y} = \pm\frac{1}{6}, \text{ and } \frac{1}{x} + \frac{1}{y} = \frac{1}{2},$$

whence $x = 3$ or 6, and $y = 6$ or 3.

(54). $x^4 - y^4 = 369$, and $x^2 - y^2 = 9$,

$$\frac{x^4 - y^4}{x^2 - y^2} = x^2 + y^2 = 41, \text{ but } x^2 - y^2 = 9;$$

$$\therefore\ x = \pm 5, \text{ and } y = \pm 4.$$

(55). $x + y = 11$, and $x^3 + y^3 = 341$,

$$\frac{x^3 + y^3}{x + y} = x^2 - xy + y^2 = 31, \text{ and } x^2 + 2xy + y^2 = 121;$$

therefore by subtraction $xy = 30$,

and $x^2 - 2xy + y^2 = 121 - 120 = 1$;

$\therefore\ x - y = \pm 1$, and $x + y = 11$;

$\therefore\ x = 6$ or 5, and $y = 5$ or 6.

(56). $x - y = 8$, and $x^4 - y^4 = 14560$,

let $x = m + n$, $y = m - n$; $\therefore\ x - y = 2n = 8$; $\therefore\ n = 4$,

and $(x^2 + y^2) = 2(m^2 + 16) : x^2 - y^2 = 16m$;

$\therefore\ (x^4 - y^4) = (x^2 - y^2)(x^2 + y^2) = 32m(m^2 + 16) = 14560$,

whence $m^3 + 16m = 455$,

and $m^3 - 343 + 16(m - 7) = 0$, or $m = 7$;

$\therefore\ x = 11$, and $y = 3$,

and $m^2 + 7m + 49 + 16 = 0$,

whence $m = -\frac{7}{2} \pm \frac{1}{2}\sqrt{(-211)}$,

also $m^4 + 16m^2 = 455m = 65 \times 7m$;

$$\therefore\ m^4 + 65m^2 + \left(\frac{65}{2}\right)^2 = 49m^2 + 455m + \left(\frac{65}{2}\right)^2;$$

$$\therefore\ m^2 = -\frac{65}{2} \pm \left(7m + \frac{65}{2}\right); \quad \therefore\ m = 7 \text{ as before.}$$

(57). $x^2 + xy + y^2 = 14x$, and $x^4 + x^2y^2 + y^4 = 84x^2$,

$$\frac{x^4 + x^2y^2 + y^4}{x^2 + xy + y^2} = x^2 - xy + y^2 = 6x,$$

also $x^2 + xy + y^2 = 14x$; $\therefore\ xy = 4x$, and $y = 4$,

from (1) $x^2 - 10x + 25 = -16 + 25 = 9$; $\therefore\ x = 8$ or 2.

(58). $x + y = 14$, and $x^4 - y^4 = 14560$,

as in (56) if $x = m + n$, and $y = m - n$,

then $x + y = 2m = 14$, and $m = 7$;

$\therefore\ (x^2 - y^2 (x^2 + y^2) = 2(49 + n^2)\, 28n = 14560$,

$$\text{whence } n^4 + 65n^2 + \left(\frac{65}{2}\right)^2 = 16n^2 + 260n + \left(\frac{65}{2}\right)^2,$$

$$\text{or } n^2 = -\frac{65}{2} \pm \left(4n + \frac{65}{2}\right) \text{ as above.}$$

(59). $x^4 + y^4 = 641$, and $x^3y + y^3x = 290$,

from (2) $x^2 + y^2 = \frac{290}{xy}$; $\therefore\ x^4 + 2x^2y^2 + y^4 = \left(\frac{290}{xy}\right)^2$,

and subtracting (1)

$$2x^2y^2 = \frac{84100}{x^2y^2} - 641;$$

$$\therefore\ x^4y^4 + \frac{641}{2}x^2y^2 + \left(\frac{641}{4}\right)^2 = \frac{84100}{2} + \left(\frac{641}{4}\right)^2 = \frac{1083681}{16};$$

$$\therefore\ x^2y^2 = -\frac{641}{4} \pm \frac{1041}{4} = 100 \text{ or } -\frac{841}{2};\ \therefore\ xy = 10,$$

and $x^2 - 2xy + y^2 = 29 - 20 = 9$; $\therefore\ x - y = \pm 3$,

and $x^2 + 2xy + y^2 = 49$; $\therefore\ x + y = \pm 7$; $\therefore\ x = \pm 5$, and $y = \pm 2$.

(60). $(x^2 + y^2)(x^3 + y^3) = 455$, and $x + y = 5$,

$$\frac{(x^2 + y^2)(x^3 + y^3)}{x + y} = \{(x^2 + y^2)\}(x^2 - xy + y^2) = 91,$$

but $x^2 + y^2 = 25 - 2xy$, and $x^2 - xy + y^2 = 25 - 3xy$;

$$\therefore\ (25 - 2xy)(25 - 3xy) = 625 - 125xy + 6x^2y^2 = 91;$$

$$\therefore\ x^2y^2 - \frac{125xy}{6} + \left(\frac{125}{12}\right)^2 = -\frac{534}{6} + \left(\frac{125}{12}\right)^2 = \frac{2809}{144};$$

$$\therefore\ xy = \frac{125}{12} \pm \frac{53}{12} = \frac{89}{12} \text{ or } 6,$$

from $xy = 6$, $x^2 - 2xy + y^2 = 1$; $\therefore\ x - y = 1$, and $x + y = 5$;

$$\therefore\ x = 3, \text{ or } y = 2.$$

(61). $x^4 + y^4 = 97$, and $x + y = 5$,

from (2) $x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4 = 625$,

whence $x^4 + 4xy(x^2 + 2xy + y^2) - 2x^2y^2 + y^4 = 625$,

but $x^4 + y^4 \qquad = 97$;

$\therefore\ 4xy(25) - 2x^2y^2 \qquad = 528$;

$$\therefore\ x^2y^2 - 50xy + (25)^2 = -264 + 625 = 361;$$

$$\therefore\ xy = 25 \pm 19 = 44 \text{ or } 6,$$

if $xy = 6$, $x^2 - 2xy + y^2 = 1$; $\therefore\ x - y = \pm 1$, but $x + y = 5$;

$$\therefore\ x = 3 \text{ or } 2. \text{ and } y = 2 \text{ or } 3.$$

(62). $x^2 + xy + 4y^2 = 6$, and $3x^2 + 8y^2 = 11$,

$$\text{from (2) } 4y^2 = \frac{11 - 3x^2}{2}; \ \therefore\ 2x^2 + 2xy + 11 - 3x^2 = 12;$$

$$\therefore\ y = \frac{x^2 + 1}{2x}, \text{ and } 3x^2 + \frac{2(x^4 + 2x^2 + 1)}{x^2} = 11,$$

$$\text{whence } x^4 - \frac{7x^2}{5} + \left(\frac{7}{10}\right)^2 = \frac{9}{100}; \ \therefore\ x = 1 \text{ or } \pm\frac{\sqrt{2}}{\sqrt{(5)}};$$

$$\therefore\ y = 1 \text{ or } \frac{7}{2\sqrt{10}}.$$

(63). $x^4 + y^4 = 17$, and $x + y = 3$,

$$\text{as in 61, } 4xy \times 9 - 2x^2y^2 = 64,$$

$$\text{whence } x^2y^2 - 18xy + 81 = -32 + 81 = 49; \ \therefore\ xy = 16 \text{ or } 2,$$

$$\text{from } xy = 2,\ x^2 - 2xy + y^2 = 1; \ \therefore\ x - y = \pm 1, \text{ but } x + y = 3;$$

$$\therefore\ x = 2 \text{ or } 1, \text{ and } y = 1 \text{ or } 2.$$

(64). $x^5 + y^5 = 33$, and $x + y = 3$,

$$\frac{x^5 + y^5}{x + y} = x^4 - x^3y + x^2y^2 - xy^3 + y^4 = 11,$$

$$\text{but } x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4 = 81;$$

$$\therefore\ 5x^3y + 5x^2y^2 + 5xy^3 = 70,$$

$$\text{or } xy(x^2 + 2xy + y^2) - x^2y^2 = 14;$$

$$\therefore\ x^2y^2 - 9xy + \left(\frac{9}{2}\right)^2 = -\frac{14}{1} + \frac{81}{4} = \frac{25}{4}; \ \therefore\ xy = 7 \text{ or } 2,$$

$$\text{from } xy = 2,\ x^2 - 2xy + y^2 = 1; \ \therefore\ x - y = \pm 1, \text{ and } x + y = 3;$$

$$\therefore\ x = 2 \text{ or } 1, \text{ and } y = 1 \text{ or } 2.$$

(65). $\dfrac{x^2 + xy + y^2}{x^2 - xy + y^2} = \dfrac{7}{3}$, and $x + y = \dfrac{3xy}{2}$,

$$\text{from (1) } \frac{x^2 + y^2}{2xy} = \frac{5}{4}; \ \therefore\ \frac{(x+y)^2}{(x-y)^2} = \frac{9}{1}, \text{ and } \frac{x+y}{x-y} = \frac{3}{1}; \ \therefore\ x = 2y;$$

$$\therefore\ 3y = 3y^2; \ \therefore\ y = 1, \text{ and } x = 2.$$

(66). $2x^2 + 3xy + y^2 = 15$, and $5x^2 + 4y^2 = 24$,

$$\text{from (2) } y^2 = 6 - \frac{5x^2}{4}, \text{ from (1) } 2x^2 + 3xy + 6 - \frac{5x^2}{4} = 15;$$

$$\therefore\ y = \left(9 - \frac{3x^2}{4}\right)\frac{1}{3x};$$

$$\therefore \text{ from (2) } 5x^2 + \frac{4}{9x^2}\left(81 - \frac{27x^2}{2} + \frac{9x^4}{16}\right) = 24,$$

$$\text{whence } x^4 - \frac{40x^2}{7} + \left(\frac{20}{7}\right)^2 = \frac{64}{49};\ \therefore\ x = \pm 2 \text{ or } \pm 2\sqrt{\left(\frac{3}{7}\right)},$$

$$\text{from } x = 2,\ y = 1.$$

(67). $2x^2 + 3y^2 = 5 = 5(2x + 3y)$,

$$\text{from (2) } x = \frac{1 - 3y}{2};\ \therefore \text{ from (1) } 2(1 - 6y + 9y^2) + 12y^2 = 20,$$

$$\text{whence } y^2 - \frac{2y}{5} + \left(\frac{1}{5}\right)^2 = \frac{16}{25}, \text{ and } y = 1 \text{ or } -\frac{3}{5},$$

$$\text{from } y = 1,\ x = -1, \text{ and from } y = -\frac{3}{5},\ x = \frac{7}{5}.$$

(68). $x^4 - x^2 + y^4 - y^2 = 312$, and $x^2 + 2xy + y^2 = 49$,

from (2) $x^2 + y^2 = 49 - 2xy$, also $x^4 + y^4 = (49 - 2xy)^2 - 2x^2y^2$;

$$\therefore\ 49^2 - 196xy + 2x^2y^2 - 49 + 2xy = 312,$$

$$\text{whence } x^2y^2 - 97xy + \left(\frac{97}{2}\right)^2 = \frac{5329}{4};\ \therefore\ xy = 85 \text{ or } 12,$$

from $xy = 12$, $x^2 - 2xy + y^2 = 1$; $\therefore\ x - y = 1$, and $x + y = 7$;

$$\therefore\ x = 4,\ y = 3.$$

(69). $x^3 + y^3 + xy(x + y) = 13$, and $(x^2 + y^2)\,x^2y^2 = 468$,

from (1) $(x + y)(x^2 - xy + y^2 + xy) = (x + y)(x^2 + y^2)\ (A) = 13$;

$$\therefore\ \frac{(x^2 + y^2)\,x^2y^2}{(x + y)(x^2 + y^2)} = \frac{468}{13}, \text{ or } \frac{x^2y^2}{36} = x + y,$$

$$\text{and } x^2 + 2xy + y^2 = \left(\frac{x^2y^2}{36}\right)^2;\ \therefore\ x^2 + y^2 = \left(\frac{x^2y^2}{36}\right)^2 - 2xy;$$

$$\therefore \text{ from } (A)\ \frac{x^2y^2}{36}\left\{\frac{x^4y^4}{(36)^2} - 2xy\right\} = 13,$$

$$\text{or } x^6y^6 - 2x^3y^3(36)^2 + (36)^4 = 13 \times 36^3 + 36^4 = 36^3 \times 49;$$

$$\therefore\ x^3y^3 = 36^2 \pm 6^3 \times 7 = -216 \text{ or } 2808,$$

from $x^3y^3 = -216$, we have $xy = -6$, $x^2y^2 = 36$, and $x + y = 1$, also $x^2 - 2xy + y^2 = 25$; $\therefore\ x - y = 5$, and $x = 3$, and $y = -2$.

(70). $x^3 + xy^2 = y$, and $y^3 - x = x^2y$,

$$x^4 + x^2y^2 - xy = 0, \text{ also } y^4 - x^2y^2 - xy = 0;$$

$$\therefore\ x^4 - y^4 + 2x^2y^2 = 0, \text{ and } x^4 + 2x^2y^2 + y^4 = 2y^4,$$

$$\text{and } x^2 = y^2(-1 \pm \sqrt{2}), \text{ or } x = y\sqrt{(-1 \pm \sqrt{2})},$$

by substitution

$$y^4(3 \mp 2\sqrt{2}) + y^4(-1 \pm \sqrt{2}) = y^2\sqrt{(-1 \pm \sqrt{2})};$$

$$\therefore\ y^2 = \sqrt{\left(\frac{1}{2} \pm \frac{1}{\sqrt{2}}\right)},$$

$$\text{and } y = \sqrt[4]{\left(\frac{1}{2} \pm \frac{1}{\sqrt{2}}\right)}, \text{ and } x = \sqrt[4]{\left(\frac{1}{\sqrt{2}} \mp \frac{1}{2}\right)}.$$

(71). $x^2 - 2xy + 3y^2 = 9$, and $x^2 - 4xy + 5y^2 = 5$,

from (1) $x = y \pm \sqrt{(9 - 2y^2)}$, from (2) $x = 2y \pm \sqrt{(5 - y^2)}$;

$$\therefore y \pm \sqrt{(5 - y^2)} = \pm\sqrt{(9 - 2y^2)}, \text{ or } 5 \pm 2y\sqrt{(5 - y^2)} = 9 - 2y^2;$$

$$\therefore\ y^2(5 - y^2) = 4 - 4y^2 + y^4,$$

$$\text{whence } y^4 - \frac{9y^2}{2} + \left(\frac{9}{4}\right)^2 = -2 + \frac{81}{16} = \frac{49}{16},$$

$$\text{and } y^2 = 4 \text{ or } \frac{1}{2};\ \therefore\ y = \pm 2 \text{ or } \pm\sqrt{\left(\frac{1}{2}\right)};$$

$$\therefore\ x = \pm 3 \text{ or } 1, \text{ or } \frac{1 \pm 4}{\sqrt{(2)}}.$$

(72). $x^2 - 2xy - y^2 = 31$, and $x^2 + 4xy - 2y^2 = 202$,

by subtraction

$$x = \frac{171 + y^2}{6y};$$

therefore, by substitution in (1),

$$47y^4 + 2826y^2 = 29241,$$

$$\text{and } y^4 + \frac{2826}{47}y^2 + \left(\frac{1413}{47}\right)^2 = \frac{29241}{(47)} + \frac{(1413)^2}{(47)^2} = \frac{3370896}{(47)^2};$$

$$\therefore\ y^2 = -\frac{1413}{47} \pm \frac{1836}{47} = 9 \text{ or } \frac{3249}{47};\ \therefore\ y = \pm 3,$$

$$\text{and } x = \frac{171 + 9}{\pm 18} = \pm 10.$$

(73). $x^4 + y^4 - 3x^2y^2 - 2xy = 1$, and $x^3 + y^3 - 2xy^2 - 2y^2 - x = 1$,

from (1) $x^4 - 2x^2y^2 + y^4 = x^2y^2 + 2xy + 1$; $\therefore\ x^2 = y^2 \pm (xy + 1)$;

$$\therefore\ x^2 - xy + y^2 = 2y^2 + 1, \text{ and } x^3 + y^3 = (2y^2 + 1)(x + y);$$

$$\therefore \text{ from (2) } 2y^2x + x + 2y^3 + y - 2xy^2 - 2y^2 - x = 1,$$

$$\text{or } 2y^2(y - 1) + y - 1 = 0;\ \therefore\ y = 1, \text{ and } x = 2 \text{ or } -1,$$

$$\text{also } y^2 = -\frac{1}{2};\ \therefore\ y = \pm\frac{1}{\sqrt{(-2)}}, \text{ and } x = \frac{1}{2\sqrt{(-2)}} \pm \sqrt{\left(\frac{3}{2}\right)}.$$

(74). $x^2 + xy + y^2 = 52$, and $xy - x^2 = 8$,

from (2) $y = \dfrac{x^2 + 8}{x}$; $\therefore$ from (1) $x^2 + x^2 + 8 + \dfrac{x^4 + 16x^2 + 64}{x^2} = 52$,

$$\text{whence } x^4 - \frac{28x^2}{3} + \left(\frac{14}{3}\right)^2 = \frac{4}{9};$$

$$\therefore\ x^2 = 4 \text{ or } \frac{16}{3}, \text{ and } x = \pm 2 \text{ or } \pm\frac{4}{\sqrt{(3)}},$$

$$\text{from } x = \pm 2,\ y = \pm 6.$$

(75). $x^3 - y^3 = 56$, and $xy(x - y) = 16$,

$$\frac{x^3 - y^3}{xy(x - y)} = \frac{x^2 + xy + y^2}{xy} = \frac{7}{2};\ \therefore\ \frac{(x + y)^2}{(x - y)^2} = 9;\ \therefore\ \frac{x + y}{x - y} = 3,$$

$$\text{and } x = 2y;\ \therefore \text{ from (2) } 2y^3 = 16, \text{ and } y = 2, \text{ and } x = 4.$$

(76). $x^2 + y(xy - 1) = 0$, and $y^2 = x(xy + 1)$,

$$\text{from (1) } x^2 + xy^2 + \frac{y^4}{4} = y + \frac{y^4}{4};\ \therefore\ x = -\frac{y^2}{2} \pm \frac{1}{2}\sqrt{(4y + y^4)},$$

$$\text{from (2) } x^2 + \frac{x}{y} + \frac{1}{4y^2} = \frac{y^2}{y} + \frac{1}{4y^2};\ \therefore\ x = -\frac{1}{2y} \pm \frac{1}{2y}\sqrt{(4y^3 + 1)},$$

$$\text{whence } y^3 - 1 = y\sqrt{(4y + y^4)} - \sqrt{(4y^3 + 1)},$$

$$\text{and } 5y^2 = \sqrt{\{(4y + y^4)(4y^3 + 1)\}};$$

$$\therefore \text{ whence } y^6 - 2y^3 + 1 = 0, \text{ or } y^3 = 1;\ \therefore\ y = 1,$$

$$\text{and from (1) } x = \frac{1}{2}(-1 \pm \sqrt{5}), \text{ see VIII. (145).}$$

(77). $x + y = 4$, and $(x^2 + y^2)(x^3 + y^3) = 280$,

$$\text{since } x + y = 4; \therefore x^2 + y^2 = 16 - 2xy,$$

$$\text{and } x^2 - xy + y^2 = 16 - 3xy;$$

$$\therefore \text{ from (2) } (16 - 2xy) \times 4 \times (16 - 3xy) = 280,$$

$$\text{or } 256 - 80xy + 6x^2y^2 = 70,$$

$$\text{or } x^2y^2 - \frac{40xy}{3} + \left(\frac{20}{3}\right)^2 = -\frac{93}{3} + \frac{400}{9} = \frac{121}{9}; \therefore xy = \frac{31}{3} \text{ or } 3,$$

$$\text{from } xy = 3,\ x^2 - 2xy + y^2 = 4;$$

$$\therefore x - y = 2, \text{ and } x + y = 4; \therefore x = 3, y = 1.$$

(78). $x + y - z = 6$, $x + z - y = 4$, and $y + z - x = 2$,

$$\text{from (1) - (2) } x + y - z - (x + z - y) = 2, \text{ or } y - z = 1,$$

$$\text{from (1) + (3) } y = 4, \text{ and } z = y - 1 = 3, \text{ also } x = z - y + 6 = 5.$$

(79). $x + y + z = 29$, $x + 2y + 3z = 62$, and $6x + 4y + 3z = 120$,

$$\text{from (2) - (1) } y + 2z = 33, \text{ from (1) - (3) } 2y + 3z = 54;$$

$$\therefore 2y + 4z = 66, \text{ and } 2y + 3z = 54; \therefore z = 12,$$

$$y = 33 - 2z = 9, \text{ and } x = 29 - 12 - 9 = 8.$$

(80). $5x - 11y^{\frac{1}{2}} + 13z^{\frac{1}{3}} = 22$, $4x + 6y^{\frac{1}{2}} + 5z^{\frac{1}{3}} = 31$, and $x - y^{\frac{1}{2}} + z^{\frac{1}{3}} = 2$,

$$\text{from (1) - (3) } 4z^{\frac{1}{3}} - 3y^{\frac{1}{2}} = 6, \text{ from (2) - (3) } 10y^{\frac{1}{2}} + z^{\frac{1}{3}} = 23;$$

$$\therefore z^{\frac{1}{3}} = \frac{6 + 3y^{\frac{1}{2}}}{4} = 23 - 10y^{\frac{1}{2}}; \therefore y^{\frac{1}{2}} = 2, \text{ and } y = 4;$$

$$\therefore z = 27,\ x = 1.$$

(81). $x + \frac{z}{4} = \frac{41}{2}$, $y + \frac{x}{2} = 41$, and $\frac{5y + z}{5} = 34$,

from (1) and (2)

$$x = \frac{41}{2} - \frac{z}{4} = 82 - 2y; \therefore y = \frac{123}{4} + \frac{z}{8} = 34 - \frac{z}{5},$$

$$\text{whence } 1230 + 5z = 1360 - 8z; \therefore z = \frac{130}{13} = 10,$$

$$y = 34 - 2 = 32,\ x = 82 - 64 = 18.$$

(82). $\frac{x+y}{44} = \frac{z}{17}$, $\frac{x-y}{20} = \frac{z}{17}$, and $\frac{x^2-y^2}{3520} = \frac{z}{17}$,

from (1) and (2)

$$\frac{x+y}{11} = \frac{x-y}{5}; \therefore x = \frac{8y}{3}; \therefore \frac{8y-3y}{60} = \frac{z}{17} \text{ or } y = \frac{12z}{17},$$

$$\text{also } x^2 - y^2 = \frac{z^2}{17^2} \times 880 = \frac{z}{17} \times 3520;$$

$$\therefore z = 68, \ y = 48, \ x = 128.$$

(83). $x + 2y + 3z = 17$, $2x + 3y + z = 12$, and $3x + y + 2z = 13$,

$$\text{from } 2(1) - (2) \ y + 5z = 22,$$

$$\text{also from (1) and (3) } 5y + 7z = 38;$$

$$\therefore 22 - 5z = \frac{38 - 7z}{5}, \text{ and } z = \frac{72}{18} = 4, \ y = 2, \ x = 1.$$

(84). $\frac{xy}{x+y} = 1$, $\frac{xz}{x+z} = 2$, and $\frac{yz}{y+z} = 3$,

$$\text{from (1) and (2) } x = \frac{y}{y-1} = \frac{2z}{z-2}; \therefore y = \frac{2z}{z+2} = \frac{3z}{z-3};$$

$$\therefore 2z - 6 = 3z + 6, \text{ and } z = -12, \ y = 2\tfrac{2}{5}, \ x = 1\tfrac{5}{7}.$$

(85). $\frac{1}{x} + \frac{1}{y} = \frac{3}{20}$, $\frac{1}{x} + \frac{1}{z} = \frac{7}{60}$, and $\frac{1}{y} + \frac{1}{z} = \frac{1}{6}$,

$$\text{from (1) and (2) } \frac{1}{y} - \frac{1}{z} = \frac{1}{30}, \text{ also } \frac{1}{y} + \frac{1}{z} = \frac{1}{6};$$

$$\therefore \frac{2}{z} = \frac{2}{15}; \therefore z = 15,$$

$$\text{and } y = 10, \ x = 20.$$

(86). $x + y + z = 15$, $x + y - z = 3$, and $x - y + z = 5$,

$$\text{from } (1) - (2) \ z = 6; \therefore x + y = 9, \text{ and } x - y = -1;$$

$$\therefore x = 4, \ y = 5.$$

(87). $x + y + z = 31$, $x + y - z = 25$, and $x - y - z = 9$,

$$\text{from } (1) - (2) \ z = 3; \therefore x + y = 28, \ x - y = 12;$$

$$\therefore x = 20, \ y = 8.$$

(88). $x + y + z = 9$, $x + 3y - 3z = 7$, and $x - 4y + 8z = 8$,

from (1) - (2) $2z - y = 1$, from (1) - (3) $5y - 7z = 1$; $\therefore y = \frac{3z}{2}$;

$\therefore z = 2, y = 3, x = 4.$

(89). $x + 2y + 3z = 22$, $2x + z + 3y = 17$, and $3x + y + 2z = 21$,

from 2 (1) - (2) $y + 5z = 27$, from 3 (1) - (3) $5y + 7z = 45$,

whence $z = 5, y = 2, x = 3.$

(90). $3x + 2y = 32$, $3y + 2z = 25$, and $2x + 3z = 18$,

from (1) - (3) $4y - 9z = 10$, and $3y + 2z = 25$;

$\therefore z = 2, y = 7, x = 6.$

(91). $x + 30 = y + z$, $x + 2z = y + 30$, and $3x + y = z + 30$,

from (1) - (2) $30 - 2z = z - 30$; $\therefore z = 20$,

from (1) + (3) $4x + 30 = 2z + 30$; $\therefore x = \frac{z}{2} = 10$, and $y = 20$.

(92). $2x + y = 3y + z = 2x + z = 120$,

$2x + y = 3y + z$; $\therefore z = 2(x - y)$, also $y = z$; $\therefore x = \frac{3z}{2}$;

$\therefore 3z + z = 120$, and $z = 30, y = 30, x = 45.$

(93). $xy = 3(x + y)$, $xz = 4(x + z)$, and $yz = 6(y + z)$,

$x = \frac{3y}{y-3} = \frac{4z}{z-4}$; $\therefore 3yz - 12y = 4zy - 12z$; $\therefore y = \frac{12z}{z+12}$;

whence $y = \frac{6z}{z-6} = \frac{12z}{z+12}$; $\therefore z + 12 = 2z - 12$;

$\therefore z = 24, y = 8, x = 4\frac{4}{5}.$

(94). $\frac{2}{x} + \frac{3}{y} - \frac{4}{z} = \frac{1}{12}$, $\frac{3}{x} - \frac{4}{y} + \frac{5}{z} = \frac{19}{24}$, and $\frac{4}{x} - \frac{5}{y} + \frac{1}{2} = \frac{6}{z}$,

from (1) $\frac{1}{x} + \frac{3}{2y} - \frac{2}{z} = \frac{1}{24}$,

from (2) $\frac{1}{x} - \frac{4}{3y} + \frac{5}{3z} = \frac{19}{72}$; $\therefore \frac{11}{3z} - \frac{17}{6y} = \frac{2}{9}$,

also $\frac{1}{x} - \frac{5}{4y} - \frac{3}{2z} = -\frac{1}{8}$; $\therefore \frac{11}{4y} - \frac{1}{2z} = \frac{1}{6}$;

$\therefore y = 12, z = 8, x = 6.$

(95). $\frac{2}{x} - \frac{5}{3y} + \frac{1}{z} = 3\frac{4}{27}$, $\frac{1}{4x} + \frac{1}{y} + \frac{2}{z} = 6\frac{11}{72}$, and $\frac{5}{6x} - \frac{1}{y} + \frac{4}{z} = 12\frac{1}{36}$,

$$\frac{2}{x} - \frac{5}{3y} + \frac{1}{z} = \frac{85}{27},$$

$$\text{from (2) } \frac{1}{8x} + \frac{1}{2y} + \frac{1}{z} = \frac{443}{144}; \ \therefore \ \frac{15}{8x} - \frac{13}{6y} = \frac{31}{432},$$

$$\text{from (3) } \frac{5}{24x} - \frac{1}{4y} + \frac{1}{z} = \frac{433}{144}; \ \therefore \ \frac{1}{12x} - \frac{3}{4y} = -\frac{10}{144},$$

whence $y = 9$, $x = 6$, and $z = \frac{1}{3}$.

(96). $\frac{1}{x} + \frac{1}{y} = \frac{1}{a}$, $\frac{1}{x} + \frac{1}{z} = \frac{1}{b}$, and $\frac{1}{y} + \frac{1}{z} = \frac{1}{c}$,

$$\text{from (1) − (2) } \frac{1}{y} - \frac{1}{z} = \frac{1}{a} - \frac{1}{b}, \text{ also } \frac{1}{y} + \frac{1}{z} = \frac{1}{c};$$

$$\therefore \frac{2}{y} = \frac{1}{a} - \frac{1}{b} + \frac{1}{c},$$

and $y = \frac{2abc}{ab + bc - ac}$, and $z = \frac{2abc}{ab + ac - bc}$, $x = \frac{2abc}{ac + bc - ab}$.

(97). $4x - 5y + mz = 7x - 11y + nz$, and $7x - 11y + nz = x + y + pz = 3$,

$$\therefore 28x - 35y + 7mz = 21,$$

also $28x - 44y + 4nz = 12$; $\therefore (7m - 4n) z + 9y = 9$ (4),

from (1) − (3) $9y + (4p - m) z = 9$ (5);

$\therefore$ (4) − (5) $(8m - 4x - 4p) z = 0$; $\therefore z = 0$;

$\therefore 4x - 5y = 3$, and $4x + 4y = 12$; $\therefore y = 1$, and $x = 2$.

(98). $x + y + z = 0$, $(a + b) x + (a + c) y + (b + c) z = 0$,

and $abx + acy + bcz = 1$,

$(a + b)(x + y + z) = 0$, also $(a + b) x + (a + c) y + (b + c) z = 0$;

$$\therefore (b - c) y + (a - c) z = 0,$$

also $ab (x + y + z) = 0$, but $abx + acy + bcz = 1$;

$$\therefore a (b - c) y + b (a - c) z = -1,$$

whence $z = \frac{1}{(a - b)(a - c)}$, $y = \frac{1}{(a - b)(c - b)}$, $x = \frac{1}{(b - c)(a - c)}$.

(99). $\frac{x}{2}+\frac{y}{3}+\frac{z}{4}=62$, $\frac{x}{3}+\frac{y}{4}+\frac{z}{5}=47$, and $\frac{x}{4}+\frac{y}{5}+\frac{z}{6}=38$,

$$\text{from (2)} - (1)\ \frac{y}{12}+\frac{z}{10}=17,\ \text{from (3)} - (1)\ \frac{y}{30}+\frac{z}{24}=7,$$

$$\text{whence } y=204-\frac{6z}{5}=\text{also } 210-\frac{5z}{4};$$

$$\therefore z=120,\ y=60,\ x=24.$$

(100). $\frac{3}{x}-\frac{4}{5y}+\frac{1}{z}=7\frac{3}{5}$, $\frac{1}{2y}-\frac{61}{6}+\frac{z+6x}{3xz}=0$,

$$\text{and } \frac{4}{5x}-\frac{161}{10}=\frac{1}{2y}-\frac{4}{z},$$

$$\text{from (1)} - (2)\ \frac{6}{x}-\frac{8}{5y}+\frac{2}{z}-\left(\frac{1}{2y}-\frac{61}{6}+\frac{1}{3x}+\frac{2}{z}\right)=\frac{76}{5},$$

$$\text{or } \frac{17}{3x}-\frac{21}{10y}=\frac{151}{30}\ (A),$$

from (1) – (3)

$$\frac{12}{x}-\frac{16}{5y}+\frac{4}{z}-\left(\frac{4}{5x}-\frac{1}{2y}+\frac{4}{z}\right)=\frac{152}{5}-\frac{161}{10}=\frac{143}{10},$$

$$\text{or } \frac{56}{5x}-\frac{27}{10y}=\frac{143}{10}\ (B),$$

$$\text{from } A-B,\ x=\frac{1}{2},\ \text{and } y=\frac{1}{3},\ \text{and } z=\frac{1}{4}.$$

(101). $x^{-2}y^{-1}z=\frac{3}{2}$, $x^{-1}yz^2=18$, and $xy^2=108z^{-3}$,

$$\text{from (1) } z=\frac{3}{2}x^2y,\ \text{from (2) } z^2=\frac{18x}{y},\ \text{from (3) } z^3=\frac{108}{xy^2};$$

$$\therefore \frac{9x^4y^2}{4}=\frac{18x}{y},\ \text{or } x^3y^3=8,\ \text{also } \frac{27}{8}x^6y^3=\frac{108}{xy^2};\ \therefore x^7y^5=32;$$

$$\therefore x^4y^2=4,\ \text{and } x^2y=\pm 2;\ \therefore z=\pm\frac{3}{2}\times 2=\pm 3,$$

$$\text{but } z^2=9=\frac{18x}{y};\ \therefore y=2x;\ \therefore 2x^3=2,\ \text{and } x=1,\ \text{and } y=2.$$

(102). $\frac{105}{xyz}=1$, $35x=3yz$, and $7xy=15z$,

$$\text{from (1) and (2) } yz=\frac{105}{x}=\frac{35x}{3};\ \therefore x^2=9,\ \text{and } x=\pm 3,$$

$$\text{from (2) and (3) } \pm 35=yz \text{ and } \pm 7y=5z;\ \therefore z=7,\ \text{and } y=5.$$

(103). $x^2y^3 = az^{-4}$, $x^3y^4 = bz^{-2}$, and $x^4y^2 = cz^{-3}$,

$$\text{from (1) and (2) } \frac{x^3y^4}{x^2y^3} = \frac{bz^2}{a}, \text{ or } x = \frac{bz^2}{ay},$$

$$\text{from (1) and (3) } x^2 = \frac{cyz}{a} = \frac{b^2z^4}{a^2y^2}; \therefore y^3 = \frac{b^2z^3}{ac},$$

$$\text{from (2) and (3) } x = \frac{cy^2}{bz} = \frac{bz^2}{ay}; \therefore y^3 = \frac{b^2z^3}{ac} \text{ (4)},$$

$$\text{from (1) and (4) } x^2 = \frac{a}{z^4} \times \frac{ac}{b^2z^3} = \frac{cz}{a} \times z \frac{b^{\frac{2}{3}}}{(ac)^{\frac{1}{3}}};$$

$$\therefore z = (a^{10}b^{-8}c)^{\frac{1}{27}}, \ y = (ab^{10}c^{-8})^{\frac{1}{27}}, \ x = (a^{-8}bc^{10})^{\frac{1}{27}}.$$

(104). $x + 2y + 3z + 4u = 27$, $3x + 5y + 7z + u = 48$,

$5x + 8y + 10z - 2u = 65$, and $7x + 6y + 5z + 4u = 53$,

$$\left.\begin{array}{r}\text{from (1) – (2) (5) } y + 2z + 11u = 33\\ \text{from (1) – (3) } 2y + 5z + 22u = 70\end{array}\right\} \text{whence } z = 4,$$

from (1) – (4) $y + 3u = 9$, also from (5) $y + 11u = 25$;

$$\therefore u = \frac{16}{8} = 2,$$

and from (5) $y = 33 - 22 - 8 = 3$, and $x = 1$.

(105). $x + y + z + u = 1$, $16x + 8y + 4z + 2u = 9$,

$81x + 27y + 9z + 3u = 36$, and $256x + 64y + 16z + 4u = 100$,

from (1) $2x + 2y + 2z + 2u = 2$,

also $16x + 8y + 4z + 2u = 9$; $\therefore 14x + 6y + 2z = 7$ (A),

also $3x + 3y + 3z + 3u = 3$,

and $81x + 27y + 9z + 3u = 36$; $\therefore 78x + 24y + 6z = 33$ (B);

again $4x + 4y + 4z + 4u = 4$,

and $256x + 64y + 16z + 4u = 100$;

$\therefore 252x + 60y + 12z = 96$ (C),

from A and B $6x + y = 2$, from A and C $28x + 4y = 9$,

$$\text{whence } x = \frac{1}{4}, \ y = \frac{1}{2}, \ z = \frac{1}{4}, \ u = 0.$$

(106). $x - 9y + 3z - 10u = 21$, $2x + 7y - z - u = 683$,

$3x + y - 5z + 2u = 325$, and $4x - 6y - 2z - 9u = 516$,

$$\left.\begin{array}{l}\text{from (1) and (2) } 25y - 7z + 19u = 641 \\ \text{from (1) and (3) } 14y - 7z + 16u = 131\end{array}\right\} \text{whence } 11y + 3u = 510 \text{ (5)},$$

from (1) and (4) $30y - 14z + 31u = 432$,

hence $2y - u = 170$ (6);

$\therefore$ from (5) and (6) $u = -\dfrac{5y}{6}$; $\therefore 2y + \dfrac{5y}{6} = 170$, and $y = 60$;

$$\therefore u = -\frac{5}{6} \times 60 = -50,\ z = -13,\ x = 100.$$

(107). $xyz = 231$, $xyw = 420$, $yzw = 1540$, and $xzw = 660$,

$$x = \frac{231}{yz} = \frac{420}{yw} = \frac{660}{zw};\ \therefore y = \frac{7w}{20},\ \text{and } z = \frac{11y}{7} = \frac{11w}{20};$$

$$\therefore yzw = \frac{7}{20} \times \frac{11}{20} w^3 = 1540,\ \text{or } w = 20;$$

$$\therefore y = 7,\ z = 11,\ x = 3.$$

(108). $7x - 2z + 3u = 17$, $4y - 2z + t = 11$, $5y - 3x - 2u = 8$,

$4y - 3u + 2t = 9$, and $3z + 8u = 33$,

from (1) and (2) $7x + 3u - 4y - t = 6$ (6),

from (1) and (5) $21x + 25u = 117$ (7),

from (2) and (4) $14x + 3u - 4y = 21$ (8),

from (3) and (8) $58x + 7u = 137$ (9),

from (7) and (9) $\dfrac{117 - 25u}{21} = \dfrac{137 - 7u}{58}$, and $u = \dfrac{3909}{1303} = 3$,

$$x = 2,\ y = 4,\ t = 1,\ z = 3.$$

(109). $x - ay + a^2z = a^3$, $x - by + b^2z = b^3$, and $x - cy + c^2z = c^3$,

from (1) and (2) $(a + b)z - y = a^2 + ab + b^2$,

from (1) and (3) $(a + c)z - y = a^2 + ac + c^2$;

$$\therefore (b - c)z = a(b - c) + (b^2 - c^2);$$

$$\therefore z = a + b + c,\ y = ab + ac + bc,\ x = abc.$$

(110). $bz(a-x) = ay(c-z) = cx(b-y)$,

$$\text{and } xyz = (a-x)(b-y(c-z),$$

$$z = \frac{ayc}{b(a-x)+ay} = \frac{cx(b-y)}{b(a-x)},$$

$$\text{also } xy \times \frac{cx(b-y)}{b(a-x)} = (a-x)(b-y) \times \frac{cx(b-y)}{ay},$$

$$\text{whence } y^2 + \frac{by}{ax}(a-x)^2 + (\quad)^2 = \frac{b^2(a^2-x^2)^2}{4a^2x^2},$$

$$\text{and } y = -\frac{b(a-x)}{2ax}\{a - x \mp (a+x)\} = \frac{b(a-x)}{a} \text{ or } \frac{b(x-a)}{x},$$

if $ay = b(a-x)$, from $ay(c-z) = bz(a-x)$, we have $z = \frac{c}{2}$,

$$\text{also } x = \frac{a}{2} \text{ and } -a, \text{ and } y = \frac{b}{2} \text{ or } 2b.$$

(111). $x + y + z = 3m$, $xy + xz + yz = 3m^2$, and $xyz = m^3$,

$$\text{from (2) and (3) } m^3 + (x+y)z^2 = 3m^2z \text{ (4)},$$

$$\text{from (1) and (4) } m^3 + 3(m-z)z^2 = 3m^2z;$$

$$\therefore z^3 - 3mz^2 \pm 3m^2z - m^3 = 0; \therefore z = m,$$

$$\text{and } x + y = 2m, \text{ and } xy = m^2;$$

$$\therefore x^2 - 2xy + y^2 = 0; \therefore x - y = 0,$$

$$\text{and } x = m, \text{ and } y = m.$$

(112). $x^2 + xy + y^2 = 37$, $x^2 + xz + z^2 = 28$, and $y^2 + yz + z^2 = 19$,

from (1) – (2) $y^2 - z^2 + x(y-z) = 9$; $\therefore x + y + z = \frac{9}{y-z}$,

from (1) – (3) $x^2 - z^2 + y(x-z) = 18$; $\therefore x + y + z = \frac{18}{x-z}$;

$$\therefore x - z = 2y - 2z; \therefore y = \frac{x+z}{2},$$

and x, y, z form an arithmetic series, let then

$$x = m + n, \ y = m, \text{ and } z = m - n;$$

$$\therefore (m+n)^2 + (m^2 - n^2) + (m-n)^2 = 28,$$

$$\text{also } (x+y+z)(y-z) = 9,$$

$$\text{whence } 3m^2 + n^2 = 28\,;\ \therefore\ mn = 3, \text{ and } m = \frac{3}{n}\,;$$

$$\therefore\ 27 + n^4 = 28n^2;\ \therefore\ n = 1, \text{ and } m = 3;$$

$$\therefore\ x = 4,\ y = 3,\ z = 2.$$

(113). $x + y + z = 11,\ x^2 + y^2 + z^2 = 49, \text{ and } yz + 3xy = 3xz,$

$$\text{from } (1)^2 - (2)\ xy + xz + yz = 36;$$

$$\therefore\ x = 11 - (y + z)\ (4) = \frac{36 - yz}{y + z}\ (5) = \frac{-yz}{3(y - z)}\ (6);$$

$$\therefore \text{ from (4) and (5) } 11(y + z) - (y + z)^2 = 36 - yz,$$

$$\text{or } 33(y + z) - 3(y + z)^2 = 108 - 3yz,$$

$$\text{also from (4) and (6) } 33(y - z) - 3(y^2 - z^2) = -yz;$$

$$\therefore\ 33y - 3y^2 - yz = 54;$$

$$\therefore\ z = \frac{33y - 3y^2 - 54}{y}\ (7),$$

$$\text{from (5) and (6) } 2yz^2 - y^2z - 54z + 54y = 0;$$

$$\therefore\ z = \frac{y^2 + 54}{4y} \pm \frac{1}{4y}\sqrt{\{y^4 - 324y^2 + (54)^2\}}\ (8),$$

$$\text{from (7) and (8) } (13y^2 - 132y + 270)^2 = y^4 - 324y^2 + (54)^2,$$

$$\text{whence } 7y^4 - 143y^3 + 1032y^2 - 2970y + 2916 = 0,$$

$$\text{and by trial } y = 3;$$

$$\therefore \text{ from (7) } z = 33 - 9 - 18 = 6, \text{ from (1) } x = 2,$$

$$\text{or let } y = mx,\ z = nx;\ \therefore\ (1 + m + n)^2 x^2 = 121,$$

$$\text{and } (1 + m^2 + n^2)\,x^2 = 49;\ \therefore\ (m + n + mn)\,x^2 = 36,$$

$$\text{also } mn = 3(n - m);\ \therefore\ m = \frac{3n}{n + 3},$$

$$\text{and if } n = 1,\ m = \frac{3}{4};\ \text{if } n = 3,\ m = \frac{3}{2},$$

which latter values satisfy the conditions

$$\left(1 + \frac{9}{4} + 9\right) x^2 = 49;\ \therefore\ x = 7 \times \frac{2}{7} = 2,\ y = 3,\ z = 6,$$

$$\text{also } \left(1 + \frac{3}{2} + 3\right) x = 11, \text{ or } x = 11 \times \frac{2}{11} = 2,\ y = 3,\ z = 6.$$

(114). $x + y + z = 10$, $x^2 + y^2 + z^2 = 38$, and $xz = y^2 + 1$,

$$x^2 + y^2 + z^2 + 2xy + 2xz + 2yz = 100,$$

$$x^2 + y^2 + z^2 \qquad = 38;$$

$$\therefore\ xy + yz + xz \qquad = 31,$$

but $xz = y^2 + 1$; $\therefore\ y^2 + 1 + y(x + z) = 31$,

or $y^2 + y(10 - y) = 30$; $\therefore\ y = 3$;

$\therefore\ x + z = 7$, and $xz = 10$, whence $x = 5$, $z = 2$.

(115). $x^2 + xy + y^2 = 13$, $x^2 + xz + z^2 = 31$, and $y^2 + yz + z^2 = 49$,

from (1) − (2) $x(y - z) + y^2 - z^2 = -18$,

or $(z - y)(x + z + y) = 18$,

from (2) − (3) $x^2 - y^2 + z(x - y) = -18$,

or $(y - x)(x + y + z) = 18$,

whence $2y = z + x$,

or x, y, z form an arithmetic series. Let therefore r be the common difference, and $y - r$, y, and $y + r$ the series,

from (2) $(y - r)^2 + y^2 - r^2 + (y + r)^2 = 31$, or $3y^2 + r^2 = 31$,

from (3), similarly, $3y^2 + 3ry + r^2 = 49$; $\therefore\ ry = 6$;

$$\therefore\ 3y^2 + \frac{36}{y^2} = 31, \text{ whence } y = \pm 3, \text{ and } r = \pm 2;$$

$\therefore\ x = 1$, $y = 3$, $z = 5$.

(116). $x + y + z = 13$, $x^2 + y^2 + z^2 = 91$, and $y^2 = xz$,

from $(1)^2 - (2)$ $xy + yz + xz = 39$, but $xz = y^2$;

$\therefore\ y^2 + y(x + z) = y^2 + y(13 - y) = 39$; $\therefore\ y = 3$,

and $x + z = 10$, and $xy = 9$; $\therefore\ x - z = 8$, and $x = 9$, and $z = 1$.

(117). $xy + z = 5$, $xyz = 4$, and $x^2 - 2y = y^4 - 2xy^2$,

$$\text{from (1) and (2) } x = \frac{5 - z}{y} = \frac{4}{yz}, \text{ whence } z = 4 \text{ or } 1,$$

$$\text{from (2)} = (4)\ xy = 1;\ \therefore\ x^2 - \frac{2}{x} = \frac{1}{x^4} - \frac{2}{x};$$

$\therefore\ x^6 = 1$, $x = 1$, and $y = 1$, see VIII. (145),

$$z = 5 - xy = \frac{4}{xy}; \ \therefore \ x^2y^2 - 5xy + \frac{25}{4} = -4 + \frac{25}{4} = \frac{9}{4};$$

$$\therefore \ xy = \frac{5}{2} \pm \frac{3}{2} = 4 \text{ or } 1;$$

$$\therefore \text{ from (2) } \frac{16}{y^2} - 2y = y^4 - 8y;$$

$$\therefore \ y^6 - 6y^3 + 9 = 25; \ \therefore \ y^3 = 3 \pm 5 = 8 \text{ or } -2;$$

$$\therefore \ y = 2, \ x = 2, \ z = 1.$$

(118). $x + y + z = 13, \ x^2 + y^2 + z^2 = 61$, and $xy + xz = 2yz$,

from $(1)^2 - (2)$ $xy + xz + yz = 54$, but $xy + xz = 2yz$,

$3yz = 54$, and $yz = 18$,

from (1) and (2), $\because \ y^2 + 2yz + z^2 = (13 - x)^2$,

we have $x^2 - 13x = -36$, whence $x = 9$ or 4,

from $x = 4$; $\therefore \ y + z = 9$, and $yz = 18$;

$\therefore \ y - z = \pm 3$, and $y = 6$, and $z = 3$,

and other values may be found from $x = 9$,

for $y + z = 4$, and $y - z = \pm 2\sqrt{(-14)}$;

$\therefore \ y = 2 \pm \sqrt{(-14)}, \ z = 2 \mp \sqrt{(-(14)}$.

(119). $xy + yz + xz = 11, \ 2xy + 3xz + 5yz = 31$,

and $7xy + 5xz + 7yz = 71$,

from (1) $2xy + 2xz + 2yz = 22$,

from (2) $2xy + 3xz + 5yz = 31$; $\therefore \ xz + 3yz = 9$,

so from (1) and (3) $xz = 3$; $\therefore \ yz = 2$; $\therefore$ from (1) $xy = 6$,

whence $z = \frac{3}{x}$; $\therefore \ y = \frac{2}{z} = \frac{2x}{3} = \frac{6}{x}$; $\therefore \ x^2 = 9$,

and $x = \pm 3$, and $z = \pm 1$, and $y = \pm 2$.

(120). $x + y + z = 14, \ x^2 + y^2 + z^2 = 84$, and $xz = y^2$,

from $(1)^2 - (2)$ $xy + xz + yz = 56$, but $xz = y^2$;

$\therefore \ y^2 + y(x + z) = y^2 + y(14 - y) = 56$; $\therefore \ y = 4$, and $xz = 16$;

$\therefore$ from (2) $x^2 - 2xz + z^2 = 84 - 16 - 32 = 36$; $\therefore \ x - z = \pm 6$,

but $x + z = 10$; $\therefore \ x = 8$ or 2, $z = 2$ or 8.

(121). $x + 2y + z = 19$, $x^2 + 4y^2 + z^2 = 133$, and $xz = 4y^2$,

from $(1)^2 - (2)$ $xz + 2y(x + z) = 114$, but $xz = 4y^2$;

$\therefore\ 4y^2 + 2y(19 - 2y) = 114$; $\therefore\ y = 3$, and $xz = 36$;

$\therefore\ x + z = 13$, and $x - z = \pm 5$; $\therefore\ x = 9$ or 4, and $z = 4$ or 9.

(122). $x + y + z = \frac{19}{6}$, $x^2 + y^2 + z^2 = \frac{133}{36}$, and $xz = y^2$,

from $(1)^2 - (2)$ $xy + yz + xz = \frac{19}{6}$, but $xz = y^2$;

$$\therefore\ y^2 + y(x + z) = \frac{19}{6};$$

$$\therefore\ y^2 + y\left(\frac{19}{6} - y\right) = \frac{19}{6};\ \therefore\ y = 1 = xz;$$

$$\therefore\ x + z = \frac{13}{6};\ \therefore\ x^2 - 2xz + z^2 = \frac{169}{36} - 4 = \frac{25}{36};$$

$$\therefore\ x - z = \frac{5}{6};\ \therefore\ x = \frac{3}{2}, \text{ and } z = \frac{2}{3}.$$

(123). $xz = y^2$, $(x + y)^2 = z(x + y) - 3$, and $(x + y + z)(z - x - y) = 7$,

from (1) and (2) $x^2 + 2xy + y^2 - y^2 - yz + 3 = 0$;

$$\therefore\ z = \frac{x^2 + 2xy + 3}{y},$$

from (2) $z = x + y + \frac{3}{x + y}$, from (3) $z^2 = (x + y)^2 + 7$,

whence $x + y = \pm 3$; $\therefore\ z = 3 + 1 = 4$, and $x = \frac{y^2}{4}$,

and $y^2 + 4y + 4 = 16$; $\therefore\ y = 2$ or -6; $\therefore\ x = 1$ or 9.

(124). $x + y + z = m$, $ax + by + cz = 0$, and $a^2x + b^2y + c^2z = 0$,

from (1) $ax + ay + az = am$, also $ax + by + cz = 0$;

$$\therefore\ (a - b)y + (a - c)z = am,$$

also $a^2x + a^2y + a^2z = a^2m$, and $a^2x + b^2y + c^2z = 0$;

$$\therefore\ (a^2 - b^2)y + (a^2 - c^2)z = a^2m,$$

whence $z = \frac{mab}{(a - c)(b - c)}$, $y = \frac{mac}{(a - b)(c - b)}$, $x = \frac{mbc}{(a - b)(a - c)}$.

(125). $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$, and $\frac{1}{x}+\frac{1}{a}=\frac{1}{y}+\frac{1}{b}=\frac{1}{z}+\frac{1}{c}$,

from (1) – (2) $\frac{2}{y}+\frac{1}{z}=\frac{2}{a}+\frac{1}{c}$, also $\frac{1}{y}+\frac{2}{z}=\frac{2}{a}+\frac{1}{b}$,

whence

$$z=\frac{3abc}{2bc+2ac-ab},\ y=\frac{3abc}{2ab+2bc-ac},\ x=\frac{3abc}{2ab+2ac-bc}.$$

(126). $ay+bx=c$, $cy+bz=a$, and $az+cx=b$,

$acy+bcx=c^2$, also $acy+abz=a^2$; $\therefore\ bcx-abz=c^2-a^2$,

but $abz+bcx=b^2$;

$$\therefore\ x=\frac{c^2+b^2-a^2}{2bc},\ y=\frac{a^2+c^2-b^2}{2ac},\ z=\frac{a^2+b^2-c^2}{2ab}.$$

(127). $xy=a(x+y)$, $xz=b(x+z)$, and $yz=c(y+z)$,

$$x=\frac{ay}{y-a}=\frac{bz}{z-b};\ \therefore\ y=\frac{abz}{bz-az+ab}=\frac{cz}{z-c},$$

whence $z=\frac{2abc}{ab+ac-bc}$, $y=\frac{2abc}{ab+bc-ac}$, $x=\frac{2abc}{ac+bc-ab}$.

(128). $\frac{xy+xz-yz}{a^2}=\frac{xy+yz-xz}{b^2}=\frac{xz+yz-xy}{c^2}=1$,

$xy-yz+xz=a^2$, also $xy+yz-xz=b^2$; $\therefore\ 2xz-2yz=a^2-b^2$,

but $xz+yz-xy=c^2$; $\therefore\ 2xz=a^2+c^2$;

$$\therefore\ z=\frac{a^2-b^2}{2(x-y)}=\frac{a^2+c^2}{2x}=\frac{a^2-xy}{x-y};$$

$$\therefore\ y=\frac{a^2+b^2}{2x}=\frac{(b^2+c^2)x}{a^2+c^2};\ \therefore\ x=\sqrt{\left\{\frac{(a^2+b^2)(a^2+c^2)}{2(b^2+c^2)}\right\}},$$

$$y=\sqrt{\left\{\frac{(a^2+b^2)(b^2+c^2)}{2(a^2+b^2)}\right\}},\ z=\sqrt{\left\{\frac{(a^2+c^2)(a^2+c^2)}{2(a^2+b^2)}\right\}}.$$

(129). $x(x+y+z)=a^2$, $y(x+y+z)=b^2$, and $z(x+y+z)=c^2$,

$$\frac{a^2}{x}=\frac{b^2}{y};\ \therefore\ y=\frac{b^2x}{a^2},\ \text{also}\ \frac{a^2}{x}=\frac{c^2}{z};\ \therefore\ z=\frac{c^2x}{a^2};$$

$$\therefore\ \text{from (1)}\ x\left(x+\frac{b^2x}{a^2}+\frac{c^2x}{a^2}\right)=a^2;$$

$$\therefore\ x=\frac{\pm a^2}{a^2+b^2+c^2},\ y=\frac{\pm b^2}{a^2+b^2+c^2},\ z=\frac{\pm c^2}{a^2+b^2+c^2}.$$

(130). $x+y+z=a,\ x^2+y^2+z^2=a^2$, and $xy+zy=2xz$,

$$\text{from } (1)^2-(2),\ xy+yz+xz=0,$$

$$\text{also } xy+yz-2xz=0;$$

$$\therefore\ 3xz=0, \text{ and } x=0, \text{ and } z=0;$$

$$\therefore\ y=a.$$

(131). $4xy\,(y+x-z)=3,\ 2yz\,(z+y-x)=15$, and $zx\,(x+z-y)=3$,

$$\text{let } y=mx,\ z=nx;\ \therefore\ 4mx^3\,(m+1-n)=3,$$

$$\text{and } 2mnx^3\,(n+m-1)=15, \text{ whence } \frac{m+1-n}{n+m-1}=\frac{n}{10},$$

$$\text{whence } m=\frac{10+n}{10-n}\times(n-1),$$

$$\text{and if } n=1,\ m=0, \text{ if } n=2,\ m=\frac{3}{2}, \text{ \&c.},$$

which latter values satisfy the conditions for

$$4xy\,(y+x-z)=4x^3\times\frac{3}{2}\left(\frac{3}{2}+1-2\right)=3x^3=3;$$

$$\therefore\ x=1,\ y=1\tfrac{1}{2},\ z=2.$$

(132). $\dfrac{a^3x}{y^2z^2}=\dfrac{b^3y}{z^2x^2}=\dfrac{c^3z}{x^2y^2}=1$,

$$\therefore\ x=\frac{y^2z^2}{a^3}, \text{ and } x^2=\frac{c^3z}{y^2}=\frac{b^3y}{z^2};\ \therefore\ y^3=\frac{c^3z^3}{b^3}, \text{ or } y=\frac{cz}{b},$$

$$\text{also } y^6=\frac{c^3a^6}{z^3};\ \therefore\ y=\frac{c^{\frac{1}{2}}a}{z^{\frac{1}{2}}}, \text{ and } \frac{c^{\frac{1}{2}}a}{z^{\frac{1}{2}}}=\frac{cz}{b};\ \therefore\ z=\left(\frac{a^2b^2}{c}\right)^{\frac{1}{3}},$$

$$y=\left(\frac{a^2c^2}{b}\right)^{\frac{1}{3}},\ x=\left(\frac{b^2c^2}{a}\right)^{\frac{1}{3}}.$$

(133). $\left(\dfrac{x}{y}\right)^{\frac{1}{2}}+\left(\dfrac{y}{x}\right)^{\frac{1}{2}}=\dfrac{61}{\sqrt{(xy)}}+1$, and $\sqrt[4]{(x^3y)}+\sqrt[4]{(xy^3)}=78$;

$$\therefore\ x+y=61+\sqrt{(xy)}, \text{ and } \sqrt{(xy)}\,\{x+2\sqrt{(xy)}+y\}=(78)^2;$$

$$\therefore\ \sqrt{(xy)}\,\{61+3\sqrt{(xy)}\}=78^2,$$

$$\text{or } xy+\frac{61}{3}\sqrt{(xy)}+\left(\frac{61}{6}\right)^2=\frac{76729}{36};\ \therefore\ \sqrt{(xy)}=36 \text{ or } -\frac{169}{3},$$

from $\sqrt{(xy)}=36,\ x+y=97$, and $x-y=65$; $\therefore\ x=81,\ y=16$.

(134). $x^2 - 6\sqrt{(x^2y)} = 27$, and $x - 2\sqrt{(xy)} = 3$,

$$\text{from (1) and (2) } \sqrt{(y)} = \frac{x^2 - 27}{6x} = \frac{x-3}{2\sqrt{(x)}};$$

$$\therefore\ x^2 - 27 = (x-3)\,3\sqrt{(x)} = 3x^{\frac{3}{2}} - 9\sqrt{(x)},$$

$$\text{or } x^{\frac{3}{2}}(x^{\frac{1}{2}} - 3) + 9(x^{\frac{1}{2}} - 3) = 0;\ \therefore\ x^{\frac{1}{2}} = 3, \text{ or } x = 9, \text{ and } y = 1,$$

$$\text{and } x^{\frac{3}{2}} = -9;\ \therefore\ x = \sqrt[3]{(81)}.$$

(135). $\dfrac{x - \sqrt{(x^2 - y^2)}}{x + \sqrt{(x^2 - y^2)}} = x$, and $\dfrac{x}{y} = \sqrt{\left(\dfrac{1+x}{1-y}\right)}$,

$$\text{from (1) } \frac{x}{\sqrt{(x^2 - y^2)}} = \frac{1+x}{1-x} \text{ or } \frac{x^2}{x^2 - y^2} = \frac{(1+x)^2}{(1-x)^2};$$

$$\therefore\ \frac{y^2}{x^2} = \frac{4x}{(1+x)^2},$$

$$\text{from (2) } \frac{y^2}{x^2} = \frac{1-y}{1+x};\ \therefore\ y = \frac{x}{1+x};$$

$$\therefore\ \frac{y^2}{x^2} = \frac{1}{(1+x)^2} = \frac{4x}{(1+x)^2};\ \therefore\ x = \frac{1}{4},$$

$$\text{and } y = \frac{1}{4\left(1 + \frac{1}{4}\right)} = \frac{1}{5}.$$

(136). $\sqrt{(ax)} + \sqrt{(by)} = \frac{1}{2}(x+y)$, and $(x+y) = 2(a+b)$,

$$\text{from (1) and (2) } \sqrt{(ax)} = (a+b) - \sqrt{(by)};$$

$$\therefore\ ax = (a+b)^2 - 2(a+b)\sqrt{(by)} + by,$$

$$\text{but } ax + ay = 2a^2 + 2ab;$$

$$\therefore\ ay = a^2 - b^2 + 2(a+b)\sqrt{(by)} - by,$$

$$\text{or } y - 2\sqrt{(by)} + b = a - b + b;$$

$$\therefore\ \sqrt{(y)} = \sqrt{(b)} \pm \sqrt{(a)};\ \therefore\ y = \{\sqrt{(b)} \pm \sqrt{(a)}\}^2,$$

$$\text{and } x = \{\sqrt{(a)} \mp \sqrt{(b)}\}^2.$$

(137). $(x+y)^2 = x^4 + x^2y^2 + y^4$, and $x^4 + 4x^3y = 8xy^3 - 4y^4$,

$$\text{from (2) } x^4 - 4x^2y^2 + 4y^4 + 4xy(x^2 - 2y^2) + 4x^2y^2 = 0;$$

$$\therefore\ x^2 - 2y^2 + 2xy = 0,$$

$$\text{and } x^2 + 2xy + y^2 = 3y^2; \;\therefore\; x = y\{-1 \pm \sqrt{(3)}\},$$

$$\therefore\; xy = y^2\{-1 \pm \sqrt{(3)}\},$$

and by substitution in (1), we have

$$3y^2 = y^4\{33 \mp 18\sqrt{(3)}\};$$

$$\therefore\; y = \pm\sqrt{\left\{\frac{1}{11 \mp 6\sqrt{(3)}}\right\}} = \sqrt{\left\{\frac{11 \pm 6\sqrt{(3)}}{13}\right\}},$$

$$\text{and } x = \pm\sqrt{\left\{\frac{8 \pm 2\sqrt{(3)}}{13}\right\}}.$$

(138). $\sqrt{(y)} - \sqrt{(a-x)} = \sqrt{(y-x)}$,

$$\text{and } \sqrt{(y-x)} + \sqrt{(a-x)} : \sqrt{(a-x)} :: 5 : 2,$$

$$\text{from (2) } \sqrt{(y-x)} : \sqrt{(a-x)} :: 3 : 2; \;\therefore\; \sqrt{(y-x)} = \frac{3}{2}\sqrt{(a-x)},$$

$$\text{from (1) } \sqrt{(y)} = \sqrt{(a-x)} + \frac{3}{2}\sqrt{(a-x)} = \frac{5}{2}\sqrt{(a-x)},$$

$$\text{or } y = \frac{25}{4}(a-x);$$

$$\therefore\; y - x = \frac{25a - 29x}{4} = \frac{9}{4}(a-x); \;\therefore\; x = \frac{4a}{5},\; y = \frac{5a}{4}.$$

(139). $yx + y\sqrt{(x^2 - y^2)} = a\{\sqrt{(x+y)} + \sqrt{(x-y)}\}$,

$$\text{and } \sqrt[4]{(x+y)} + \sqrt[4]{(x-y)} = y,$$

$$\text{from (1) } y\{2x + 2\sqrt{(x^2 - y^2)}\} = y\{\sqrt{(x-y)} + \sqrt{(x+y)}\}^2$$

$$= 2a\{\sqrt{(x+y)} + \sqrt{(x-y)}\};$$

$$\therefore\; y\{\sqrt{(x-y)} + \sqrt{(x+y)}\} = 2a \;(3),$$

$$\text{from (2) } \sqrt{(x+y)} + \sqrt{(x-y)} + 2\sqrt[4]{(x^2 - y^2)} = y^2,$$

$$\therefore\; \text{from (3) } 2\sqrt[4]{(x^2 - y^2)} = y^2 - \frac{2a}{y};$$

$$\therefore\; 4\sqrt{(x^2 - y^2)} = \left(y^2 - \frac{2a}{y}\right)^2,$$

$$\text{from (3)}^2\; 2\sqrt{(x^2 - y^2)} = \frac{4a^2}{y^2} - 2x, \text{ or } 4\sqrt{(x^2 - y^2)} = 4\left(\frac{2a^2}{y^2} - x\right);$$

$$\therefore\; y^4 - 4ay + \frac{4a^2}{y^2} = \frac{8a^2}{y^2} - 4x; \;\therefore\; x = \frac{a^2}{y^2} + ay - \frac{y^4}{4} \;(4);$$

$$\therefore\ (x^2 - y^2) = \left(\frac{a^2}{y^2} + ay - \frac{y^4}{4}\right)^2 - y^2 = \frac{1}{16}\left(y^2 - \frac{2a}{y}\right)^4,$$

$$\text{whence } y = a\ \sqrt[3]{\left(\frac{4}{a^2+1}\right)},$$

$$\text{and from (4) } x = \frac{(1+a^2)^{\frac{3}{2}}}{2\sqrt[3]{(2)}}\left\{1 + \left(\frac{2a}{1+a^2}\right)^2\right\}.$$

(140). $x^4 + y^4 - 1 = 2xy + 3x^2y^2$, and $x^3 + y^3 - 1 = x + 2y^2 + 2xy^2$, see (73).

(141). $x^{\frac{1}{4}} + y^{\frac{1}{5}} = 6$, and $x^{\frac{3}{4}} + y^{\frac{3}{5}} = 126$,

$$\text{from (1) } x^{\frac{3}{4}} + y^{\frac{3}{5}} + 3x^{\frac{1}{4}}y^{\frac{1}{5}}(x^{\frac{1}{4}} + y^{\frac{1}{5}}) = 216;$$

therefore, by substitution,

$$3x^{\frac{1}{4}}y^{\frac{1}{5}} \times 6 = 216 - 126 = 90;$$

$$\therefore\ x^{\frac{1}{4}}y^{\frac{1}{5}} = 5, \text{ whence } x^{\frac{1}{4}} - y^{\frac{1}{5}} = \pm 4, \text{ and } x^{\frac{1}{4}} + y^{\frac{1}{5}} = 6;$$

$$\therefore\ x = 625, \text{ and } y = 1.$$

(142). $x^2y + y^2 = 116$, and $xy^{\frac{1}{2}} + y = 14$,

$$\text{from (2) } x = \frac{14 - y}{y^{\frac{1}{2}}};\ \therefore\ 196 - 28y + y^2 + y^2 = 116;$$

$$\therefore\ y^2 - 14y + 49 = 9, \text{ and } y = 10 \text{ or } 4, \text{ and } x = 2\sqrt{\left(\frac{2}{5}\right)} \text{ or } 5.$$

(143). $\sqrt[3]{(x)} + \sqrt[3]{(y)} = 6$, and $x + y = 72$,

$$x + y + 3x^{\frac{1}{3}}y^{\frac{1}{3}}(x^{\frac{1}{3}} + y^{\frac{1}{3}}) = 216;$$

therefore, by substitution,

$$3x^{\frac{1}{3}}y^{\frac{1}{3}} \times 6 = 216 - 72 = 144;\ \therefore\ x^{\frac{1}{3}}y^{\frac{1}{3}} = 8, \text{ and } xy = 512,$$

$$\text{whence } x - y = 56;\ \therefore\ x = 64,\ y = 8.$$

(144). $\sqrt[3]{(x)} + \sqrt[3]{(y)} = 3$, and $x + y = 9$,

$$\text{as in (143) } x^{\frac{1}{3}}y^{\frac{1}{3}} = 2;\ \therefore\ xy = 8,$$

$$\text{and } x - y = \pm 7;\ \therefore\ x = 8, \text{ and } y = 1.$$

(145). $\sqrt{(4y - x)} + \sqrt{(y - x)} = 2\sqrt{(2y - x)}$,

$$\text{and } 3\sqrt{(x^2 - 6y)} + 4\sqrt{(y^2 - 9x)} : 4\sqrt{(x^2 - 6y)} :: 7 : 4,$$

$$\text{from (2) } \sqrt{(y^2 - 9x)} = \sqrt{(x^2 - 6y} \ (A),$$

$$\text{from (1) } 2\sqrt{(4y^2 - 5xy + x^2)} = 3y - 2x,$$

$$\text{or } 16y^2 - 20xy + 4x^2 = 9y^2 - 12xy + 4x^2; \ \therefore y = \frac{8x}{7},$$

$$\text{and from } (A) \therefore \frac{64x^2}{49} - 9x = x^2 - \frac{48x}{7}; \ \therefore x = 7, \text{ and } y = 8$$

(146). $\dfrac{\sqrt[3]{(x+y)}}{8y} + \dfrac{\sqrt[3]{(x+y)}}{8x} = \dfrac{8}{63}$, and $\dfrac{\sqrt[3]{(x-y)}}{y} + \dfrac{\sqrt[3]{(x-y)}}{x} = \dfrac{32}{63}$,

$$\text{from (1) } (x+y)\sqrt[3]{(x+y)} = \frac{64xy}{63},$$

$$\text{from (2) } (x+y)\sqrt[3]{(x-y)} = \frac{32xy}{63};$$

$$\therefore \frac{x+y}{x-y} = (2)^3 = 8; \ \therefore x = \frac{9y}{7};$$

$$\therefore \text{from (1) } \left(\frac{16y}{7}\right)^{\frac{4}{3}} = \frac{64}{63} \times \frac{9y^2}{7}; \ \therefore y = \frac{7}{2}, \text{ and } x = \frac{9}{2}.$$

(147). $(x^2 - y^2) = 17ay + 3y^2 + 4x\sqrt{(2ay - y^2)}$,

$$\text{and } x^2 + y^2 = 2ay + 8\sqrt{(y)}\,\{x\sqrt{(a)} - y\sqrt{(y)}\},$$

$$\text{from (1) } x^2 - 4x\sqrt{(2ay - y^2)} + 4(2ay - y^2) = 25ay;$$

$$\therefore x = 2\sqrt{(2ay - y^2)} \pm 5\sqrt{(ay)},$$

$$\text{from (2) } x = 4\sqrt{(ay)} \pm 3\sqrt{(2ay - y^2)};$$

$$\therefore \sqrt{(ay)} = \sqrt{(2ay - y^2)};$$

$$\therefore ay = 2ay - y^2, \text{ and } y = a;$$

$$\therefore x = 4a \pm 3a = 7a \text{ or } a.$$

(148). $x^{\frac{4}{3}} + y^{\frac{2}{5}} = 20$, and $x^{\frac{2}{3}} + y^{\frac{1}{5}} = 6$,

$$x^{\frac{4}{3}} + y^{\frac{2}{5}} = 20, \text{ also } x^{\frac{4}{3}} + 2x^{\frac{2}{3}}y^{\frac{1}{5}} + y^{\frac{2}{5}} = 36; \ \therefore x^{\frac{2}{3}}y^{\frac{1}{5}} = 8,$$

$$\text{whence } x^{\frac{2}{3}} - y^{\frac{1}{5}} = \pm 2; \ \therefore x^{\frac{2}{3}} = 4; \ \therefore x = 8, \text{ and } y = 32.$$

(149). $(x^2 + y^2)^2 + 4xy\,(x + y)^2 = 1396$, and $x - y = 4$,

from (2) $x^2 + y^2 = 16 + 2xy$;

therefore, by substitution in (1),

$$(16 + 2xy)^2 + 4xy\,(16 + 4xy) = 1396,$$

$$x^2y^2 + \frac{32xy}{5} + \left(\frac{16}{5}\right)^2 = \frac{1681}{25}; \;\therefore\; xy = 5 \text{ or } -\frac{57}{5},$$

from $xy = 5$, $x + y = 6$; $\therefore$ $x = 5$, and $y = 1$.

(150). $\dfrac{\sqrt{(x)} + \sqrt{(x - y)}}{\sqrt{(x)} - \sqrt{(x - y)}} = 4$, and $y = 4\sqrt{(x)}$,

$$\text{from (1) } \frac{x}{x - y} = \frac{25}{9}; \;\therefore\; y = \frac{16x}{25} = 4\sqrt{(x)};$$

$$\text{from (2) } \therefore\; x = \frac{625}{16},\; y = 25.$$

(151). $x + y = a$, and $x^4 + y^4 = b^4$,

from (1) $x^4 + 4xy\,(x^2 + 2xy + y^2) - 2x^2y^2 + y^4 = a^4$;

$$\therefore\; 4xy\,(a^2) - 2x^2y^2 = a^4 - b^4;$$

$$\therefore\; x^2y^2 - 2a^2xy + a^4 = \frac{b^4 - a^4}{2} + a^4 = \frac{b^4 + a^4}{2};$$

$$\therefore\; xy = a^2 \pm \frac{1}{2}\sqrt{\{2\,(a^4 + b^4)\}},$$

$$\text{whence } x^2 - 2xy + y^2 = a^2 - 4a^2 \pm \frac{1}{2}\sqrt{\{2\,(a^4 + b^4)\}};$$

$$\therefore\; x - y = -3a^2 \mp \frac{1}{2}\sqrt{\{2\,(a^4 + b^4)\}}^{\frac{1}{2}},$$

$$\text{and } x = \frac{1}{2}a \pm \sqrt{\left[-3a^2 = \frac{1}{2}\sqrt{\{2\,(a^4 + b^4)\}}\right]},$$

$$y = \frac{1}{2}a \mp \sqrt{\left[-3a^2 = \frac{1}{2}\sqrt{\{2\,(a^4 + b^4)\}}\right]}.$$

(152). $x^4y^3 - x^3y^4 = 216$, and $xy\,(x - y) = 6$,

$$\frac{x^3y^3\,(x - y)}{xy\,(x - y)} = 36, \text{ or } xy = \pm 6;$$

$$\therefore\; x - y = 1, \text{ and } x + y = \pm 5; \;\therefore\; x = 3,\; y = 2.$$

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(153). $x^2y^2 + 4xy = 96$, and $x + y = 6$,

from (1) $x^2y^2 + 4xy + 4 = 100$; $\therefore\ xy = 8$ or -12,

and $x + y = 6$; $\therefore\ x - y = 2$; $\therefore\ x = 4$, and $y = 2$.

(154). $x\{x + \sqrt[3]{(xy^2)}\} = 208$, and $y\{y + \sqrt[3]{(x^2y)}\} = 1053$,

$$\frac{x^{\frac{4}{3}}(x^{\frac{2}{3}} + y^{\frac{2}{3}})}{y^{\frac{4}{3}}(x^{\frac{2}{3}} + y^{\frac{2}{3}})} = \frac{16}{81};\ \therefore\ x = \frac{8y}{27};$$

$$\therefore\ \frac{8y}{27}\left(\frac{8y}{27} + \frac{2y}{3}\right) = 208;$$

$\therefore\ y^2 = 27^2$, and $y = \pm 27$, and $x = \pm 8$.

(155). $x^2 + xy = 6$, and $xy - y^2 = 1$,

from (2) $x = \dfrac{y^2 + 1}{y}$; $\therefore\ (y^2 + 1)^2 + y^2(y^2 + 1) = 6y^2$,

whence $y^2 = \dfrac{3}{4} \pm \dfrac{1}{4} = 1$ or $\dfrac{1}{2}$; $\therefore\ y = 1$ or $2^{-\frac{1}{2}}$,

and $x = \pm 2$ or $\left(\dfrac{3}{2}\right)^{\frac{1}{2}}$.

(156). $x^2 + y^2 + xy(x + y) = 68$, and $x^2(x - 3) + y^2(y - 3) = 12$,

from (1) $3(x^2 + y^2) + 3xy(x + y) = 204$,

also $x^3 + y^3 - 3(x^2 + y^2) = 12$; $\therefore\ x^3 + 3x^2y + 3xy^2 + y^3 = 216$,

and $x + y = 6$; $\therefore\ x^2 + y^2 = 36 - 2xy$,

and from (1) $36 - 2xy + 6xy = 68$; $\therefore\ xy = 8$,

and $x - y = \pm 2$; $\therefore\ x = 4$ or 2, and $y = 2$ or 4.

(157). $x^2 + y^2 - (x + y) = 18$, and $x + y = 19 - xy$,

from (2) $2(x + y) + 2xy = 38$,

also $x^2 + y^2 - (x + y) = 18$; $\therefore\ (x + y)^2 + x + y + \dfrac{1}{4} = \dfrac{225}{4}$;

$\therefore\ x + y = 7$ or -8; $\therefore\ xy = 12$ or 27;

$\therefore\ x - y = \pm 1$, and $x = 4$ or 3, also $y = 3$ or 4.

(158). $\frac{x^2+y^2}{10} = \frac{x+y}{3}$, and $xy = 8$,

$$\text{from (1) } x^2 + y^2 - \frac{10(x+y)}{3} = 0, \text{ but } 2xy = 16;$$

$$\therefore (x+y)^2 - \frac{10(x+y)}{3} + \frac{25}{9} = 16 + \frac{25}{9} = \frac{169}{9};$$

$$\therefore x + y = 6 \text{ or } -\frac{8}{3},$$

$$\text{from } (x+y) = 6, \text{ and } xy = 8; \therefore x - y = \pm 2;$$

$$\therefore x = 4 \text{ or } 2, \text{ and } y = 2 \text{ or } 4.$$

(159). $xy(x+y) = 84$, and $x^2 + y^2 = 3600x^{-2}y^{-2}$,

$$\text{from (1) } x^2 + y^2 = \frac{(84)^2}{x^2y^2} - 2xy, \text{ and } x^2 + y^2 = \frac{3600}{x^2y^2};$$

$$\therefore x^3y^3 = 1728, \text{ and } xy = 12;$$

$$\therefore x + y = 7, \text{ and } x - y = \pm 1, \text{ and } x = 4 \text{ or } 3, \text{ and } y = 3 \text{ or } 4.$$

(160). $x^2 + y^2 = c^2$, and $n(x+y) = m(x-y)$,

$$\text{from (2) } \frac{x+y}{x-y} = \frac{m}{n}; \therefore x = \frac{m+n}{m-n}y,$$

$$\text{and } \frac{(m+n)^2}{(m-n)^2}y^2 + y^2 = 2(m^2+n^2)y^2 = c^2(m-n)^2;$$

$$\therefore y = \pm c\frac{m-n}{\sqrt{(m^2+n^2)}}, \text{ and } x = \pm c\frac{m+n}{\sqrt{(m^2+n^2)}}.$$

(161). $y\{x - \sqrt{(x^2-y^2)}\} = a\{\sqrt{(x+y)}\} + \sqrt{(x-y)}$,

$$\text{and } (x+y)^{\frac{3}{2}} - (x-y)^{\frac{3}{2}} = b,$$

$$\text{let } x + y = 2r, \ x - y = 2s; \ \therefore x = r + s, \ y = r - s,$$

$$\text{and from (1) } (r-s)\{r + s - 2\sqrt{(rs)}\} = a\sqrt{2}\,(r^{\frac{1}{2}} + s^{\frac{1}{2}});$$

$$\therefore (r^{\frac{1}{2}} - s^{\frac{1}{2}})(r^{\frac{1}{2}} - s^{\frac{1}{2}})^2 = (r^{\frac{1}{2}} - s^{\frac{1}{2}})^3 = a\sqrt{2};$$

$$\therefore r^{\frac{1}{2}} - s^{\frac{1}{2}} = a^{\frac{1}{3}}2^{\frac{1}{6}} \ (A),$$

$$\text{from (2) } 2^{\frac{3}{2}}(r^{\frac{3}{2}} - s^{\frac{3}{2}}) = b \ (B);$$

$$\therefore \frac{B}{A} = \frac{r^{\frac{3}{2}} - s^{\frac{3}{2}}}{r^{\frac{1}{2}} - s^{\frac{1}{2}}} = r + \sqrt{(rs)} + s = \frac{b}{a^{\frac{1}{3}} 2^{\frac{5}{3}}},$$

$$\text{and from } (A)\ r - 2\sqrt{(rs)} + s = a^{\frac{2}{3}} 2^{\frac{1}{3}};$$

$$\therefore \sqrt{(rs)} = \frac{1}{3}\left(\frac{b - 4a}{a^{\frac{1}{3}} 2^{\frac{5}{3}}}\right);$$

$$\therefore r + 2\sqrt{(rs)} + s = \frac{2^{\frac{1}{3}}(b - a)}{a^{\frac{1}{3}}};$$

$$\therefore r^{\frac{1}{2}} + s^{\frac{1}{2}} = \pm \frac{2^{\frac{1}{6}}\sqrt{(b - a)}}{a^{\frac{1}{6}}}, \text{ and } r^{\frac{1}{2}} - s^{\frac{1}{2}} = a^{\frac{1}{3}} 2^{\frac{1}{6}},$$

$$\text{whence } r^{\frac{1}{2}} = \frac{\sqrt{(a)} \pm \sqrt{(b - a)}}{a^{\frac{1}{6}} 2^{\frac{5}{6}}}; \ \therefore r = \frac{b \pm 2\sqrt{(ab - a^2)}}{2^{\frac{5}{3}} a^{\frac{1}{3}}},$$

$$\text{and so also } s = \frac{b \mp 2\sqrt{(ab - a^2)}}{2^{\frac{5}{3}} a^{\frac{1}{3}}};$$

$$\therefore x = r + s = \frac{b}{\sqrt[3]{(4a)}},$$

$$\text{and } y = \sqrt[6]{(4a)} \times \sqrt{(b - a)}.$$

(162). $x^2 + xy + y^2 = 7(x + y)$, and $(x^2 - xy + y^2) = 9(x - y)$,

from (1) $x^3 - y^3 = 7(x^2 - y^2)$, from (2) $x^3 + y^3 = 9(x^2 - y^2)$;

$$\therefore \frac{x^3 + y^3}{x^3 - y^3} = \frac{9}{7}; \ \therefore x^3 = 8y^3, \text{ or } x = 2y;$$

$$\therefore 4y^2 + 2y^2 + y^2 = 7(3y); \ \therefore y = 3, \ x = 6.$$

(163). $\dfrac{xy}{x + y} = a$, and $\dfrac{x^2y^2}{x^2 + y^2} = b^2$,

$$x^2 = \frac{a^2y^2}{(y - a)^2} = \frac{b^2y^2}{y^2 - b^2}, \text{ whence } (a^2 - b^2)\,y^2 + 2ab^2y = 2a^2b^2,$$

$$\text{and } y = \frac{ab\,\{-b \pm \sqrt{(2a^2 - b^2)}\}}{a^2 - b^2}, \ x = \frac{ab}{a^2 - b^2}\{-b \mp \sqrt{(2a^2 - b^2)}\}.$$

(164). $\frac{1}{x} + \frac{1}{y} = m$, and $\frac{1}{x^2} + \frac{1}{y^2} = n^2$,

$$\frac{1}{x^2} = m^2 + \frac{1}{y^2} - \frac{2m}{y}, \text{ also } \frac{1}{x^2} = -\frac{1}{y^2} + n^2;$$

$$\therefore \frac{2}{y^2} - \frac{2m}{y} = n^2 - m^2,$$

$$\text{and } \frac{1}{y^2} - \frac{m}{y} + \frac{m^2}{4} = \frac{n^2 - m^2}{2} + \frac{m^2}{4} = \frac{2n^2 - m^2}{4};$$

$$\therefore \frac{1}{y} = \frac{m}{2} \pm \frac{1}{2}\sqrt{(2n^2 - m^2)},$$

$$\text{and } y = \frac{m \mp \sqrt{(2n^2 - m^2)}}{m^2 - n^2}, \text{ also } x = \frac{m \pm \sqrt{(2n^2 - m^2)}}{m^2 - n^2}.$$

(165). $\frac{1+x}{1+y} + \frac{1-y}{1-x} = \frac{4}{13}$, and $\frac{1+x}{1-y} + \frac{1+y}{1-x} = \frac{9}{13}$,

$$\frac{1-x^2}{1-y^2} + 1 = \frac{4}{13}\,\frac{1-x}{1-y}, \text{ also } \frac{1-x^2}{1-y^2} + 1 = \frac{9}{13}\,\frac{1-x}{1+y};$$

$$\therefore \frac{4}{1-y} = \frac{9}{1+y}; \therefore y = \frac{5}{13}, \ 1 - y = \frac{8}{13}, \ 1 + y = \frac{18}{13};$$

$$\therefore \frac{13(1+x)}{8} + \frac{18}{13}\,\frac{1}{1-x} = \frac{9}{13};$$

$$\therefore \text{ whence } x = \frac{36}{169} \pm \frac{205}{169} = \frac{241}{169} \text{ or } -1.$$

(166). $\frac{x + y - \sqrt{(x^2 + y^2)}}{x + y + \sqrt{(x^2 + y^2)}} = \frac{2x}{a}$, and $x\sqrt{(a - y)} = y\sqrt{(a + x)}$,

$$\text{from (1) } \frac{x+y}{\sqrt{(x^2 + y^2)}} = \frac{a + 2x}{a - 2x},$$

$$\text{from (2) } \frac{x^2}{y^2} = \frac{a+x}{a-y}; \ \therefore \frac{x^2 - y^2}{x^2} = \frac{x+y}{a+x},$$

$$\text{or } x - y = \frac{x^2}{a+x}; \ \therefore y = \frac{ax}{a+x}.$$

$$\text{Again, } \frac{x^2 + 2xy + y^2}{x^2 + y^2} = \left(\frac{a+2x}{a-2x}\right)^2;$$

$$\therefore \frac{2xy}{(x+y)^2} = \frac{8ax}{(a+2x)^2} \text{ or } \frac{y}{(x+y)^2} = \frac{4a}{(a+2x)^2};$$

$$\therefore \frac{ax(a+x)^2}{(a+x)(2ax+x^2)^2} = \frac{4a}{(a+2x)^2}; \therefore \frac{(a+x)}{x(2a+x)^2} = \frac{4}{(a+2x)^2},$$

$$\text{whence } x^2 + \frac{11ax}{8} + \frac{121a^2}{256} = \frac{153a^2}{256}; \therefore x = \frac{-11 \pm \sqrt{(153)}}{16} \times a,$$

$$\text{and } y = \frac{13 \mp \sqrt{(153)}}{8} \times a.$$

(167). $\frac{x^2}{y^2} + \frac{y^2}{x^2} + \frac{x}{y} + \frac{y}{x} = \frac{27}{4}$, and $x - y = 2$,

$$\left(\frac{x}{y} + \frac{y}{x}\right)^2 + \left(\frac{x}{y} + \frac{y}{x}\right) + \frac{1}{4} = \frac{36}{4}; \therefore \frac{x}{y} + \frac{y}{x} = \frac{5}{2} \text{ or } -\frac{7}{2};$$

$$\therefore x^2 + y^2 = \frac{5xy}{2}, \text{ but } x^2 + y^2 = 4 + 2xy; \therefore xy = 8,$$

whence $x + y = \pm 6$; $\therefore x = 4$ or -2, and $y = 2$ or -4.

(168). $\frac{\sqrt{(3x)}}{\sqrt{(x+y)}} - 2 + \frac{\sqrt{(x+y)}}{\sqrt{(3x)}} = 0$, and $xy - 54 = x + y$,

$$\frac{3x}{x+y} - \frac{2\sqrt{(3x)}}{\sqrt{(x+y)}} + 1 = 0; \therefore \frac{\sqrt{(3x)}}{\sqrt{(x+y)}} = 1; \therefore y = 2x,$$

from (2) $2x^2 - 54 = 3x$; $\therefore x = \frac{3}{4} \pm \frac{21}{4} = 6$ or $-\frac{9}{2}$,

and $y = 12$ or -9.

(169). $\sqrt{(x)} - \sqrt{(y)} = \sqrt{(y+2)}$, and $\frac{x+8}{8} = \sqrt{(y+2)}$,

from (2) $y + 2 = \left(\frac{x}{8} + 1\right)^2$;

$\therefore$ from (1) $x - 2\sqrt{(xy)} + y = y + 2$; $\therefore x - 2\sqrt{(xy)} = 2$,

and $x^2 - 4x + 4 = 4xy = \frac{x^3}{16} + x^2 - 4x$;

$\therefore x^3 = 64$, and $x = 4$, and $y = \frac{1}{4}$.

(170). $3x^2 + 4y^2 = 7xy$, and $x^{\frac{3}{2}} - yx^{\frac{1}{2}} = \dfrac{2y^2}{9}$,

$$\text{from (1) } x^2 - \frac{7xy}{3} + \left(\frac{7y}{6}\right)^2 = \frac{y^2}{36}; \ \therefore\ x = \frac{4y}{3} \text{ or } y,$$

$$\text{from } x = y,\ (x^{\frac{3}{2}} - x^{\frac{3}{2}}) = 2x^2; \ \therefore\ x = 0, \text{ and } y = 0,$$

$$\text{from } x = \frac{4y}{3},\ y^{\frac{1}{2}} = \frac{9}{2}\left\{\left(\frac{4}{3}\right)^{\frac{3}{2}} - \left(\frac{4}{3}\right)^{\frac{1}{2}}\right\}; \ \therefore\ y = 3, \text{ and } x = 4.$$

(171). $5ax + 12y(a - x) = 0$, and $x^2 + a^2 = y^2$,

$$\text{from (1) } y = \frac{5ax}{12(x - a)},$$

$$\text{from (2) } (x^2 + a^2) \times 144(x - a)^2 = 25a^2x^2;$$

$$\therefore\ 144(x^2 + a^2)^2 - 288ax(x^2 + a^2) + 144a^2x^2 = 169a^2x^2;$$

$$\therefore\ x^2 + a^2 = \frac{(12 \pm 13)ax}{12} = \frac{25ax}{12} \text{ or } \frac{ax}{12},$$

$$\text{whence } x = \frac{4a}{3} \text{ or } \frac{3a}{4}, \text{ and } y = \frac{5a}{3} \text{ or } -\frac{5a}{4}.$$

(172). $x^2y - 4 = 4y\sqrt{(x)} - \dfrac{y^3}{4}$, and $\{\sqrt{(x)} - \sqrt{(y)}\}\sqrt{(xy)} = x^{\frac{3}{2}} - 3$,

$$\text{from (1) } x^2y + xy^2 + \frac{y^3}{4} = xy^2 + 4y\sqrt{(x)} + 4;$$

$$\therefore\ x\sqrt{(y)} + \frac{y^{\frac{3}{2}}}{2} = \pm\{y\sqrt{(x)} + 2\},$$

$$\text{or } y^{\frac{3}{2}} + 2x\sqrt{(y)} - 2y\sqrt{(x)} - 4 = 0,$$

$$\text{and } x^{\frac{3}{2}} - 3 - x\sqrt{(y)} + y\sqrt{(x)} = 0;$$

$$\therefore\ y^{\frac{3}{2}} - 3y\sqrt{(x)} + 3x\sqrt{(y)} - x^{\frac{3}{2}} = 1,$$

$$\text{and } y^{\frac{1}{2}} - x^{\frac{1}{2}} = 1; \ \therefore \text{ from (2) } x^{\frac{3}{2}} + \sqrt{(xy)} - 3 = 0,$$

and by substitution,

$$x^{\frac{3}{2}} + \sqrt{x}(\sqrt{x} + 1) = 3, \text{ or } x^{\frac{3}{2}} + x + \sqrt{x} - 3 = 0;$$

$$\therefore\ x(x^{\frac{1}{2}}-1)+(x-1)+\sqrt{x}-1=0,\ \sqrt{x}-1=0;$$

$$\therefore\ x=1, \text{ and } y=4,$$

$$\text{also } x+x^{\frac{1}{2}}+2=0,$$

$$\text{whence } x=\frac{1}{2}\{-3\mp\sqrt{(-7)}\}, \text{ and } y=\frac{1}{2}\{-3\pm\sqrt{(-7)}\}.$$

(173). $x^2+xy+y^2=a^2$, and $x+\sqrt{(xy)}+y=b$,

$$\frac{x^2+xy+y^2}{x+\sqrt{(xy)}+y}=x-\sqrt{(xy)}+y=\frac{a^2}{b}, \text{ and } x+\sqrt{(xy)}+y=b;$$

$$\therefore\ \sqrt{(xy)}=\frac{b^2-a^2}{2b};\ \therefore \text{ from (2) } x+y=\frac{a^2+b^2}{2b},$$

$$\text{and } x-y=\frac{1}{2b}\sqrt{\{4a^2b^2-3(b^2-a^2)\}};$$

$$\therefore\ x=\frac{1}{4b}[b^2+a^2+\sqrt{\{4a^2b^2-3(b^2-a^2)^2\}}],$$

$$\text{and } y=\frac{1}{4b}[b^2+a^2-\sqrt{\{4a^2b^2-3(b^2-a^2)^2\}}].$$

(174). $(x^2+y^2)(x^3+y^3)=455$, and $x+y=5$,

$$\text{from (1) } (x^2-xy+y^2)(x+y)(x^2+y^2)=455,$$

$$\text{from (2) } x^2+y^2=25-2xy;$$

therefore, by substitution,

$$(25-3xy)\times 5\times(25-2xy)=455, \text{ whence } xy=6 \text{ or } \frac{89}{6};$$

$$\therefore\ x-y=\pm 1, \text{ and } x=3 \text{ or } 2, \text{ and } y=2 \text{ or } 3.$$

(175). $3(x^2+y^2)y=26x$, and $2x(x^2-y^2)=15y$,

$$\frac{x^2+y^2}{x^2-y^2}=\frac{52x^2}{45y^2}, \text{ whence } 52x^4-97x^2y^2=45y^4,$$

$$\text{and } x^2=\frac{9y^2}{4} \text{ or } -\frac{5y^2}{13},$$

$$\text{from } x=+\frac{3y}{2},\ \frac{9y^2}{4}-y^2=15y\times\frac{1}{3y}=5;$$

$$\therefore\ y=\pm 2, \text{ and } x=\pm 3.$$

(176). $\sqrt{(x)} - \sqrt{(y)} = x + \sqrt{(xy)}$, and $(x+y)^2 = 2(x-y)^2$,

$$\text{from (2) } \frac{x+y}{x-y} = \sqrt{2}\,; \;\therefore\; x = \frac{\sqrt{2}+1}{\sqrt{2}-1}\,y = (3 + 2\sqrt{2})\,y,$$

by substitution in (1),

$$(3 + 2\sqrt{2})\,y + \sqrt{y} + (\sqrt{y} - 1)(1 + \sqrt{2}) = 0,$$

whence $y = 1$ or $3 - 2\sqrt{(2)}$, and $x = 3 + 2\sqrt{2}$ or 1.

(177). $(x-2)\,y - 2y^2 + x = \sqrt{(xy)}\,(y^2 - 1)$,

$$\text{and } xy\,(xy - 18) = 4\,\{\sqrt{(xy)} - 12\},$$

$$\text{from (2) } x^2y^2 - 14xy + 49 = 4xy + 4\sqrt{(xy)} + 1\,;$$

$$\therefore\; xy - 7 = \pm\,\{2\sqrt{(xy)} + 1\}\,;$$

$$\therefore\; xy - 2\sqrt{(xy)} + 1 = 9, \text{ and } \sqrt{(xy)} = 4 \text{ or } -2,$$

$$\text{from (1), if } xy = 16,\; 3y^3 + y^2 - 10y - 8 = 0,$$

$$\text{whence } y^3 - 8 + 2y^2(y-2) + 5y\,(y-2) = 0\,; \;\therefore\; y = 2,$$

$$\text{and } y^2 + \frac{7y}{3} + \left(\frac{7}{6}\right)^2 = \frac{1}{36}; \;\therefore\; y = -\frac{7}{6} \pm \frac{1}{6} = -1 \text{ or } -\frac{4}{3},$$

$$\text{and } x = 8,\; -16,\; -12,$$

and from $xy = 4$ other values may be obtained.

(178). $\dfrac{x}{y} - \dfrac{y}{x} = \dfrac{x+y}{x^2+y^2}$, and $\dfrac{x^2}{y^2} - \dfrac{y^2}{x^2} = \dfrac{x-y}{y^2}$,

$$\text{from (1) } \frac{x-y}{xy} = \frac{1}{x^2+y^2}, \text{ from (2) } \frac{x+y}{x^2} = \frac{1}{x^2+y^2};$$

$$\therefore\; \frac{x-y}{xy} = \frac{x+y}{x^2}, \text{ whence } x = y\,(1 \pm \sqrt{2}),$$

$$\text{from (1) } 1 \pm \sqrt{2} + 1 \mp \sqrt{2} = \frac{2 \pm \sqrt{2}}{(4 \pm 2\sqrt{2})\,y};$$

$$\therefore\; y = \frac{1}{4}, \text{ and } x = \frac{1}{4}(1 \pm \sqrt{2}).$$

(179). $\dfrac{\sqrt{(x^2+y^2)} + \sqrt{(x^2-y^2)}}{\sqrt{(x^2+y^2)} - \sqrt{(x^2-y^2)}} = \dfrac{5 + \sqrt{(7)}}{5 - \sqrt{(7)}}$, and $\dfrac{8y}{3} - 12 = \dfrac{x^2}{16}$,

$$\frac{\sqrt{(x^2+y^2)}}{\sqrt{(x^2-y^2)}} = \frac{5}{\sqrt{(7)}}, \text{ and } \frac{x^2+y^2}{x^2-y^2} = \frac{25}{7};$$

$$\therefore \frac{x^2}{y^2} = \frac{16}{9}; \ \therefore x = \pm \frac{4y}{3},$$

$$\text{from (2) } \frac{x^2}{16} - 2x + 16 = 4;$$

$$\therefore x = 24 \text{ or } 8, \text{ and } y = 18 \text{ or } 6.$$

(180). $\{\sqrt{(x)} - 3\sqrt{(y)}\}^2 + 5 - 2\sqrt{(y+2)} = \dfrac{9x^2}{64},$

$$\text{and } 7 - 10\sqrt{(xy)} = y(x - 16),$$

$$\text{from (2) } xy + 10\sqrt{(xy)} + 25 = 16(y+2);$$

$$\therefore \sqrt{(xy)} = -5 \pm 4\sqrt{(y+2)} \ (3),$$

by substitution in (1),

$$x + 9y + 6\{5 - 4\sqrt{(y+2)}\} + 5 - 2\sqrt{(y+2)} = \frac{9x^2}{64},$$

$$\text{whence } 9(y+2) - 26\sqrt{(y+2)} + \frac{169}{9} = \frac{9x^2}{64} - x + \frac{16}{9},$$

$$\text{and } 3\sqrt{(y+2)} = \frac{13}{3} \pm \left(\frac{3x}{8} - \frac{4}{3}\right); \ \therefore \sqrt{(y+2)} = \frac{x}{8} + 1,$$

$$\text{and } y = \frac{x^2}{64} + \frac{x}{4} - 1;$$

and by substitution in (3), we have

$$\sqrt{\left(\frac{x^3}{64} + \frac{x^2}{4} - x\right)} = -5 \pm \left(\frac{x}{2} + 4\right) = \frac{x}{2} - 1,$$

$$\text{and } \frac{x^3}{64} + \frac{x^2}{4} - x = \frac{x^2}{4} - x + 1; \ \therefore x^3 = 64, \text{ and } x = 4,$$

$$\text{and } y = \frac{1}{4}.$$

(181). $a(1 - xy) = x\sqrt{(1 - y^2)}$, and $\sqrt{(x)}\,(1 - xy) = y - x$,

$$\text{from (2) } y = \frac{x + x^{\frac{1}{2}}}{x^{\frac{3}{2}} + 1} = \frac{x^{\frac{1}{2}}}{x - x^{\frac{1}{2}} + 1},$$

$$\text{whence } 1 - xy = \frac{x - x^{\frac{1}{2}} + 1 - x^{\frac{3}{2}}}{x - x^{\frac{1}{2}} + 1} = \frac{(1+x)(1 - x^{\frac{1}{2}})}{x - x^{\frac{1}{2}} + 1},$$

$$\text{and } 1 - y^2 = \frac{(x - x^{\frac{1}{2}} + 1)^2 - x}{(x - x^{\frac{1}{2}} + 1)^2} = \frac{(x + 1)(x^{\frac{1}{2}} - 1)^2}{(x - x^{\frac{1}{2}} + 1)^2};$$

$$\therefore \text{ from (1) } - x\sqrt{(x + 1)} = a(1 + x), \text{ or } - x = a\sqrt{(1 + x)};$$

$$\therefore x = -1, \text{ and } y = -1,$$

$$\text{also } x^2 = a^2x \pm a^2, \text{ whence } x = \frac{1}{2}\{a^2 + a\sqrt{(a^2 + 4)}\}, \text{ and } y = \frac{a^2 - 1}{a^2 + 1}.$$

(182). $x^n + y^n = a^n$, and $xy = b^2$,

$$x^{2n} + 2x^ny^n + y^{2n} = a^{2n}, \text{ but } x^ny^n = b^{2n};$$

$$\therefore x^{2n} - 2x^ny^n + y^{2n} = a^{2n} - 4b^{2n};$$

$$\therefore x^n - y^n = \pm\sqrt{(a^{2n} - 4b^{2n})},$$

$$\text{and } x^n + y^n = a^{2n}; \ \therefore x = \sqrt[n]{\left[\frac{1}{2}\{a^{2n} \pm \sqrt{(a^{2n} - 4b^{2n})}\}\right]},$$

$$\text{and } y = \sqrt[n]{\left[\frac{1}{2}\{x^{2n} \mp \sqrt{(a^{2n} - 4b^{2n})}\}\right]}.$$

(183). $x + \sqrt{(3y^2 - 11 + 2x)} = 7 + 2y - y^2$, and $\sqrt{(3y - x + 7)} = \dfrac{x + y}{x - y}$,

$$\text{from (1) } 3y^2 + 2x - 11 + 2\sqrt{(3y^2 + 2x - 11)} + 1 = y^2 + 4y + 4;$$

$$\therefore \sqrt{(3y^2 + 2x - 11)} = \pm(y + 2) - 1 = y + 1 \text{ or } -(y + 3)\ (B);$$

$$\therefore 3y^2 + 2x - 11 = y^2 + 2y + 1\ (A), \text{ or } y^2 + 6y + 9),$$

$$\text{from } (A)\ x = 6 + y - y^2;$$

therefore, by substitution in (2),

$$\sqrt{(y^2 + 2y + 1)} = \frac{6 + 2y - y^2}{6 - y^2} = y + 1,$$

whence $y = \pm 2$, and $x = 4$, also $y = 0$, and $x = 6$,

and the other values may be obtained from (B).

(184). $x^3 - y^3 : (x - y)^3 :: 61 : 1$, and $xy = 320$,

$$\text{from (1) } x^2 + xy + y^2 : x^2 - 2xy + y^2; \ \therefore 61 : 1;$$

$$\therefore \frac{3xy}{(x - y)^2} = \frac{60}{1}, \text{ but } xy = 320;$$

$$\therefore x - y = \pm 4, \text{ and } x^2 + 2xy + y^2 = 1296; \ \therefore x + y = \pm 36;$$

$$\therefore x = 20 \text{ or } -20, \text{ and } y = 16 \text{ or } -16.$$

(185). $2(x^2+y^2)(x+y)=15xy$, and $4(x^4-y^4)(x^2+y^2)=75x^2y^2$,

$$\frac{(x^2+y^2)(x^2-y^2)(x^2+y^2)}{(x^2+y^2)(x+y)}=\frac{5xy}{2},$$

$$\text{or } (x^2+y^2)(x-y)=\frac{5xy}{2}\ (A);$$

$$\therefore \text{ from (1) } \frac{x+y}{x-y}=3;\ \therefore\ x=2y,$$

$$\text{from } (A)\ 5y^2\times y=5y^2;\ \therefore\ y=1, \text{ and } x=2.$$

(186). $(x^2-y^2)(x-y)=3xy$, and $(x^4-y^4)(x^2-y^2)=45x^2y^2$,

$$\frac{(x^4-y^4)(x^2-y^2)}{(x^2-y^2)(x-y)}=15xy, \text{ or } (x^2+y^2)(x+y)=15xy\ (A);$$

$$\therefore\ \frac{(x^2+y^2)(x+y)}{(x^2-y^2)(x-y)}=\frac{x^2+y^2}{(x-y)^2}=5;\ \therefore\ \frac{2xy}{(x-y)^2}=4,$$

$$\text{whence } \left(\frac{x+y}{x-y}\right)^2=9, \text{ and } \frac{x+y}{x-y}=3;\ \therefore\ x=2y,$$

$$\text{and from } (A)\ 5y^2\times 3y=30y^2;\ \therefore\ y=2, \text{ and } x=4.$$

(187). $\sqrt{(y)}+\sqrt{(x)} : \sqrt{(y)}-\sqrt{(x)} :: \sqrt{(x)}+2 : 1$,

and $\sqrt{(y)}+2-\sqrt{(x)} : \sqrt{(x)} :: 3x+\sqrt{(x)}+\sqrt{(y)} : \sqrt{(xy)}$,

$$\text{from (1) } \sqrt{y} : \sqrt{x} :: \sqrt{x}+3 : \sqrt{x}+1;\ \therefore\ \sqrt{y}=\frac{(x^{\frac{1}{2}}+3)\,x^{\frac{1}{2}}}{x^{\frac{1}{2}}+1},$$

by substitution in (2)

$$2\times\frac{(2x^{\frac{1}{2}}+1)}{x^{\frac{1}{2}}+1} : x^{\frac{1}{2}} :: \frac{3x^{\frac{3}{2}}+5x+4x^{\frac{1}{2}}}{x^{\frac{1}{2}}+1} : x\times\frac{x^{\frac{1}{2}}+3}{x^{\frac{1}{2}}+1},$$

$$\text{whence } 2\times\frac{2x^{\frac{1}{2}}+1}{x^{\frac{1}{2}}+1}\times\frac{x^{\frac{1}{2}}+3}{1}=3x+5x^{\frac{1}{2}}+4;$$

$$\therefore\ 2(2x+7x^{\frac{1}{2}}+3)=3x^{\frac{3}{2}}+8x+9x^{\frac{1}{2}}+4;$$

$$\therefore\ 3x^{\frac{3}{2}}+4x-5x^{\frac{1}{2}}-2=0;$$

$$\therefore\ 3x\,(x^{\frac{1}{2}}-1)+7x^{\frac{1}{2}}\,(x^{\frac{1}{2}}-1)+2\,(x^{\frac{1}{2}}-1)=0;$$

$$\therefore\ x=1,\ \text{and}\ y=4,$$

$$\text{and}\ 3x+7x^{\frac{1}{2}}+2=0,$$

$$\text{whence}\ x=\frac{1}{9}\ \text{or}\ 4,\ y=\frac{5}{2}\ \text{or}\ 2.$$

(188). $x-8y\,\sqrt{(x^2-9xy^2)}=(9-16x)\,y^2$, and $5x-4=25y^2$,

$$\text{from (2)}\ x=5y^2+\frac{4}{5},$$

$$\text{from (1)}\ x-9y^2-8y\,\sqrt{\{x\,(x-9y^2)\}}+16xy^2=0;$$

$$\therefore\ \sqrt{(x-9y^2)}=4y\,\sqrt{(x)},\ \text{and}\ x-9y^2=16xy^2;$$

$$\therefore\ x=\frac{9y^2}{1-16y^2}=5y^2+\frac{4}{5},\ \text{and}\ y^2=-\frac{21}{200}\pm\frac{29}{200}=\frac{1}{25}\ \text{or}\ -\frac{1}{4};$$

$$\therefore\ y=\pm\frac{1}{5}\ \text{or}\ \pm\frac{1}{2}\sqrt{(-1)},\ \text{and}\ x=1\ \text{or}\ -\frac{9}{20}.$$

(189). $\sqrt{\{5\,\sqrt{(x)}+5\,\sqrt{(y)}\}}=10-\{\sqrt{(x)}+\sqrt{(y)}\}$, and $x^{\frac{5}{2}}+y^{\frac{5}{2}}=275$

$$\text{from (1)}\ \sqrt{x}+\sqrt{y}+\sqrt{5}\,\sqrt{(\sqrt{x}+\sqrt{y})}+\frac{5}{4}=10+\frac{5}{4}=\frac{45}{4};$$

$$\therefore\ \sqrt{(\sqrt{x}+\sqrt{y})}=\frac{\sqrt{5}}{2}\pm\frac{3\,\sqrt{5}}{2}=2\,\sqrt{5}\ \text{or}\ -\sqrt{5};$$

$$\therefore\ \sqrt{x}+\sqrt{y}=20\ \text{or}\ 5,$$

taking the latter value, we have, from (2),

$$\frac{x^{\frac{5}{2}}+y^{\frac{5}{2}}}{x^{\frac{1}{2}}+y^{\frac{1}{2}}}=x^2-x^{\frac{3}{2}}y^{\frac{1}{2}}+xy-x^{\frac{1}{2}}y^{\frac{3}{2}}+y^2=55,$$

$$\text{or}\ (x+y)^2-x^{\frac{1}{2}}y^{\frac{1}{2}}\,(\sqrt{x}+\sqrt{y})^2+xy=55,$$

$$\text{or}\ \{25-2\,\sqrt{(xy)}\}^2-x^{\frac{1}{2}}y^{\frac{1}{2}}\times 25+xy=55,$$

$$\text{whence}\ xy-25\,\sqrt{(xy)}+114=0;$$

$$\therefore\ \sqrt{(xy)}=\frac{25}{2}\pm\frac{13}{2}=19\ \text{or}\ 6,$$

and from $\sqrt{(xy)}=6$, $\sqrt{x}-\sqrt{y}=1$; $\therefore\ x=9$, and $y=4$, and other values of x and y may be found from

$$\sqrt{(xy)} = 19, \text{ and } \sqrt{x} + \sqrt{y} = 20,$$

$$\text{and } x = 361, \text{ and } y = 1.$$

(190). $2\sqrt{\{6\sqrt{(x)} + 6\sqrt{(y)}\}} + \sqrt{(x)} = 18 - \sqrt{(y)}$, and $x - y = 12$,

$$\text{from (1) } \sqrt{x} + \sqrt{y} + 2\sqrt{6}\sqrt{(\sqrt{x} + \sqrt{y})} + 6 = 24;$$

$$\therefore \sqrt{(\sqrt{x} + \sqrt{y})} = -\sqrt{6} \pm 2\sqrt{6} = \sqrt{6} \text{ or } -3\sqrt{6},$$

$$\text{and } \sqrt{x} + \sqrt{y} = 6 \text{ or } 54,$$

from $\sqrt{x} + \sqrt{y} = 6$, we have $\dfrac{x - y}{\sqrt{x} + \sqrt{y}} = 2$; $\therefore \sqrt{x} - \sqrt{y} = 2$,

$$\text{whence } x = 16, \text{ and } y = 4, \text{ also } \sqrt{x} - \sqrt{y} = \frac{2}{9},$$

$$\text{whence } x = \frac{100}{81}, \text{ and } y = \frac{64}{81}.$$

(191). $\sqrt{\{(1 + x)^2 + y^2\}} + \sqrt{\{(1 - x)^2 + y^2\}} = 4$, and $(4 - x^2)^2 + 4y^2 = 18$,

from (1) $(1 + x)^2 + y^2 = 16 - 8\sqrt{(1 - x)^2 + y^2\}} + (1 - x)^2 + y^2$;

$$\therefore x - 4 = -2\sqrt{\{(1 - x)^2 + y^2\}},$$

$$\text{or } x^2 - 8x + 16 = 4(1 - 2x + x^2) + 4y^2;$$

$$\therefore 4y^2 = 12 - 3x^2;$$

$$\therefore \text{from (2) } 16 - 8x^2 + x^4 + 12 - 3x^2 = 18;$$

$$\therefore x^4 - 11x^2 + \frac{121}{4} = \frac{81}{4};$$

$$\therefore x^2 = 10 \text{ or } 1, \text{ and } x = 1 \text{ or } \pm\sqrt{(10)};$$

$$\therefore y = \pm\frac{3}{2} \text{ or } \pm 3\sqrt{\left(-\frac{1}{2}\right)}.$$

(192). $\dfrac{x + \sqrt{(x^2 - y^2)}}{x - \sqrt{(x^2 - y^2)}} + \dfrac{x - \sqrt{(x^2 - y^2)}}{x + \sqrt{(x^2 - y^2)}} = 4\frac{1}{4}$,

$$\text{and } x(x + y) + \sqrt{(x^2 + xy + 4)} = 52,$$

$$\text{from (1) } \{x^2 + \sqrt{(x^2 - y^2)}\}^2 + \{x - \sqrt{(x^2 - y^2)}\}^2 = \frac{17y^2}{4},$$

$$\text{or } 4x^2 = 2y^2 + \frac{17y^2}{4} = \frac{25y^2}{4}; \ \therefore x = \pm\frac{5y}{4},$$

$$\text{from (2) } x^2 + xy + 4 + \sqrt{(x^2 + xy + 4)} + \frac{1}{4} = \frac{225}{4};$$

$$\therefore\ \sqrt{(x^2 + xy + 4)} = 7 \text{ or } -8;$$

$$\therefore\ x^2 + xy + 4 = 49 \text{ or } 64;$$

$$\therefore\ \frac{25y^2}{16} + \frac{5y^2}{4} = \frac{45y^2}{16} = 45;\ \therefore\ y = \pm 4,$$

$$\text{and } x = \pm 5.$$

(193). $\sqrt{\left(\dfrac{x+y^2}{4x}\right)} + \dfrac{y}{\sqrt{(y^2+x)}} = \dfrac{y^2}{4}\sqrt{\left(\dfrac{4x}{y^2+x}\right)}$,

and $\sqrt{(x)} + \sqrt{(x-y-1)} = (y+1)\{\sqrt{(x)} - \sqrt{(x-y-1)}\}$,

from (2) $\dfrac{\sqrt{x} + \sqrt{(x-y-1)}}{\sqrt{x} - \sqrt{(x-y-1)}} = y + 1;\ \therefore\ \dfrac{\sqrt{x}}{\sqrt{(x-y-1)}} = \dfrac{y+2}{y}$,

and $\dfrac{x}{x-y-1} = \dfrac{(y+2)^2}{y^2};\ \therefore\ \dfrac{y+1}{x} = \dfrac{4(y+1)}{(y+2)^2};\ \therefore\ x = \dfrac{1}{4}(y+2)^2$,

$$\text{from (1) } \frac{y^2}{4x} + \frac{y}{2\sqrt{(x)}} + \frac{1}{4} = \frac{y^2}{4};$$

$$\therefore\ y = \left(-\frac{1}{2} + \frac{y}{2}\right) 2\sqrt{(x)};\ \therefore\ x = \frac{y^2}{(y-1)^2};$$

$$\therefore\ (y+2)^2 (y-1)^2 = 4y^2, \text{ or } y^2 + y - 2 = \pm 2y;$$

$$\therefore\ y^2 - y + \frac{1}{4} = \frac{9}{4};\ \therefore\ y = 2 \text{ or } -1,$$

$$\text{also } x = 4 \text{ or } \frac{1}{4}.$$

(194). $(x-2)y - \sqrt{(xy)}(y^2 - 1) = 2y^2 - x$,

$$\text{and } xy(xy - 18) = 4\sqrt{(xy)} - 48,$$

$$\text{from (2) } x^2y^2 - 14xy + 49 = 4xy + 4\sqrt{(xy)} + 1;$$

$$\therefore\ xy - y = \pm\{2\sqrt{(xy)} + 1\};$$

$$\therefore\ xy - 2\sqrt{(xy)} + 1 = 9;\ \therefore\ \sqrt{(xy)} = 4 \text{ or } -2,$$

$$\text{from } \sqrt{(xy)} = 4 \text{ in (1) } 16 - 2y - 4y^2 + 4 = 2y^2 - \frac{16}{y},$$

$$3y^3 + y^2 - 10y - 8 = 0;$$

$$\therefore\ 3y^2(y-2) + 7y(y-2) + 4(y-2) = 0;$$

$$\therefore\ y = 2, \text{ and } x = 8;$$

$$\text{also } y^2 + \frac{7y}{3} + \left(\frac{7}{6}\right)^2 = \frac{1}{36};$$

$$\therefore\ y = -1 \text{ or } -\frac{4}{3}, \text{ and } x = -16 \text{ or } -12.$$

(195). $xy + xy^2 = 12$, and $x + xy^3 = 18$,

$$\therefore\ \frac{x(1+y^3)}{xy(1+y)} = \frac{1-y+y^2}{y} = \frac{3}{2},$$

$$\text{whence } y = \frac{5}{4} \pm \frac{3}{4} = 2 \text{ or } \frac{1}{2}, \text{ and } x = \frac{12}{y(1+y)} = 2 \text{ or } 16.$$

(196). $\dfrac{x+\sqrt{(x^2-y^2)}}{x-\sqrt{(x^2-y^2)}} = \dfrac{17}{4} - \dfrac{x-\sqrt{(x^2-y^2)}}{x+\sqrt{(x^2-y^2)}}$,

$$\text{and } 2x^2 + 6xy - \sqrt{(x^2+3xy-21)} = 162,$$

$$\text{see (192) from (1) } x = \pm\frac{5y}{4};$$

$\therefore$ from (2)

$$x^2 + 3xy - 21 - \frac{1}{2}\sqrt{(x^2+3xy-21)} + \frac{1}{16} = 60 + \frac{1}{16} = \frac{961}{16};$$

$$\therefore\ \sqrt{(x^2+3xy-21)} = 8 \text{ or } -\frac{15}{2};$$

$$\therefore\ x^2 + 3xy = 85, \text{ and } y = \frac{85-x^2}{3x} = \frac{4x}{5},$$

$$\text{whence } x \pm 5, \text{ and } y = \pm 4.$$

(197). $2x + y = 26 - \sqrt{(2x+y+4)}$,

$$\text{and } \frac{2x+\sqrt{(y)}}{2x-\sqrt{(y)}} = \frac{16}{15} + \frac{2x-\sqrt{(y)}}{2x+\sqrt{(y)}};$$

$$\therefore\ 2x + y + 4 - \sqrt{(2x+y+4)} + \frac{1}{4} = 30 + \frac{1}{4} = \frac{121}{4},$$

$$\text{whence } 2x + y + 4 = 36 \text{ or } 25;\ \therefore\ y = 32 - 2x \text{ or } 21 - 2x,$$

$$\text{from (2) } \left(\frac{2x+\sqrt{y}}{2x-\sqrt{y}}\right)^2 - \frac{16}{15}\,\frac{2x+\sqrt{y}}{2x-\sqrt{y}} + \left(\frac{8}{15}\right)^2 = 1 + \frac{64}{225} = \frac{289}{225};$$

$$\therefore\ \frac{2x+\sqrt{y}}{2x-\sqrt{y}} = \frac{5}{3} \text{ or } \frac{3}{5};\ \therefore\ x = 2\sqrt{y};$$

$$\therefore\ 2x = 4\sqrt{y} = 32 - y,$$

$$\text{whence } y = 16 \text{ or } 64, \text{ and } x = 8 \text{ or } 16.$$

(198). $\dfrac{x - \sqrt{(x^2 - y^2)}}{x + \sqrt{(x^2 - y^2)}} = x$, and $\dfrac{x}{y} = \sqrt{\left\{\dfrac{5}{2}\left(\dfrac{1 - x^2}{1 - y^2}\right)\right\}}$,

$$\frac{x}{\sqrt{(x^2 - y^2)}} = \frac{1 + x}{1 - x}; \;\therefore\; \frac{x^2}{x^2 - y^2} = \left(\frac{1 + x}{1 - x}\right)^2,$$

$$\text{and } \frac{x^2}{y^2} = \frac{(1 + x)^2}{4x}; \;\therefore\; y = \pm\frac{2x^{\frac{3}{2}}}{1 + x},$$

$$\text{from (2) } \frac{2x^2}{5y^2} = \frac{1 - x^2}{1 - y^2};$$

$$\therefore\; 2x^2 = 5y^2 - 3x^2y^2 = (5 - 3x^2)\frac{4x^3}{(1 + x)^2};$$

$$\therefore\; 1 + 2x + x^2 = 10x - 6x^3,$$

$$\text{whence } 6x(x^2 - 1) + x(x - 1) - (x - 1) = 0;$$

$$\therefore\; x = 1 \text{ and } y = 1,$$

$$\text{also } 6x^2 + 7x - 1 = 0, \text{ whence } x = \frac{1}{12}\{-7 \pm \sqrt{(73)}\}.$$

(199). $\sqrt{(x)} + \sqrt{(y)} = 4\{\sqrt{(x)} - \sqrt{(y)}\}$, and $x^2 - y^2 = 544$,

$$\frac{\sqrt{x} + \sqrt{y}}{\sqrt{x} - \sqrt{y}} = 4; \;\therefore\; \sqrt{x} = \frac{5}{3}\sqrt{y}, \text{ or } x = \frac{25y}{9};$$

$$\therefore\; \frac{625y^2}{81} - y^2 = \frac{544y^2}{81} = 544; \;\therefore\; y = \pm 9 \text{ and } x = \pm 25.$$

(200). $\dfrac{2xy + y\sqrt{(x^2 - y^2)}}{14} = \sqrt{\left(\dfrac{x + y}{2}\right)} + \sqrt{\left(\dfrac{x - y}{2}\right)}$,

$$\text{and } \left(\frac{x + y}{2}\right)^{\frac{3}{2}} + \left(\frac{x - y}{2}\right)^{\frac{3}{2}} = 9,$$

$$\text{let } x + y = 2m, \; x - y = 2n;$$

$$\therefore\; x = m + n, \; y = m - n,$$

$$\text{and } 2(m^2 - n^2) + 2(m - n)\sqrt{(mn)} = 14(m^{\frac{1}{2}} + n^{\frac{1}{2}}),$$

$$\text{or } (m^{\frac{1}{2}} - n^{\frac{1}{2}})\{m + n + \sqrt{(mn)}\} = m^{\frac{3}{2}} - n^{\frac{3}{2}} = 7,$$

$$\text{also } m^{\frac{3}{2}} + n^{\frac{3}{2}} = 9,$$

$$\text{and } m^{\frac{3}{2}} - n^{\frac{3}{2}} = 7; \;\therefore\; m^{\frac{3}{2}} = 8, \text{ and } m = 4 \text{ and } n = 1.$$

$$\therefore\; x = 5, \; y = 3.$$

(201). $\sqrt{(x)} - \sqrt{(y)} = 21$, and $\sqrt[4]{(x)} + \sqrt[4]{(y)} = 7$,

$$(\sqrt[4]{x} - \sqrt[4]{y})(\sqrt[4]{x} + \sqrt[4]{y}) = \sqrt[4]{x} + \sqrt[4]{y} \times 7 = 21;$$

$$\therefore \frac{\sqrt[4]{x} + \sqrt[4]{y}}{\sqrt[4]{x} - \sqrt[4]{y}} = \frac{7}{3}, \text{ and } \sqrt{x} = \frac{25\sqrt{y}}{4};$$

$$\therefore 25\sqrt{y} - 4\sqrt{y} = 21 \times 4; \quad \therefore y = 16 \text{ and } x = 625.$$

(202). $xy + \sqrt{(x^2y^2 - y^4)} = 8\{\sqrt{(x+y)} + \sqrt{(x-y)}\}$,

$$\text{and } (x+y)^{\frac{3}{2}} - (x-y)^{\frac{3}{2}} = 26,$$

$$\text{let } x + y = 2m, \ x - y = 2n; \ \therefore x = m + n, \ y = m - n,$$

$$\text{and } y\{x + \sqrt{(x^2 - y^2)}\} = 8\{\sqrt{(x+y)} + \sqrt{(x-y)}\},$$

$$\text{or } (m-n)\{m + n + 2\sqrt{(mn)}\} = 8\sqrt{2}\,(m^{\frac{1}{2}} + n^{\frac{1}{2}});$$

$$\therefore (m^{\frac{1}{2}} - n^{\frac{1}{2}})(m^{\frac{1}{2}} + n^{\frac{1}{2}})^2 = 8\sqrt{(2)}\ (A),$$

$$\text{from (2) } 2^{\frac{3}{2}}(m^{\frac{3}{2}} - n^{\frac{3}{2}}) = 26\ (B);$$

$$\therefore \frac{A}{B} = \frac{m + n + 2\sqrt{(mn)}}{m + n + \sqrt{(mn)}} = \frac{16}{13},$$

$$\text{whence } m^{\frac{1}{2}} = 3n^{\frac{1}{2}} \text{ and } 3m^{\frac{1}{2}} = n^{\frac{1}{2}};$$

$$\therefore 2^{\frac{3}{2}}(m^{\frac{3}{2}} - n^{\frac{3}{2}}) = 2^{\frac{3}{2}} \times 26n^{\frac{3}{2}} = 26; \ \therefore n = \frac{1}{2} \text{ and } m = \frac{9}{2};$$

$$\therefore x = m + n = \frac{1}{2} + \frac{9}{2} = 5 \text{ and } y = 4.$$

(203). $\sqrt{(x+y)} - \sqrt{(x-y)} = a$, and $\sqrt[4]{(x+y)} + \sqrt[4]{(x-y)} = b$,

$$\text{from (2) } 2x + 4\sqrt[4]{(x^2-y^2)}\{\sqrt{(x+y)} + \sqrt{(x-y)} + 2\sqrt[4]{(x^2-y^2)}\} - 2\sqrt{(x^2-y^2)} = b^4;$$

$$\therefore 2x + 4\sqrt[4]{(x^2-y^2)}\,(b^2) - 2\sqrt{(x^2-y^2)} = b^4,$$

$$\text{from (1) } 2x - 2\sqrt{(x^2-y^2)} = a^2\ (A);$$

$$\therefore 4b^2\sqrt[4]{(x^2-y^2)} = b^4 - a^2;$$

$$\therefore \sqrt{(x^2-y^2)} = \left(\frac{b^4 - a^2}{4b^2}\right)^2;$$

$$\text{from } (A) \therefore x = \sqrt{(x^2 - y^2)} + \frac{a^2}{2} = \frac{(b^4 - a^2)^2}{16b^4} + \frac{a^2}{2};$$

$$\therefore x^2 - y^2 = \left(x - \frac{a^2}{2}\right)^2 = \left\{\frac{(b^4 - a^2)^2}{16b^4}\right\}^2,$$

$$\text{whence } x = \frac{(b^4 - a^2)^2}{16b^4} + \frac{a^2}{2},\ y = \frac{a}{4}\left(b^2 + \frac{a^2}{b^2}\right).$$

(204). $3\sqrt{(x+y)} = 4 + 4(x+y)^{-\frac{1}{2}}$, and $\sqrt{(x+y)} + \sqrt{(x-y)} = 5$,

$$\text{from (1) } x + y - \frac{4}{3}\sqrt{(x+y)} + \frac{4}{9} = \frac{4}{3} + \frac{4}{9} = \frac{16}{9};$$

$$\therefore \sqrt{(x+y)} = 2 \text{ or } -\frac{1}{3}; \ \therefore \sqrt{(x-y)} = 5 - 2 = 3 \text{ or } \frac{16}{3}.$$

$$\text{whence } x + y = 4 \text{ and } x - y = 9; \ \therefore x = \frac{13}{2} \text{ and } y = -\frac{5}{2}.$$

(205). $3x^2 + 4y^2 = 7xy$, and $x^{\frac{3}{2}} - \frac{2}{9}y^2 = yx^{\frac{1}{2}}$,

$$\text{from (1) } y^2 - \frac{7xy}{4} + \left(\frac{7x}{8}\right)^2 = \frac{x^2}{64}; \ \therefore y = x \text{ or } \frac{3x}{4},$$

from (1), if $y = x$, $y = 0$ and $x = 0$,

$$\text{if } y = \frac{3x}{4},\ \frac{2}{9} \times \frac{9x^2}{16} + \frac{3x^{\frac{3}{2}}}{4} = x^{\frac{3}{2}}; \ \therefore x = 4 \text{ and } y = 3.$$

(206). $a^2x^{2n} - b^2y^{2n} = c^2 - d^2$, and $ax^n + by^n = c + d$,

$$\therefore \frac{a^2x^{2n} - b^2y^{2n}}{ax^n + by^n} = ax^n - by^n = c - d,$$

$$\text{and } ax^n + by^n = c + d;$$

$$\therefore x = (a^{-1}c)^{\frac{1}{n}},\ y = (b^{-1}d)^{\frac{1}{n}}.$$

(207). $x^2 + x\sqrt[3]{(xy^2)} = 208$, and $y^2 + y\sqrt[3]{(x^2y)} = 1053$,

$$\frac{x^{\frac{4}{3}}(x^{\frac{2}{3}} + y^{\frac{2}{3}})}{y^{\frac{4}{3}}(x^{\frac{2}{3}} + y^{\frac{2}{3}})} = \frac{208}{1053} \text{ or } y^{\frac{4}{3}} = \frac{81x^{\frac{4}{3}}}{16},$$

$$\text{whence } y = \frac{27x}{8}; \ \therefore x^2 + \frac{9}{4}x^2 = 208;$$

$$\therefore x = \pm 8 \text{ and } y = \pm 27.$$

K 2

(208). $2\sqrt{(x^2-y^2)}+x+y=2(x-1)$, and $15(x^2+y^2)=34xy$,

$$\text{from (2) } x^2-\frac{34xy}{15}+\left(\frac{17y}{15}\right)^2=\frac{64y^2}{225};\ \therefore\ x=\frac{5}{3}y \text{ or } \frac{3}{5}y,$$

$$\text{from } x=\frac{5y}{3} \text{ in (1), } \frac{8y}{3}+\frac{8y}{3}=2\left(\frac{5y-3}{3}\right),$$

$$\text{or } y=-1 \text{ and } x=-\frac{5}{3},$$

$$\text{from } x=\frac{3y}{5},\ y=\frac{5}{17}\{4\sqrt{(-1)}-1\} \text{ and } x=\frac{3}{17}\{4\sqrt{(-1)}-1\}.$$

(209). $\sqrt{(x)}-\sqrt{(y)}=x+\sqrt{(xy)}$, and $(x+y)^2=2(x-y)^2$,

$$\text{from (1) } \frac{\sqrt{(x)}-\sqrt{(y)}}{\sqrt{(x)}+\sqrt{(y)}}=\sqrt{x};\ \therefore\ \sqrt{y}=\sqrt{x}\,\frac{1-\sqrt{x}}{1+\sqrt{x}},$$

$$\text{from (2) } \frac{x+y}{x-y}=\sqrt{2};\ \therefore\ y=\frac{\sqrt{2}-1}{\sqrt{2}+1}x=x\left(\frac{1-\sqrt{x}}{1+\sqrt{x}}\right)^2,$$

$$\text{or } \frac{1+\sqrt{x}}{1-\sqrt{x}}=\sqrt{2}+1;\ \therefore\ \sqrt{x}=\frac{\sqrt{2}}{\sqrt{2}+2}=\sqrt{2}-1;$$

$$\therefore\ x=3-2\sqrt{(2)} \text{ and } y=(\sqrt{2}-1)^4=17-12\sqrt{(2)}.$$

(210). $(x+y)^m-(x+y)^{-m}=a$, and $x-y=b$,

$$\text{from (1) } (x+y)^{2m}-a(x+y)^m+\frac{a^2}{4}=\frac{a^2}{4}+1;$$

$$\therefore\ (x+y)^m=\frac{a}{2}\pm\frac{1}{2}\sqrt{(a^2+4)};$$

$$\therefore\ x+y=\left[\frac{1}{2}\{a\pm\sqrt{(a^2+4)}\}\right]^{\frac{1}{m}} \text{ and } x-y=b;$$

$$\therefore\ x=\frac{b}{2}+\sqrt[m]{\left[\frac{1}{2}\{a\pm\sqrt{(a^2+4)}\}\right]},$$

$$\text{and } y=-\frac{b}{2}+\sqrt[m]{\left[\frac{1}{2}\{a\pm\sqrt{(a^2+4)}\}\right]}.$$

(211). $x^ma^n+y^nb^m=2\sqrt{\{(ax)^m(by)^n\}}$, and $xy=ab$,

$$\text{from (2) } y=\frac{ab}{x};$$

$$\therefore\ \text{from (1) } x^{m+n}-2a^{\frac{m-n}{2}}b^nx^{\frac{m+n}{2}}+(\quad)=a^{m-n}b^{2n}-a^nb^{m+n};$$

$$\therefore x = [b^n \{\sqrt{(a^{m-n})} \pm \sqrt{(a^{m-n} - a^n b^{m-n})}\}]^{\frac{2}{m+n}},$$

$$\text{and } y = [a^m \{\sqrt{(b^{n-m})} \mp \sqrt{(b^{n-m} - a^{n-m})}\}]^{\frac{2}{m+n}}.$$

(212). $x^m y^n = (ac)^m b^n$, and $x^n y^m = (bc)^n a^m$,

$$x = acb^{\frac{n}{m}} y^{-\frac{n}{m}} = bca^{\frac{m}{n}} y^{-\frac{m}{n}};$$

$$\therefore y^{\frac{m^2-n^2}{mn}} = b^{\frac{m-n}{m}} a^{\frac{m-n}{n}}, \text{ or } y = (a^m b^n)^{\frac{1}{m+n}},$$

$$\text{and } x = c\,(a^m b^n)^{\frac{1}{m+n}}.$$

(213). $x^2 y - 4y\sqrt{(x)} = 4 - \dfrac{1}{4} y^3$, and $x^{\frac{3}{2}} - \sqrt{(xy)}\,\{\sqrt{(x)} - \sqrt{(y)}\} = 3$, see Ex. (172).

(214). $\dfrac{x^2}{y^2} + \dfrac{2x + y}{\sqrt{(y)}} = 20 - \dfrac{y^2 + x}{y}$, and $4y - x = 8$,

$$\frac{x^2}{y^2} + \frac{2x}{\sqrt{(y)}} + y + \frac{x}{y} + \sqrt{y} + \frac{1}{4} = \frac{81}{4},$$

$$\text{whence } \frac{x}{y} + \sqrt{y} = 4 \text{ or } -5;$$

$$\therefore x = 4y - y^{\frac{3}{2}} = 4y - 8;\ \therefore y = 4 \text{ and } x = 8,$$

$$\text{also } x = -5y - y^{\frac{3}{2}} = 4y - 8, \text{ whence } y\,(y^{\frac{1}{2}} + 1) + 8\,(y - 1) = 0;$$

$$\therefore y = 1 \text{ and } x = 4,$$

$$\text{also } y + 8y^{\frac{1}{2}} + 16 = 8 + 16 = 24;$$

$$\therefore y = (-4 \pm 2\sqrt{6})^2 \text{ and } x = 34 \mp 16\sqrt{6}.$$

(215). $x + y = a$, and $x^4 + y^4 = d^4$,

$$(x + y)^4 = x^4 + y^4 + 4xy\,(x^2 + 2xy + y^2) - 2x^2y^2 = a^4,$$

$$\text{or } x^4 + y^4 + 4a^2xy - 2x^2y^2 = a^4,$$

$$\text{but } x^4 + y^4 \qquad\qquad = b^4;$$

$$\therefore 2x^2y^2 - 4a^2xy = b^4 - a^4,$$

$$\text{or } x^2y^2 - 2a^2xy + a^4 = \frac{b^4 + a^4}{2};$$

$$\therefore\ xy = a^2 \pm \sqrt{\left(\frac{b^4 + a^4}{2}\right)},$$

$$\text{and } x - y = \sqrt{[-3a^2 \pm 2\sqrt{\{2(a^4 + b^4)\}}]};$$

$$\therefore\ x = \frac{1}{2}[a + \sqrt{\{-3a^2 \mp 2\sqrt{(2a^4 + b^4)}\}}],$$

$$\text{and } y = \frac{1}{2}[a \mp \sqrt{\{-3a^2 \mp \sqrt{(2(a^4 + b^4))}\}}].$$

(216). $\{\sqrt{(x)} + \sqrt{(y)}\}^2 + \{\sqrt[4]{(x)} + \sqrt[4]{(y)}\}^2 = 210 + \sqrt[4]{(xy)}\{\sqrt[4]{(xy)} + 1\}$,

and $\{\sqrt{(x)} - \sqrt{(y)}\}^2 + \{\sqrt[4]{(x)} - \sqrt[4]{(y)}\}^2 = 126 - \sqrt[4]{(xy)}\{\sqrt[4]{(xy)} + 1\}$

by subtraction,

$$4\sqrt{(xy)} + 4\sqrt[4]{(xy)} = 84 + 2\sqrt{(xy)} + 2\sqrt[4]{(xy)},$$

$$\text{whence } \sqrt{(xy)} + \sqrt[4]{(xy)} + \frac{1}{4} = 42 + \frac{1}{4} = \frac{169}{4};$$

$$\therefore\ \sqrt[4]{(xy)} = -\frac{1}{2} \pm \frac{13}{2} = 6 \text{ or } -7,$$

from $\sqrt[4]{(xy)} = 6$ by substitution,

$$x + y + \sqrt{x} + \sqrt{y} = 168;$$

$$\therefore\ (\sqrt{x} + \sqrt{y})^2 + \sqrt{x} + \sqrt{y} + \frac{1}{4} = 240 + \frac{1}{4} = \frac{961}{4};$$

$$\therefore\ \sqrt{x} + \sqrt{y} = 15 \text{ or } -16,$$

from $\sqrt{(x)} + \sqrt{(y)} = 15$, $\sqrt{x} - \sqrt{y} = \pm 9$; $\therefore\ x = 144$, and $y = 9$.

(217). $\dfrac{x + y + \sqrt{(x^2 - y^2)}}{x + y - \sqrt{(x^2 - y^2)}} = \dfrac{9}{8y}(x + y)$,

$$\text{and } (x^2 + y)^2 + x - y = 2x(x^2 + y) + 506,$$

from (2) $(x^2 + y)^2 - 2x(x^2 + y) + x^2 - (x^2 + y - x) + \dfrac{1}{4} = \dfrac{2025}{4}$;

$$\therefore\ x^2 + y - x = 23 \text{ or } -22\ (A),$$

from (1) $\dfrac{x + y}{\sqrt{(x^2 - y^2)}} = \dfrac{9y + 17y}{9x + y}$, or $\dfrac{x + y}{x - y} = \left(\dfrac{9x + 17y}{9x + y}\right)^2$,

$$\text{whence } \frac{x}{y} = \frac{162x^2 + 324xy + 290y^2}{288xy + 288y^2},$$

$$\text{and } x^2 - \frac{2xy}{7} + \frac{y^2}{49} = \frac{1024}{441}; \therefore x = \frac{5y}{3} \text{ or } -\frac{29y}{21},$$

from the positive value in (A),

$$\frac{25y^2}{9} - \frac{2y}{3} + \frac{1}{25} = \frac{576}{25}; \therefore y = 3 \text{ and } x = 5.$$

(218). $\frac{x+y-\sqrt{(x^2+y^2)}}{x+y+\sqrt{(x^2+y^2)}} = \frac{2x}{a}$, and $\frac{x}{y} = \sqrt{\left(\frac{a+x}{a-y}\right)}$,

$$\text{from (1) } \frac{x+y}{\sqrt{(x^2+y^2)}} = \frac{a+2x}{a-2x} \; (A),$$

$$\text{from (2) } \frac{x^2-y^2}{x^2} = \frac{x+y}{a+x}, \text{ or } x-y = \frac{x^2}{a+x};$$

$$\therefore y = \frac{ax}{a+x}, \text{ and } x+y = \frac{2ax+x^2}{a+x}, \text{ and } x^2+y^2 = \frac{a^2x^2+x^2(a+x)^2}{(a+x)^2};$$

$$\text{from } (A) \; \frac{(2a+x)^2}{a^2+(a+x)^2} = \frac{(a+2x)^2}{(a-2x)^2}; \therefore \frac{2a^2+2ax}{(2a+x)^2} = \frac{8ax}{(a+2x)^2},$$

$$\text{whence } x^2 + \frac{11ax}{8} + \left(\frac{11a}{16}\right)^2 = \frac{153a^2}{256};$$

$$\therefore x = \frac{-11 \pm \sqrt{(153)}}{16} \times a \text{ and } y = \frac{13 \pm \sqrt{(153)}}{8} \times a.$$

(219). $\frac{1}{x}\left(1 - \frac{4}{y}\right) + \frac{1}{y}\left(1 - \frac{1}{x}\right) = 1$,

$$\text{and } x + y + 12 = 2xy,$$

$$\text{from (1) } x = \frac{y-5}{y-1}, \text{ from (2)} = \frac{y+12}{2y-1};$$

$$\therefore 2y^2 - 11y + 5 = y^2 + 11y - 12;$$

$$\therefore y^2 - 22y + 121 = 104; \therefore y = 11 \pm 2\sqrt{(26)},$$

$$\text{and } x = 11 \mp 2\sqrt{(26)}.$$

(220). $x^4 + y^4 - 3x^2y^2 - 2xy = 1$, and $x^3 + y^3 - 2y^2x - 2y^2 = x + 1$, see Ex. 140.

(221). $\frac{1}{x} + \frac{1}{y} = \frac{1}{m}$, and $\frac{1}{x^3} + \frac{1}{y^3} = \frac{1}{n^3}$,

$$\left(\frac{1}{x^3} + \frac{1}{y^3}\right) \div \left(\frac{1}{x} + \frac{1}{y}\right) = \frac{1}{x^2} - \frac{1}{xy} + \frac{1}{y^2} = \frac{m}{n^3},$$

$$\text{also } \frac{1}{x^2} + \frac{2}{xy} + \frac{1}{y^2} = \frac{1}{m^2}; \ \therefore \ \frac{3}{xy} = \frac{n^3 - m^3}{m^2 n^3},$$

$$\text{whence } \frac{1}{x} - \frac{1}{y} = \sqrt{\left(\frac{4m^3 - n^3}{3m^2 n^3}\right)},$$

and

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{m}; \ \therefore \ \frac{2}{x} = \frac{1}{m} + \sqrt{\left(\frac{4m^3 - n^3}{3m^2 n^3}\right)} = \frac{\sqrt{(3n^3)} + \sqrt{(4m^3 - n^3)}}{m\sqrt{(3n^3)}};$$

$$x = \frac{2m\sqrt{(3n^3)}}{\sqrt{(3n^3)} + \sqrt{(4m^3 - n^3)}}, \ y = \frac{2mn\sqrt{(3n)}}{n\sqrt{(3n)} \mp \sqrt{(4m^3 - n^3)}}.$$

(222). $x^4 = mx + ny$, and $y^4 = my + nx$,

$$\because \ x^4 - y^4 = m(x - y) + n(x - y);$$

$$\therefore \ x - y = 0, \text{ and } x = y,$$

$$\text{and } x^3 = m + n; \ \therefore \ x = \sqrt[3]{(m + n)} = y,$$

$$\text{also } \frac{x^4}{y^4} = \frac{mx + ny}{my + nx}, \text{ let } x = yz;$$

$$\therefore \ z^4 = \frac{mz + n}{m + nz}, \text{ whence } mz(z^3 - 1) + n(z^5 - 1) = 0;$$

$$\therefore \ z - 1 = 0, \text{ and } x = y,$$

$$\text{also } n(z^4 + z^3 + z^2 + z + 1) + mz(z^2 + z + 1) = 0;$$

$$\therefore n\left(z^2 + z + 1 + \frac{1}{z} + \frac{1}{z^2}\right) + m\left(z + 1 + \frac{1}{z}\right) = 0;$$

$$\text{or } \left(z + \frac{1}{z}\right)^2 + \frac{m + n}{n}\left(z + \frac{1}{z}\right) + \left(\frac{m + n}{2n}\right)^2 = \frac{m^2 - 2mn + 5n^2}{4n^2},$$

$$z + \frac{1}{z} = -\frac{m + n}{2n} \pm \frac{\sqrt{(m^2 - 2mn + 5n^2)}}{2n} = a;$$

$$\therefore \ z^2 - az + \frac{a^2}{4} = \frac{a^2 - 4}{4}; \ \therefore z = \frac{a}{2} \pm \frac{\sqrt{(a^2 - 4)}}{2}.$$

By substitution in (1)

$$\left(\frac{a \pm \sqrt{a^2 - 4}}{2}\right) y^3 = m . \frac{a}{2} \pm \frac{\sqrt{(a^2 - 4)}}{2} + n;$$

$$\therefore \ y = \sqrt[3]{\left[m + \frac{n}{2}\{a \pm \sqrt{(a^2 - 4)}\}\right]},$$

$$\text{and } x = \sqrt[3]{\left[m + \frac{n}{2}\{a \mp \sqrt{(a^2 - 4)}\}\right]}.$$

(223). $x^2 - y^2 = a^2$, and $(x + y + b)^2 + (x - y + b)^2 = c^2$,

$$\text{from (2) } 2x^2 + 2y^2 + 4bx = c^2 - 2b^2,$$
$$\text{from (1) } 2x^2 - 2y^2 \qquad = 2a^2;$$
$$\therefore\ 4x^2 + 4bx + b^2 = 2a^2 + c^2 - b^2;$$
$$\therefore\ x = \frac{1}{2}\{-b \pm \sqrt{(2a^2 + c^2 - b^2)}\},$$
$$\text{and } y = \pm\frac{1}{2}\sqrt{[\{c^2 - 2a^2 \pm \sqrt{(2a^2 + c^2 - b^2)}\}]}.$$

(224). $a(1 - xy) = x\sqrt{(1 - y^2)}$, and $\sqrt{(x)}(1 - xy) + x = y$,

$$\text{from (2) } y = \frac{x + \sqrt{(x)}}{1 + x^{\frac{3}{2}}} = \frac{\sqrt{x}}{1 - x^{\frac{1}{2}} + x};$$
$$\therefore\ 1 - y^2 = \frac{(1 - x^{\frac{1}{2}} + x)^2 - x}{(1 - x^{\frac{1}{2}} + x)^2} = \frac{(1 + x)(1 - x^{\frac{1}{2}})^2}{(1 - x^{\frac{1}{2}} + x)^2},$$
$$1 - xy = \frac{1 - x^{\frac{1}{2}} + x - x^{\frac{3}{2}}}{(1 - x^{\frac{1}{2}} + x)} \quad \frac{(1 + x)(1 - x^{\frac{1}{2}})}{1 - x^{\frac{1}{2}} + x};$$
$$\therefore\ a.\frac{(1 + x)(1 - x^{\frac{1}{2}})}{1 - x^{\frac{1}{2}} + x} = \frac{x(1 - x^{\frac{1}{2}})(1 + x)^{\frac{1}{2}}}{1 - x^{\frac{1}{2}} + x},$$
$$\text{hence } x^{\frac{1}{2}} - 1 = 0,\ x = 1, \text{ and } y = 1,$$
$$\text{also } (x + 1) = 0;\ \therefore\ x = -1, \text{ and } y = -1,$$
$$\text{also } a(1 + x)^{\frac{1}{2}} = x;\ \therefore\ a^2 + a^2x = x^2;$$
$$\therefore\ x^2 - a^2x + \frac{a^4}{4} = a^2 + \frac{a^4}{4} = \frac{4a^2 + a^4}{4};$$
$$\therefore\ x = \frac{a}{2}\{a \pm \sqrt{(a^2 + 4)}\} = m;$$
$$\therefore\ y = \frac{\sqrt{m}}{1 + m - \sqrt{(m)}}.$$

(225). $\dfrac{y - x + \sqrt{(2xy - 3x^2)}}{3(y - 2x)} = \dfrac{(2y - 3x)^{\frac{3}{2}} + x^{\frac{3}{2}}}{(2y - 3x)^{\frac{3}{2}} - x^{\frac{3}{2}}}$,

$$\text{and } \frac{y}{x^2} + \frac{16}{81}\left\{x - \sqrt{(x)} - \frac{3}{4}\right\} = \frac{4}{9x}\{2\sqrt{(xy)} - \sqrt{(y)}\},$$

$$\text{from (1) } \frac{\{\sqrt{(2y-3x)} + x^{\frac{1}{2}}\}^2}{6(y-2x)} = \frac{(2y-3x)^{\frac{3}{2}} + x^{\frac{3}{2}}}{(2y-3x)^{\frac{3}{2}} - x^{\frac{3}{2}}};$$

$$\therefore \frac{\sqrt{(2y-3x)} + \sqrt{(x)}}{6(y-2x)} = \frac{2y - 2x - \sqrt{(2xy - 3x^2)}}{(2y-3x)^{\frac{3}{2}} - x^{\frac{3}{2}}},$$

$$\text{whence } \frac{(2y-3x) - x}{6(y-2x)} = \frac{2(y-x) - \sqrt{(2xy-3x^2)}}{2(y-x) + \sqrt{(2xy-3x^2)}};$$

$$\therefore y - x = \sqrt{(2xy - 3x^2)}; \ \therefore y^2 - 4y + 4x^2 = 0, \text{ and } y = 2x,$$

$$\text{from (2) } \frac{2}{x} - \frac{4\sqrt{2}}{9\sqrt{x}}(2\sqrt{x} - 1) + \frac{4}{81}(2\sqrt{x} - 1)^2 = \frac{16}{81},$$

$$\text{whence } 4x - 2\sqrt{x} = 9\sqrt{2}, \text{ or } 4x - 6\sqrt{(x)} = 9\sqrt{2}),$$

from the latter

$$x = \frac{9}{8}\{1 + 2\sqrt{2} \pm \sqrt{(4\sqrt{2} + 1)}\} \text{ and } y = 2x.$$

(226). $(x^6 + 1)y = (y^2 + 1)x^3$, and $(y^6 + 1)x = 9y^3(x^2 + 1)$,

$$\text{from (1) } x^3 + \frac{1}{x^3} = y + \frac{1}{y}, \text{ from (2) } \frac{1}{3}\left(y^3 + \frac{1}{y^3}\right) = 3\left(x + \frac{1}{x}\right);$$

$$\therefore y^3 + \frac{1}{y^3} + 3\left(y + \frac{1}{y}\right) = \left\{x^3 + \frac{1}{x^3} + 3\left(x + \frac{1}{x}\right)\right\} \times 3;$$

$$\therefore y + \frac{1}{y} = \left(x + \frac{1}{x}\right)\sqrt[3]{(3)} = x^3 + \frac{1}{x^3};$$

$$\therefore x^2 - 1 + \frac{1}{x^2} = \sqrt[3]{(3)}, \text{ and } x^2 + 1 = 0; \ \therefore x = \pm\sqrt{(-1)},$$

$$\text{also } x^2 + 2 + \frac{1}{x^2} = \sqrt[3]{(3)} + 3;$$

$$\therefore x + \frac{1}{x} = \pm\sqrt{\{3 + \sqrt[3]{(3)}\}},$$

$$\text{and so also } x - \frac{1}{x} = \pm\sqrt{\{-1 + \sqrt[3]{(3)}\}};$$

$$\therefore x = \frac{1}{2}[\pm\sqrt{(3 + \sqrt[3]{3})} \pm \sqrt{\{-1 + \sqrt[3]{(3)}\}}];$$

$$\therefore\ y + \frac{1}{y} = \left(x + \frac{1}{x}\right) \sqrt[3]{(3)} = \pm \sqrt{(3 + \sqrt[3]{3})} \times \sqrt[3]{3},$$

$$\text{whence } y = \frac{1}{2}\left[\sqrt{(3 + \sqrt[3]{3})} \times \sqrt[3]{3} \pm \sqrt{\{3\sqrt[2]{(9)} - 1\}}\right].$$

(227). $x^2 + 6x\sqrt[3]{(y^2)} + \sqrt[3]{(y^4)} = 128$, and $\sqrt{(x^3)}\sqrt[3]{(y)} + y\sqrt{(x)} = 32$,

$$\text{from (1) } x^2 + 6x\sqrt[3]{(y^2)} + \sqrt[3]{(y^4)} = 148\ (A),$$

$$\text{from (2) } x^{\frac{1}{2}} y^{\frac{1}{3}} \{x + \sqrt[3]{(y^2)}\} = 32;$$

$$\therefore\ x^2 + 2x\sqrt[3]{(y^2)} + \sqrt[3]{(y^4)} = \frac{1024}{x\sqrt[3]{(y^2)}};$$

$$\text{from } (A)\ \therefore\ 4x\sqrt[3]{(y^2)} = 128 - \frac{1024}{x\sqrt[3]{(y^2)}};$$

$$\therefore\ 4x^2\sqrt[3]{(y^4)} - 128x\sqrt[3]{(y^2)} + (32)^2 = 0;$$

$$\therefore\ x\sqrt[3]{(y^2)} = 16 \text{ and from } (A)$$

$$\therefore\ x^2 - 2x\sqrt[3]{(y^2)} + \sqrt[3]{(y^4)} = 0, \text{ and } x = \sqrt[3]{(y^2)};$$

$$\therefore\ x \times x = 16, \text{ and } x = \pm 4, \text{ and } y = \pm 8.$$

(228). $x^5 + y^5 = (x + y)^3 \times xy$, and $y^2\sqrt{(x)} = (x + y)^{\frac{3}{2}}$,

$$\frac{x^5 + y^5}{(x + y)^3} = \frac{x^4 - x^3y + x^2y^2 - xy^3 + y^4}{x^2 + 2xy + y^2} = xy,$$

$$\text{or } x^4 - 2x^3y - x^2y^2 - 2xy^3 + y^4 = 0,$$

$$\text{whence } \frac{x^2}{y^2} + 2 + \frac{y^2}{x^2} - 2\left(\frac{x}{y} + \frac{y}{x}\right) + 1 = 4;$$

$$\therefore\ \frac{x}{y} + \frac{y}{x} = 1 \pm 2 = 3 \text{ or } -1,$$

$$\text{multiplying by } \frac{x}{y},\ \frac{x^2}{y^2} - \frac{3x}{y} + \frac{9}{4} = -1 + \frac{9}{4} = \frac{5}{4};$$

$$\therefore\ x = \frac{3 \pm \sqrt{(5)}}{2} \times y,$$

$$\text{from (2) } xy^4 = (x + y)^3 \text{ or } \frac{3 \pm \sqrt{5}}{2} \times y^5 = (5 \pm \sqrt{5})^3 \frac{y^3}{8};$$

$$\therefore y^2 = \frac{5}{4}\left\{\frac{(5 \pm \sqrt{5})^3}{(3 \pm \sqrt{5})}\right\} = \frac{10}{1} \times \frac{5 \pm \sqrt{5}}{3 \pm \sqrt{5}} = \frac{5}{2}(5 \pm \sqrt{5});$$

$$\therefore y = \sqrt{\left\{\frac{5}{2}(5 \pm \sqrt{5})\right\}},$$

$$\text{and } x = \sqrt{\left\{\left(\frac{3 \pm \sqrt{5}}{2}\right)^2 \times \frac{5}{2}(5 \pm \sqrt{5})\right\}} = \sqrt{\left\{\frac{5 \times (50 \pm 10\sqrt{5})}{4}\right\}}$$

$$= \frac{5}{2}\sqrt{(10 \pm 2)} = \frac{5}{2}\sqrt{(10 \pm 2\sqrt{5})}.$$

(229). $x^3y^2 + xy^4 = 156$, and $2x^3y^2 - x^2y^3 = 144$,

$$\text{let } x = ry; \; \therefore \frac{y^5 r(r^2+1)}{y^5 r^2(2r-1)} = \frac{13}{12}; \; \therefore 14r^2 - 13r = 12,$$

$$\text{and } r^2 - \frac{13r}{14} + \left(\frac{13}{28}\right)^2 = \frac{12}{14} + \left(\frac{13}{28}\right)^2 = \frac{841}{(28)^2}; \; \therefore r = \frac{3}{2} \text{ or } -\frac{4}{7},$$

from $r = \frac{3}{2}$,

$$x = \frac{3y}{2}, \text{ and } \left(\frac{27}{8} + \frac{3}{2}\right) y^5 = \frac{39y^5}{8} = 156;$$

$$\therefore y^5 = 8 \times 4, \text{ and } y = 2 \text{ and } x = 3,$$

$$\text{from } r = -\frac{4}{7}, \; y^5 = -\frac{343}{132} \times 156,$$

$$\text{and } y = \sqrt[5]{\left(-\frac{4459}{11}\right)} \text{ and } x = 4\sqrt[5]{\left(\frac{13}{539}\right)}.$$

(230). $(x^2 + y^2)\,xy = 78$, and $x^4 + y^4 = 97$,

$$\text{from (1) } x^4 + 2x^2y^2 + y^4 = \frac{6084}{x^2y^2} \text{ and } x^4 + y^4 = 97;$$

$$\therefore x^4y^4 + \frac{97x^2y^2}{2} + \left(\frac{97}{4}\right)^2 = \left(\frac{241}{4}\right)^2; \; \therefore x^2y^2 = 36 \text{ or } -\frac{119}{2};$$

$$\therefore xy = \pm 6,$$

$$\text{and } x^2 + 2xy + y^2 = 13 + 12;$$

$$\therefore x + y = \pm 5 \text{ and } x - y = \pm 1;$$

$$\therefore x = 3 \text{ and } y = 1.$$

(231). $x^4 + y^4 = 17$, and $2xy(x^2+y^2) + 3x^2y^2 = 32$,

$$\text{from (1) + (2) } x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4 = 81;$$

$$\therefore x + y = \pm 3,\ x^2 + y^2 = 9 - 2xy;$$

$$\therefore 2xy(9 - 2xy) + 3x^2y^2 = 32,$$

$$x^2y^2 - 18xy + 81 = -32 + 81 = 49,$$

$$xy = 9 \pm 7 = 16 \text{ or } 2,$$

$$\text{whence } x^2 - 2xy + y^2 = 1 \text{ or } -55;$$

$$\therefore x - y = \pm 1 \text{ or } \pm\sqrt{(-55)};$$

$$\therefore x = \pm 2 \text{ or } \frac{1}{2}\{\pm 3 \pm \sqrt{(-55)}\},\ y = \pm 1 \text{ or } \frac{1}{2}\{\pm 3 \mp \sqrt{(-55)}\}.$$

(232). $(x+y)(x^3+y^3) = 76$, and $(x+y)^3 = 64(x-y)$,

$$\text{let } x = m + n,\ y = m - n;\ \therefore x + y = 2m,\ x - y = 2n;$$

$$\therefore 2m(2m^3 + 6mn^2) = 76, \text{ or } m^4 + 3m^2n^2 = 19\ (A),$$

$$\text{also } 8m^3 = 64 \times 2n;\ \therefore m = 2\sqrt[3]{(2n)};$$

$$\therefore \text{from } (A)\ 16 \times 2^{\frac{4}{3}} n^{\frac{4}{3}} + 3n^2 \times 4 \times 2^{\frac{2}{3}} n^{\frac{2}{3}} = 19,$$

$$\text{whence } n^{\frac{8}{3}} + \frac{4 \times 2^{\frac{2}{3}}}{3} n^{\frac{4}{3}} + (\quad)^2 = \frac{121}{4 \times 9 \times 2^{\frac{2}{3}}};$$

$$\therefore n^{\frac{4}{3}} = -\frac{2 \times 2^{\frac{2}{3}}}{3} \pm \frac{11}{6 \times 2^{\frac{1}{3}}} = \frac{1}{2 \times 2^{\frac{1}{3}}} \text{ or } -\frac{19}{6 \times 2^{\frac{1}{3}}};$$

$$\therefore n^4 = \frac{1}{16} \text{ and } n = \frac{1}{2},$$

$$\text{from } n = \frac{1}{2},\ m = 2;\ \therefore x = \frac{5}{2},\ y = \frac{3}{2},$$

and other values may be found from

$$n = \sqrt[4]{\left\{\frac{(-19)^3}{6^3 \times 2}\right\}}.$$

(233). $5 - 2\sqrt{(y+2)} = \dfrac{9x^2}{64} - \{\sqrt{(x)} - 3\sqrt{(y)}\}^2$,

$$\text{and } \frac{7}{y} - 10\sqrt{\left(\frac{x}{y}\right)} = x - 16,$$

$$\text{from (2) } x + 10\sqrt{\left(\frac{x}{y}\right)} + \frac{25}{y} = \frac{32 + 16y}{y};$$

$$\therefore \sqrt{(xy)} = -5 \pm 4\sqrt{(y+2)}\ (A),$$

from (1) $x + 9y - 6\sqrt{(xy)} - 2\sqrt{(y+2)} + 5 = \frac{9x^2}{64}$;

$$\therefore x + 9y - 26\sqrt{(y+2)} + 35 = \frac{9x^2}{64} \text{ from } (A),$$

$$y + 2 - \frac{26}{9}\sqrt{(y+2)} + \left(\frac{13}{9}\right)^2 = \frac{x^2}{64} - \frac{x}{9} + \frac{16}{81};$$

$$\therefore \sqrt{(y+2)} = \frac{13}{9} \pm \left(\frac{x}{8} - \frac{4}{9}\right) = \frac{x}{8} + 1 \text{ or } -\left(\frac{x}{8} - \frac{17}{9}\right);$$

$$\therefore y = \frac{x^2}{64} + \frac{x}{4} - 1, \text{ and } \sqrt{(xy)} = \frac{x}{2} + 4 - 5 = \frac{x}{2} - 1;$$

$$\therefore y = \frac{x}{4} - 1 + \frac{1}{x},$$

whence $\frac{x^2}{64} + \frac{x}{4} = \frac{x}{4} + \frac{1}{x}$; $\therefore x = 4$ and $y = \frac{1}{4}$.

(234). $3x + \frac{2}{3}\sqrt{(xy^2 + 9x^2y)} = \left(x - \frac{1}{3}\right)y$, and $6x + y : y :: x + 5 : 3$,

from (2) $6x : y :: x + 2 : 3 :: y = \frac{18x}{x+2}$;

$$\therefore 9x(x+2) + 2 \times 9x\sqrt{\{4x + 2x(x+2)\}} = (3x-1)\,18x,$$

or $2\sqrt{(2x^2 + 8x)} = 5x - 4$,

whence $x^2 - \frac{72x}{17} + \frac{36}{(17)^2} = \frac{1024}{(17)^2}$; $\therefore x = 4$ or $\frac{4}{17}$;

and $y = 12$ or $\frac{36}{19}$,

or from (1) $y + 9x + 2\sqrt{(xy)}\sqrt{(y+9x)} + xy = 4xy$;

$\therefore \sqrt{(y+9x)} = \sqrt{(xy)}$ or $-3\sqrt{(xy)}$, &c.

(235). $3x - x\sqrt{\left(\frac{5x^2}{4} - 2y + 8\right)} = 2 - y$,

and $\frac{\sqrt{(x+y)}}{2x} - \frac{3x}{4} = \frac{2x-3}{\sqrt{(x+y)}} - \frac{3y}{2x}$,

from (1)

$$(5x^2 - 8y + 32) + 4x\sqrt{(5x^2 - 8y + 32)} + 4x^2 = 9x^2 + 24x + 16;$$

$$\therefore \sqrt{(5x^2 - 8y + 32)} = -2x \pm (3x + 4)$$
$$= x + 4 \text{ or } 4 - 5x,$$

from the first value

$$5x^2 - 8y + 32 = x^2 + 8x + 16,$$

$$\text{whence } y = \frac{x^2 - 2x + 4}{2};$$

$$\therefore\ x + y = \frac{x^2 - 2x + 4}{2} + x = \frac{x^2 + 4}{2};$$

$$\therefore \text{ from (2) } \frac{\sqrt{(x^2+4)}}{2\sqrt{2x}} - \frac{3x}{4} = \frac{(2x-3)\sqrt{2}}{\sqrt{(x^2+4)}} - \frac{3x}{4} + \frac{3}{2} - \frac{3}{x},$$

$$\text{whence } 4 - 7x^2 + 12x = 3(x-2)\sqrt{(2)}\sqrt{(x^2+4)},$$

$$\text{or } 3(x-2)\sqrt{2}\sqrt{(x^2+4)} + 7x(x-2) + 2(x-2) = 0;$$

$$\therefore\ x = 2 \text{ and } y = 2,$$

$$\text{also } 3\sqrt{2}\sqrt{(x^2+4)} + 7x + 2 = 0, \text{ or } 7x + 2 = -3\sqrt{(2)}\sqrt{(x^2+4)},$$

$$\text{whence } 31x^2 + 28x + 76 = 0, \text{ and } x = \frac{34}{31},\ y = \frac{1446}{961}.$$

(236). $1 + \frac{x^2}{5y^2} + \frac{x^4}{25y^4} + \text{\&c. to inf.} = \frac{5x^2}{y^2},$

$$\text{and } 1 + \frac{1}{2}(x+y)^{-1} + \frac{1.3}{2\cdot 4}(x+y)^{-2} + \text{\&c.} = \sqrt{(1\cdot 25)},$$

$$\text{from (1) } \frac{5x^2}{y^2} = \frac{1}{1 - \frac{x^2}{5y^2}} = \frac{5y^2}{5y^2 - x^2};$$

$$\therefore\ x^4 - 5x^2y^2 + \frac{25y^4}{4} = +\frac{9y^4}{4};\ \therefore\ \frac{x^2}{y^2} = \frac{5 \pm \sqrt{(21)}}{2},$$

$$\text{from (2) } \left(1 - \frac{1}{x+y}\right)^{-\frac{1}{2}} = \left(\frac{x+y-1}{x+y}\right)^{-\frac{1}{2}};$$

$$\therefore\ \frac{x+y-1}{x+y} = \frac{100}{125} = \frac{4}{5};\ \therefore\ x + y = 5, \text{ and } x = 5 - y,$$

$$y^2\{3 \pm \sqrt{(21)}\} + 20y = 50,$$

$$\text{whence } y^2 - \frac{5\{3 \mp \sqrt{(21)}\}y}{3} + (\ \ )^2 = \frac{50\{3 \mp \sqrt{(21)}\}}{-12} + (\ \ )^2 = \frac{300}{36};$$

$$\therefore\ y = \frac{5(3 \mp \sqrt{21})}{6} \pm \frac{10\sqrt{3}}{6} = \frac{5}{6}\{3 \pm 2\sqrt{(3)} \mp \sqrt{21}\},$$

$$\text{and } x = \frac{5}{6}\{3 \mp 2\sqrt{(3)} \pm \sqrt{(21)}\}.$$

(237). $x^2 + xy + y^2 = a^2$, and $x + x^{\frac{1}{2}}y^{\frac{1}{2}} + y = b$,

$$\frac{x^2 + xy + y^2}{x + \sqrt{(xy)} + y} = x - \sqrt{(xy)} + y = \frac{a^2}{b},$$

$$\text{and } x + \sqrt{(xy)} + y = b; \ \therefore\ 2\sqrt{(xy)} = \frac{b^2 - a^2}{b},$$

$$\text{and } \sqrt{x} + \sqrt{y} = \sqrt{\left(\frac{3b^2 - a^2}{2b}\right)},\ \sqrt{x} - \sqrt{y} = \sqrt{\left(\frac{3a^2 - b^2}{2b}\right)};$$

$$\therefore\ x = \frac{1}{4b}\left[a^2 + b^2 + \sqrt{\{(3b^2 - a^2)(3a^2 - b^2)\}}\right],$$

$$\text{and } y = \frac{1}{4b}\left[a^2 + b^2 - \sqrt{\{(3b^2 - a^2)(3a^2 - b^2)\}}\right].$$

(238). $xy - \dfrac{1200}{x} = 56\left(\dfrac{y}{x}\right)^{\frac{1}{3}} - (xy^2)^{\frac{1}{3}}$,

$$\text{and } 2x^{\frac{4}{3}} + 16y^{\frac{4}{3}} + 42(xy)^{\frac{2}{3}} = 4\sqrt[3]{(xy)}\,(5x^{\frac{2}{3}} + 11y^{\frac{2}{3}}),$$

$$\text{let } x = m^3y; \ \therefore \text{ from (2)} \because\ x^{\frac{1}{3}} = my^{\frac{1}{3}},$$

$$\text{we have } 3m^4y^{\frac{4}{3}} + 16y^{\frac{4}{3}} + 42m^2y^{\frac{4}{3}} = 20my^{\frac{4}{3}} + 44my^{\frac{4}{3}},$$

and by trial

$$m^3(3m - 2) - 6n^2(3m - 2) + 10m(3m - 2) - 8(3m - 2) = 0;$$

$$\therefore\ 3m - 2 = 0 \text{ and } m = \frac{2}{3},$$

$$\text{from (1) } x^2y + x^{\frac{4}{3}}y^{\frac{2}{3}} - 56x^{\frac{2}{3}}y^{\frac{1}{3}} - 1200 = 0, \text{ and if } x = m^3y,$$

$$m^6y^3 - 56m^2y + m^4y^2 - 1200 = 0;$$

therefore, by trial,

$$m^4y^2(m^2y - 12) + 13m^2y(m^2y - 12) + 100(m^2y - 12) = 0;$$

$$\therefore\ m^2y = 12, \text{ and } y = 12 \times \frac{9}{4} = 27; \ \therefore\ x = 8,$$

$$\text{also } m^4y^2 + 13m^2y + 100 = 0,$$

of which the roots are impossible.

(239). $\frac{x^{\frac{1}{3}}}{y^{\frac{1}{3}}}(x^{\frac{2}{3}}-1)+\frac{y^{\frac{1}{3}}}{x^{\frac{1}{3}}}(2x^{\frac{2}{3}}-1)=\frac{4y^{\frac{1}{3}}}{x^{\frac{1}{3}}}(y^{\frac{1}{3}}+x^{\frac{1}{3}})+\frac{3y}{x^{\frac{1}{3}}}+2,$

$$\text{and } \frac{x^{\frac{4}{3}}}{y^{\frac{4}{3}}}-\frac{2x^{\frac{2}{3}}}{y}-\frac{2x^{\frac{1}{3}}}{y^{\frac{1}{3}}}=\frac{133}{36}\times\frac{1}{y^{\frac{2}{3}}}-\frac{2}{x^{\frac{1}{3}}}-\frac{y^{\frac{2}{3}}}{x^{\frac{2}{3}}},$$

for $x^{\frac{1}{3}}$, $y^{\frac{1}{3}}$ write x and y,

$$\text{then } x^2(x^2-1)+y^2(2x^2-1)=4y^2(x+y)+3y^4+2xy,$$

$$\text{or } x^4+2x^2y^2+y^4=x^2+2xy+y^2+4y^2(x+y)+4y^4,$$

$$\text{or } x^2+y^2=x+y+2y^2,$$

$$\text{or } x^2-y^2=x+y;\ \therefore\ x+y=0, \text{ and } x-y=1\ (A),$$

$$\text{from (2) } x^6-2x^4y-2x^3y^3=\frac{133}{36}x^2y^2-2xy^4-y^6,$$

$$\text{or } (x^3-y^3)^2-2xy(x^3-y^3)+x^2y^2=\frac{169x^2y^2}{36};$$

$$\therefore\ x^3-y^3=xy\pm\frac{13xy}{6}=\frac{19xy}{6} \text{ or } -\frac{7xy}{6},$$

and dividing by (A)

$$x^2+xy+y^2=\frac{19xy}{6}, \text{ or } (x-y)^2=\frac{xy}{6};$$

$$\therefore\ (x-y)^2=1=\frac{xy}{6};\ \therefore\ xy=\pm 6;$$

$$\therefore\ x+y=\pm 5, \text{ and } x=3 \text{ or } -2 \text{ and } y=2 \text{ or } -3,$$

and the cubes of these values are

27 or − 8, and 8 or − 27.

(240). $xy=a$, $xz=b$, $xu=c$, and $xyzu=d$,

$$x=\frac{a}{y}=\frac{b}{z}=\frac{c}{u}=\frac{d}{uyz};$$

$$\therefore\ y=\frac{az}{b}=\frac{au}{c}=\frac{d}{bu}=\frac{d}{cz};$$

$$\therefore\ z=\frac{bu}{c}=\frac{d}{au}=\pm\sqrt{\left(\frac{db}{ac}\right)},\ u=\pm\sqrt{\left(\frac{dc}{ab}\right)},$$

$$y = \frac{az}{b} = \pm \frac{a}{b} \sqrt{\left(\frac{bd}{ac}\right)} = \pm \sqrt{\left(\frac{ad}{bc}\right)},$$

$$x = \frac{x}{y} = \pm a \sqrt{\left(\frac{bc}{ad}\right)} = \sqrt{\left(\frac{abc}{d}\right)}.$$

(241). $\sqrt[3]{\left\{\frac{27y^{\frac{3}{2}} - 1}{x^3 + 3y^2 - 2xy^{\frac{3}{2}}}\right\}} = 3 \sqrt{\left(\frac{x}{y}\right)},$

$$\text{and } 3x^2 + 42xy + 16y^2 = 4 \sqrt{(xy)} (5x + 11y),$$

$$\text{from (2) } 3x^2 - 20x^{\frac{3}{2}}y^{\frac{1}{2}} + 42xy - 44x^{\frac{1}{2}}y^{\frac{3}{2}} + 16y^2 = 0.$$

$$\text{Let } x = m^2y; \ \therefore\ 3m^4 - 20m^3 + 42m^2 - 44m + 16 = 0;$$

$$\therefore\ (3m-2)\,m^3 - (3m-2)\,6m^2 + (3m-2)\,10m - (3m-2)\,8 = 0;$$

$$\therefore\ m = \frac{2}{3},$$

$$\text{also } m^3 - 6m^2 + 10m - 8 = 0;$$

$$\therefore\ (m-4)\,m^2 - 2m\,(m-4) + 2\,(m-4) = 0;$$

$$\therefore\ m = 4,$$

$$\text{and } m^2 - 2m + 2 = 0;$$

$$\therefore\ m = 1 \pm \sqrt{(-1)},$$

which is an impossible value; therefore substituting in (1) cubed

$$y = \frac{9x}{4},$$

$$\text{we have } 8x^3 - 54x^{\frac{5}{2}} + \frac{243}{2}x^2 - \frac{729}{8}x^{\frac{3}{2}} = -1,$$

and taking the cube root

$$2x - \frac{9x^{\frac{1}{2}}}{2} = -1;$$

$$\therefore\ x - \frac{9x^{\frac{1}{2}}}{4} = -\frac{1}{2},$$

$$\text{whence } x = 4 \text{ or } \frac{1}{16} \text{ and } y = 9 \text{ or } \frac{9}{64}.$$

(242). $2(x^2+y^2)=2x+a$, and $x^3(x-2)+6y^2x(x-1)+y^4=\dfrac{b-2x}{2}$,

$$\text{from (1) } x=\frac{1}{2}\pm\frac{1}{2}\sqrt{(2a-4y^2+1)}\ (A),$$

$$\text{from (2) } x=\frac{1}{2}\pm\frac{1}{2}\sqrt{\{-12y^2+3\pm 2\sqrt{(2b+32y^4-12y^2+1)}\}}\ (B),$$

equating (A) and (B) and squaring and arranging

$$16y^4-4(2a+1)y^2=a^2-2a-2b,$$

$$\text{whence } 4y^2=\frac{2a+1}{2}\pm\frac{\sqrt{(8a^2-4a-8b+1)}}{2},$$

$$\text{and } y=\pm\frac{1}{4}\sqrt{\{4a+2\pm 2\sqrt{(8a^2-4a-8b+1)}\}},$$

$$\text{and } x=\frac{1}{2}\pm\frac{1}{4}\sqrt{\{4a+2\mp 2\sqrt{(8a^2-4a-8b+1)}\}}.$$

(243). $(1-x^2)^2(1+y^2)-(1+x^2)^2(1-y^2)=4x^2\sqrt{(1+y^4)}$,

$$\text{and } \frac{4xy}{1-y^2}=\sqrt{(2)}\,(1-x^2),$$

from (1)

$$(1-2x^2+x^4)(1+y^2)-(1+2x^2+x^4)(1-y^2)=4x^2\sqrt{(1+y^4)},$$

$$\text{whence } y^2(1+x^4)=2x^2\{1+\sqrt{(1+y^4)}\}\ (A),$$

$$\text{from (2) } (1-x^2)^2=\frac{8x^2y^2}{(1-y^2)^2};$$

$$\therefore\ (1+x^4)=\frac{8x^2y^2}{(1-y^2)^2}+2x^2=\frac{2x^2(1+y^2)^2}{(1-y^2)^2}$$

$$=\frac{2x^2\{1+\sqrt{(1+y^4)}\}}{y^2}\ \text{from } (A);$$

$$\therefore\ \frac{(1+y^2)^2}{(1-y^2)^2}=\frac{1+\sqrt{(1+y^4)}}{y^2},$$

$$\text{whence } 3y^8+6y^4=1,$$

$$\text{whence } y^2=\frac{\sqrt{3}-1}{\sqrt{(2\sqrt{3})}} \text{ and } y=\sqrt{\left\{\frac{\sqrt{3}-1}{\sqrt{(2\sqrt{3})}}\right\}},$$

$$\text{and } x^2+\frac{1}{x^2}+2=\frac{1+\sqrt{(1+y^4)}}{y^2}+2=m^2;$$

$$\therefore\ x + \frac{1}{x} = \pm m,$$

$$\text{and } x = -\frac{m}{2} \pm \frac{1}{2}\sqrt{(m^2 - 4)}.$$

(244). $m^{x^2}.n^{z^2} = a$, and $x : z :: r : s$,

$$x^2 \log m + z^2 \log n = \log a,$$

$$\text{and } z = \frac{xs}{r};$$

$$\therefore\ x^2\left(\log m + \frac{s^2}{r^2}\log n\right) = \log a;$$

$$\therefore\ x = \pm\frac{r\sqrt{(\log a)}}{\sqrt{(r^2\log m + s^2\log n)}},$$

$$\text{and } z = \frac{s\sqrt{(\log a)}}{\sqrt{(r^2\log m + s^2\log n)}}.$$

(245). $(2 + 4xy - 3x^2)^2 = 2 - 4x^2y^2 + 3x^4$,

and $(x^2 - 1)^2 = (2y^2 + x^2 + 1)(2y^2 - x^2 - 1)$,

from (2) we find

$$2y^4 - x^4 = 1 \text{ ....} (3).$$

Then, by (1) and (3),

$$(2 + 4xy - 3x^2)^2 = 3x^4 - 4x^2y^2 + 2(2y^4 - x^4)$$

$$= x^4 - 4x^2y^2 + 4y^4;$$

$$\therefore\ 2 + 4xy - 3x^2 = \pm(x^2 - 2y^2) \text{............} (4).$$

Taking – in (4) gives

$$x - y = \pm 1 \text{........................} (5).$$

Substituting from (5) in (3), we get

$$2(x \mp 1)^4 - x^4 = 1,$$

$$\text{or } \left(x + \frac{1}{x} \mp 4\right)^2 = 6;$$

$$\therefore\ x = \frac{1}{2}[4 \pm \sqrt{(6)} \pm \sqrt{\{18 \pm 8\sqrt{(6)}\}}];$$

$$\text{and } y = \frac{1}{2}[2 \pm \sqrt{(6)} \pm \sqrt{\{18 \pm 8\sqrt{(6)}\}}].$$

Taking + in (4), other values may be determined.

(246). $\frac{x^3}{y} - \frac{9}{x} = \frac{8}{x^{\frac{3}{2}}}\sqrt{\left(\frac{x^5}{y}\right)}$, and $\frac{y}{x^2} + \frac{x}{y\sqrt{(x)}} \cdot \sqrt{\left(\frac{y}{x}\right)} = \frac{10x^2}{81y^2}$,

$$\text{from (1) } x^2 - 9y - 8x^2 y^{\frac{1}{2}} = (x^2 - 9\sqrt{y})(x^2 + \sqrt{y}) = 0;$$

$$\text{we have } x = \pm 3\sqrt[4]{y}, \text{ or } x = \pm\sqrt{(-\sqrt{y})},$$

$$\text{by substituting } x^2 = 9\sqrt{y} \text{ in (2)},$$

$$\text{we have } y^2 + 9y = 10, \text{ or } y^2 - 1 + 9(y-1) = 0; \therefore y = 1 \text{ or } -10;$$

$$\therefore x^2 = 9;$$

$$\therefore x = \pm 3 \text{ or } \pm\sqrt[4]{(-10)}.$$

(247). $\left\{\sqrt{\left(\frac{x}{y}\right)} - \sqrt{\left(\frac{y}{x}\right)}\right\}^2 + \sqrt{(y)}\left\{\sqrt{(x)} - \frac{\sqrt{(y)}}{2}\right\}$

$$= \frac{x}{\sqrt{(y)}}\left\{\sqrt{(x)} - \frac{y}{\sqrt{(2)}}\right\},$$

$$\text{and } 9\sqrt{\left(\frac{x}{y}\right)} + 3\sqrt{\left(\frac{y}{x}\right)} = \frac{21\sqrt{(2x)} - 1}{2}\sqrt{\left(\frac{y}{x}\right)} + \frac{1}{2\sqrt{(xy)}},$$

$$\text{from (2) } 9x + 3y = \frac{21\sqrt{(2x)}}{2}y - \frac{y}{2} + \frac{1}{2};$$

$$\therefore 9x - \frac{21\sqrt{(x)}}{\sqrt{(2)}}y + \frac{49y^2}{8} = \frac{49y^2}{8} - \frac{7y}{2} + \frac{1}{2},$$

$$\text{or } 3\sqrt{x} = \frac{7y}{2\sqrt{2}} \pm \frac{1}{\sqrt{2}}\left(\frac{7y}{2} - 1\right) = \frac{7y-1}{\sqrt{(2)}} \text{ or } \frac{1}{\sqrt{2}}\,(A);$$

$$\therefore x = \frac{1}{18} \text{ or } \frac{7y-1}{\sqrt{(2)}},$$

$$\text{from (1) } \frac{(x-y)^2}{xy} - \frac{x^{\frac{1}{2}}}{y^{\frac{1}{2}}}(x-y) + \frac{x^2}{4} = \frac{x^2}{4} - \frac{xy^{\frac{1}{2}}}{\sqrt{2}} + \frac{y}{2};$$

$$\therefore \frac{x-y}{\sqrt{(xy)}} = \frac{x}{2} \pm \left\{\frac{x}{2} - \frac{\sqrt{(y)}}{\sqrt{(2)}}\right\} = x - \frac{\sqrt{y}}{\sqrt{2}} \text{ or } \frac{\sqrt{(y)}}{\sqrt{2}},$$

from the latter value

$$y = \frac{x\sqrt{(2)}}{\sqrt{(x)} + \sqrt{2}} = \frac{3\sqrt{(2x)} + 1}{7} \text{ from } (A),$$

$$\text{whence } \sqrt{x} = \frac{7}{8\sqrt{2}} \pm \frac{9}{8\sqrt{2}} = \sqrt{2} \text{ or } -\frac{1}{4\sqrt{(2)}};$$

$$\therefore x = 2 \text{ or } \frac{1}{32} \text{ and } y = 1 \text{ or } \frac{1}{4}.$$

(248). $\frac{x^2y^2}{2} + 4 - 40y^2 = 140 - y^2\sqrt{\left(x^2 - \frac{272}{y^2}\right)}$,

$$\text{and } x^2 - \frac{2}{y}\left(15x + \frac{3}{y}\right) = \frac{30}{y^2} + \frac{5x}{y},$$

$$\text{from (2) } x^2y^2 - 30xy - 6 = 30 + 5xy;$$

$$\therefore x^2y^2 - 35xy - 36 = (xy+1)(xy-36) = 0;$$

$$\therefore xy = 36 \text{ or } -1,$$

by substitution in (1),

$$648 + 4 - 40y^2 = 140 - y\sqrt{(1024)},$$

$$\text{or } 5y^2 - 4y - 64 = 0;$$

$$\therefore 4(y^2 - 16) + y(y-4) = 0, \text{ and } y = 4, \text{ and } x = 9,$$

$$\text{or } y = -\frac{16}{5}, \text{ and } x = -\frac{45}{4}.$$

(249). $x^2(b-y) = (y-n)ay$, and $y^2(a-x) = (x-n)bx$,

by addition,

$$x^2y + y^2x = n(ay + bx),$$

$$\text{or } x^2y + (y^2 - nb)x - nay = 0 \ldots\ldots\ldots\ldots (A).$$

$$\text{From (1), } x^2 = \frac{y-n}{b-y}ay;$$

$\therefore$ substituting in (A),

$$ay^2 \times \frac{y-n}{b-y} + (y^2 - nb)\sqrt{\left(\frac{y-n}{b-n}\right)}\sqrt{(ay)} - nay = 0,$$

or $ay^3 - any^2 + (y^2 - nb)\sqrt{\{(y-n)(b-y)\}}\sqrt{(ay)} - naby + nay^2 = 0;$

$\therefore ay(y^2 - nb) + (y^2 - nb)\sqrt{\{(y-n)(b-y)\}}\sqrt{(ay)} = 0 \ldots (B);$

$$\therefore y^2 - nb = 0, \text{ and } y = \pm\sqrt{(nb)},$$

and the corresponding values of x are

$$x^2 = \frac{\sqrt{(nb)} - n}{b - \sqrt{(nb)}}a\sqrt{(nb)} = an\frac{\sqrt{(b)} - \sqrt{n}}{\sqrt{(b)} - \sqrt{(n)}} = an; \ \therefore x = \pm\sqrt{an}.$$

$$\text{Again from } (B),\ ay + \sqrt{\{(y-n)(b-y)\sqrt{ay}\}} = 0;$$

$$\therefore ay = (y-n)(b-y),$$

$$\text{or } ay = -y^2 + ny + by - nb, \text{ or } y^2 + (a - b - n)y = -nb,$$

$$\text{whence } y = -\frac{1}{2}(a-b-n) \pm \frac{1}{2}\sqrt{\{(a-b-n)^2 - 4nb\}}$$

$$\text{and } x = \frac{1}{2}(a-b-n) \pm \frac{1}{2}\sqrt{\{(a-b-n)^2 - 4nb\}}.$$

(250). $y^4 = x^2(ay - bx)$, and $x^2 = ax - by$,

$$\text{from (1), } \frac{y^4}{x^2} = ay - bx,$$

$$\text{from (2), } x^2 = ax - by;$$

$$\therefore \frac{y^4}{x^4} = \frac{ay - bx}{ax - by} = \frac{a\frac{y}{x} - b}{a - b.\frac{y}{x}}.$$

$$\text{Let } \frac{y}{x} = t; \therefore t^4 = \frac{at - b}{a - bt},$$

$$\text{or } at(t^3 - 1) - b(t^5 - 1) = 0;$$

$$\therefore t - 1 = 0, \text{ and } t = 1 \text{ or } y = x \text{ ....}(A);$$

$$\text{also } bt^4 + (b - a)t^3 + (b - a)t^2 + (b - a)t + b = 0;$$

$$\text{whence } \left(t + \frac{1}{t}\right)^2 - \left(\frac{a - b}{b}\right)\left(t + \frac{1}{t}\right) = \frac{a + b}{b};$$

$$\therefore t + \frac{1}{t} = \frac{a - b \pm \sqrt{(a^2 + 2ab + 5b^2)}}{2b} = m;$$

$$\text{then } t^2 - mt = -1, \text{ or } t = \frac{m}{2} \pm \sqrt{\left(\frac{m^2}{4}\right) - 1}\text{...}(B),$$

$$\text{from (2), if } x = y, \text{ then } x = a - b;$$

$$\text{if } t = \frac{m}{2} \pm \sqrt{\left(\frac{m^2}{4}\right) - 1}, \; x = a - \frac{b}{2}\{m \pm \sqrt{(m^2 - 4)}\};$$

$$\text{and } y = tx = \frac{1}{2}\left(a - \frac{b}{2}\right)\{m \pm \sqrt{(m^2 - 4)}\}\{m \pm \sqrt{(m^2 - 4)}\}.$$

(251). $\dfrac{3 + 2x^2 - 4x^4}{x^2 - 1} = y^2(1 - 2y^2)$, and $(2x^2 - 1)(2y^2 - 1) = 3$,

For x^2 write u, and for y^2 write v,

from (2), $4uv - 2(u + v) = 2$, or $2uv - (u + v) = 1 \text{ ... } (A)$,

from (1), $4u^2 - 2u - 3 = v(2v - 1)(u - 1)$,

from (A), $v(2uv - u - 2v + 1) = v(2 - v)$;

$\therefore 4u^2 - 4uv + v^2 = 1$, and $v = 2u \mp 1 \text{ ......}(4)$,

and substituting in (A),

$$v = 2u - 1, \; u = \frac{5}{4} \text{ or } 0,$$

$$\text{or } v = 2u + 1,\ 4u^2 - u = 2;$$

$$\text{whence } u = \frac{1 \pm \sqrt{(33)}}{8},$$

$$\text{and } x = \pm \frac{1}{2}\sqrt{5} \text{ or } 0,\ y = \pm \frac{1}{2}\sqrt{6}, \text{ \&c. \&c.}$$

(252). $x^5 + y^5 = xy(x+y)^3$, and $y^2\sqrt{(x)} = (x+y)^{\frac{3}{2}}$.
see Ex. (228).

(253). $a\{\sqrt{(x+y)} + \sqrt{(x-y)}\} = xy - y\sqrt{(x^2 - y^2)}$,

$$\text{and } \sqrt[4]{(x+y)} + \sqrt[4]{(x-y)} = b,$$

let $x + y = 2m$, $x - y = 2n$; $\therefore x = m + n$, and $y = m - n$,

from (1) and $a\sqrt{2}\,(m^{\frac{1}{2}} + n^{\frac{1}{2}}) = m^2 - n^2 - (m-n)\,2\sqrt{(mn)}$;

$$\therefore a\sqrt{2} = (m^{\frac{1}{2}} - n^{\frac{1}{2}})\{m + n - 2\sqrt{(mn)}\} = (m^{\frac{1}{2}} - n^{\frac{1}{2}})^3;$$

$$\therefore m^{\frac{1}{2}} - n^{\frac{1}{2}} = a^{\frac{1}{3}}2^{\frac{1}{6}}, \text{ or } (m^{\frac{1}{4}} + n^{\frac{1}{4}}) \times (m^{\frac{1}{4}} - n^{\frac{1}{4}}) = a^{\frac{1}{3}}2^{\frac{1}{6}},$$

$$\text{from (2) } m^{\frac{1}{4}} - n^{\frac{1}{4}} = \frac{a^{\frac{1}{3}}2^{\frac{1}{6}}}{b},$$

$$\text{but } m^{\frac{1}{4}} + n^{\frac{1}{4}} = b;$$

$$\therefore m^{\frac{1}{4}} = \frac{1}{2}\left\{\frac{\sqrt[3]{(4a)}}{b} + b\right\}, \text{ and } n^{\frac{1}{4}} = \frac{1}{2}\left\{b - \frac{\sqrt[3]{(4a)}}{b}\right\};$$

$$\therefore m = \frac{1}{16}\left\{b + \frac{\sqrt[3]{(4a)}}{b}\right\}^4, \text{ and } n = \frac{1}{16}\left\{b - \frac{\sqrt[3]{(4a)}}{b}\right\},$$

and by expansion and addition and subtraction,

$$\because m = \frac{1}{16}\left\{b^4 + 4b^2\sqrt[3]{(4a)} + 6(4a)^{\frac{2}{3}} + \frac{16a}{b^2} + \frac{(4a)^{\frac{4}{3}}}{6^4}\right\},$$

$$x = m + n = \frac{1}{16}\left\{b^4 + 6(4a)^{\frac{2}{3}} + \frac{(4a)^{\frac{4}{3}}}{6^4}\right\},$$

$$y = m - n = \frac{b^2}{2}(4a)^{\frac{1}{3}} + \frac{a}{b^2}.$$

(254). $2 + 4xy - 3x^2 = \sqrt{(2 - 4x^2y^2 + 3x^4)}$,

$$\text{and } 5x^2y^2 + \frac{27x^4}{32} = \frac{9x^3y}{2} + 2xy + 1,$$

from (1)

$$4 + 16xy + 16x^2y^2 - 12x^2 - 24x^2y + 9x^4 = 2 - 4x^2y^2 + 3x^4,$$

$$\text{whence } 1 + 8xy + 10x^2y^2 - 6x^2 - 12x^3y + 3x^4 = 0,$$

$$\text{or } x^4 - 4x^3y + 4x^2y^2 - 2(x^2 - 2xy) + 1 = \frac{2x^2y^2}{3} + \frac{4xy}{3} + \frac{2}{3}$$

$$= \frac{2}{3}(xy + 1)^2,$$

$$\text{or } x^2 - 2xy = 1 + \sqrt{\frac{2}{3}} \times (xy + 1) \text{ ....} (A),$$

$$\text{and from (2) } \frac{9x^4}{16} - 3x^3y + \frac{10x^2y^2}{3} = \frac{4xy}{3} + \frac{2}{3};$$

$$\therefore \frac{9x^4}{16} - 3x^3y + 4x^2y^2 = \frac{2x^2y^2}{3} + \frac{4xy}{3} + \frac{2}{3} = \frac{2}{3}(xy + 1)^2;$$

$$\therefore \frac{3x^2}{4} - 2xy = \sqrt{\frac{2}{3}} \times (xy + 1) = x^2 - 2xy - 1;$$

$$\therefore 3x^2 = 4x^2 - 4, \text{ and } x = \pm 2,$$

$$\text{from } (x = 2) \text{ in } (A) \ 4 - 4y = 1 + \sqrt{\frac{2}{3}}(2y + 1);$$

$$\therefore y = \frac{3 - \sqrt{\frac{2}{3}}}{2\sqrt{\frac{2}{3}} + 4} = \frac{3\sqrt{3} - \sqrt{2}}{2(\sqrt{2} + 2\sqrt{3})} = 1 - \frac{\sqrt{(6)}}{4}.$$

(255). $5y + \dfrac{\sqrt{(x^2 - 15y - 14)}}{5} = \dfrac{x^2}{3} - 36$,

$$\text{and } \frac{x^2}{8y} + \frac{2x}{3} = \sqrt{\left(\frac{x^3}{3y} + \frac{x^2}{4}\right)} - \frac{y}{2},$$

$$\text{from (1) } x^2 - 15y - 14 - \frac{3}{5}\sqrt{(x^2 - 15y - 14)} + \frac{9}{100} = \frac{9409}{100};$$

$$\therefore \sqrt{(x^2 - 15y - 14)} = \frac{3}{10} \pm \frac{97}{10} = 10 \text{ or } -\frac{47}{5},$$

$$\therefore x^2 - 15y - 14 = 100, \text{ and } y = \frac{x^2 - 114}{15} (A).$$

Let $x = my$,

$$\text{then from (2) } \frac{m^2}{8} + \frac{2m}{3} + \frac{1}{2} = \sqrt{\left(\frac{m^3}{3} + \frac{m^2}{4}\right)},$$

$$\text{or } \left\{\sqrt{\left(\frac{4m}{3} + 1\right)} - \frac{m}{2}\right\}^2 = 0;$$

$$\therefore \frac{4m}{3} + 1 = \frac{m^2}{2},$$

$$\text{whence } m = 6 \text{ or } -\frac{2}{3},$$

by substitution in (A),

$$36y^2 - 15y = 114, \text{ and } y = \frac{5}{24} \pm \frac{43}{24} = 2 \text{ or } -\frac{19}{24};$$

$$\therefore x = my = 12 \text{ or } -\frac{19}{2}.$$

(256). $(x - 2)y + x - 2y^2 = \sqrt{(xy)}(y^2 - 1)$,

and $xy(xy - 18) = 4\{\sqrt{(xy)} - 12\}$,

see Ex. (194).

(257). $(xy^2 + x)^{\frac{1}{2}} + x^{\frac{1}{2}} = y(x + 9)^{\frac{1}{2}} + 3y$,

and $x(y + 1)^2 = 4(9y^3 + 16)$,

$$\text{from (1) } \frac{x^{\frac{1}{2}} \times y^2}{\sqrt{(y^2 + 1)} - 1} = \frac{y \times x}{\sqrt{(x + 9)} - 3},$$

$$\text{or } \frac{y}{\sqrt{(y^2 + 1)} - 1} = \frac{x^{\frac{1}{2}}}{\sqrt{(x + 9)} - 3},$$

and multiplying this by (1),

$$\frac{\{\sqrt{(y^2 + 1)} + 1\}}{\sqrt{(y^2 + 1)} - 1} = \frac{\sqrt{(x + 9)} + 3}{\sqrt{(x + 9)} - 3};$$

$$\therefore \frac{\sqrt{(y^2 + 1)}}{1} = \frac{\sqrt{(x + 9)}}{3}; \therefore 9y^2 + 9 = x + 9,$$

$$\text{or } x = 9y^2,$$

$$\text{from (2) } 9y^2(y^2 + 2y + 1) = 36y^3 + 64;$$

$$\therefore 9y^2(y^2 - 2y + 1) = 64; \therefore 3y(y-1) = \pm 8;$$

$$\therefore y^2 - y + \frac{1}{4} = \pm\frac{8}{3} + \frac{1}{4} = \frac{35}{12} \text{ or } -\frac{29}{12};$$

$$\therefore y = \frac{1}{2}\left\{1 \pm \sqrt{\left(\frac{35}{3}\right)}\right\}, \text{ or } \frac{1}{2}\left\{1 - \sqrt{\left(-\frac{29}{3}\right)}\right\},$$

$$\text{and } x = 9y^2 = \frac{3}{2}\{19 \pm \sqrt{(105)}\}, \text{ or } \frac{3}{2}\{48 - \sqrt{(-87)}\}.$$

(258). $\sqrt[3]{(x+y)} + \sqrt[3]{(x-y)} = a^{\frac{1}{3}}$, and $(x^2+y^2)^{\frac{1}{3}} + (x^2-y^2)^{\frac{1}{3}} = a^{\frac{2}{3}}$,

$$\text{cubing (1) } 2x + 3\sqrt[3]{(x^2-y^2)}\{(x+y)^{\frac{1}{3}} + (x-y)^{\frac{1}{3}}\} = a,$$

$$\text{or } 2x + 3a^{\frac{1}{3}}\sqrt{(x^2-y^2)} = a; \therefore \sqrt[3]{(x^2-y^2)} = \frac{a-2x}{3\sqrt[3]{(a)}};$$

$$\therefore \text{from (1) } \sqrt[3]{(x^2+y^2)} = a^{\frac{2}{3}} - \frac{a-2x}{3a^{\frac{1}{3}}} = \frac{2(a+x)}{3a^{\frac{1}{3}}};$$

$$\therefore x^2 + y^2 \pm x^2 - y^2 = \frac{1}{27a}\{8(a+x)^3 \pm (a-2x)^3\},$$

$$\text{whence } 2x^2 = \frac{1}{27a}(9a^3 + 18a^2x + 36ax^2),$$

$$\text{and } x^2 - ax + \frac{a^2}{4} = \frac{a^3}{2} + \frac{a^2}{4} = \frac{3a^2}{4};$$

$$\therefore x = \frac{a}{2}\{1 \pm \sqrt{(3)}\},$$

$$\text{and } x^2 + y^2 = \frac{(3 \pm \sqrt{3})^3 a^3}{27a} = \frac{(\sqrt{3} \pm 1)^3 a^2}{3\sqrt{3}};$$

$$\therefore y^2 = \frac{\{10 \pm 6\sqrt{(3)}\}a^2}{3\sqrt{3}} - \frac{a^2}{4}\{4 + 2\sqrt{(3)}\};$$

$$\therefore y = \sqrt{\left(\frac{11 + 6\sqrt{3}}{12\sqrt{3}}\right)} = \sqrt{\left\{\frac{1}{2} + \frac{11\sqrt{(3)}}{18}\right\}}.$$

(259). $(x^2 + y^2 + c^2)^{\frac{1}{3}} + (x - y + c)^{\frac{2}{3}} = (32xy)^{\frac{1}{3}}$,

$$\text{and } xy = c(x-y),$$

$$\text{from (2) } 2cx - 2cy - 2xy = 0,$$

$$\text{also } (x - y + c)^2 = x^2 + y^2 + c^2 + 2cx - 2cy - 2xy;$$

$$\therefore x^2 + y^2 + c^2 = (x - y + c)^2;$$

$$\therefore \text{from (1) } (x - y + c)^2 = 4xy = 4c(x - y),$$

$$\text{and } (x - y - c)^2 = 0;$$

$$\therefore x - y = c, \text{ or } y = x - c,$$

$$\text{and from (2) } y = \frac{cx}{x + c} = x - c,$$

$$\text{whence } x = \frac{c}{2}\{1 \pm \sqrt{(5)}\}, \text{ and } y = \frac{c}{2}(-1 \pm \sqrt{5}).$$

(260). $\frac{y}{x}\sqrt{\left(\frac{x}{y}\right)} + \frac{1}{2}\sqrt{\left(\frac{x}{y}\right)} \times \sqrt[4]{\left(\frac{y^3}{x^3}\right)} = 5,$

$$\text{and } \frac{2x^2}{y} - \frac{x}{3\sqrt{(y)}} = \frac{1}{3},$$

$$\text{from (1) } \sqrt{\left(\frac{y}{x}\right)} + \frac{1}{2}\sqrt[4]{\left(\frac{y}{x}\right)} + \frac{1}{16} = \frac{81}{16};$$

$$\therefore \sqrt[4]{\frac{y}{x}} = -\frac{1}{4} \pm \frac{9}{4} = 2 \text{ or } -\frac{5}{2};$$

$$\therefore y = 16x \text{ or } \frac{625}{8}x,$$

$$\text{from (2) } \frac{x^2}{y} - \frac{x}{6\sqrt{y}} + \frac{1}{144} = \frac{25}{144}; \ \therefore \frac{x}{\sqrt{(y)}} = \frac{1}{2} \text{ or } -\frac{1}{3};$$

$$\therefore y = 4x^2 \text{ or } 9x^2,$$

$$\text{if } y = 4x^2 = 16x, \text{ then } x = 4 \text{ and } y = 64,$$

$$\text{if } y = 9x^2 = 16x, \text{ then } x = \frac{16}{9} \text{ and } y = \frac{256}{9}.$$

(261). $\sqrt{\{x^2 + \sqrt[3]{(x^4y^2)}\}} + \sqrt{\{y^2 + \sqrt[3]{(x^2y^4)}\}} = a$, and $x + y + 3\sqrt[3]{(bxy)} = b$,

$$\text{from (1) } \sqrt{\{x^{\frac{4}{3}}(x^{\frac{2}{3}} + y^{\frac{2}{3}})\}} + \sqrt{\{y^{\frac{4}{3}}(x^{\frac{2}{3}} + y^{\frac{2}{3}})\}} = a;$$

$$\therefore \sqrt{(x^{\frac{2}{3}} + y^{\frac{2}{3}})}\,(x^{\frac{2}{3}} + y^{\frac{2}{3}}) = (x^{\frac{2}{3}} + y^{\frac{2}{3}})^{\frac{3}{2}} = a;$$

$$\therefore x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}},$$

$$\text{also } (x^{\frac{1}{3}} + y^{\frac{1}{3}})^3 = x + y + 3x^{\frac{1}{3}}y^{\frac{1}{3}}(x^{\frac{1}{3}} + y^{\frac{1}{3}});$$

$$\therefore \text{from (2) } x^{\frac{1}{3}} + y^{\frac{1}{3}} = b^{\frac{1}{3}},$$

$$\text{and } x^{\frac{2}{3}} + 2(xy)^{\frac{1}{3}} + y^{\frac{2}{3}} = b^{\frac{2}{3}};$$

$$\therefore 2(xy)^{\frac{1}{3}} = b^{\frac{2}{3}} - a^{\frac{2}{3}}, \text{ and } x^{\frac{2}{3}} - 2(xy)^{\frac{1}{3}} + y^{\frac{2}{3}} = 2a^{\frac{2}{3}} - b^{\frac{2}{3}};$$

$$\therefore x^{\frac{1}{3}} - y^{\frac{1}{3}} = \pm\sqrt{(2a^{\frac{2}{3}} - b^{\frac{2}{3}})}; \quad \therefore x^{\frac{1}{3}} = \frac{1}{2}\{b^{\frac{1}{3}} \pm \sqrt{(2a^{\frac{2}{3}} - b^{\frac{2}{3}})}\};$$

$$\therefore x = \frac{1}{8}\{b^{\frac{1}{3}} \pm \sqrt{(2a^{\frac{2}{3}} - b^{\frac{2}{3}})}\}^3 \text{ and } y = \frac{1}{8}\{b^{\frac{1}{3}} \mp \sqrt{(2a^{\frac{2}{3}} - b^{\frac{2}{3}})}\}^3.$$

(262). $6x - x\sqrt{\{5x^2 - 8(y-4)\}} = 2(2-y)$,

$$\text{and } \frac{\sqrt{(x+y)}}{2x} - \frac{3x}{4} = \frac{2x-3}{\sqrt{(x+y)}} - \frac{3y}{2x},$$

see Ex. (235).

(263). $\sqrt{\left(\frac{x}{6} - 18 + 7y^4\right)} - \frac{11}{13}\sqrt{(9y^2 - x)} = \frac{x}{120} + \frac{41}{10} + \frac{7y^4}{20}$,

$$\text{and } y\sqrt{(x+9)} - \sqrt{(xy^2 + x)} = x^{\frac{1}{2}} - 3y,$$

from (2)

$$xy^2 + 9y^2 = xy^2 + x + x + 9y^2 - 6x^{\frac{1}{2}}y + 2\sqrt{(xy^2 + x)}\,(x^{\frac{1}{2}} - 3y),$$

$$x^{\frac{1}{2}} - 3y = -\sqrt{(y^2+1)}\,(x^{\frac{1}{2}} - 3y); \quad \therefore x = 9y^2,$$

$$\text{and } y^2 + 1 = 1; \quad \therefore y = 0,$$

$$\text{from (1) } 7y^4 - 18 + \frac{x}{6} - 20\sqrt{\left(7y^4 - 18 + \frac{x}{6}\right)} + 100 = 0;$$

$$\therefore \sqrt{\left(7y^4 - 18 + \frac{x}{6}\right)} = 10;$$

$$\therefore 7y^4 - 18 + \frac{x}{6} = 100;$$

$$\therefore y^4 + \frac{3y^2}{14} + \left(\frac{9}{28}\right)^2 = \frac{118}{7} + \frac{9}{784} = \frac{1125}{784};$$

$$\therefore y^2 = \frac{3}{28} \pm \frac{115}{28} = 4 \text{ or } \frac{59}{14};$$

$$\therefore = y \pm 2 \text{ or } \pm\sqrt{\left(\frac{59}{14}\right)};$$

$$\therefore x = 36 \text{ or } \frac{531}{14}.$$

(264). $30\sqrt{\left\{\frac{x^{\frac{2}{3}}+y^{\frac{2}{3}}}{x^{\frac{4}{3}}y^{\frac{4}{3}}}\right\}}+40\sqrt{\left\{\frac{x^{\frac{2}{3}}y^{\frac{2}{3}}}{x^{\frac{2}{3}}+y^{\frac{2}{3}}}\right\}}=241,$

$$\text{and}\left[1+\left(\frac{y}{x}\right)^{\frac{2}{3}}\right]\times\left[3x^{\frac{4}{3}}y^{\frac{2}{3}}+\frac{91}{216}\sqrt{(x^2+x^{\frac{4}{3}}y^{\frac{2}{3}})}\right]$$

$$=\frac{125-216(x^2+y^2)}{216},$$

$$\text{from (2)}\ \ 3x^{\frac{4}{3}}y^{\frac{2}{3}}+3x^{\frac{2}{3}}y^{\frac{4}{3}}+\frac{91}{216}(x^{\frac{2}{3}}+y^{\frac{2}{3}})^{\frac{3}{2}}=\frac{125}{216}-x^2-y^2,$$

$$\text{and}\ x^2+3x^{\frac{4}{3}}y^{\frac{2}{3}}+3x^{\frac{2}{3}}y^{\frac{4}{3}}+y^2+\frac{91}{216}(x^{\frac{2}{3}}+y^{\frac{2}{3}})^{\frac{3}{2}}=\frac{125}{216},$$

$$\text{or}\ (x^{\frac{2}{3}}+y^{\frac{2}{3}})^3+\frac{91}{216}(x^{\frac{2}{3}}+y^{\frac{2}{3}})^{\frac{3}{2}}+\left(\frac{91}{432}\right)^2=\frac{116281}{(432)^2};$$

$$\therefore (x^{\frac{2}{3}}+y^{\frac{2}{3}})^{\frac{3}{2}}=-\frac{91}{432}\pm\frac{341}{432}=\frac{125}{216}\ \text{or}\ -1;$$

$$\therefore x^{\frac{2}{3}}+y^{\frac{2}{3}}=\frac{25}{36}\ \text{or}\ 1\ (A),$$

$$\text{from (1)}\ \ 30(x^{\frac{2}{3}}+y^{\frac{2}{3}})+40xy=241\sqrt{\{(x^{\frac{2}{3}}+y^{\frac{2}{3}})\}}\times x^{\frac{2}{3}}y^{\frac{2}{3}};$$

$$\therefore \text{from (2)}\ \frac{125}{6}+40xy=241x^{\frac{2}{3}}y^{\frac{2}{3}}\times\frac{5}{6},$$

$$25+48xy=241x^{\frac{2}{3}}y^{\frac{2}{3}},$$

and by trial

$$25-x^{\frac{2}{3}}y^{\frac{2}{3}}=48x^{\frac{2}{3}}y^{\frac{2}{3}}(5-x^{\frac{1}{3}}y^{\frac{1}{3}});$$

$$\therefore 5+x^{\frac{1}{3}}y^{\frac{1}{3}}=48x^{\frac{2}{3}}y^{\frac{2}{3}}\ \text{and}\ xy=125,$$

$$\text{also}\ 48x^{\frac{2}{3}}y^{\frac{2}{3}}-x^{\frac{1}{3}}y^{\frac{1}{3}}=5,$$

$$\text{from}\ x^{\frac{1}{3}}y^{\frac{1}{3}}=5\ \text{in}\ (A),\ x^{\frac{2}{3}}-2x^{\frac{1}{3}}y^{\frac{1}{3}}+y^{\frac{2}{3}}=-\frac{335}{36};$$

$$\therefore x^{\frac{1}{3}}-y^{\frac{1}{3}}=\frac{1}{6}\sqrt{(-335)}\ (B),$$

$$x^{\frac{2}{3}}y^{\frac{2}{3}} - \frac{x^{\frac{1}{3}}y^{\frac{1}{3}}}{48} + \left(\frac{1}{96}\right)^2 = \frac{25}{48} + \frac{1}{(96)^2} = \frac{961}{(96)^2};$$

$$\therefore\ x^{\frac{1}{3}}y^{\frac{1}{3}} = \frac{1}{96} \pm \frac{31}{96} = \frac{1}{3} \text{ or } -\frac{5}{16};$$

$$\therefore\ x^{\frac{2}{3}} - 2x^{\frac{1}{3}}y^{\frac{1}{3}} + y^{\frac{2}{3}} = \frac{25}{36} - \frac{2}{3} = \frac{1}{36};$$

$$\therefore\ x^{\frac{1}{3}} - y^{\frac{1}{3}} = \frac{1}{6};\ \therefore \text{ also } x^{\frac{1}{3}} + y^{\frac{1}{3}} = \frac{7}{6};$$

$$\therefore\ x^{\frac{1}{3}} = \frac{2}{3}, \text{ and } x = \frac{8}{27}, \text{ also } y^{\frac{1}{3}} = \frac{1}{2};\ \therefore\ y = \frac{1}{8}.$$

and other values may be found from (B).

PROBLEMS PRODUCING SIMPLE EQUATIONS.

X. (1). What number is that, from the treble of which if 48 be subtracted, the remainder is 42?

Let x = the number,

$$\text{then } 3x - 48 = 42;\ \therefore\ x = \frac{90}{3} = 30.$$

(2). To determine two numbers, such that their difference may be 5, and the difference of their squares 75.

Let x and $x + 5$ be the numbers,

$$\text{then } (x+5)^2 - x^2 = 10x + 25 = 75;\ \therefore\ x = 5,$$

and the numbers are 10 and 5.

(3). Find a number such that its third part being added to it, the sum is less than 9 by as much as the number itself is greater than 5.

Let x = the number,

$$\text{then } 9 - \left(x + \frac{x}{3}\right) = x - 5,$$

$$\text{or } 27 - 4x = 3x - 15;\ \therefore\ x = \frac{42}{7} = 6.$$

(4). What number is that, the double of which exceeds four-fifths of its half by 40?

Let x = the number,

$$\text{then } 2x - \frac{4}{5} \times \frac{x}{2} = 40;$$

$$\therefore\ 10x - 2x = 200, \text{ and } x = \frac{200}{8} = 25.$$

(5). Find two consecutive numbers such that the half and fifth part of the less may together be equal to the sum of the third and fourth parts of the greater.

Let x and $x + 1$ be the numbers,

$$\text{then } \frac{x}{2} + \frac{x}{5} = \frac{x+1}{3} + \frac{x+1}{4};$$

$$\therefore\ 30x + 12x = 20x + 20 + 15x + 15;\ \therefore\ x = \frac{35}{7} = 5;$$

$\therefore$ 5 and 6 are the numbers.

(6). To find two numbers with these conditions, viz. that half the first with a third part of the second may make 9, and that a fourth part of the first with a fifth part of the second may make 5.

Let x and y be the numbers,

$$\frac{x}{2} + \frac{y}{3} = 9;\ \therefore\ x + \frac{2y}{3} = 18,$$

$$\text{also } \frac{x}{4} + \frac{y}{5} = 5;\ \therefore\ x + \frac{4y}{5} = 20,$$

$$\text{whence } \left(\frac{4}{5} - \frac{2}{3}\right) y = 2;\ \therefore\ y = 15;\ \therefore\ x = 8.$$

(7). Of 3200*l.*, *A* has 400*l.* more than *B*, and *B* has 200*l.* more than *C*: find the share of each.

x = *B*'s share; $\therefore$ $x + 400$ = *A*'s, $x - 200$ = *C*'s;

$\therefore$ $3x + 200 = 3200$, and x = 1000*l.* = *B*'s share,

A's share = 1400, and *C*'s share = 800.

(8). Find two numbers in the proportion of 9 : 7, such that the square of their sum shall be equal to the cube of their difference.

Let $9x$ and $7x$ be the numbers;

$$\therefore \text{ then } (16x)^2 = (2x)^3, \text{ or } 256x^2 = 8x^3; \ \therefore \ x = 32,$$

and the numbers are 288 and 224.

(9). Required the number of which $\frac{1}{2}$, $\frac{1}{3}$ and $\frac{1}{4}$ together are as much greater than 233, as $\frac{1}{5}$, $\frac{1}{6}$ and $\frac{1}{7}$ of it together are less than the same.

Let x = the number,

$$\text{then } \frac{x}{2} + \frac{x}{3} + \frac{x}{4} - 223 = 223 - \frac{x}{5} - \frac{x}{6} - \frac{x}{7},$$

then, multiply by 12,

$$6x + 4x + 3x = 446 \times 12 - \frac{12x}{5} - 2x - \frac{12x}{7},$$

$$\text{and } 15x \times 35 + 84x + 60x = 446 \times 12 \times 35;$$

$$\therefore \ x = \frac{446 \times 12 \times 35}{669} = 280.$$

(10). To divide the number 2 into two such parts, that a third of the one part added to a fifth of the other may make $\frac{3}{5}$.

Let x = one part; $\therefore$ $2 - x$ = the other,

$$\text{then } \frac{x}{3} + \frac{2 - x}{5} = \frac{3}{5}; \ \therefore \ 5x + 6 - 3x = 9;$$

$$\therefore \ x = \frac{3}{2} \text{ and } \frac{1}{2} \text{ is the other.}$$

(11). *A* and *B* began to play with equal sums; *A* won 1*l.* 10*s.*, and then their money was in the proportion of 13 : 7. How much had each when they left off playing?

Let $xs.$ = sum each had,

$$\text{then } x + 30 : x - 30 :: 13 : 7,$$

$$\text{or } x : 30 : 20 : 6; \ \therefore \ x = \frac{30 \times 20}{6} = 100s. = 5l.;$$

$\therefore$ *A* had 6*l.* 10*s.*, *B* 3*l.* 10*s.*

(12). The ages of two brothers differ by 20 years, and one is as much above 25 as the other is below 25; what are their ages?

Let x = age of younger, and $x + 20$ of the elder,

then $x + 20 - 25 = 25 - x$; $\therefore\ x = 15$, and the elder is 35.

(13). Find a number of which the cube root is $\frac{1}{5}$ the square root.

Let x = the number,

then $x^{\frac{1}{3}} = \frac{1}{5}(x)^{\frac{1}{2}}$, and $x^2 \times 5^6 = x^3$; $\therefore\ x = 15625$.

(14). Find a fraction which becomes $\frac{4}{5}$ when unity is added to its numerator, and $\frac{1}{4}$ if unity be added to its denominator.

Let $\frac{x}{y}$ be the fraction,

$$\text{then } \frac{x+1}{y} = \frac{4}{5} \text{ and } \frac{x}{y+1} = \frac{1}{4},$$

$$\text{whence } y = \frac{5(x+1)}{4}, \text{ and } y = 4x - 1;$$

$$\therefore\ 16x - 4 = 5x + 5 \text{ and } x = \frac{9}{11},$$

$$\text{and } y = \frac{36 - 11}{11} = \frac{25}{11}.$$

(15). To find three numbers, such that the sum of the first and second shall be 7, the sum of the first and third 8, and the sum of the second and third 9.

Let x = the first number; $\therefore\ 7 - x$ = second, $8 - x$ = third;

$\therefore\ 15 - 2x = 9$; $\therefore\ x = 3$, and the numbers are 3, 4, 5.

(16). Divide 64*l.* among 3 persons, so that the first may have 3 times as much as the second; and the third, one third as much as the first and second together.

Let $3x$ = sum of the first, x of the second, $\frac{4x}{3}$ of the third,

$$\text{then } 3x + x + \frac{4x}{3} = 64l.;\ \therefore\ x = \frac{64 \times 3}{16} = 12l.,$$

and the sums are 36*l.*, 12*l.*, and 16*l.*

(17). A person, dying, bequeathed his fortune, which was 2800*l.*, to his son and daughter, in this manner; that for every half-crown the son might have, the daughter was to have a shilling. What, then, were their two shares?

Let x = number of half-crowns and shillings,

$$\frac{5x}{2} + x = 2800 \times 20s.; \;\therefore\; x = \frac{2800 \times 40}{7} = 16000s. = 800l.,$$

$$\text{and } \frac{5x}{2} = 400 \times 5 = 2000l. = \text{son's share.}$$

(18). A gamester at one sitting lost $\frac{1}{5}$ of his money, and then won 10*s.*; at a second, he lost $\frac{1}{3}$ of the remainder, and then won 3*s.*; after which he had 3 guineas left. How much money had he at first?

Let x = shillings he had at first,

$$\text{then } x - \frac{x}{5} + 10 = \frac{4x}{5} + 10 = \text{money he had for second start,}$$

$$\text{and } \frac{2}{3}\left(\frac{4x}{5} + 10\right) + 3 = 63s.,$$

$$\text{whence } \frac{4x}{5} + 10 = 90s., \text{ and } x = 100s. = 5l.$$

(19). Three persons, *A*, *B*, *C*, make a joint contribution, which in the whole amounts to 400*l.*; of which sum *B* contributes twice as much as *A* and 20*l.* more; and *C* as much as *A* and *B* together. What sum did each contribute?

Let x = *A*'s contribution; $\therefore$ $2x + 20$ = *B*'s, or $3x + 20$ = *C*'s;

$$\therefore\; 6x + 40 = 400l., \text{ and } x = 60l.,$$

$$B\text{'s} = 140l.,\; C\text{'s} = 200l.$$

(20). A person being asked the hour of the day, answered thus:—If $\frac{3}{8}$ of the number of hours remaining till midnight be multiplied by 4, the product will as much exceed 12 hours, as $\frac{1}{2}$ of the present hour from noon is below 4. What was the hour after noon?

Let the time be x hours past 12 o'clock at noon,

$$\text{then } 4\left\{\frac{3}{8}(12-x)\right\} - 12 = 4 - \frac{x}{2},$$

$$18 - \frac{3x}{2} - 12 = 4 - \frac{x}{2}; \ \therefore\ x = 2 \text{ o'clock.}$$

(21). Two coaches start at the same time from York and London, a distance of 200 miles: the one from London travels at $9\frac{1}{4}$ miles an hour, and that from York at $10\frac{3}{4}$. Where will they meet, and in what time from starting?

Let x = number of hours till they meet,

$$\text{then } \frac{37x}{4} + \frac{43x}{4} = 200, \text{ or } x = \frac{200 \times 4}{80} = 10 \text{ hours};$$

$$\therefore \text{ they meet } \frac{370}{4} = 92\tfrac{1}{2} \text{ miles from London.}$$

(22). A person paid a bill of 100*l.* with half-guineas and crowns, using in all 202 pieces; how many pieces were there of each sort?

Let x = number of half-guineas, y = number of crowns,

$$\text{then } x + y = 202,$$

$$\text{and } \frac{21x}{2} + 5y = 100 \times 20 = 2000;$$

$$\therefore\ 5x + 5y = 1010,$$

$$\text{and } \frac{21x}{2} - 5x = 990; \ \therefore\ x = \frac{990 \times 2}{11} = 180,$$

$$\text{and } y = 202 - 180 = 22.$$

(23). *A* and *B* begin to play with equal sums; *A* won 5*l.*, and then three times *A*'s money was equal to eleven times *B*'s. What had each at first?

Let £x. = the money of each,

$$\text{then } 3(x+5) = 11(x-5); \ \therefore\ x = \frac{70}{8} = 8l.\ 15s.$$

(24). Says A to B, If you give me 10 guineas of your money, I shall then have twice as much as you will have left. But says B to A, Give me 10 of your guineas, and then I shall have three times as many as you. How many had each?

$$\text{Let } x = A\text{'s guineas, } y = B\text{'s},$$

$$\text{then } x + 10 = 2(y - 10), \text{ also } 3(x - 10) = y + 10;$$

$$\therefore x = 2y - 30 = \frac{y+40}{3}, \text{ whence } x = 22 \text{ guineas, } y = 26 \text{ guineas.}$$

(25). A messenger starts on an errand at the rate of 4 miles an hour; another is sent an hour and a half after to overtake him; the latter walks at the rate of $4\frac{3}{4}$ miles an hour: when will he overtake the former?

$$\text{Let } x = \text{the number of hours,}$$

$$\text{and gain per hour} = \frac{3}{4} \text{ mile};$$

$$\therefore x = 6 \div \frac{3}{4} = \frac{24}{3} = 8 \text{ hours.}$$

(26). A garrison of 500 men was victualled for 48 days; after 15 days it was reinforced, and then the provisions were exhausted in 11 days. Required the number of men in the reinforcement.

$$\text{Let } x = \text{number of men in reinforcement,}$$

$$\text{then } x \times 11 + 26 \times 500 = 48 \times 500;$$

$$\therefore x = \frac{(48 - 26)\,500}{11} = 1000 \text{ men.}$$

(27). A and B together possess 150*l.* and C has 50*l.* more than D; also A has twice as much as C, and B thrice as much as D. Required the money of each.

$$\text{Let } 2x = A\text{'s money, } 150 - 2x = B\text{'s},$$

$$x = C\text{'s money, } \frac{150 - 2x}{3} = D\text{'s},$$

$$\text{then } x - 50 = \frac{150 - 2x}{3}, \text{ or } x = 60l.,$$

$$A\text{'s} = 120l., \ B\text{'s} = 30l., \ C\text{'s} = 60l., \ D\text{'s} = 10l.$$

(28). What number is that, of which the half, the fifth, and the seventh parts are together equal to 59?

Let x = the number,

$$\text{then } \frac{x}{2} + \frac{x}{5} + \frac{x}{7} = 59, \text{ or } x = \frac{59 \times 70}{59} = 70.$$

(29). In the election of a Member of Parliament, $\frac{1}{16}$ of the constituency refuse to vote, and of two candidates the one who is supported by $\frac{19}{40}$ of the whole constituency is returned by a majority of 5: find the number for each candidate.

$$\text{Number of voters} = \frac{15}{16} \text{ of constituency} = \frac{15x}{16},$$

$$\text{then } \frac{19x}{40} - \left(\frac{15x}{16} - \frac{19x}{40}\right) = 5; \ \therefore \ x = 400,$$

and the numbers are 190 and 185.

(30). Required a number from which if 84 be taken, three times the remainder will exceed the required number by a fourth of itself.

Let x = the number,

$$\text{then } 3(x - 84) = x + \frac{x}{4}, \text{ whence } x = 144.$$

(31). A person spends 2*s.* at a tavern: he then borrows as much money as he has left, and spends 2*s.* at another tavern: borrowing again as much as was left, he spends 2*s.* at a third tavern; and repeating this, he spends 2*s.*, all he now has, at a fourth tavern: what had he at first?

Let $xs.$ = money he had,

$x - 2$ = money left;

$\therefore \ 2(x - 2) - 2$ = money left in second case $= 2\{(x - 3)\}$,

or $4(x - 3) - 2 = 4x - 14$ = money left in third case,

and $2(4x - 14) - 2 = 0; \ \therefore \ x = 3s.\ 9d.$

(32). The greater of two numbers is equal to four times the less, and the excess of the greater over the less is 24; required the two numbers.

$$x = \text{one number},\ 4x = \text{the other};$$

$$\therefore\ 4x - x = 24;\ \therefore\ x = 8, \text{ and the other is } 32.$$

(33). A person pays an income-tax of 7*d.* in the pound, and a poor-rate exceeding it by 22*l.* 10*s.*, and has 486*l.* left: find his income.

$$\text{Let } \pounds x. = \text{his income};\ \therefore\ \text{his income-tax} = \frac{7x}{240} l.,$$

$$\text{then } x - \frac{7x}{240} - \frac{7x}{240} - 22\tfrac{1}{2} = 486,$$

$$\text{whence } \frac{226x}{240} = \frac{1017}{2};\ \therefore\ x = \frac{1017 \times 240}{113 \times 4} = 540l.$$

(34). Required a number such, that if it be multiplied by 11, and 320 be taken from the product, the tenth part of the remainder will be 20 less than the number itself.

$$\text{Let } x = \text{the number},$$

$$\text{then } \frac{11x - 320}{10} = x - 20;$$

$$\therefore\ 11x - 320 = 10x - 200, \text{ and } x = 120.$$

(35). There are two kinds of coin, of which *a* and *b* pieces respectively are equivalent to 1*l.*: how many pieces of each kind must be taken so that *c* pieces together may be equivalent to 1*l.*?

Let x and y be the number of each kind of coin,

$$\text{then } \frac{20}{a} s. \text{ and } \frac{20}{b} s. = \text{their values},$$

$$\text{then } bx + ay = ab, \text{ and } x + y = c, \text{ or } ax + ay = ac;$$

$$\therefore\ x = \frac{ac - ab}{a - b} = \frac{a(c - b)}{a - b}, \text{ and } y = \frac{b(a - c)}{a - b}.$$

(36). If A and B have between them 1200*l.*, A and C 1400*l.*, B and C 1500.; how much has each?

Let $x = A$'s money, then $1200 - x = B$'s, $1400 - x = C$'s,

then $2600 - 2x = 1500$; $\therefore\ x = 550l. = A$'s,

B's $= 650l.$, C's $= 850l.$

(37). A tradesman, after expending 100*l.* a year, augments the remainder of his property by one third part of it, and at the end of 3 years his original property is doubled: what had he at first?

Let x = money he had, then $x - 100$ = money left;

$$\therefore\ x - 100 + \frac{x - 100}{3} = 4\left(\frac{x - 100}{3}\right) = \text{first year's result},$$

$$\text{and } \frac{4(x - 100)}{3} - 100 + \frac{4x - 700}{9} = \text{second year's result},$$

$$\text{and } \frac{4}{27}(16x - 3700) = 2x = \text{third year's result},$$

whence $x = 1480l.$

(38). A common of 864 acres is to be divided among three landowners, A, B, C, so that A's share shall be to B's as 5 to 11, and that C shall receive as much as A and B together; required the share of each.

Let $5x$ and $11x$ be A's share and B's respectively,

then $16x = C$'s;

$$\therefore\ 32x = 864, \text{ and } x = \frac{108}{4} = 27;$$

$\therefore\ A$'s share = 135 acres, B's = 297 acres, C's = 432 acres.

(39). Divide 30 into two such parts, that one may be the square of the other.

Let x = one part, then $30 - x$ = the other,

$$\text{and } 30 - x = x^2;\ \therefore\ x^2 + x + \frac{1}{4} = \frac{121}{4}, \text{ and } x = -\frac{1}{2} \pm \frac{11}{2} = 5;$$

$\therefore\ 30 - x = 25$ the other part.

(40). A farmer engaged a labourer on condition of paying him 1*s.* 4*d.* a day for every day he should work, and of charging him 9*d.* for his board every day he should be idle. Now, at the end of a year (313 days) the man was entitled to 19*l.* 10*s.* 3*d.* How many days did he work?

Let x = number of days of work, $313 - x$ = number of idle days,

$$\text{then } x \times \frac{4}{3} - (313 - x) \times \frac{3}{4} = 390\tfrac{1}{4}s.,$$

$$\text{whence } x = \frac{7500}{25} = 300 \text{ days of work.}$$

(41). Find two numbers, the sum of whose squares is 100, and their product 48.

Let x and y be the numbers,

$$\text{then } x^2 + y^2 = 100, \text{ and } xy = 48;$$

$$\therefore x^2 + 2xy + y^2 = 196, \text{ and } x^2 - 2xy + y^2 = 4;$$

$$\therefore x + y = 14, \text{ and } x - y = 2; \ \therefore x = 8, \ y = 6.$$

(42). Find two numbers whose product is equal to the difference of their squares, and the sum of their squares to the difference of their cubes.

Let x and y be the numbers,

$$\text{then } xy = x^2 - y^2, \text{ and } x^2 + y^2 = x^3 - y^3;$$

$$\therefore x^2 - xy + \frac{y^2}{4} = \frac{5y^2}{4}; \ \therefore x = \frac{1}{2}(1 \pm \sqrt{5})\,y,$$

$$\text{and } \frac{1}{2}(3 \pm \sqrt{5})\,y^2 + y^2 = (2 \pm \sqrt{5})\,y^3 - y^3;$$

$$\therefore y = \frac{5 \pm \sqrt{5}}{2(1 \pm \sqrt{5})} = \pm\frac{\sqrt{5}}{2}, \text{ and } x = \frac{5 \pm \sqrt{5}}{4}.$$

(43). A general after detaching $\frac{5}{13}$ of his army to take possession of a height, and $\frac{5}{18}$ of the remainder to reconnoitre the enemy, had 1280 men left; what was his whole force?

$$\text{Let } x = \text{his army}; \ \therefore x - \frac{5x}{13} = \text{first remainder};$$

$$\therefore\ x - \frac{5x}{13} - \frac{5}{18}\left(x - \frac{5x}{13}\right) = 1280;$$

$$\therefore\ \frac{13}{18}\left(x - \frac{5x}{13}\right) = 1280;$$

$$\therefore\ x = 2880.$$

(44). Divide 10 into three such parts, that when the first is multiplied by 2, the second by 3, and the third by 4, the three products may be equal.

Let x, y, z be the parts,

then $x + y + z = 10$,

$2x = 3y = 4z$;

$$\therefore\ x + \frac{2x}{3} + \frac{x}{2} = 10, \text{ or } x = \frac{60}{13},\ y = \frac{40}{13},\ z = \frac{30}{13}$$

(45). Let 10 be divided into 4 parts, such that when they are respectively divided by 2, 3, 4, and 5, the quotients will be in the same proportion as 6, 7, 8, and 9.

Let x, y, z, and w be the parts,

then $x + y + z + w = 10$,

then $\frac{x}{2} : \frac{y}{3} :: 6 : 7$; $\therefore\ y = \frac{7x}{4}$, also $\frac{x}{2} : \frac{z}{4} :: 6 : 8$; $\therefore\ z = \frac{8x}{3}$,

also $\frac{x}{2} : \frac{w}{5} :: 6 : 9$; $\therefore\ w = \frac{15x}{4}$,

whence $x + \frac{7x}{4} + \frac{8x}{3} + \frac{15x}{4} = 10$;

$$\therefore\ x = \frac{12}{11},\ y = \frac{21}{11},\ z = \frac{32}{11},\ w = \frac{45}{11}.$$

(46). Find the fraction which, if 1 be added to its numerator, becomes $\frac{1}{3}$, but if 1 be added to its denominator, becomes $\frac{1}{4}$.

Let $\frac{x}{y}$ be the fraction, then $\frac{x+1}{y} = \frac{1}{3}$, but $\frac{x}{y+1} = \frac{1}{4}$,

whence $y = 3x + 3 = 4x - 1$; $\therefore\ x = 4$ and $y = 15$.

(47). A person distributed p shillings among n persons, giving 9$d.$ to some, and 15$d.$ to the rest; how many were there of each?

Let x persons receive 9$d.$ each, then $n - x$ receive 15$d.$,

$$\text{then } 9x + (n - x)\,15 = 12p;\ \therefore\ x = \frac{1}{2}(5n - 4p),$$

$$\text{and } n - x = \frac{1}{2}(4p - 3n).$$

(48). A cask, which held 60 gallons, was filled with a mixture of brandy, wine, and cyder, in this manner, viz. the cyder was 6 gallons more than the brandy, and the wine as much as the cyder and $\frac{1}{5}$ of the brandy; how much was there of each?

Let x = gallons of brandy, $x + 6$ = gallons of cyder,

$$x + 6 + \frac{x}{5} = \text{gallons of wine};$$

$$\therefore\ 3x + \frac{x}{5} + 12 = 60,\ \text{and } x = 15 \text{ gallons of brandy,}$$

21 gallons of cyder, 24 of wine.

(49). A cistern is filled in 24 minutes by 3 pipes, one conveying 8 gallons more, and another 7 gallons less, than the third, every 3 minutes; the cistern holds 1088 gallons: how much flows through each pipe in a minute?

Let x = gallons per 3 minutes by third pipe,

$x + 8$, $x - 7$ gallons by the two others;

$$\therefore\ \frac{3x + 1}{3} \times 24 = 1088,\ \text{or } 3x = 136 - 1;\ \therefore\ x = 45 \text{ gallons};$$

$\therefore$ gallons by first pipe = 53, by second = 38 gallons.

(50). A farmer buys a sheep for £P, and sells b of them at a gain of 5 per cent.: at what price ought he to sell the remainder to gain 10 per cent. on the whole?

$$\text{Price per sheep} = \frac{P}{a}\text{£, price that } b \text{ sold for} = b\left(\frac{P}{a} + \frac{P}{20a}\right),$$

$$\text{to gain £10 per cent. on the whole, price} = P + \frac{P}{10} = \frac{11P}{10}\text{£};$$

$$\therefore\ \frac{11P}{10} - \frac{21bP}{20a} = \text{price for } (a - b) \text{ sheep};$$

$$\therefore\ \text{price of each} = \frac{P}{20a}\,\frac{22a - 21b}{a - b}\text{£}.$$

(51). A general, disposing his army into a square form, finds that he has 284 men more than a perfect square; but increasing the side by 1 man, he then wants 25 men to complete the square; how many men had he under his command?

Let x = side of square,

then $x^2 + 284$ = number of army, also $(x+1)^2 - 25$ = army;

$$\therefore\ x^2 + 2x - 24 = x^2 + 284;$$

$\therefore\ x = 154$, and number of men $= 155^2 - 25 = 24000$.

(52). A boy at a fair spends his money in oranges; if he had received 5 more for his money, they would have averaged a half-penny each less, if 3 less, a half-penny each more: how much did he spend?

Let x = pence he had, y = number of oranges bought,

then $\frac{x}{y}$ = price of each, also $\frac{x}{y+5} = \frac{x}{y} - \frac{1}{2}$, $\frac{x}{y-3} = \frac{x}{y} + \frac{1}{2}$,

whence $x = \frac{y^2 + 5y}{10} = \frac{y^2 - 3y}{6}$, and $y = 15$, and $x = 30d.$

(53). The stock of three traders amounted to 760*l.*; the shares of the first and second exceeded that of the third by 240*l.*, and the sum of the second and third exceeded the first by 360*l.*; what was the share of each?

Let x, y, z be their shares;

$\therefore\ x + y + z = 760l.$, also $x + y = 240 + z$, $y + z = x + 360$,

whence $z = \frac{760 - 240}{2} = 260l.$, $x = \frac{760 - 360}{2} = 200l.$, $y = 300l.$

(54). *A* sets out from *C* to go to *D*, at the same time that *B* sets out from *D* to go to *C*; *A* arrives at *D* *a* hours, and *B* at *C* *b* hours after they meet: in what time did each perform the journey?

Let x = hours till they meet, A = distance of C and D,

$x + a$ = hours by A, $x + b$ = hours by B;

$\therefore\ \frac{A}{x+a}$, $\frac{A}{x+b}$ = speed of each per hour;

$$\therefore \text{ dist. by } A \text{ in } x \text{ hours} = \frac{Ax}{x+a} = \text{by } B \text{ in } b \text{ hours} = \frac{Ab}{x+b};$$

$$\text{or } x^2 + bx = bx + ab, \text{ whence } x = \pm \sqrt{(ab)},$$

$$\text{and times} = a + \sqrt{(ab)}, \text{ and } b + \sqrt{(ab)}.$$

(55). A garrison of 1000 men was victualled for 30 days; after 10 days it was reinforced, and then the provisions were exhausted in 5 days; required the number of men in the reinforcement.

Let x = number of men in reinforcement,

$$x \times 5 + 1000 \times 5 = 1000 \times 20; \; \therefore \; x = 3000 \text{ men.}$$

(56). What are those two numbers, of which the greater is to the less as their sum is to 20, and as their difference is to 10?

Let x and y be the numbers,

$$\text{then } x : y :: x + y : 20; \; \therefore \; x = \frac{y^2}{20 - y},$$

$$\text{also } x : y :: x - y : 10; \; \therefore \; x = \frac{y^2}{y - 10};$$

$$\therefore \; y = 15, \text{ and } x = 45.$$

(57). To find three numbers in arithmetical progression, of which the first is to the third as 5 to 9, and the sum of all three is 63.

Let $x - y$, x, and $x + y$ be the numbers;

$$\therefore \; 3x = 61, \text{ and } x = 21,$$

$$\text{then } 21 - y : 21 + y :: 5 : 9,$$

$$\text{or } 21 : y :: 14 : 4; \; \therefore \; y = 6,$$

and the numbers are 15, 21, 27.

(58). There was a run, during the late panic, on two bankers, A and B; B stopped payment at the end of three days; in consequence of which the alarm increased, and the daily demand for cash on A being tripled, A failed at the end of two more days; but if A and B had joined their capitals, they might both have stood the run as it was at first, for 7 days, at the end of which time B would have been indebted to A 4000*l.* What was the daily demand for cash on A's bank at the beginning of the run?

Let x = daily demand on A at first,

$$3x + 6x = \text{what he paid} = 7x + 4000;$$

$$\therefore\ x = 2000l.$$

(59). What two numbers are those, whose difference, sum and product, are to each other as the three numbers 2, 3, 5?

Let x and y be the numbers,

$$\text{then } x - y : x + y : xy :: 2 : 3 : 5,$$

$$\text{whence } x : y :: 5 : 1;\ \therefore\ x = 5y,$$

$$\text{also } 5x + 5y = 3xy;\ \therefore\ y = 2, \text{ and } x = 10.$$

(60). A bill of 26*l.* 5*s.* was paid with half-guineas and crowns, and twice the number of half-guineas exceeded three times the number of crowns by 17; how many were there of each?

Let x = number of half-guineas, y = number of crowns,

$$\text{then } \frac{21x}{2} + 5y = 525, \text{ also } 2x = 3y + 17;$$

$$\therefore\ \frac{1050 - 10y}{21} = \frac{3y + 17}{2}, \text{ whence } y = 21, \text{ and } x = 40.$$

(61). One-third of a ship belongs to A, and one-fifth to B, and A's part is worth 1000*l.* more than B's; required the value of the ship.

Let x = value of ship,

$$\frac{x}{3} = \frac{x}{5} + 1000;\ \therefore\ x = 7500l.$$

(62). A person sets out from A, and travels towards B at the rate of $3\frac{1}{2}$ miles an hour; 40 minutes afterwards another sets out from B to meet him, travelling at the rate of $4\frac{1}{2}$ miles an hour, and he goes half a mile beyond the middle of the distance before he meets the first traveller; find the distance between A and B.

Let $2x$ = the distance, $\dfrac{x-\frac{1}{2}}{3\frac{1}{2}}$ = hours of A, $\dfrac{x+\frac{1}{2}}{4\frac{1}{2}}$ = hours of B,

$$\text{but } \frac{2x-1}{7} = \frac{2x+1}{9} + \frac{2}{3}, \text{ whence } 2x = 29 \text{ miles.}$$

(63). It is required to divide 252 into three parts, such that one-third of the first, one-fourth of the second, and one-fifth of the third, shall all be equal to one another.

Let x, y, and $252 - (x + y)$ be the parts,

$$\frac{x}{3} = \frac{y}{4} = \frac{252-(x+y)}{5}; \ \therefore\ x = \frac{3y}{4},$$

whence $x = 63$, $y = 84$, and the third $= 105$.

(64). Two labourers, A and B, received 5*l.* 17*s.* for their wages, A having been employed 15, and B 14 days, and A received for working four days 11*s.* more than B did for three days; what were their daily wages?

Let $x = A$'s daily pay, $y = B$'s,

then $15x + 14y = 117$, also $4x = 3y + 11$;

$$\therefore\ x = \frac{117-14y}{15} = \frac{3y+11}{4}, \text{ whence } x = 5s., \ y = 3s.$$

(65). A person expends half-a-crown in apples and pears, buying his apples at 4, and his pears at 5 a penny; and afterwards accommodates his neighbour with half his apples and one-third of his pears, for 13 pence. How many did he buy of each?

Let x = number of apples, y = number of pears,

$$\frac{x}{4} + \frac{y}{5} = 30d., \text{ also } \frac{x}{8} + \frac{y}{15} = 13,$$

whence $x = 72$, $y = 60$.

(66). To find a number such that if it be multiplied by 10, and the product be divided by 13, the quotient, increased by the number itself, and by 80, will amount to 1000.

Let x = the number,

$$\text{then } \frac{10x}{13} + x + 80 = 1000, \text{ whence } x = 520.$$

(67). A person travels a journey at a certain rate; had he travelled half a mile an hour faster he would have performed the journey in $\frac{4}{5}$ of the time, but had he travelled half a mile an hour slower, he would have been $2\frac{1}{2}$ hours longer on the road; find the distance, and his rate of travelling.

Let x = number of miles per hour, and y = the distance,

$\frac{y}{x}$ = number of hours, but $\frac{y}{x+\frac{1}{2}} = \frac{4}{5}\left(\frac{y}{x}\right)$, also $\frac{y}{x-\frac{1}{2}} = \frac{y}{x} + \frac{5}{2}$,

whence $x = 2$, and $y = 15$ miles.

(68). Required a number to which if one-half of itself, one-third of that half, and one-fourth of that third, be added, the sum will be 287.

Let x = the number,

$$x + \frac{x}{2} + \frac{x}{6} + \frac{x}{24} = 287, \text{ whence } x = 168.$$

(69). A person had two casks, the larger of which he filled with ale, and the smaller with cyder. Ale being half-a-crown, and cyder 11*s.* per gallon, he paid 8*l.* 6*s.*; but had he filled the larger with cyder, and the smaller with ale, he would have paid 11*l.* 5*s.* 6*d.*: how many gallons did each hold?

Let x = gallons in larger, y = gallons in smaller,

$$\text{then } \frac{5x}{2} + 11y = 166, \text{ but } 11x + \frac{5y}{2} = 225\tfrac{1}{2},$$

whence $x = 18$, $y = 11$.

(70). A father bequeaths to his three sons 7800*l.*, in such a manner that if the share of the eldest be multiplied by 4, that of the second by 6, and that of the third by 8, the products are all equal; what are their shares?

Let x, y, and z be the shares;

$$\therefore\ x + y + z = 7800l., \text{ and } 4x = 6y = 8z;$$

$$\therefore\ x + \frac{2x}{3} + \frac{x}{2} = 7800, \text{ whence } x = 3600l., y = 2400l., z = 1800l.$$

(71). A person rows from Cambridge to Ely (a distance of 20 miles) and back again in 10 hours, the stream flowing uniformly in the same direction the whole time; and he finds that he can row 2 miles against the stream in the same time that he rows 3 miles with it; find the rate of the stream, and the time of his going and returning.

Let x = hours in going, $10 - x$ = hours in returning;

$$\therefore \text{ respective speeds are } \frac{20}{x} \text{ and } \frac{20}{10-x},$$

$$\text{but } \frac{20}{x} : \frac{20}{10-x} :: 3 : 2; \ \therefore x = 4; \ \therefore 10 - x = 6 \text{ hours},$$

$$\text{rate of stream} = \frac{1}{2}\left(\frac{20}{4} - \frac{20}{6}\right) = \frac{1}{2} \times \frac{10}{6} = \frac{5}{6} \text{ of a mile per hour.}$$

(72). The number of a gentleman's horses is two-fifths of the number of his black cattle, and for every four of the latter he has 11 sheep; required the number of each, the number of the sheep exceeding that of the horses by 141.

$$\text{Let } x = \text{number of black cattle, } \frac{2x}{5} = \text{number of horses,}$$

$$\frac{11x}{4} = \text{number of sheep,}$$

$$\text{then } \frac{11x}{4} - \frac{2x}{5} = 141, \text{ whence } x = 60, \ \frac{2x}{5} = 24, \ \frac{11x}{4} = 165.$$

(73). A gamester lost, first, the 6th part, and secondly, the 10th part of a certain sum of money; he then gained the third part of the same sum: supposing that his gain exceeded his loss by 3*l.*, what was the sum?

Let £x = the sum;

$$\therefore \frac{x}{3} = \frac{x}{6} + \frac{x}{10} + 3, \text{ whence } x = 45l.$$

(74). A pedestrian finding that he could walk four times as fast forwards as he could backwards, undertook to walk a certain distance $\left(\frac{1}{4}\text{ of it backwards}\right)$ in a certain time. But the ground being bad, he found that his rate per hour backwards was $\frac{1}{5}$ of a mile less than he had supposed, and that to have won his wager he must have walked forwards 2 miles an hour faster than he did: what is his rate per hour backwards?

Let $4x$ = his rate forwards, and x = his rate backwards,

and $4A$ = distance,

$$\text{then } \frac{3A}{4x} + \frac{A}{x} = \text{time} = \frac{3A}{4x+2} + \frac{A}{x - \frac{1}{5}}, \text{ whence } x = 1.$$

(75). A farm was rated at 3*s*. an acre, and the tenant, on receiving back at his rent-day 10 per cent. of his rent, found that the sum returned amounted to 6*l*. more than the whole rate. The next year the rates were doubled, and he received back 15 per cent. of his rent; but he now found that the sum returned only just paid for the whole rate. What was the rent of the farm, and of how many acres did it consist?

Let x = number of acres, y = £ per acre; ∴ £xy = rent,

$$\text{and } \frac{xy}{10} = \frac{3x}{20} + 6, \text{ also } \frac{3xy}{20} = \frac{3x}{10},$$

whence $y = 2l.$, $x = 120$ acres; ∴ rent $= 240l.$

(76). Two women, at the distance of 150 miles, set out to meet each other—one goes 3 miles in the time the other goes 7; what part of the distance does each travel?

Let $3x$ and $7x$ be the distances,

then $10x = 150$, and $3x = 45$, and $7x = 105$.

(77). Two persons *A* and *B*, can perform a piece of work in 16 days. They work together for 4 days, when *A* being called off, *B* is left to finish it, which he does in 36 days more: in what time would each do it separately?

Let *A*'s daily part $= \frac{1}{x}$, *B*'s $= \frac{1}{y}$;

$$\therefore \frac{1}{x} + \frac{1}{y} = \frac{1}{16}, \text{ and the remainder after 4 days} = \frac{3}{4};$$

$$\therefore \frac{36}{y} = \frac{3}{4}, \text{ whence } y = 48 \text{ days, and } x = 24 \text{ days.}$$

(78). A company of 90 persons consists of men, women, and children; the men are 4 in number more than the women, and the children exceed the number of men and women by 10. How many men, women, and children are there in the company?

Let x = number of women, $x + 4$ = number of men;

$\therefore 2x + 4 + 10$ = number of children,

and $4x + 18 = 90$; $\therefore x = 18$, number of men = 22,

number of children = 50.

(79). The owner of a balloon calculated that if he filled the enclosure which he had hired for the day for 5*l.* with spectators at 2*s.* each, and two persons ascended with him, he should gain $\frac{7}{5}$ of his outlay. The gas and the weather proving bad, he pays but half the price of inflating, and ascends alone from the enclosure a fourth part full, and loses $\frac{1}{3}$ of his outlay. He ascends on the next day with a full balloon, the enclosure $\frac{3}{4}$ filled, and a companion with him. By the whole speculation he gained 10*l.*: what did it cost him to fill his balloon?

Let $2x$ = number of persons enclosure will hold,

£y = price paid by each companion, and £z = price of gas,

$$\text{then } 4x + 40y = \frac{12}{5}(100 + 20z), \text{ or } x + 10y = 60 + 12z;$$

$$\text{again, } x = \frac{2}{3}(100 + 10z), \text{ whence } 10y = \frac{16z - 20}{3},$$

$$\text{also } 3x + 20y = 100 + 20z + \frac{1}{3}(100 + 10z) + 200,$$

$$\text{whence } 3x + 20y = \frac{1000 + 70z}{3}, \text{ and } 10y = \frac{200 + 5z}{3},$$

$$\text{wherefore } 16z - 20 = 200 + 5z; \ \therefore\ z = \frac{220}{11} = 20l, \text{ price of gas.}$$

(80). A farm of 864 acres is divided between 3 persons. C has as many acres as A and B together; and the portions of A and B are in the proportion of 5 : 11. How many acres has each?

Let $5x$ and $11x$ be A's and B's share; $\therefore\ 32x = 864$, and $x = 27$; the shares are 135, 297, 432 acres.

(81). There is a cistern, into which water is admitted by three cocks, two of which are of exactly the same dimensions: when they are all open, five-twelfths of the cistern is filled in 4 hours; and if one of the equal cocks be stopped, seven-ninths of the cistern is filled in 10 hours and 40 minutes. In how many hours would each cock fill the cistern?

Let x = hours by equal cocks, y = hours by the other, A gallons = cistern;

$$\therefore\ \frac{A}{x} \text{ and } \frac{A}{y} = \text{gallons per hour, then } \frac{8}{x} + \frac{4}{y} = \frac{5}{12},$$

$$\text{also } \frac{32}{3x} + \frac{32}{3y} = \frac{7}{9}, \text{ whence } x = 32, \text{ and } y = 24 \text{ hours.}$$

(82). A labourer is engaged for 48 days on the following conditions: for every day on which he works he is to receive 2*s.* and his board, but for every day on which he does not work he is to pay 1*s.* for his board. If at the close of his engagement he receives 2*l.* 2*s.*, on how many days must he have worked, and on how many has he been idle?

Let x = days he worked;

$$\therefore\ 2x - (48 - x) = 42; \ \therefore\ x = 30 \text{ days' work.}$$

(83). A merchant wishing to buy a certain quantity of pimento, the price of which he calculates at the rate of 8*l.* for 5 bags, transmits to his foreign agent the requisite sum of money. Before the order arrives, pimento has risen in value, and the money is sufficient only to buy a quantity less by 18 bags than that which the merchant intended. It appears also that $5\frac{1}{4}$ bags more than $\frac{1}{3}$ of the original quantity will now cost 10*l.* 7*s.* more than they would have done had the price not varied: what was the quantity intended to be purchased?

Let $5x$ = number of bags; $\therefore$ £$8x$ money sent,

and $\frac{8}{5}$ £ price per bag intended,

and $5x - 18$ = number of bags bought,

then $\frac{8x}{5x-18}$ = price of bags bought,

$$\text{whence } \left(\frac{5x}{3} + 5\tfrac{1}{4}\right) \times \left(\frac{8x}{5x-18} - \frac{8}{5}\right) = \frac{207}{20},$$

$$\text{or } (20x + 63) \times 144 = 621\,(5x - 18);$$

$$\therefore\ 5x = \frac{2250}{5} = 450 \text{ bags}.$$

(84). A charitable person distributed 5*l.* 14*s.* amongst some poor women and children, giving to each woman 6*s.*, and to each child two; and the number of women was to the number of children as 4 : 7; how many were relieved?

Let $4x$ and $7x$ be the respective numbers,

$$\text{then } 24x + 14x = 114;\ \therefore\ x = \frac{114}{38} = 3,$$

and the numbers are 12 women and 21 children.

(85). Some hours after a courier had been sent from A to B, which are 147 miles distant, a second was sent, who wished to overtake him just as he entered B; to do which he found he must perform the journey in 28 hours less than the first did. Now the time in which the first travels 17 miles added to the time in which the second travels 56 miles is 13 hours and 40 minutes: how many miles does each go per hour?

Let x = miles per hour by A;

$$\therefore \frac{147}{x} = \text{hours, and } \frac{147 - 28x}{x} = \text{hours by } B;$$

$$\therefore \frac{147x}{147 - 28x} = \text{miles per hour by } B,$$

$$\text{then } \frac{17}{x} + \frac{56}{147x} \times (147 - 28x) = 13\tfrac{2}{3} = \frac{41}{3},$$

$$\text{whence } x = \frac{219}{73} = 3 \text{ miles per hour for } A,$$

$$\text{and } B\text{'s speed} = \frac{147 \times 3}{147 - 84} = 7 \text{ miles.}$$

(86). The sum of three numbers is 70; and if the second is divided by the first, the quotient is 2, and the remainder 1; but if the third is divided by the second, the quotient is 3, and the remainder 3: what are the numbers?

Let x, y, z be the numbers, then $x + y + z = 70$,

then $y = 2x + 1$, $z = 3y + 3$;

$\therefore$ by substitution,

$$x + 2x + 1 + 3 + 3(2x + 1) = 70;$$

$$\therefore x = \frac{63}{9} = 7, \; y = 15, \; z = 48.$$

(87). A and B start to run a race to a certain post and back again. A returning meets B 90 yards from the post, and arrives at the starting place 3 minutes before him. If he had returned immediately to meet B, he would have met him at $\frac{1}{6}$ of the distance between the post and starting place: find the length of the course and the duration of the race.

Let x = length of course,

then $x + 90 : x - 90 :: A$'s speed : B's speed, or $A\text{'s} = \frac{x + 90}{x - 90} B\text{'s}$,

also $2x + \frac{x}{6} : 2x - \frac{x}{6} :: A$'s speed : B's speed, or $A\text{'s} = \frac{13B\text{'s}}{11}$;

$$\therefore 11(x + 90) = 13(x - 90); \; \therefore x = \frac{24 \times 90}{2} = 1080 \text{ yards,}$$

$$A\text{'s whole gain} : 180 :: 2160 : 990;$$

$$\therefore A\text{'s whole gain} = \frac{4320}{11}, \text{ which } B \text{ does in 3 minutes;}$$

$$\therefore B\text{'s whole time} = \frac{2160 \times 11 \times 3}{4320} = \frac{33}{2} = 16\tfrac{1}{2} \text{ minutes.}$$

(88). *A* and *B* begin trade, *A* with triple the stock of *B*. They each gain 50*l*., which makes their stocks in the proportion of 7 to 3. What were their original stocks?

$$3x = A\text{'s stock}, \ x = B\text{'s},$$

$$\text{then } 3x + 50 : x + 50 :: 7 : 3;$$

$$\therefore x = \frac{200}{2} = 100l. \ B\text{'s stock, and } 300l. = A\text{'s stock.}$$

(89). Two loaded waggons were weighed, and their weights were found to be in the ratio of 4 to 5. Parts of their loads, which were in the proportion of 6 to 7, being taken out, their weights were then found to be in the ratio of 2 to 3; and the sum of their weights was then 10 tons. What were the weights at first?

Let $4x$ and $5x$ be their original weights,

and $6y$ and $7y$ the weights taken out,

then $4x - 6y : 5x - 7y :: 2 : 3$ or $x = 2y$

and $9x - 13y = 5y = 10$; $\therefore y = 2$ tons, $x = 4$ tons;

$\therefore$ original weights were 16 and 20 tons.

(90). Two canal boats are despatched from the same place, the first at 6 o'clock in the morning, the other at 4 in the afternoon; the first goes 4 miles an hour, the second 9. How many hours will the second boat take to overtake the first?

Distance gone by first boat in 10 hours = 40 miles;

$$\therefore \text{second boat overtakes it in } \frac{40}{5} = 8 \text{ hours.}$$

(91). The upper spokes *R* and *r* of the hind and fore wheels of a carriage are vertical at starting. After *r* has made one revolution, its direction is at right angles to the spoke next before R; and when R has made $\frac{9}{8}$ of a revolution, r ascending through its second revolution makes the same angle with the horizontal line through the axle as the spoke next before it. Given that the diameter of the fore wheel : difference of the heights of the axles as the number of the spokes in the fore wheel : 2; compare the magnitudes of the wheels, and find the number of spokes in each.

Let $4R$ and $4r$ be the number of spokes, x and y the radii;

$$\text{then } 3R - 1 = 4r;$$

$$\text{also } 2y : x - y :: 4r : 2; \ \therefore \ x = \frac{r+1}{r}.y;$$

$$\text{circumference of large wheel} = 2\pi y.\frac{r+1}{r};$$

$$\therefore \text{ distance of spokes at circumference} = \frac{3\pi y}{16r+4}.\frac{r+1}{r};$$

$$\text{by first condition, } 2\pi y = \frac{3}{4} \times 2\pi.\frac{r+1}{r}.y - \frac{3\pi y}{16r+4}.\frac{r+1}{r},$$

$$\text{whence } 8r = 16, \ r = 2, \text{ and } R = 3;$$

$$\therefore \text{ number of spokes are 12 and 8};$$

$$\odot \text{ of larger wheel} : \odot \text{ smaller} :: 2\pi y\,\frac{r+1}{r} : 2\pi y$$

$$:: 3 : 2.$$

The second conditions in the problem are unnecessary for the solution.

(92). There are two numbers in the proportion of $\frac{1}{2}$ to $\frac{2}{3}$, which being increased respectively by 6 and 5, are in the proportion of $\frac{2}{5}$ to $\frac{1}{2}$; required the numbers.

Let x and y be the numbers,

$$\text{then } x : y :: \frac{1}{2} : \frac{2}{3} :: 3 : 4, \text{ and } x = \frac{3y}{4},$$

$$\text{also } x + 6 : y + 5 :: \frac{2}{5} : \frac{1}{2} :: 4 : 5;$$

$$\therefore\ 3y + 24 : 4(y + 5) : 4 : 5;$$

$$\therefore\ y = 40,\ x = 30.$$

(93). A gentleman gave away a certain sum in charity to 14 men and 15 women. Had the sum been less by 12*s.*, and only half the number of men relieved, the rest being divided amongst the women, each woman would have received 2*s.* more than each man did; but if there had been only 8 women, and the rest had been divided amongst the men, each man would have received twice as much as each woman. How much money was given away?

$$\text{Let } xs. = \text{sum},$$

$$ys. = \text{sum for each man},\ zs. = \text{sum for each woman},$$

$$\text{then } 14y + 15z = x,$$

$$\text{but } 7y + 15(y + 2) = x - 12,\ \text{also } 14 \times 2z + 8z = x;$$

$$\therefore\ x = 36z,\ \text{and } 14y = x - 15z = 21z;\ \therefore\ y = \frac{3z}{2},$$

$$\text{and } 22 \times \frac{3z}{2} = 36z - 42;\ \therefore\ z = 14,$$

$$\text{and } x = \frac{36 \times 14}{21} = 24 \text{ guineas.}$$

(94). The garrison of a town threatened with siege abandoned the town, and retreated at the rate of 27 miles per day. Two days afterwards a corps was sent in pursuit, with orders to overtake the fugitives in 6 days. How many miles per day must the second party march to accomplish their orders?

$$\text{Let } x = \text{miles per day},$$

$$\text{then } 6x = 6 \times 27 + 54;\ \therefore\ x = 36 \text{ miles per day.}$$

(95). A farmer's rent was 50*l.* a year, and his annual expenditure (including the assessed taxes, which amounted to $\frac{1}{6}$ of his expenses) was such that he was able to pay his landlord only 30*l.* The year following his rent was lowered 20 per

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cent.; the taxes also were reduced one-half, and agricultural produce increased in value $\frac{1}{3}$; in consequence, he was enabled to pay his rent and former debt, and to lay by 5*l.* What was his expenditure and the value of his produce each year?

Let x = his yearly expenditure, y = value of produce,

$$\text{then } x + 30 = y,$$

$$\text{second year's rent} = 40l.,$$

$$\text{and expenditure and taxes} = \frac{5x}{6} + \frac{x}{12} = \frac{11x}{12},$$

$$\text{then } \frac{11x}{12} + 40 + 20 + 5 = y + \frac{y}{3}; \ \therefore \ y = \frac{11x + 780}{16},$$

$$\text{whence } x = 60l., \text{ and } y = 90l. \text{ for first year;}$$

$$\therefore \ 120l. \text{ is value of produce for second year}$$

(96). A person engaged to reap a field of corn for 5*s.* an acre, but leaving 6 acres not reaped, he received 2*l.* 10*s.* Of how many acres did the field consist?

$$\text{Let } x = \text{number of acres},$$

$$\text{then } (x - 6)\,5 = 50s.; \ \therefore \ x = 16 \text{ acres}.$$

(97). When wheat was 5*s.* a bushel, and rye 3*s.*, a man wanted to fill his sack with a mixture of rye and wheat, for the money he had in his purse. If he bought 7 bushels of rye, and laid out the rest of his money in wheat, he would want 2 bushels to fill his sack; but if he bought 6 bushels of wheat, and filled his sack with rye, he would have 6*s.* left. How must he lay out his money, and fill his sack?

$$\text{Let } x = \text{bushel of wheat}, \ y = \text{bushel of rye},$$

$$\text{then } 5x + 3y = \text{his money},$$

$$\text{and } 7 + \frac{5x + 3y - 2}{5} = x + y - 2, \text{ and } y = 12,$$

$$\text{but } 30 + (x + y - 6)\,3 = 5x + 3y - 6; \ \therefore \ x = 9.$$

(98). From the first of two mortars in a battery 36 shells are thrown before the second is ready for firing. Shells are then thrown from both in the proportion of 8 from the first to 7 from the second; the second mortar requiring as much powder for 3 charges as the first does for 4, it is required to determine after how many discharges of the second mortar the quantity of powder consumed by it is equal to the quantity consumed by the first.

Let $7x$ = times the second was fired;

$\therefore\ 8x + 36$ = times the first,

then $(8x + 36)\,3 = 28x$; $\therefore\ x = 27$,

and $7x = 189$ times the second was fired.

(99). Two persons, A and B, start at the same time for a race, which lasted 6 minutes. Now after galloping 4 minutes at the same uniform pace at which each started, the distance between them is $\frac{1}{440}$ of the whole length of the course. They continue to run for 1 minute more at the same speed as at first; and then B, who is last, quickens the speed of his horse 20 yards a minute, and comes in exactly 2 yards before A, whose horse has run at the same uniform pace throughout. What is the length of the course?

Let x yards = length of course,

and $\frac{x-2}{6} = A$'s speed per minute;

$\therefore$ distance by A in $4' = \frac{2(x-2)}{3}$;

$\therefore$ distance by B in $4' = \frac{2x}{3} - \frac{4}{3} - \frac{x}{440} = \frac{877x}{1340} - \frac{4}{3}$;

$\therefore\ B$'s speed per minute $= \frac{877x}{5280} - \frac{1}{3}$;

$\therefore\ \left(\frac{877x}{5280} - \frac{1}{3}\right) \times 6 + 20 = x$; $\therefore\ x = 3 \times 1760 = 3$ miles

(100). A stage coach carries 6 inside, the fare outside is 13*s.*, and one-third of the sum of the outside fares exceeds one-sixth of those inside by 1*l.* 5*s.* 4*d.* An opposition arising, the coachman loses three outside and two inside passengers, and also reduces the inside fare by 5*s.*, and halves the outside; and then the whole loss is 7*l.* 0*s.* 6*d.* Find the number of outside places, and the fare inside.

Let x = number of outside places, and y*s.* = inside fare,

$$\text{then } 13x + 6y = \text{sum of fares, but } \frac{13x}{3} = y + \frac{76}{3},$$

$$\text{again } (x - 3) \times \frac{13}{2} + 4(y - 5) = 13x + 6y - 140\tfrac{1}{2};$$

$$\therefore\ 13x + 4y = 202;\ \therefore\ y = \frac{101}{2} - \frac{13x}{4};$$

$$\therefore\ x = 10, \text{ and } y = 18.$$

(101). In a sea fight, the number of ships taken was 7 more, and the number burnt 2 fewer, than the number sunk; 15 escaped, and the fleet consisted of 8 times the number sunk. Of how many did the fleet consist?

Let x = number sunk, $x + 7$ = number taken,

$x - 2$ = number burnt;

$\therefore\ 3x + 5 + 15 = 8x;\ \therefore\ x = 4$, and 32 = number in the fleet.

(102). A gentleman being asked the age of his two sons, answered, that if to the sum of their ages 18 be added, the result will be double the age of the elder; but if 6 be taken from the difference of their ages, the remainder will be equal to the age of the younger. What then were their ages?

Let x = age of elder, and y of the younger,

$$\text{then } x + y + 18 = 2x,$$

$$\text{also } x - y - 6 = y;$$

$$\therefore\ y = x - 18 = \frac{x - 6}{2};\ \therefore\ x = 30, \text{ and } y = 12.$$

(103). A cistern is filled by three pipes, A, B, C; the pipes A, B together fill the cistern in 70 minutes; A, C together in 84 minutes; B, C together in 140 minutes. In what time will each pipe fill the cistern, and in what time will it be filled if all three pipes are open?

Let x, y, z be galls. per minute respectively by A, B and C, then $70(x+y) = 84(x+z) = 140(y+z) =$ galls. in cask $= m$;

$$\therefore x = 5y - 6z = y + 2z = \frac{5y+2z}{3},$$

$$\text{whence } y = 2z, \text{ and } x = 4z,$$

$$\text{but } x + y + z = \frac{m}{140} + \frac{m}{168} + \frac{m}{280} = 7z;$$

$$\therefore z = \frac{28 \times m}{7 \times 1680} = \frac{m}{420}; \ \therefore \text{ time by } C = 420',$$

by $B = 210'$, by $A = 105'$, and by $A + B + C = 60'$.

(104). A entered into a canal speculation with 14 others, and the profits in this concern amounted in all to 595*l.* more than 5 times the price of an original share. Seven of his former partners joined him in a scheme for navigating the canal with steam-boats; each venturing a sum less than his former gains by 173*l.* But the steam-boats blowing up, A found he had lost 419*l.* by them; for the company not only never recovered the money advanced, but lost all they had gained by digging the canal, and 368*l.* besides. What were the prices of shares in the two concerns originally?

Let £x = price of original share; $\therefore 5x + 595l. =$ first gain,

$$\text{each man's second venture} = \frac{5x+595}{15} - 173 = \frac{x-400}{3},$$

$$\text{then } \frac{8(x-400)}{3} + \frac{8(5x+595)}{15} + 368 = 8 \times 419,$$

$$\text{whence } x = 700l., \text{ and } \frac{x-400}{3} = 100l.$$

(105). To find four numbers such, that the sum of the first, second, and third shall be 13; the sum of the first, second, and fourth, 15; the sum of the first, third, and fourth, 18; and, lastly, the sum of the second, third, and fourth, 20.

Let x, y, z, w be the numbers,

$$\text{then } x + y + z = 13,\ x + z + w = 18,$$
$$x + y + w = 15,\ y + z + w = 20$$
$$\therefore\ w - z = 2,\qquad z - y = 3;$$
$$\therefore\ y = 20 - w - z = z - 3;\ \therefore\ w = 23 - 2z = 2 + z;$$
$$\therefore\ 3z = 21 \text{ or } z = 7,\ w = 9,\ y = 4,\ x = 2.$$

(106). Supposing that 32 lbs. of sea water contain 1 lb. of salt how much fresh water must be mixed with these 32 lbs. in order that 32 lbs. of the mixture may contain only 2 oz. of salt, or $\frac{1}{8}$ of the former quantity?

$$32 : x :: \frac{1}{16} : \frac{1}{2};\ \therefore\ x = 16 \times 16 = 256;$$
$$\therefore\ 256 - 32 \text{ lbs.} = 224 \text{ lbs. of fresh water.}$$

(107). Two clocks are striking the hour together, and are heard to strike 19 times. There is a difference of 2 seconds in their time, and one strikes every 3, the other every 4 seconds. What is the hour they strike; it being observed that, when the clocks strike in the same second, the sounds cannot be distinguished, so as to determine whether one or both strike in that second? and that this is the case with the last stroke of the faster clock?

The clock which is 2 seconds the faster and strikes every 3 seconds, strikes first, and it will in 6 seconds strike the third time and the other will strike its second at the same time. Let it be x o'clock, then when they first strike together, the faster has $x - 3$ strikes to strike, and they will strike together every 12 seconds after the first time, and $x - 3$ strikes is the number remaining to the faster clock, after they first struck together; therefore $\frac{x-3}{4} + 1$ will be the number of times they strike together, but $2x - \left(\frac{x-3}{4} + 1\right) = 19$, whence $x = 11$ o'clock.

(108). To divide 48 into 4 such parts that the first increased by 3, the second diminished by 3, the third multiplied by 3, and the fourth divided by 3, may be all equal to each other.

Let x, y, z, w be the 4 parts; $\therefore\ x + y + z + w = 48$,

$$\text{then } x + 3 = y - 3 = 3z = \frac{w}{3}, \text{ then } y = x + 6,\ z = \frac{x+3}{3};$$

$$\therefore\ x + x + 6 + \frac{x+3}{3} + 3x + 9 = 48;$$

$$\therefore\ x = 6,\ y = 12,\ z = 2,\ w = 27.$$

(109). A number is expressed by 3 figures; the sum of the figures is 13; the figure which expresses the simple units of the number is triple the figure which expresses the hundreds; and if 396 is added to the number, the sum is the number reversed. Required the number.

Let $100x + 10y + z$ be the number, then $x + y + z = 13$, and $z = 3x$; $\therefore\ 100x + 10y + 3x + 396 = 300x + 10y + x$;

$$\therefore\ 200x - 2x = 396, \text{ or } x = 2,\ z = 6,\ y = 5,$$

and the number is 256.

(110). The gas contractors engage to light a shop with 5 large and 3 small burners; but having by them only one large burner, supply the deficiency with 5 small ones. The shopkeeper, not finding this light sufficient, procures two more small burners, and at the same time agrees for the lights to burn double the usual time on Saturday nights, for which additional gas he was to pay 1*l.* 11*s.* How much did he pay a year altogether?

4 large burners are equal to 5 small, let then $5xs.$ = price of large, $4x$ = price of small ones per night, and for 1 large and 8 small ones for 52 nights, and 2 small for 52 weeks he pays 31*s.* extra;

$$\therefore\ 5x \times 52 + 32x \times 52 + 8x \times 7 \times 52 = 31s.;$$

$$\therefore\ 52x = \frac{31}{93} = \frac{1}{3}s.;$$

$\therefore$ whole sum paid $= 5x \times 6 \times 52 + 32x \times 6 \times 52 + 31s.$

$= 10 + 64 + 31s. = 5l.\ 5s.$

(111). A cistern is to be filled with water from three different cocks; from the first it can be filled in 4 hours, from the second in 10, and from the third in 15. How soon would they all together fill it?

The first fills $\frac{1}{4}$, the second $\frac{1}{10}$, the third $\frac{1}{15}$ per hour;

$$\therefore \text{ if } x = \text{number of hours } x\left(\frac{1}{4} + \frac{1}{10} + \frac{1}{15}\right) = 1,$$

$$\text{or } x = \frac{60}{25} = \frac{12}{5} = 2 \text{ hours } 24 \text{ minutes.}$$

(112). A and B are two towns situated on the banks of a river which runs at the rate of 4 miles an hour. A waterman rows from A to B and back again, and finds that he is 39 minutes longer on the water than he would have been had there been no stream. The next day he repeats his voyage with another waterman, with whose assistance he can row half as fast again; and they find that they are only 8 minutes longer in performing their voyage than they would have been had there been no stream. Required the rate at which the waterman would row by himself.

Let x miles = distance of A and B,

and y = miles per hour of rowing,

then $\frac{x}{y+4}$ and $\frac{x}{y-4}$ = times of going and returning,

and $\frac{2x}{y}$ = time without stream;

$$\therefore \frac{x}{y+4} + \frac{x}{y-4} - \frac{2x}{y} = \frac{39}{60}, \text{ whence } x = \frac{13y}{640}(y^2 - 16),$$

and miles per hour by the two $= y + \frac{y}{2} = \frac{3y}{2}$;

$$\therefore \frac{x}{\frac{3y}{2}+4} + \frac{x}{\frac{3y}{2}-4} - \frac{4x}{3y} = \frac{2}{15}, \text{ or } x = \frac{9y^3 - 64y}{640},$$

$$\text{whence } 13y^2 - 13 \times 16 = 9y^2 - 64; \ \therefore y^2 = \pm 6,$$

and the rate = 6 miles per hour.

(113). Three brothers, A, B, C, buy a house for 2000*l.*; C can pay the whole price if B give him half of his money; B can pay the whole price if A give him one-third of his money; A can pay the whole price if C give him one-fourth of his money. How much has each?

Let x, y, and z be the money of each,

$$\text{then } z + \frac{y}{2} = 2000 = y + \frac{x}{3} = x + \frac{z}{4};$$

$$\therefore x = 2000 - \frac{z}{4} = 6000 - 3y = 3z - \frac{3y}{2};$$

$$\therefore y = \frac{1}{3}\left(4000 + \frac{z}{4}\right) = \frac{2}{3}(6000 - 3z); \ \therefore z = 1280l.,$$

$$y = 1440l., \ x = 1680l.$$

(114). Shew at what periods the hands of a watch will be together between 7 and 8 o'clock.

At 7 o'clock the hour-hand is 35 divisions before the minute-hand, and the minute-hand moves 12 times as fast as the hour-hand; if then the hour-hand be x minute divisions past seven at the time

$$12x = 35 + x; \ \therefore x = \frac{35}{11} = 3\tfrac{2}{11} \text{ minutes spaces};$$

$\therefore$ the time is $38\tfrac{2}{11}$ minutes past seven.

(115). A farmer sold a certain number of bushels of barley, and ten bushels of wheat, for 7*l.* 19*s.* Now each bushel of wheat cost within 3 shillings as much as two bushels of barley. He afterwards sold as many bushels of barley and four more, and fifteen bushels of wheat, and received two shillings per bushel more for his wheat and barley than he did before; when he found that if he had received 1*l.* 4*s.* more, he should just have received twice as much as he did before. How many bushels of barley did he sell the first time; and what were the prices per bushel of the wheat and barley?

Let x = bushels of barley sold,

and y = price of barley per bushel;

$\therefore\ 2y - 3$ = price of wheat per bushel;

$$\therefore\ xy + 10(2y - 3) = 159;\ \therefore\ x = \frac{189 - 20y}{y};$$

again $(x+4)(y+2) + 15(2y-1) = 318 - 24;\ \therefore\ x = \dfrac{301 - 34y}{y+2}$,

whence $y = \dfrac{38}{7} \pm \dfrac{11}{7} = 7s.$ for barley, $11s.$ for wheat,

and 7 bushels of barley.

(116). A farmer laid up a stock of corn, expecting to sell it in six months at three shillings per bushel more than he gave for it. But the price of corn falling one shilling per bushel, he found that by selling it he should lose the price of five bushels. He therefore kept it till the end of the year, and selling it at two shillings per bushel under prime cost, found his loss to be ten shillings less than his expected gain. Required the quantity of corn laid up, and the price per bushel, allowing 5 per cent. simple interest.

Let x = bushels of corn, $ys.$ = price per bushel,

then price at six months' end $= x \times (y + 3)$,

$$\text{expected gain} = xy + 3x - xy - \frac{xy}{40} = \frac{120x - xy}{40},$$

$$\text{also } xy - x(y - 1) + \frac{xy}{40} = 5y, \text{ or } x = \frac{200y}{40 + y},$$

price sold for $= x(y - 2)$;

$$\therefore \text{loss} = xy - x(y - 2) + \frac{xy}{20} = \frac{40x + xy}{20},$$

$$\text{then } \frac{40x + xy}{20} = \frac{120x - xy}{40} - 10, \text{ whence } x = \frac{400}{40 - 3y},$$

$$\therefore\ \frac{y}{40 + y} = \frac{2}{40 - 3y}, \text{ whence } y = 10 \text{ and } x = 40.$$

(117). Two men, A and B, set out from the same place to travel. A goes in 6 days twice as many miles as B goes in 5 days, but does not arrive at the end of his journey till 5 days after B has arrived at the end of his, when he finds that he has travelled 259 miles more than B. But had B gone 2 miles per day more than he did, and A stopped 6 days sooner, A would then have gone only 37 miles more than B. How many miles did each travel per day, and how many days did they travel?

Let A's rate $= x$ miles per day; $\therefore$ B's rate $= \frac{3x}{5}$,

and y = number of days of travelling of A, $y - 5$ of B,

$$\text{then } xy = \frac{3x}{5}(y-5) + 259,$$

$$\text{whence } y = \frac{259 \times 5 - 15x}{2x};$$

$$\text{again } x(y-6) = \frac{3x+10}{5} \times (y-5) + 37, \quad \text{or } y = \frac{15x+135}{2x-10};$$

$$\therefore \frac{259-3x}{x} = \frac{3x+27}{x-5}, \text{ whence } x = 35,\ y = 11;$$

$\therefore$ A travels 11 days at 35 miles per day,
B travels 6 days at 21 miles per day.

(118). Bacchus caught Silenus asleep by the side of a full cask, and seized the opportunity of drinking, which he continued for two-thirds of the time that Silenus would have taken to empty the whole cask. After that, Silenus awoke, and drank what Bacchus had left. Had they drank both together, it would have been emptied two hours sooner, and Bacchus would have drank only half what he left Silenus. Required the time in which they would empty the cask separately.

Let x = hours in which Bacchus, y = hours in which Silenus could have drank it, a = contents of cask; $\therefore \frac{a}{x}$ and $\frac{a}{y}$ are the hourly consumption of each; $\therefore \frac{a(x+y)}{xy}$ = hourly consumption of the two;

$$\therefore \frac{xy}{x+y} = \text{hours by the two,}$$

$$\text{and } \frac{a}{x} \times \frac{2y}{3} = \text{quantity first drank by Bacchus;}$$

$$\therefore a - \frac{2ay}{3x} = \frac{3ax-2ay}{3x} = \text{quantity left by Bacchus for Silenus;}$$

$$\frac{a}{x} \cdot \frac{xy}{x+y} = \text{quantity consumed in second case by Bacchus;}$$

$$\text{(and by problem)} = \frac{1}{2} \cdot \frac{3ax-2ay}{3x},$$

$$\text{or } \frac{y}{x+y} = \frac{3x-2y}{6x};$$

$$\therefore 6xy = 3x^2 - 2xy + 3xy - 2y^2,$$

$$\text{and } x = \frac{5y}{6} \pm \frac{7y}{6} = 2y.$$

$$\text{Then } \frac{xy}{x+y} + 2 = \frac{2y}{3} + \frac{3x-2y}{3x} \cdot y,$$

whence $y = 3$, and $x = 6$ hours.

(119). A silversmith has three bars composed of silver, copper, and tin, mixed in different proportions. The pound (avoirdupois) of the first bar contains 7 oz. of silver, 3 oz. of copper, and 6 oz. of tin; the pound of the second contains 12 oz. of silver, 3 oz. of copper, and 1 oz. of tin; and the pound of the third contains 4 oz. of silver, 7 oz. of copper, and 5 oz. of tin. How much of each of these 3 bars must be taken to form a fourth, the pound weight of which shall contain 8 oz. of silver, $3\frac{3}{4}$ oz. of copper, and $4\frac{1}{4}$ oz. of tin?

Let x be the oz. of the first bar, y of the second, z of the third,

then silver in one oz. of first $= \frac{7x}{16}$, in second $= \frac{3y}{4}$,

in third $= \frac{z}{4}$ oz., and so for copper and tin;

$$\therefore \frac{7x}{16} + \frac{3y}{4} + \frac{z}{4} = 8 \ (1), \quad \frac{3x}{16} + \frac{3y}{16} + \frac{7z}{16} = \frac{15}{4} \ (2),$$

$$\frac{3x}{8} + \frac{y}{16} + \frac{5z}{16} = \frac{17}{4} \ (3),$$

from $(2)-(1)\ \frac{5x}{64} + \frac{3z}{8} = \frac{7}{4}$, from $(3)-(1)\ \frac{65x}{16} + \frac{7z}{2} = 43$,

$$\text{whence } z = \frac{14}{3} - \frac{5x}{24} = \frac{86}{7} - \frac{65x}{56};$$

$$\therefore x = 8, \ z = 3, \ y = 5.$$

(120). Two master bricklayers undertake to lay the foundation of a new court, each taking a certain part, and begin at the same time. If they had continued to work together until the whole was finished, it would have required only $\frac{4}{5}$ of the time it actually took; and in this case B would do enough to occupy A 3 months, and A enough to occupy B 12 months, which is 36 yards more than A contracted to do. How many yards did the foundation contain?

Let $x = A$'s work in 1 month,

$y = B$'s work in 1 month,

z = whole time they were about it = B's time,

v = time A worked.

Then $\frac{4}{5}(x+y)z = xv + yz$ ....(1),

$\frac{4}{5}xz = 12y$ ....(2),

$\frac{4}{5}yz = 3x$ ....(3),

$\frac{4}{5}xz - 36 = xv$ ....(4),

from $(2) \div (3)$; $\therefore 3x^2 = 12y^2$; $\therefore x = 2y$;

$$\therefore \frac{4}{5}z = 6,$$

$$z = \frac{15}{2},$$

$$\text{from (1) } 6(x + y) = xv + \frac{15y}{2}, \text{ and } x = 2y;$$

$$\therefore v = \frac{21}{4},$$

$$\text{from (4) } 6x - 36 = \frac{21x}{4}; \ \therefore x = 48;$$

$$\therefore y = 24,$$

and whole work $= xv + yz$

$$= \frac{48}{1} \times \frac{21}{4} + \frac{24}{1} \times \frac{15}{2} = 432 \text{ yards.}$$

(121). A labourer engages to work for 3*s.* 6*d.* a day and his board, but to allow 9*d.* for his board each day that he is unemployed. At the end of 24 days he has to receive 3*l.* 2*s.* 9*d.* How many days did he work?

Let x = days he worked, $24 - x$ = days he did not work,

$$\text{then } \frac{7x}{2} - \frac{24 - x}{1} \times \frac{3}{4} = 62\tfrac{3}{4}, \text{ whence } x = 19 \text{ days.}$$

(122). *A* and *B* playing at billiards, *A* bet 5 shillings to 4 on every game, and found that after a certain number of games he had won 10 shillings. Had *B* won one game more, the number won by him would have been to the number won by *A* as 3 to 4. How many did each win?

Let x = games won by *A*, y = games won by *B*,

$$\text{then } y + 1 : x - 1 :: 3 : 4; \ \therefore y = \frac{3x - 7}{4},$$

and $4x - 5y = 10$, or $16x - 15x + 35 = 40$; $\therefore x = 5$, $y = 2$.

(123). A revenue cutter observes a smuggler q leagues directly to windward; and gives chase, sailing at $5\frac{1}{3}$ points from the wind, and making tacks of $4p$ miles. The smuggler immediately lies off on the other tack at $2\frac{2}{3}$ points, making tacks of $\frac{p}{\sqrt{3}}$ miles, its rate of sailing being to the cutter's as 1 : 4 $\sqrt{3}$. They sail half the above distances before the first tack. In what tack will the smuggler first come within range of the cutter's guns, which carry r miles?

It is obvious from the conditions of the question, that they both tack at the *same time*, and that the smuggler is directly to windward of the cutter in the *end* of each tack; moreover, in each tack the cutter goes to windward, $2p$ miles $(= 4p \cos 60°)$, and the smuggler $\frac{1}{2}\,p$ miles $\left\{= \frac{p}{\sqrt{(3)}} \cos 30°\right\}$. Hence, from the commencement of the chase to the end of the first tack, the cutter gains $\frac{3}{2}p$ miles on the smuggler, and continues to gain the same from this to the end of each succeeding tack: and therefore the number of the tack at the end of which the cutter will be within r miles of the smuggler, is the integer $= \dfrac{3q - r}{\frac{3}{2}p}$, or $= \dfrac{2q}{p} - \dfrac{2r}{3p}$.

124). Three workmen are employed to dig a ditch of 191 yards in length. A can dig 27 yards in 4 days, B 35 yards in 6 days, and C 40 yards in 12 days; in what time could they do it if they worked simultaneously?

$$\text{Daily work of } A = \frac{27}{4}, \text{ of } B = \frac{35}{6}, \text{ of } C = \frac{10}{3} \text{ yards;}$$

$$\therefore \text{ daily work of all} = \frac{81 + 70 + 40}{12} = \frac{191}{12};$$

$$\therefore \text{ time} = \frac{191 \times 12}{191} = 12 \text{ days, or if } x = \text{number of days,}$$

$$x \times \frac{191}{12} = 191; \;\therefore\; x = 12.$$

(125). A besieged garrison had such a quantity of bread, as would, if distributed to each at 10 ounces a day, last 6 weeks; but having lost 1200 men in a sally, the governor was enabled to increase the allowance to 12 ounces per day for 8 weeks. Required the number of men at first in the garrison.

Let x = number of men in the garrison;

$$\therefore\ x \times 10 \times 42 = \text{oz. of bread,}$$

$$\text{but } (x - 1200) \times 12 \times 56 = \text{oz. of bread} = x \times 420;$$

$$\therefore\ (x - 1200) \times 8 = 5x;\ \ \therefore\ x = \frac{1200 \times 8}{3} = 3200 \text{ men.}$$

(126). A man who is not aware that his watch gains uniformly, engages to ride from Cambridge to London in 9 hours, and sets his watch by St. Mary's at the time of starting. Upon looking at his watch after having gone half way, he supposes it necessary to increase his pace in the ratio of 4 : 3; in consequence of which he arrives in London a quarter of an hour within the time agreed on. But if the watch had lost at the same rate, and he had looked at it at the 14th milestone, and then regulated his pace accordingly, he would have been in London too late by 7 minutes. Find at what rate he set out, and the distance from Cambridge to London by the road he travelled.

Let x = miles per hour,

y = distance in miles between the two places;

$$\therefore\ \frac{y}{2x} = \text{time in going the first half way,}$$

$$\text{then second rate} : x :: 4 : 3;\ \ \therefore\ \text{second rate} = \frac{4x}{3};$$

$$\therefore\ \frac{y}{2} \times \frac{3}{4x} = \frac{3y}{8x} = \text{time in second half;}$$

$$\therefore\ \frac{y}{2x}\left(1 + \frac{3}{4}\right) = \frac{7y}{8x} = 8\tfrac{3}{4} \text{ hours;}\ \ \therefore\ y = 10x.$$

Let his watch gain t'' per hour;

$$\therefore \text{ in first half way it gains } \frac{y}{2x} \times \frac{t''}{60} = \frac{ty}{120x} = \frac{1}{4};$$

$$\therefore\ t = \frac{30x}{y} = 3''.$$

In the second supposition, the time of going 14 miles $= \frac{14}{x}$,

and in that time the watch would lose $\frac{14}{x} \times t''$;

$$\therefore \frac{14t}{x} = 7; \;\therefore x = 2t = 6 \text{ miles},$$

$$\text{and } y = 10x = 60 \text{ miles}.$$

(127). *A* and *B* travelled on the same road, and at the same rate, from *H* to *L*. At the 50th milestone from *L*, *A* overtook a drove of geese, which were proceeding at the rate of 3 miles in 2 hours, and two hours afterwards met a stage-waggon which was moving at the rate of 9 miles in 4 hours. *B* overtook the same drove of geese at the 45th milestone, and met the same stage-waggon exactly 40 minutes before he came to the 31st milestone. Where was *B* when *A* reached *L*?

Let x = miles per hour of *A* and *B*;

since the geese travel $1\frac{1}{2}$ mile per hour,

there were $\frac{10}{3}$ hours between *A* and *B* passing them;

$$\therefore A \text{ was } \frac{10x}{3} - 5 \text{ miles in advance of } B \text{ .... (1)}.$$

The speed of the waggon is $\frac{9}{4}$ miles per hour;

and when *A* met it, it was $50 - 2x$ miles from London;

the waggon had travelled, when *B* met it,

$$\left\{\left(31 + \frac{2}{3}x\right) - (50 - 2x) = \left(\frac{8x}{3} - 19\right)\right\} \text{ miles};$$

and the time between *A* and *B* meeting it, was

$$\left(\frac{8x}{3} - 19\right)\frac{4}{9} \text{ hours};$$

and *A* had, since he met the waggon, travelled

$$\left(\frac{8x}{3} - 19\right)\frac{4}{9}x \text{ miles};$$

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$\therefore$ when B met the waggon, they were

$$\left(\frac{8x}{3} - 19\right)\frac{4x}{9} + \left(\frac{8x}{3} - 19\right) \text{ miles apart;}$$

$$\therefore \text{ from (1);}$$

$$\therefore \frac{10x}{3} - 5 = \frac{8x}{3} - 19 + \left(\frac{8x}{3} - 19\right)\frac{4x}{9};$$

$$\text{or } 32x^2 - 228x - 18x = 378,$$

$$16x^2 - 123x + \left(\frac{123}{8}\right)^2 = 189 + \left(\frac{123}{8}\right)^2$$

$$= \frac{27225}{64};$$

$$\therefore 4x = \frac{123}{8} \pm \frac{165}{8} = 36; \therefore x = 9,$$

$$\text{and } \frac{10x}{3} - 5 = 30 - 5 = 25 \text{ miles,}$$

the distance of B from London.

(128). A composition of copper and tin containing 100 cubic inches weighed 505 ounces; how many ounces of each metal did it contain, supposing a cubic inch of copper to weigh $5\frac{1}{4}$ ounces, and a cubic inch of tin to weigh $4\frac{1}{4}$ ounces?

Let x = oz. of copper, $505 - x$ = oz. of tin;

$$\text{then } x \times \frac{4}{21} + \frac{505 - x}{1} \times \frac{4}{17} = 100,$$

$$\text{or } 17x + (505 - x) \times 21 = 25 \times 17 \times 21;$$

$$\therefore x = 420 \text{ oz. of copper and } 85 \text{ oz. of tin.}$$

(129). A landlord agrees with his steward to allow him a certain per centage on the rents collected, on condition that he returns half the same per centage on the rents not paid. The first year the steward's income amounts to 6 per cent. on the whole rental; but in the following he finds it necessary, in order to make up the deficiency from his last year's income, to make a return of rents received 270*l.* under their actual value. In the third year, though the rents are reduced $7\frac{1}{2}$ per cent., the amount of rents not paid is the same as in the second year; the steward's income is only $\frac{3}{4}$ of his first year's income, and to make up the deficiency he doubles the amount of his last year's fraud. Required the rental of the estate.

Let £a = whole rental,

$2x$ = rate per cent. allowed him,

y = second year's receipts.

Then since in the second year he declared 270*l.* less than he received, he had to pay a per centage on $a - y + 270$, and to receive it on $y - 270$. Therefore by question,

$$\frac{2x}{100}(y - 270) - \frac{x}{100}(a - y + 270) + 270 = \frac{6}{100}a,$$

$$\text{or } 3xy - 810x - ax + 27000 = 6a \text{ ....(1).}$$

Now in the third year the rental unpaid is still $a - y$, and therefore the deficiency which is $\frac{7\frac{1}{2}}{100}a$ must fall on the rent received, and reduce them to $y - \frac{7\frac{1}{2}a}{100}$; but the steward declares them to be respectively $y - \frac{7\frac{1}{2}a}{100} - 540$ and $a - y - 540$, and therefore, by question,

$$\frac{2x}{100}\left(y - \frac{7\frac{1}{2}a}{100} - 540\right) - \frac{x}{100}(a - y + 540) + 540 = \frac{6}{100}a,$$

$$\text{or } 3xy - 1620x - \frac{115}{100}ax + 54000 = 6a \text{......(2),}$$

he ought to have had.

$$\frac{2x}{100}\left(y - \frac{7\frac{1}{2}a}{100}\right) - \frac{x}{100}(a - y) = \frac{3}{4}\left(\frac{6a}{100}\right) \text{ by question,}$$

$$\text{or } 3xy - \frac{115}{100}ax = \frac{9a}{2} \text{ .... (3).}$$

Take (1) from (2), $810x + \frac{15}{100}ax = 27000$ ...(4),

and (2) from (3), $1620x = 54000 - \frac{3a}{2}$ ......(5),

$$a = 36000 - 1080x.$$

Put this value of a in (4), which then becomes

$$810x + 5400x - 162x^2 = 27000;$$

$$\therefore\ 3x^2 - 115x = -500,$$

$$\text{whence } x = 5,$$

$$\text{and } a = 36000 - 1080x = 30600l.$$

(130). There are two towns, A and B, which are 131 miles distant from each other. A coach sets out from A, at 6 o'clock in the morning, and travels at the rate of 4 miles an hour without intermission, in the direct road towards B. At 2 o'clock in the afternoon of the same day a coach sets out from B to go to A, and goes at the rate of 5 miles an hour constantly. Where will they meet?

Let x = hours of A on the road;

$\therefore\ 4x$ = distance travelled by A, $x - 8$ = hours by B,

$(x - 8)\,5$ = distance by B;

$$\therefore\ 4x + 5x - 40 = 131;\ \therefore\ x = 19 \text{ hours},$$

$$\text{or } 4 \times 19 = 76 \text{ miles from } A.$$

(131). A lent B a sum of money to be repaid with interest at the end of a year, and received as security Spanish 5 per cent. bonds to such an amount that their interest was equal to the interest of the debt. At the year's end B proved insolvent, and Spanish bonds having fallen 40 per cent. A found that he had lost 400l. Had they not fallen in value, he would have been enabled to repay himself, and to return to B 250l.; and had he been at liberty to have sold them out when they were at 50 (which was before the interest upon them was payable) he would have lost only 300l. Required the amount of the debt, and its interest, and the price of Spanish bonds at the beginning of the year.

Let x = the debt, y the rate of interest,

z = the price of the bonds,

then the sum due at the end of the year $= x\left(1 + \dfrac{y}{100}\right)$.

Now, if A be the whole amount of the bonds advanced, the interest upon them would be $\frac{5A}{z}$, which must $= \frac{xy}{100}$;

$$\therefore \frac{A}{z} = \frac{xy}{500}.$$

Hence from the question we have the three conditions,

$$\text{I. } (z-40)\frac{A}{z} + \frac{5A}{z} = (x-400) + \frac{xy}{100},$$

$$\text{or } (z-40)\frac{xy}{500} = (x-400;$$

$$\therefore xyz - 40xy = 500x - 200000 \ldots\ldots\ldots\ldots(\alpha).$$

$$\text{II. } \frac{xyz}{500} + \frac{xy}{100} = x + 250 + \frac{xy}{100};$$

$$\therefore xyz = 500x + 125000 \ldots\ldots\ldots\ldots\ldots(\beta).$$

$$\text{III. } 50\frac{A}{z} = x + \frac{xy}{100} - 300, \text{ or } \frac{xy}{10} = x + \frac{xy}{100} - 300;$$

$$\therefore 10xy = 100x + xy - 30000; \therefore 9xy - 100x = -30000 \ldots(\gamma).$$

From (α) and (β), $40xy = 325000$; $\therefore xy = \frac{32500}{4} = 8125$;

$$\therefore 9xy = 73125;$$

$\therefore$ from (γ), $100x = 103125$, $x = 1031\frac{25}{100} = 1031\frac{1}{4} = 1031l.\ 5s.$;

$$\therefore y = \frac{7312500}{928125} = \frac{292500}{37125} = 7\frac{29}{33}.$$

From (α) $8125z - 40 \times 8125 = 500 \times 1031\frac{1}{4} - 200000$;

$$\therefore z = 78\frac{11}{13}.$$

Ans. in Book, 1031*l.* 4*s.* Should be 1031*l.* 5*s.*

(132). A sets out to ride from Newmarket to London, at the same time that B and C leave Hockeril and London to ride to Newmarket. A meets B 4 hours before C overtakes B; but A on his return from London, meets C 1 hour before he meets B, on their way back from Newmarket. It was observed that A rode 10 miles an hour, and met B at the same place going and returning. It is required to find the rates of travelling of B and C, and the distance from London to Newmarket; it being given that Hockeril is equally distant from each.

Let $x = B$'s rate, $y = C$'s rate per hour,

and z = distance from L to H;

$$\therefore \frac{z}{x+10} = \text{time till } A \text{ meets } B,$$

$$\frac{z}{y-x} = \text{time till } C \text{ overtakes } B;$$

$$\therefore \frac{z}{x+10} + 4 = \frac{z}{y-x} \ldots\ldots\ldots\ldots\ldots(1);$$

$$\frac{6z}{y+10} = \text{time till } A \text{ meets } C \text{ second time},$$

$$\frac{5z}{y+10} = \text{time till } A \text{ meets } B \text{ second time};$$

$$\therefore \frac{6z}{y+10} + 1 = \frac{5z}{x+10} \ldots\ldots\ldots\ldots\ldots(2);$$

$$\frac{10z}{x+10} = \text{distance from } N \text{ where } A \text{ met } B;$$

$$\therefore \frac{50z}{x+10} = \text{distance till } A \text{ meets } B \text{ second time};$$

$$\therefore \frac{10z}{x+10} + \frac{50z}{x+10} = 4z,$$

whence $x = 5$.

From (1), $\frac{z}{15} + 4 = \frac{z}{y-5}$;

$\therefore$ whence $z = \frac{60.(y-5)}{20-y}$,

also from (2) $\frac{6z}{y+10} + 1 = \frac{z}{3}$,

and substituting for z, $7y^2 - 90y = -200$,

whence $y = 10$, $z = 30$, $x = 5$.

(133). *A* and *B* row between two places, *B* in a time during which the minute hand of his watch moves over a certain space: but when the minute hand of *A*'s watch has described an equal space, he is obliged to relax his speed, and for the rest of the distance moves only $\frac{2}{3}$ as fast as before. When the stream which flows at a given rate (a) is in their favour, the first part of the distance takes A 6 times as long as the last, but, when the stream is against him, the two parts are performed in equal times; they are also performed by him in equal times, even if he increase his speed in the ratio of 7 : 5, provided he exchanged watches with B at starting. Supposing their watches to gain uniformly, find the velocities of A and B.

Let x = number of divisions of B's watch corresponding to 1 unit of time,

x' = ... A's ....

y = true velocity of B by his own exertions,

y' = ... A ....

z = whole distance.

$\frac{z}{y-a}$ = time required by B against stream,

$\frac{z}{y+a}$ = ... B with ......

$\frac{xz}{y-a}$ = divisions of B's watch going against stream,

$\frac{xz}{y+a}$ = ... with ......;

$\therefore \frac{x}{x'}\frac{z}{y+a}$ = time required by A before change of speed

$\frac{x}{6x'}\frac{z}{y+a}$ = ... after ....

going with stream,

$y'+a$ = velocity of A before change of speed

$\frac{2}{3}(y'+a)$ = ... after ....

with stream;

$$\therefore \frac{x}{x'} z . \frac{y'+a}{y+a} + \frac{x}{6x'} z . \frac{2(y'+a)}{3(y+a)} = z,$$

$$\text{whence } 10x(y'+a) = 9x'(y+a) \ldots\ldots (\alpha).$$

In a similar manner: against stream we get $\frac{x}{x'} \; \frac{z}{y-a}$ for the time A requires to perform each part, and $y'-a$ and $\frac{2}{3}(y'-a)$ his two velocities;

$$\therefore \frac{x}{x'} . \frac{z}{y-a} . \frac{5}{3}(y'-a) = z,$$

$$\text{or } 5x(y'-a) = 3x'(y-a) \ldots\ldots\ldots (\beta).$$

Now when they exchange watches we must put x for x' and x' for x, and also $\frac{7}{5}(y'-a)$ instead of $\frac{2}{3}(y'-a)$ for the last velocity, and we get

$$\frac{x'}{x} . \frac{z}{y-a} . \frac{12}{5}(y'-a) = z,$$

$$\text{or } 12x'(y'-a) = 5x(y-a) \ldots\ldots\ldots (\gamma).$$

Divide (α) by (β),

$$\frac{2(y'+a)}{y'-a} = \frac{3(y+a)}{y-a} \ldots\ldots\ldots\ldots (\delta).$$

Multiplying (β) by (γ), cancelling common fraction,

$$4(y'-a)^2 = (y-a)^2,$$

$$2(y'-a) = y-a,$$

$$y = 2y'-a.$$

Put this value in (δ),

$$\frac{2(y'+a)}{y'-a} = \frac{6y'}{2(y'-a)},$$

$$2y'+2a = 3y',$$

$$y' = 2a,$$

$$y = 3a.$$

PROBLEMS PRODUCING QUADRATIC EQUATIONS.

XI. (1). Find two numbers whose difference is 3, and the sum of their squares 89.

Let x and $x + 3$ be the numbers,

$$\text{then } (x+3)^2 + x^2 = 89;$$

$$\therefore\ x^2 + 3x + \frac{9}{4} = \frac{80}{2} + \frac{9}{4} = \frac{169}{4};$$

$$\therefore\ x = -\frac{3}{2} \pm \frac{13}{2} = 5 \text{ or } -8,$$

and the numbers are 5 and 8, or − 8 and 5.

(2). There are two numbers, whose sum is to their difference as 8 to 1, and the difference of whose squares is 128. What are the numbers?

Let x and y be the numbers,

$$\text{then } x + y : x - y :: 8 : 1,$$

$$\text{then since if } \frac{a}{b} = \frac{c}{d}, \text{ we have } \frac{a+b}{a-b} = \frac{c+d}{c-d},$$

$$\text{we have } x : y :: 9 : 7;\ \therefore\ x = \frac{9y}{7},$$

$$\text{and } x^2 - y^2 = 128;\ \therefore\ \frac{81y^2}{49} - y^2 = 128;$$

$$\therefore\ y^2 = \frac{128}{32} \times 49;\ \therefore\ y = \pm 2 \times 7 = \pm 14;$$

$$\therefore\ x = \pm 18.$$

(3). What number is that which added to its square makes 42?

Let x be the number;

$$\therefore\ x^2 + x + \frac{1}{4} = 42 + \frac{1}{4} = \frac{169}{4};$$

$$\therefore\ x = -\frac{1}{2} \pm \frac{13}{2} = 6 \text{ or } -7.$$

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(4). The sum of the squares of the digits composing a number of two places of figures is 25, and the product of the digits is 12; find the number.

Let x and y be the digits;

$$\therefore\ x^2 + y^2 = 25, \text{ and } xy = 12;$$

$$\therefore\ x^2 - 2xy + y^2 = 1;\ \therefore\ x - y = 1,$$

$$\text{also } x^2 + 2xy + y^2 = 49;\ \therefore\ x + y = 7;\ \therefore\ x = 4,\ y = 3;$$

$\therefore$ the number is 34 or 43.

(5). In a court there are two square grass-plots; a side of one of which is 10 yards longer than the side of the other; and their areas are as 25 to 9. What are the lengths of the sides?

Let x = side of one; $\therefore\ x + 10$ = side of the other,

$$\text{then } (x + 10)^2 : x^2 :: 25 : 9;\ \therefore\ 20x + 100 : x^2 :: 16 : 9;$$

$$\therefore\ 16x^2 = 180x + 900;$$

$$\therefore\ x^2 - \frac{45x}{4} + \left(\frac{45}{8}\right)^2 = \frac{5625}{64};$$

$$\therefore\ x = \frac{45}{8} \pm \frac{75}{8} = 15 \text{ or } -3\tfrac{3}{4};$$

$\therefore$ the sides are 15 and 25.

(6). To find two numbers, such that the less may be to the greater as the greater is to 12, and that the sum of their squares may be 45.

Let x and y be the numbers,

$$\text{then } x : y :: y : 12;\ \therefore\ y^2 = 12x, \text{ and } x^2 + y^2 = 45,$$

$$\text{or } x^2 + 12x + 36 = 45 + 36 = 81;$$

$$\therefore\ x = -6 \pm 9 = 3 \text{ or } -15,$$

and the numbers are 3 and 6.

(7). Find two numbers, such that three times their product is equal to the sum of their squares, and their quotient equal to the difference of their squares.

Let x and y be the numbers,

$$\text{then } 3xy = x^2 + y^2, \text{ and } \frac{x}{y} = x^2 - y^2;$$

$$\therefore x^2 - 3xy + \frac{9y^2}{4} = \frac{5y^2}{4}, \text{ and } x = \frac{y}{2}\{3 \pm \sqrt{(5)}\},$$

$$\text{and by substitution } \frac{y^2}{4}(3 \pm \sqrt{5})^2 - y^2 = \frac{3 \pm \sqrt{5}}{2};$$

$$\therefore y^2 = \frac{3 \pm \sqrt{5}}{2} \times \frac{2}{(5 \pm 3\sqrt{5})} = \pm \frac{1}{\sqrt{5}}; \ \therefore y = \pm \sqrt{\left(\pm \frac{1}{\sqrt{5}}\right)};$$

$$\therefore x = \pm \sqrt{\left(\pm \frac{1}{\sqrt{5}}\right)} \times \frac{3 \pm \sqrt{5}}{2}.$$

(8). A person bought two pieces of linen, which together measured 36 yards. Each of them cost as many shillings per yard, as there were yards in the piece; and their whole prices were in the proportion of 4 to 1. What were the lengths of the pieces?

$$\text{Let } x = \text{yards in one};$$

$$\therefore 36 - x = \text{yards in the other};$$

$$\therefore x^2 : (36 - x)^2 :: 4 : 1, \text{ or } x : 36 - x :: 2 : 1;$$

$$\therefore x : 36 :: 2 : 3; \ \therefore x = 24,$$

$$\text{and } 36 - x = 12 \text{ yards length of the other.}$$

(9). What two numbers are those, whose difference is 2, and the difference of their cubes 98?

$$\text{Let } x = \text{one}; \ \therefore x - 2 = \text{the other};$$

$$\therefore x^3 - (x - 2)^3 = 98; \ \therefore 6x^2 - 12x = 90;$$

$$\therefore x^2 - 2x + 1 = 16; \ \therefore x = 1 \pm 4 = 5 \text{ or } -3;$$

$$\therefore \text{the numbers are 5 and 3.}$$

(10). The sum of three successive biquadrate numbers is 98; find them.

$$\text{Let } (x-1)^4,\ x^4 \text{ and } (x+1)^4 \text{ be the numbers,}$$

$$\text{by addition } 3x^4 + 12x^2 + 2 = 98, \text{ whence } x^2 = 4; \ \therefore x = \pm 2,$$

$$\text{and the numbers are 1, 16, 81.}$$

(11). There are two numbers, whose sum is to the less as 5 to 2; and whose difference, multiplied by the difference of their squares, is 135. Required the numbers.

Let x and y be the numbers,

then $x + y : y :: 5 : 2$; $\therefore\ x : y :: 3 : 2$,

also $(x - y)(x^2 - y^2) = 135$, or $\because\ x = \frac{3y}{2}$,

by substitution $\frac{y}{2} \times \frac{9y^2 - 4y^2}{4} = 135$; $\therefore\ y^3 = 27 \times 8$;

$\therefore\ y = 6$ and $x = 9$.

(12). What two numbers are those, whose sum is 6, and the sum of their cubes 72?

Let x and $6 - x$ be the numbers;

$\therefore\ x^3 + (6 - x)^3 = 72$, or $216 - 108x + 18x^2 = 72$,

whence $x^2 - 6x + 9 = 1$, and $x = 3 \pm 1 = 4$ or 2,

and the numbers are 4 and 2.

(13). Find two numbers, such that their sum, product, and the difference of their squares may be all equal.

Let x and y be the numbers,

then $x + y = xy$, also $x^2 - y^2 = x + y$;

$\therefore\ x - y = 1$, or $x = y + 1$;

$\therefore$ by substitution $y + 1 + y = y(y + 1)$,

or $y^2 - y + \frac{1}{4} = \frac{5}{4}$;

$\therefore\ y = \frac{1 \pm \sqrt{5}}{2}$, $x = \frac{3 \pm \sqrt{5}}{2}$.

(14). There are two numbers, which are in the proportion of 3 to 2; the difference of whose fourth powers is to the sum of their cubes as 26 to 7. Required the numbers.

Let $3x$ and $2x$ be the numbers,

then $(3x)^4 - (2x)^4 : (3x)^3 + 2x)^3 :: 26 : 7$,

or $x = \frac{35 \times 26}{7 \times 65} = 2$,

and the numbers are 6 and 4.

(15). Find three numbers, such that if the first be multiplied by the sum of the second and third, the second by the sum of the first and third, and the third by the sum of the first and second, the products shall be 26, 50, and 56.

Let x, y, z be the numbers,

then $x(y+z)=26,\ y(x+z)=50,$ and $z(x+y)=56$;

$$\therefore\ x=\frac{26}{y+z}=\frac{50-yz}{y}=\frac{56-zy}{z},$$

whence $24y+50z-yz(y+z)=0=30z+56y-yz(y+z)$;

$$\therefore\ 20z=32y;\ \therefore\ z=\frac{8y}{5},\ \text{and}\ y+z=\frac{13y}{5};$$

$$\therefore\ \frac{26\times 5}{13y}=\frac{50}{y}-\frac{8y}{5},\ \text{whence}\ y^2=25,\ \text{and}\ y=\pm 5;$$

$$\therefore\ z=\pm 8,\ \text{and}\ x=\pm 2.$$

(16). There is a field in the form of a rectangular parallelogram, whose length is to its breadth in the proportion of 6 to 5. A part of this, equal to one-sixth of the whole, being planted, there remain for ploughing 625 square yards. What are the dimensions of the field?

Let $6x$ and $5x$ be the length and breadth;

$$\therefore\ \text{area}=30x^2;$$

$$\therefore\ 30x^2-5x^2=625,\ \text{or}\ x=\pm 5;$$

$\therefore$ the sides are 30 and 25 yards.

(17). To divide the number 11 into two such parts, that the product of their squares may be 784.

Let x and $11-x$ be the numbers;

$$\therefore\ x^2(11-x)^2=784=49\times 16;$$

$$\therefore\ x(11-x)=\pm 7\times 4=\pm 28,$$

$$\text{and}\ x^2-11x+\frac{121}{4}=-28+\frac{121}{4}=\frac{9}{4};$$

$$\therefore\ x=\frac{11}{2}\pm\frac{3}{2}=7\ \text{or}\ 4.$$

(18). The product of two numbers $=a$, their quotient $=b$; required the expressions for the numbers in terms of a, b.

$$\text{Let } xy = a,\ \frac{x}{y} = b,\ \text{ or } x = by;$$

$$\therefore\ by^2 = a, \text{ and } y = \sqrt{(ab^{-1})} \text{ and } x = \sqrt{(ab)}.$$

(19). What three numbers are those which have their differences equal, their sum 15, and the sum of their cubes 495?

Let x, y, z be the numbers;

$$\therefore \text{ then } x - y = y - z, \text{ or } y = \frac{x+z}{2},$$

$$\text{also } x + y + z = 15;\ \therefore\ \frac{3(x+z)}{2} = 15, \text{ or } x + z = 10;\ \therefore\ y = 5;$$

$$\therefore\ x^3 + y^3 + z^3 = (10 - z)^3 + z^3 + 5^3 = 495,$$

$$\text{whence } 1000 - 300z + 30z^2 = 370, \text{ and } z = 7 \text{ or } 3;$$

$$\therefore\ x = 3 \text{ or } 7.$$

(20). To divide the number 5 into two such parts, that the sum of their alternate quotients may be $4\frac{1}{4}$, that is, of the two quotients of each part divided by the other.

Let $x =$ one part, $5 - x =$ the other part,

$$\frac{x}{5-x} + \frac{5-x}{x} = \frac{17}{4};$$

$$\therefore\ \frac{x^2}{(5-x)^2} - \frac{17x}{4(5-x)} + \left(\frac{17}{8}\right)^2 = -1 + \frac{289}{64} = \frac{225}{64};$$

$$\therefore\ \frac{x}{5-x} = \frac{17}{8} \pm \frac{15}{8} = 4 \text{ or } \frac{1}{4};\ \therefore\ x = 4, \text{ and 1 is the other.}$$

(21). The sum of the squares of two numbers is 13001, and the difference of their squares is 1449. Required the numbers.

Let x and y be the numbers,

$$\text{then } x^2 + y^2 = 13001, \text{ and } x^2 - y^2 = 1449;$$

$$\therefore\ x^2 = \frac{11552}{2} = 5776;\ \therefore\ x = \pm 76,$$

$$\text{and } y^2 = 7225;\ \therefore\ y = \pm 85.$$

(22). Find three numbers in the proportion of $\frac{1}{2}$, $\frac{2}{3}$, and $\frac{3}{4}$, the sum of whose squares is 724.

Let $\frac{x}{2}$, $\frac{2x}{3}$, $\frac{3x}{4}$ be the numbers,

$$\text{then } \frac{x^2}{4} + \frac{4x^2}{9} + \frac{9x^2}{16} = 724,$$

$$\text{or } 36x^2 + 64x^2 + 81x^2 = 724 \times 144;$$

$$\therefore x^2 = \frac{724 \times 144}{181} = 144 \times 4;$$

$$\therefore x = \pm 24,$$

and the numbers are 12, 16, 18.

(23). What number added to the numerator and denominator of the fraction $\frac{2}{5}$ in succession, will make the resulting fraction in the former case $2\frac{4}{5}$ times as great as that in the latter?

Let x = the number,

$$\text{then } \frac{x+2}{5} = \frac{14}{5} \times \frac{2}{x+5}, \text{ or } x^2 + 7x + 10 = 28;$$

$$\therefore x^2 + 7x + \frac{49}{4} = \frac{121}{4}; \therefore x = -\frac{7}{2} \pm \frac{11}{2} = 2 \text{ or } -9.$$

(24). To find two numbers, such that the sum of their squares may be 89, and their sum multiplied by the greater may produce 104.

Let x and y be the numbers,

$$\text{then } x^2 + y^2 = 89, \text{ and } x(x+y) = 104;$$

$$\therefore y = \frac{104 - x^2}{x}, \text{ and } x^2 + \frac{(104 - x^2)^2}{x^2} = 89;$$

$$\therefore x^4 + 10816 - 208x^2 + x^4 = 89x^2;$$

$$\therefore x^4 - \frac{297x^2}{2} + \left(\frac{297}{4}\right)^2 = \frac{1681}{16};$$

$$\therefore x^2 = \frac{297}{4} \pm \frac{41}{4} = \frac{169}{2} \text{ or } 64;$$

$$\therefore x = \pm 8, \text{ and } y = \pm 5.$$

(25). A person bought a number of oxen for 112*l.*: if he had had two more for the money, each would have cost him 2*l.* 16*s.* less. Find the number.

Let x = the number; $\therefore \frac{112}{x}$ = price of each,

$$\text{but } \frac{112}{x+2} = \frac{112}{x} - 2\tfrac{4}{5}, \text{ whence } x^2 + 2x + 1 = 81;$$

$$\therefore x = -1 \pm 9 = 8 \text{ or } -10.$$

(26). The sum of two numbers is 16, and the quotient of the greater divided by the less is to the quotient of the less by the greater as 25 is to 9. Find them.

Let x = one number; $\therefore 16 - x$ = the other;

$$\therefore \frac{16-x}{x} : \frac{x}{16-x} :: 25 : 9;$$

$$\therefore 16 - x : x :: 5 : 3, \text{ or } 16 : x :: 8 : 3;$$

$$\therefore x = 6, \text{ and } 16 - 4 = 10 \text{ is the other number.}$$

(27). A party at a tavern owed 7*l.* 4*s.*; but in consequence of three of them having no money, each of the rest had to pay 4*s.* more than he otherwise would have done. Required their number.

Let x = the number, then $\frac{144}{x}$ = share of each,

$$\text{but } \frac{144}{x-3} = \frac{144}{x} + 4; \ \therefore x^2 - 3x = 108,$$

whence $x = 12$.

(28). What number is that, which being divided by the product of its two digits, the quotient is $5\frac{1}{3}$, but when 9 is subtracted from it, there remains a number having the same digits inverted?

Let $10x + y$ be the number,

$$\text{then } \frac{10x+y}{xy} = \frac{16}{3}, \text{ but } 10x + y - 9 = 10y + x,$$

$$\text{whence } y = \frac{30x}{16x-3} = x - 1, \text{ or } 16x^2 - 49x + 3 = 0,$$

$$\text{and } 4x = \frac{49}{8} \pm \frac{47}{8} = 12 \text{ or } \frac{1}{4};$$

$$\therefore\ x = 3, \text{ and } y = 2,$$

and the number is 32.

(29). A draper bought a number of pieces of cloth for 33*l.* 15*s.* which he sold at 2*l.* 8*s.* a piece, and gained as much as one piece cost him. How many pieces were there?

$$\text{Let } x = \text{number of pieces}; \ \therefore\ \frac{33\frac{3}{4}}{x} = \text{price of each},$$

$$\text{then } x \times 2\tfrac{2}{5} = 33\tfrac{3}{4} + \frac{33\frac{3}{4}}{x},$$

$$\text{whence } x = \frac{225}{32} \pm \frac{255}{32} = 15.$$

(30). What two fractions are those whose sum is 1, and the greater divided by the less gives the quotient 10?

$$\text{Let } x \text{ and } y \text{ be the fractions, then } x + y = 1,$$

$$\frac{x}{y} = 10; \ \therefore\ 10y + y = 1, \text{ and } y = \frac{1}{11}; \ \therefore\ x = \frac{10}{11}.$$

(31). *A* and *B* distribute 1200*l.* each among a number of persons; *A* gives to 40 more than *B*, and *B* gives 5*l.* a piece to each more than *A*. Find the numbers.

$$\text{Let } x \text{ and } x - 40 \text{ be the numbers},$$

$$\text{and } y - 5 \text{ and } y \text{ the sums given by each},$$

$$\text{then } x \times (y - 5) = (x - 40) \times y; \ \therefore\ x = 8y,$$

$$\text{also } 8y(y - 5) = 1200,$$

$$\text{whence } y = \frac{5}{2} \pm \frac{25}{2} = 15, \text{ and } x = 120;$$

$\therefore$ the numbers are 120 and 80.

(32). Divide the number 49 into two such parts, that the quotient of the greater divided by the less may be to the quotient of the less divided by the greater as $\frac{4}{3}$ to $\frac{3}{4}$.

Let x and $49 - x$ be the parts,

$$\text{then } \frac{49 - x}{x} : \frac{x}{49 - x} :: \frac{4}{3} : \frac{3}{4} :: 16 : 9,$$

$$\text{or } 49 - x : x :: 4 : 3; \ \therefore\ 49 : x :: 7 : 3;$$

$\therefore\ x = 21$ is one number, and $49 - 21 = 28$ is the other.

(33). A vintner sold 7 dozen of sherry and 12 dozen of claret for 50*l.*, and he sold 3 dozen more of sherry for 10*l.* than of claret for 6*l.* Find the price of each.

Let £x = price of sherry, £y = price of claret;

$$\therefore\ 7x + 12y = 50, \text{ also } \frac{10}{x} - \frac{6}{y} = 3, \text{ or } 10y - 6x = 3xy,$$

$$\text{whence } x = \frac{50 - 12y}{7} = \frac{10y}{3y + 6}, \text{ and } y = \frac{1}{9} \pm \frac{26}{9} = 3;$$

$\therefore$ the claret is 3*l.* per dozen and the sherry 2*l.*

(34). A detachment of soldiers from a regiment being ordered to march on a particular service, each company furnished four times as many men as there were companies in the regiment; but these being found to be insufficient, each company furnished 3 more men; when their number was found to be increased in the ratio of 17 to 16. How many companies were there in the regiment?

Let x = number of companies,

$4x^2$ = number of men first furnished,

$$\text{but } 4x^2 + 3x : 4x^2 :: 17 : 16;$$

$$\therefore\ 3x : 4x^2 :: 1 : 16,$$

$$\text{or } 3 : x :: 1 : 4; \ \therefore\ x = 12.$$

(35). A person bought silk for 240*l.*, and keeping 10 yards he sells the remainder for 205*l.*, thus gaining 2*s.* a yard upon the prime cost. Find the quantity bought.

Let x = number of yards bought;

$$\therefore\ \frac{240}{x}\,l. = \text{cost price, and } \frac{205}{x - 10} = \text{selling price,}$$

$$\frac{205}{x - 10} = \frac{240}{x} + \frac{1}{10},$$

$$\text{whence } 2050x = 2400x - 24000 + x^2 - 10x,$$

$$\text{and } x^2 + 340x + (170)^2 = 52900,$$

$$\text{and } x = -170 \pm 230 = 60 \text{ yards}.$$

(36). A charitable person distributed a certain sum amongst some poor men and women, the numbers of whom were in the proportion of 4 to 5. Each man received one-third of as many shillings as there were persons relieved; and each woman received twice as many shillings as there were women more than men. Now, the men received all together 18*s.* more than the women. How many were there of each?

Let $4x$ and $5x$ be the numbers respectively of men and women;

$$\therefore\ 9x = \text{number relieved},$$

each man received $3x$*s.*, each woman $2x$*s.*,

$$\text{then } 4x \times 3x - 5x \times 2x = 18, \text{ or } x^2 = 9;\ \therefore\ x = 3,$$

and there were 12 men and 15 women.

(37). *A* and *B* had 100 eggs, for which they received equal sums; had *A* sold as many as *B* he would have received 18*d.*, and had *B* sold as many as *A* he would have only got 8*d.* How many had each?

Let x = number *A* had; $\therefore$ $100 - x$ = number *B* had,

and y*d.* = price of *A*'s eggs, z = price of *B*'s,

$$\text{then } xy = (100 - x)z, \text{ also } (100 - x)y = 18, \text{ and } xz = 8,$$

$$\text{whence } z = \frac{xy}{100 - x} = \frac{8}{x}, \text{ and } y = \frac{18}{100 - x},$$

$$\text{and } x^2y = 800 - 8x;\ \therefore\ y = \frac{8(100 - x)}{x^2},$$

$$\text{whence } 9x^2 = 4(100 - x)^2;\ \therefore\ 3x = 2(100 - x),$$

and $x = 40$, and *B*'s number = 60.

(38). What number is that which, being added to its square three times, shall make the sum 70?

Let x = the number, then $x^2 + 3x = 70$,

$$\text{or } x^2 + 3x + \frac{9}{4} = \frac{289}{4};\ \therefore\ x = -\frac{3}{2} \pm \frac{17}{2} = 7 \text{ or } -10.$$

(39). A grazier bought a number of sheep for 75*l.*, and after losing two he sold the rest for 2*s.* 6*d.* each more than they cost, and gained the prime cost of two sheep. Find the number.

$$\text{Let } x = \text{number of sheep}; \;\therefore\; \frac{75}{x}\,\pounds. = \text{price of each,}$$

$$\text{and } \left(\frac{75}{x} + \frac{1}{8}\right) = \text{selling price,}$$

$$\text{then } (x-2)\left(\frac{75}{x} + \frac{1}{8}\right) = 75 + \frac{150}{x},$$

$$\text{whence } x^2 - 2x + 1 = 2401; \;\therefore\; x = 1 \pm 49 = 50 \text{ sheep.}$$

(40). Required that number which added to its cube shall make the sum 68.

$$\text{Let } x = \text{the number, then } x^3 + x = 68,$$

$$\text{and } x^3 - 64 + x - 4 = 0; \;\therefore\; x = 4,$$

$$\text{also } x^2 + 4x + 16 + 1 = 0;$$

$$\therefore\; x^2 + 4x + 4 = -17 + 4 = -13; \;\therefore\; x = -2 \pm \sqrt{(-13)}.$$

(41). Two sides of a rectangular plot of ground are as 1 : 3; and if the less side be diminished by 1 yard, and the greater be increased by 28 yards, the plot will be doubled. Find the sides.

$$\text{Let } x \text{ and } 3x \text{ be the sides,}$$

$$\text{then } (x-1)(3x+28) = 2 \times 3x^2,$$

$$\text{whence } x^2 - \frac{25x}{3} + \left(\frac{25}{6}\right)^2 = \frac{289}{36}; \;\therefore\; x = \frac{25}{6} \pm \frac{17}{6} = 7,$$

and the sides are 7 and 21 yards.

(42). A merchant ventured a certain sum upon a speculation, and found at the end of a year that he had gained 69*l.* This being added to his stock, at the end of another year he found he had gained exactly as much per cent. as in the year preceding. Proceeding in the same manner, each year adding to his stock the gain of the year preceding, he found at the beginning of the fifth year that his stock was to the original stock as 81 to 16. What was the sum he first laid out?

$$\text{Let } x = \text{his stock}; \ \therefore \text{ gain per cent.} = \frac{6900}{x};$$

$$\therefore x + 69 = \text{2nd year's stock};$$

$$\therefore x + 69 + (x + 69)\frac{69}{x} = \frac{(x + 69)^2}{x} = \text{3rd year's stock};$$

$$\therefore \frac{(x + 69)^4}{x^3} : x :: 81 : 16; \ \therefore x + 69 : x :: 3 : 2;$$

$$\therefore x = 138l.$$

(43). A person bought a number of 50*l.* railway shares for 900*l.*, when they were at a certain discount; and afterwards, when they were at the same premium, sold all but 10 for 550*l.* What number did he buy, and what did he give for each?

$$\text{Let } x = \text{number of shares},$$

$$\frac{900}{x} = \text{price of each, but } \frac{550}{x - 10} = \text{selling price};$$

$$\therefore 50 - \frac{900}{x} = \text{discount per share},$$

$$\text{then } \frac{550}{x - 10} = 50 + 50 - \frac{900}{x} = \frac{100x - 900}{x},$$

$$\text{whence } 2x^2 - 49x = -180, \text{ and } x = \frac{49}{4} \pm \frac{31}{4} = 20 \text{ shares},$$

$$\text{and price} = \frac{900}{20} = 45l.$$

(44). The sum of the squares of two numbers being expressed by a, and the difference of their squares by b, it is required to express the two numbers in terms of a, b.

Let x and y be the numbers, then $x^2 + y^2 = a$, and $x^2 - y^2 = b$;

$$\therefore x^2 = \frac{a + b}{2}, \text{ and } x = \pm\sqrt{\left(\frac{a + b}{2}\right)}, \text{ and } y = \pm\sqrt{\left(\frac{a - b}{2}\right)}.$$

(45). The daily receipts on a railway are 1750*l.*: on three of the trains being taken off, the receipts per train increase by 75*l.*, the total daily receipts remaining the same as before. How many trains now run?

Let x = original number of trains; $\therefore \frac{1750}{x}$ = receipt per train;

$$\text{then } (x-3)\left(\frac{1750}{x}+75\right)=1750,$$

$$\text{whence } x^2-3x=70, \text{ and } x=\frac{3}{2}\pm\frac{17}{2}=10;$$

$\therefore$ number remaining = 7 trains.

(46). A detachment from an army was marching in regular column, with 5 men more in depth than in front: but upon the enemy coming in sight, the front was increased by 845 men; and by this movement the detachment was drawn up in 5 lines. Required the number of men.

Let x = men in front, $x+5$ = men in depth,

$$\text{but } (x+845)\times 5 = x\times(x+5),$$

$$\text{or } x^2=4225; \therefore x=65,$$

and number of men $65\times 70 = 4550$ men.

(47). A sets out to walk to a town $10\frac{1}{2}$ miles distant; and 30 minutes afterwards B is sent after him, overtakes him, and then returns to the place they started from at the same time that A reaches the town. If B walk 4 miles an hour, find A's rate.

Let x = A's speed per hour; $\therefore \frac{x}{2}$ = distance in $\frac{1}{2}$ an hour,

and $4-x$ = difference of speed,

and y = hours before B overtakes A,

$$\text{then } 4y=\frac{x}{2}+xy, \text{ or } y=\frac{x}{2(4-x)};$$

$$\therefore \text{distance by } A=\frac{2x}{4-x};$$

$$\therefore \text{remaining distance}=\frac{21}{2}-\frac{2x}{4-x}=\frac{84-25x}{2(4-x)},$$

then since remaining times must be equal,

$$\frac{84-25x}{2x(4-x)} = \frac{2x}{(4-x)\times 4}, \text{ or } 84-25x = x^2;$$

$$\therefore x^2 + 25x + \left(\frac{25}{2}\right)^2 = \frac{961}{4};$$

$$\therefore x = -\frac{25}{2} \pm \frac{31}{2} = 3 \text{ miles per hour for } A.$$

(48). A number consisting of two digits, being multiplied by the digit on the left hand, produces 46; but the sum of the digits multiplied by the same digit produces only 10. Required the number.

Let $10x + y$ be the number,

then $x(10x + y) = 46$, and $x(x + y) = 10$;

$\therefore 9x^2 = 36$, or $x = \pm 2$, and $y = 3$,

and the number is 23.

(49). If two numbers are to each other as 3 to 4, and the sum of their squares is 324900; what are the numbers?

Let $3x$ and $4x$ be the numbers,

then $9x^2 + 16x^2 = 324900$, or $x^2 = 12996$, or $x = 114$,

and the numbers are 342 and 456.

(50). A vintner draws a certain quantity of wine out of a full vessel that holds 256 gallons; and then filling the vessel with water, draws off the same quantity of liquor as before, and so on, for four draughts, when there were only 81 gallons of pure wine left. How much wine did he draw each time?

Let x = gallons of wine drawn first time;

$\therefore 256 - x$ = quantity left in,

then 2nd draw : x :: $256 - x$: 256;

$$\therefore \text{2nd draw} = \frac{256-x}{6} \times \frac{x}{256},$$

$$\text{and quantity of wine left in} = \frac{(256-x)^2}{256},$$

$$3\text{rd draw} : x :: \frac{(256-x)^2}{256} :: 256;\ \therefore 3\text{rd draw} = \frac{(256-x)^2 \times x}{(256)^2},$$

$$\text{and quantity left in} = \frac{(256-x)^3}{(256)^2},$$

$$\text{and after 4th draw, quantity left in} = \frac{(256-x)^4}{(256)^3};$$

$$\therefore (256-x)^4 = (256)^3\, 81 = 4^{12} \times 81;$$

$$\therefore 256 - x = 64 \times 3, \text{ or } x = 64 \text{ gallons first draw,}$$

$$\text{and } \frac{256-64}{1} \times \frac{64}{256} = 48, \text{ 2nd draw,}$$

and 36, the 3rd draw, 27 gallons the 4th draw.

(51). From two towns, *C* and *D*, being distant 396 miles, two persons, *A* and *B*, setting out at the same time, met each other, after travelling as many days as are equal to the difference of the number of miles they travelled per day, and it appeared that *A* had travelled 216 miles. How many miles did each travel per day?

$$\text{Let } x = \text{miles per day by } A,\ y = \text{miles by } B;$$

$$\therefore x(x-y) = 216, \text{ and } y(x-y) = 180;$$

$$\therefore x^2 - 2xy + y^2 = 36, \text{ and } x - y = 6;$$

$$\therefore x = \frac{216}{6} = 36, \text{ and } y = \frac{180}{6} = 30 \text{ miles.}$$

(52). The sum of the squares of two numbers is *b*, and the first number is to the second as *m* is to *n*. What are the numbers?

Let x and y be the numbers,

$$\text{then } x^2 + y^2 = b, \text{ and } x : y :: m : n;\ \therefore y = \frac{nx}{m};$$

$$\therefore x^2 + \frac{n^2x^2}{m^2} = b;\ \therefore x = \pm m\sqrt{\left(\frac{b}{m^2+n^2}\right)}, \text{ and } y = n\sqrt{\left(\frac{b}{m^2+\ \ ^2}\right)}.$$

(53). There is a number consisting of two digits, which, w en divided by the sum of its digits, gives a quotient greater by 2 than the first digit; but if the digits be reversed, and then divided by a number greater by unity than the sum of the digits, the quotient is greater by 2 than the preceding quotient. Required the number.

Let $10x + y$ be the number,

$$\text{then } \frac{10x + y}{x + y} = x + 2, \text{ also } \frac{10y + x}{x + y + 1} = x + 4;$$

$$\therefore y = \frac{8x - x^2}{x + 1} = \frac{x^2 + 4x + 4}{6 - x}, \text{ whence } 19x^2 - 40x = -4,$$

and $x = 2$, and $y = 4$, and the number is 24.

(54). The difference between the hypothenuse and base of a right-angled triangle is 6, and the difference between the hypothenuse and the perpendicular is 3. What are the sides?

Let x = the hypothenuse;

$\therefore\ x - 6$ = base, and $x - 3$ = perpendicular;

$$\therefore\ x^2 = (x - 6)^2 + (x - 3)^2, \text{ or } 36 - 12x + x^2 - 6x + 9 = 0;$$

$$\therefore\ x^2 - 18x + 81 = 36, \text{ and } x = 9 \pm 6 = 15 \text{ or } 3;$$

$\therefore$ the sides are 15, 9, 12.

(55). There are two numbers such that the sum of the products of the first by 4, and of the second by 3, is 53, and the difference of their squares is 15. Required the two numbers.

Let x and y be the numbers, then $4x + 3y = 53$, and $x^2 - y^2 = 15$,

$$\text{whence } x = \frac{53 - 3y}{4}, \text{ or } 2809 - 318y + 9y^2 - 16y^2 = 240;$$

$$\therefore y^2 + \frac{318y}{7} = \frac{2569}{7}, \text{ whence } y = -\frac{159}{7} \pm \frac{208}{7} = 7, \text{ and } x = 8.$$

(56). In a parcel containing 24 coins of silver and copper, each silver coin is worth as many pence as there are copper coins, and each copper coin is worth as many pence as there are silver coins, and the whole is worth 18 shillings. How many are there of each?

Let x = number of silver coins;

$\therefore\ 24 - x$ = number of copper coins,

$$\text{then } 2x(24 - x) = 216, \text{ or } x^2 - 24x = -108;$$

$$\therefore\ x = 12 \pm 6 = 18 \text{ or } 6.$$

(57). A number consisting of three digits, which are in arithmetical progression, being divided by the sum of its digits, gives a quotient 48; and if 198 be subtracted from it, the digits will be inverted. Required the number.

Let $x+y$, x and $x-y$ be the digits; $\therefore$ their sum $=3x$,

$$\text{then } \frac{100(x+y)+10x+x-y}{3x}=48, \text{ whence } x=3y,$$

$$\text{but } 100(x-y)+10x+x+y=100(x+y)+10x+x-y-198$$

$$\text{whence } \because x=3y, \text{ we have } 198y=198, \text{ or } y=1,$$

and the number is 432.

(58). A horse-dealer pays a certain sum for a horse, which he afterwards sells for 144*l.*, and gains exactly as much per cent. as the horse cost him. How much did the horse cost?

Let x = price of horse; $\therefore x : 100 :: 144 : 100+x$;

$$\therefore x^2+100x+2500=16900; \therefore x=-50\pm130=80l.$$

(59). A farmer received 7*l.* 4*s.* for certain bushels of wheat, and an equal sum at a price less by 1*s.* 6*d.* per bushel for a quantity of barley, which exceeded the quantity of wheat by 16 bushels. How many bushels were there of each?

Let x = bushels of wheat, and $x+16$ = bushels of barley,

and y*s.* = price of wheat per bushel;

$$\therefore xy = \text{price of wheat, and } (x+16)\left(y-\frac{3}{2}\right)=xy=144s.;$$

$$\therefore xy-\frac{3x}{2}+16y-24=xy, \text{ or } x=\frac{32y-48}{3}=\frac{144}{y};$$

$$\therefore \text{whence } y=\frac{3}{4}\pm\frac{15}{4}=\frac{9}{2} \text{ or } -3,$$

and $x=32$, and 48 = the bushels of barley.

(60). During a scarcity, a person wished to make a mixture of 24 bushels, consisting of wheat, oats, and barley, the quantities of each forming an increasing arithmetical progression. Not being able, however, to procure any barley, he mixed additional quantities of wheat and oats, in proportion of 2 to 3, so as to complete his 24 bushels, when he found the whole quantities of wheat and oats to be in proportion of 5 to 7. How many bushels of each did he originally intend to mix?

Let $x - y$, x, and $x + y$ be the quantities of each;

$$\therefore\ 3x = 24, \text{ and } x = 8,$$

let $2z$ and $3z$ be the quantities of wheat and oats added,

$$\text{then } \frac{8 - y + 2z}{8 + 3z} = \frac{5}{7}, \text{ whence } z = 16 - 7y,$$

$$\text{but } 8 - y + 2z + 8 + 3z = 24;\ \therefore\ z = \frac{8 + y}{5};$$

$$\therefore\ 80 - 35y = 8 + y, \text{ and } y = 2,$$

and the quantities were 6, 8, 10 bushels each of wheat, oats, and barley.

(61). To divide 20 into three parts, such that the continual product of all three may be 270, and that the difference of the first and second may be 2 less than the difference of the second and third.

Let x, y, and z be the parts; $\therefore\ x + y + z = 20,\ xyz = 270,$

$$\text{and } y - x + 2 = z - y;\ \therefore\ x = 2y + 2 - z;$$

$$\therefore\ 20 - y - z = 2y + 2 - z;\ \therefore\ y = \frac{18}{3} = 6;$$

$$\therefore\ x + z = 14, \text{ and } xz = 45;$$

$$\therefore\ x^2 - 2xz + z^2 = 16;\ \therefore\ x - z = \pm 4;$$

$$\therefore\ x = 9 \text{ or } 5, \text{ and } z = 5 \text{ or } 9,$$

and the parts are 5, 6, 9.

(62). Two messengers, *A* and *B*, being dispatched at the same time to a place 90 miles distant, *A* rode one mile an hour more than *B*, and arrived at the end of his journey an hour before him. At what rate did each travel per hour?

Let $x + 1 = A$'s rate per hour; $\therefore\ x = B$'s, and $y =$ hours by *A*;

$$\therefore\ \frac{90}{x + 1} = A\text{'s time},\ \frac{90}{x} = B\text{'s time},$$

$$\text{then } \frac{90}{x+1} = \frac{90}{x} - 1; \therefore x = -\frac{1}{2} \pm \frac{19}{2} = 9 \text{ miles per hour for } B,$$

and 10 miles for A.

(63). The difference between the first and second of four numbers in geometrical progression is 36, and the difference between the third and fourth is 4. What are the numbers?

Let xy^{-1}, x, xy, and xy^2 be the numbers,

then $xy^{-1} - x = 36$, and $xy - xy^2 = 4$,

$$\text{whence } x = \frac{36y}{1-y} = \frac{4}{y(1-y)}, \text{ or } y = \frac{1}{3};$$

$\therefore x = 18$, and the numbers are 54, 18, 6, 2.

(64). To find three numbers in arithmetical progression, such that the sum of their squares may be 56, and the sum arising by adding together thrice the first, and twice the second, and thrice the third, may amount to 32.

Let $x - y$, x, and $x + y$ be the numbers,

then $(x-y)^2 + x^2 + (x+y)^2 = 3x^2 + 2y^2 = 56$,

and $3(x-y) + 2x + 3(x+y) = 8x = 32$; $\therefore x = 4$,

and $2y^2 = 56 - 48 = 8$; $\therefore y = \pm 2$,

and the series is 2, 4, 6.

(65). Bought a number of books, consisting of folios, quartos, and octavos, for 96*l.* 12*s.* Fourteen folios (which was the whole number) cost three times as much as all the quartos; and one quarto cost as many shillings as there were quartos. The number of octavos was 32, and their value was such, that 4 of them cost as much as one quarto. Required the value of each, and the number of quartos.

Let x = number of quartos; $\therefore x^2$ = cost of quartos in shillings,

$3x^2$ = cost of folios, and $8x$ = cost of octavos;

$\therefore 4x^2 + 8x = 1932$;

$\therefore x^2 + 2x + 1 = 483 + 1 = 484$,

and $x = -1 \pm 22 = 21$, the number of quartos,

and 21*s.* price of each;

$$\therefore \text{ cost of folio} = \frac{3 \times 21 \times 21}{14 \times 21} \text{ guineas} = 4\tfrac{1}{2} \text{ guineas each,}$$

$$\text{cost of each octavo} = \frac{21}{4} = 5\tfrac{1}{4}s. \text{ each.}$$

(66). To find two numbers, such that their product added to their sum may make 47, and their sum taken from the sum of their squares may leave 62.

Let x and y be the numbers,

$$\text{then } x + y + xy = 47, \text{ or } 2xy = 94 - 2(x + y),$$

$$\text{and } (x^2 + y^2) - (x + y) = 62;$$

$$\therefore (x+y)^2 + (x+y) + \frac{1}{4} = 156 + \frac{1}{4}, \text{ or } x + y = 12; \therefore xy = 35,$$

$$\text{whence } x - y = 2, \text{ and } x = 7 \text{ and } y = 5.$$

(67). To find three numbers having equal differences, and such that the square of the least added to the product of the two greater may make 28, but the square of the greatest added to the product of the two less may make 44.

Let $x - y$, x, and $x + y$ be the numbers,

$$\text{then } (x - y)^2 + x(x + y) = 28, \text{ and } (x + y)^2 + x(x - y) = 44,$$

$$\text{by subtraction } xy = 8;$$

$$\therefore \left(x - \frac{8}{x}\right)^2 + x\left(x + \frac{8}{x}\right) = 28, \text{ or } x^4 - 18x^2 = -32,$$

$$\text{whence } x^2 = 9 \pm 7 = 16, \text{ and } x = 4 \text{ and } y = 2;$$

$$\therefore \text{ the numbers are } 2, 4, 6.$$

(68). The paving of two square courtyards cost 205*l.*, a yard of each costing one-fourth of as many shillings as there were yards in a side of the other; and a side of the greater and less together measure 41 yards. Required the length of a side of each.

Let x = yards in side of one, and $41 - x$ = in side of other,

$$\text{then } \frac{x^2(41 - x)}{4} + (41 - x)^2 \frac{x}{4} = 205 \times 20,$$

$$\text{whence } x^2 - 41x + \left(\frac{41}{2}\right)^2 = -\frac{81}{4}; \therefore x = 25 \text{ or } 16.$$

(69). Three merchants, A, B, C, on comparing their gains, find that among them all they have gained 1444*l*., and that B's gain added to the square root of A's made 920*l*., but if added to the square root of C's it made 912*l*. What were their several gains?

Let $x^2 = A$'s gain; $\therefore$ B's gain $= 920 - x$, and $y^2 = C$'s gain;

$$\therefore B\text{'s gain} = 912 - y;$$

$$\therefore x - y = 8, \text{ but } x^2 + 920 - x + y^2 = 1444,$$

$$\text{whence } y = -\frac{15}{4} \pm \frac{63}{4} = 12, \text{ and } x = 20;$$

$\therefore$ A's gain is 400*l*., B's gain 900*l*., and C's gain 144*l*.

(70). A person buying a number of apples and pears, amounting together to 80, gave twice as much for the apples as pears; but had he bought as many apples as he did pears, and as many pears as he did apples, his apples would have cost 10*d*., and his pears 3*s*. 9*d*. How many did he buy of each?

Let x = number of apples, and y = price of each,

$80 - x$ = number of pears, and z = price of each,

then $xy = 2(80 - x)z$, also $(80 - x)y = 10$, and $xz = 45$,

$$\text{whence } x^2y = 2(80 - x)45, \text{ and } y = \frac{10}{80 - x},$$

$$\text{and } x^2 = 9(80 - x)^2; \ \therefore x = 3(80 - x),$$

or $x = 60$ number of apples, and $80 - 60 = 20$ number of pears.

(71). To find three numbers in arithmetical progression, so that the sum of their squares shall be 93; also if the first be multiplied by 3, the second by 4, and the third by 5, the sum of the products may be 66.

Let $x - y$, x, and $x + y$ be the numbers,

then $3x^2 + 2y^2 = 93$, and $3(x - y) + 4x + 5(x + y) = 66$,

or $12x + 2y = 66$; $\therefore$ $y = 33 - 6x$, and $3x^2 + 2(33 - 6x)^2 = 93$,

$$\text{whence } 75x^2 - 792x + 2085 = 0, \text{ or } x = \frac{132}{25} \pm \frac{7}{25} = 5;$$

$\therefore$ $y = 3$, and the numbers are 2, 5, 8.

(72). The sum of two numbers is 2, and their product is also 2· what are they? Also find two numbers whose sum is a and whose product is b^2.

Let x = one number, $2 - x$ = the other, or $2x - x^2 = 2$;

$\therefore\ x^2 - 2x + 1 = -1$, or $x = 1 \pm \surd(-1)$ is one number,

and $1 \mp \surd(-1)$ is the other.

Let x and $a - x$ be the numbers, and $ax - x^2 = b^2$;

$$\therefore\ x^2 - ax + \frac{a^2}{4} = \frac{a^2}{4} - b^2;\ \therefore\ x = \frac{a \pm \surd(a^2 - 4b^2)}{2},$$

and the other is $\dfrac{a \mp \surd(a^2 - 4b^2)}{2}$.

(73). A person exchanged a quantity of brandy for a quantity of rum and 11*l.* 5*s.*, the brandy and rum being each valued at as many shillings per gallon as there were gallons of that liquor; but had the rum been worth as many shillings per gallon as the brandy was, the whole value of the rum and brandy would have been 56*l.* 5*s.* How many gallons were there of each?

Let x = gallons of brandy, y = gallons of rum;

$\therefore\ x^2$ and y^2 are their prices;

$$\therefore\ x^2 - y^2 = 225, \text{ but } xy + x^2 = 1125, \text{ or } y = \frac{1125 - x^2}{x};$$

$$\therefore\ x^4 - (1125 - x^2)^2 = 225x^2,$$

whence $x = 25$ gallons of brandy at 25*s.*,

and $y = 20$ gallons of rum at 20*s.* per gallon.

(74). There are two rectangular vats, the greater of which contains 20 solid feet more than the other; their capacities are in the ratio of 4 to 5, and their bases are squares, a side of each of which is equal to the depth of the other. What are the depths?

Let x^2 = base of one, and y^2 = base of the other in feet,

then $x^2y - y^2x = 20$, also $x^2y : y^2x :: 5 : 4$; $\therefore\ x = \dfrac{5y}{4}$,

$$\text{and } \frac{25y^3}{16} - \frac{5y^3}{4} = 20;$$

$$\therefore\ y = 4 \text{ and } x = 5 \text{ are their respective depths.}$$

(75). A ship containing 74 sailors, and a certain number of soldiers, besides officers, took a prize. The sailors received each one-third as many pounds as there were soldiers, and the soldiers, received 3*l.* a-piece less, and 768*l.* fell to the share of the officers. Had the officers, however, received nothing, the soldiers and sailors might have received half as many pounds per man, as there were soldiers. How many soldiers were there, and how much did each receive?

Let $3x$ = number of soldiers; $\therefore$ x = what each sailor received,

and $x - 3$ what each soldier received;

$$\therefore\ 74x + 3x(x-3) + 768 = 74 \times \frac{3x}{2} + 3x \times \frac{3x}{2},$$

$$\text{whence } x = -\frac{46}{3} \pm \frac{82}{3} = 12l. \text{ what each sailor received,}$$

and $3x = 36$ number of soldiers, and $12 - 3 = 9l.$ each.

(76). A person employed three workmen, whose daily wages were in arithmetical progression. The number of days they worked was equal to the number of shillings that the second received per day. The whole amount of their wages was seven guineas, and the best workman received 28 shillings more than the worst. What were their daily wages?

Let $x - y$, x, and $x + y$ be the daily wages of each;

$\therefore$ x = shillings per day,

then $(x - y)x + x^2 + x(x + y) = 147$, or $3x^2 = 147$ or $x = 7$,

also $(x + y)x = x(x - y) + 28$; $\therefore$ $2xy = 28$, or $y = 2$;

$\therefore$ their wages were respectively 5, 7, and 9*s.* per day.

(77). The sum of 700*l.* was divided among four persons, whose shares were in geometrical progression; and the difference between the greatest and least was to the difference between the means as 37 to 12. What were their respective shares?

Let xr^{-1}, x, xr, and xr^2 be their respective shares,

$$\text{then } xr^2 - xr^{-1} : xr - x :: 37 : 12,$$

$$\text{or } r^3 - 1 : (r-1) \times r :: 37 : 12,$$

$$\text{or } r^2 + r + 1 : r : 37 : 12; \therefore (r+1)^2 : (r-1)^2 :: 49 : 1,$$

$$\text{whence } r + 1 = 7(r-1), \text{ or } r = \frac{4}{3},$$

$$\text{and } x\left(\frac{3}{4} + 1 + \frac{4}{3} + \frac{16}{9}\right) = 700; \therefore x = 144l.,$$

and the other shares are 108*l*., 192*l*., and 256*l*.

(78). A poulterer going to market to buy turkeys, met with four flocks. In the second were 6 more than three times the square root of double the number in the first; the third contained three times as many as the first and second; and the fourth contained 6 more than the square of one-third of the number in the third; and the whole number was 1938. How many were there in each flock?

Let x = number in first; $\therefore$ $3\sqrt{(2x)} + 6$ = number in second,

$$3\{x + 3\sqrt{(2x)} + 6\} = \text{number in third},$$

$$\text{and } \{x + 3\sqrt{(2x)} + 6\}^2 + 6 = \text{number in fourth};$$

$$\therefore \{x + 3\sqrt{(2x)} + 6\}^2 + 4\{x + 3\sqrt{(2x)} + 6\} + 4 = 1936,$$

$$\text{or } x + 3\sqrt{(2x)} + 6 = -2 \pm 44 = 42,$$

$$\sqrt{x} = -\frac{3\sqrt{2}}{2} \pm \frac{9\sqrt{2}}{2} = \frac{6}{\sqrt{2}}; \therefore x = 18, \text{ number in first flock,}$$

$3 \times 6 + 6 = 24$, number in second, 126 in third, 1770 in fourth.

(79). There are three numbers in geometrical progression, the sum of the first and second of which is 9, and the sum of the first and third is 15. Required the numbers.

Let xy^{-1}, x, and xy be the numbers, then $xy^{-1} + x = 9$,

$$\text{and } xy^{-1} + xy = 15; \therefore \frac{y+1}{y^2+1} = \frac{9}{15},$$

$$\text{whence } y = \frac{5}{6} \pm \frac{7}{6} = 2, \text{ and } x = 6,$$

and the numbers are 3, 6, 12.

(80). There are four numbers in arithmetical progression: the sum of the squares of the first and second is 34, and the sum of the squares of the third and fourth is 130. Required the numbers.

Let $x - 3y$, $x - y$, $x + y$, and $x + 3y$ be the numbers,

$$\text{then } (x-3y)^2 + (x-y)^2, \text{ or } 2x^2 - 8xy + 10y^2 = 34,$$

$$\text{also } (x+3y)^2 + (x+y)^2, \text{ or } 2x^2 + 8xy + 10y^2 = 130; \therefore xy = 6;$$

$$\therefore x^2 + 5y^2 = 41, \text{ and } x^4 + 180 = 41x^2,$$

$$\text{whence } x^2 = \frac{41}{2} \pm \frac{31}{2} = 36, \text{ and } x = 6; \therefore y = 1;$$

$\therefore$ the numbers are 3, 5, 7, 9.

(81). There are three numbers, the difference of whose differences is 5, their sum is 44, and continued product 1950. What are the numbers?

Let x, y, z be the numbers,

then their differences are $x - y$, and $y - z$,

$$\text{and } x - y - (y - z) = 5, \text{ and } x + y + z = 44, \text{ also } xyz = 1950;$$

$$\therefore x = 5 + 2y - z = 44 - (y + z) = \frac{1950}{yz},$$

$$\text{whence } y = 13, \text{ and } 5 - z + 26 = \frac{150}{z};$$

$$\therefore z = \frac{31}{2} \pm \frac{19}{2} = 25, \text{ and } x = 6.$$

(82). There are three numbers in geometrical progression, whose continued product is 64, and the sum of their cubes is 584. Required the numbers.

Let xy^{-1}, x, and xy be the numbers; $\therefore x^3 = 64$, and $x = 4$,

$$\text{then } 64\left(\frac{1}{y^3} + 1 + y^3\right) = 584, \text{ and } y^3 = \frac{65}{16} \pm \frac{63}{16} = 8; \therefore y = 2,$$

and the series is 2, 4, 8,

(83). A farmer buys m sheep for £p, and sells n of them at 5 per cent. profit. How must he sell the remainder so as to gain 10*l.* per cent. on the whole?

$$\text{Price of each sheep} = \frac{p}{m}\,£;$$

$$\therefore\ n \times \frac{p}{m} + \frac{np}{20m} = \frac{21np}{20m} \text{ price sold for,}$$

$$\text{amount required} = p + \frac{p}{10} = \frac{11p}{10}\,£,$$

and if $£x$ be the price of each remaining, then

$$\frac{21np}{20m} + (m - n)\,x = \frac{11p}{10};\ \therefore\ x = \frac{22m - 21n}{20m\,(m - n)} \times £p.$$

(84). There are three numbers in geometrical progression, whose sum is 14, and the sum of the first and second is to the sum of the second and third as 1 to 2. Required the numbers.

Let xy^{-1}, x, and xy be the numbers, then $x\,(1 + y + y^2) = 14y$,

$$\text{also } \frac{x}{y}\,(1 + y) : x\,(1 + y), \text{ or } 1 : y :: 1 : 2;\ \therefore\ y = 2,$$

$$\text{and } x = \frac{28}{7} = 4, \text{ and the numbers are } 2, 4, 8.$$

(85). A and B engaged to reap equal quantities of wheat, and A began half an hour before B. They stopped at 12 o'clock and rested an hour, observing that just half the work was done. B's part was finished at 7 o'clock, and A's at a quarter before ten. Supposing them to have laboured uniformly, at what time did each commence?

Let x = the hours B laboured before 12 o'clock;

and if A be the work of each, then

$$\frac{A}{x + 6} = B\text{'s hourly work, and } \frac{A}{x + 9\frac{1}{4}} = A\text{'s hourly work;}$$

$$\text{but } \frac{A}{x + 6}.x + \frac{4A}{4x + 37}.\left(x + \frac{1}{2}\right) = A$$

(or A's hourly work and B's hourly work) multiplied by the number of hours before $12 = \frac{1}{2}$ work);

$$\therefore\ 4x^2 + 37x + 4x^2 + 26x + 12 = 4x^2 + 61x + 222;$$

$$\therefore\ 4x^2 + 2x + \frac{1}{4} = 210 + \frac{1}{4} = \frac{841}{4},$$

$$2x = -\frac{1}{2} \pm \frac{29}{2} = 14;\ \therefore\ x = 7;$$

B started at 5 o'clock, and *A* at half-past 4 o'clock.

(86). A butcher bought a certain number of calves and sheep, and for each of the former gave as many shillings as there were sheep, and for each of the latter one-fourth as much. Now had he given 4 shillings more for each of the former, and 2 shillings more for each of the latter, he would have paid 7*l*. more. But had a sheep cost as much as a calf, he would have expended 56*l*. 8*s*. How many did he buy of each, and what were their prices?

Let x = number of calves, y = number of sheep,

$$\text{then } xy + \frac{y^2}{4} = x(y+4) + \left(\frac{y}{4} + 2\right)y - 140;$$

$$\therefore\ x = \frac{70 - y}{2},$$

$$\text{also } xy + y^2 = 1128;\ \therefore\ x = \frac{1128 - y^2}{y};$$

$\therefore\ y = -35 \pm 59 = 24$ sheep at 6*s*., and $x = 23$ calves at 24*s*. each.

(87). A traveller set out from a certain place, and went 1 mile the first day, 3 the second, 5 the next, and so on, going every day 2 miles more than he had gone the preceding day. After he had been gone three days, a second sets out, and travels 12 miles the first day, 13 the second, and so on. In how many days will the second overtake the first?

If x = the number of days the second travels,

$$\text{then } \{24 + (x-1)\}\frac{x}{2} = \text{his distance},$$

$$\text{and } \{2 + (x+2) \times 2\}\frac{x+3}{2} = \text{distance of the first};$$

$$\therefore\ (23 + x) \times \frac{x}{2} = (3 + x)(x + 3),$$

$$\text{whence } x = \frac{11}{2} \pm \frac{7}{2} = 9 \text{ days}.$$

(88). There are four numbers in geometrical progression, the second of which is less than the fourth by 24; and the sum of the extremes is to the sum of the means as 7 to 3. Required the numbers.

Let xy^{-3}, xy^{-1}, xy, xy^3 be the numbers,

then $xy^3 - xy^{-1} = 24$, or $x(y^4 - 1) = 24y$,

also $xy^{-3} + xy^3 : xy^{-1} + xy :: 7 : 3$,

or $y^6 + 1 : y^2(y^2 + 1)$, or $y^4 - y^2 + 1 : y^2 :: 7 : 3$;

$\therefore (y^2 - 1)^2 : y^2 :: 4 : 3$; $\therefore y^2 - 1 : y :: 2 : \sqrt{(3)}$,

$$\text{whence } y^2 - \frac{2y}{\sqrt{3}} + \frac{1}{3} = \frac{4}{3};\ \therefore y = \frac{1 \pm 2}{\sqrt{3}} = \sqrt{3} \text{ or } -\frac{1}{\sqrt{(3)}};$$

$$\therefore x = \frac{24\sqrt{3}}{9 - 1} = 3\sqrt{(3)}, \text{ and the series is } 1, 3, 9, 27.$$

(89). A farmer at a fair found the price of an ox equal to that of three sheep, and that he could just dispose of 100*l.*, buying twice as many sheep as oxen. But waiting till the evening, when the price of an ox fell 1*l.*, and of a sheep 6*s.* 8*d.*, he got for 100*l.* three times as many sheep as oxen, and increased his whole stock by ten more than he would have had in the former case. How many sheep and oxen did he buy, and what was the price of each?

Let $3x$ = price of an ox, y = number of oxen,

and $2y$ = number of sheep;

$\therefore 3xy + 2xy = 100$, or $xy = 20$.

In the evening, price of an ox $= 3x - 1$, and of a sheep $= x - \frac{1}{3}$,

then his whole stock $3y + 10$, and $\frac{3y + 10}{4}$ = number of oxen

$$\therefore \frac{3y + 10}{4}(3x - 1) + \frac{3}{4}(3y + 10)\frac{3x - 1}{3} = 100l.;$$

$$\therefore x = \frac{70 + y}{3y + 10} = \frac{20}{y},$$

and $y = -5 \pm 15 = 10$ oxen at 5*l.* each,

and 30 sheep at $1\frac{2}{3}$*l.* each.

(90). A person has two pieces of ground, one of which is in the form of an equilateral triangle, and the other of a rectangular parallelogram, one side of which is equal to a side of the triangle, and the other side is 8 yards less. These he plants with trees, at the distance of two yards from each other on the sides, and finds that there are 11 more on the rectangle than on the triangle. What are the lengths of the sides?

Let $2x$ = side of triangle;

$\therefore\ 2x$ and $2x - 8$ = sides of rectangle,

and since there will be a tree in each corner, there will be $x + 1$ in side of each figure, and in the triangle there will be one less in each row from the base;

$$\therefore \text{ number} = \{2(x+1) - x\}\frac{x+1}{2};$$

$$\therefore (x+1)(x-3) - (x+2)\frac{x+1}{2} = 11,$$

whence $x = \frac{7}{2} + \frac{13}{2} = 10$, and the sides are 20 and 12.

(91). There are four numbers in arithmetical progression, whose sum is 28, and their continual product is 585. Required the numbers.

Let $x - 3y$, $x - y$, $x + y$, $x + 3y$ be the numbers; $\therefore\ x = 7$,

$(49 - y^2)(49 - 9y^2) = 585$; $\therefore\ y^2 = \frac{245}{9} \pm \frac{209}{9} = 4$; $\therefore\ y = 2$,

and the series is 1, 5, 9, 13.

(92). When 962 men were drawn up in two solid squares, it was found that the front of one contained 18 more men than the front of the other. What was the number of men in each square?

Let x = men in front of one, and $x - 18$ in front of other,

then $x^2 + (x - 18)^2 = 962$, or $x^2 - 18x = 319$;

$\therefore\ x = 9 \pm 20 = 29$; $\therefore\ 29 - 18 = 11$;

$\therefore$ the numbers were 121 and 841.

(93). A gentleman bought a horse for a certain sum, and having re-sold it for 119*l.*, found that he had gained as much per cent. by the transaction as the horse cost him. What was the prime cost of the horse?

Let x = cost of the horse,

then $x : 100 :: 119 : 100 + x$;

$\therefore\ x = -50 \pm 120 = 70l.$

(94). A cask, whose contents is 20 gallons, is filled with brandy a certain quantity of which is then drawn off into another cask of equal size; this last cask is then filled with water; after which the first cask is filled with the mixture; and it appears, that if $6\frac{2}{3}$ gallons of the mixture be drawn off from the first into the second cask, there will be equal quantities of brandy in each. Required the quantity of brandy first drawn off.

Let x = the number of gallons first drawn off;

$\therefore\ 20 - x$ = gallons of water poured into second cask,

and if y be quantity of brandy poured back into first cask,

$$\text{then } y : x :: x : 20, \text{ then } y = \frac{x^2}{20},$$

$$\text{and } x - \frac{x^2}{20} = \text{brandy remaining in second cask;}$$

$\therefore$ quantity of brandy in first cask

$$= 20 - x + \frac{x^2}{20} = \frac{400 - 20x + x^2}{20};$$

$$\text{then, again, brandy drawn} : 6\tfrac{2}{3} :: \frac{400 - 20x + x^2}{20} : 20;$$

$$\therefore \text{ brandy drawn off} = \frac{(400 - 20x + x^2)}{60};$$

$$\therefore \text{ brandy in each} = \frac{400 - 20x + x^2}{30} = \frac{20x - x^2}{20} + \frac{400 - 20x + x^2}{60},$$

$$\text{or } 60x - 3x^2 = 400 - 20x + x^2;$$

$$\therefore\ x^2 - 20x + 100 = 0;\ \therefore\ x = 10 \text{ gallons.}$$

(95). There are two numbers, such that 3 times the sum of their squares multiplied by the less is equal to 26 times the greater, and twice the difference of their squares multiplied by the greater is equal to 15 times the less. Required the two numbers.

Let x and y be the numbers,

$$\text{then } 3(x^2+y^2)\,y = 26x, \text{ and } 2(x^2-y^2)\,x = 15y,$$

$$\text{then } \frac{x}{y} = \frac{3(x^2+y^2)}{26} = \frac{15}{2(x^2-y^2)}, \text{ or } x^4 - y^4 = 65,$$

multiplying (1) by (2)

$$45y^4 + 97x^2y^2 = 52x^4,$$

$$\text{whence } y^2 = -\frac{97x^2}{90} \pm \frac{137x^2}{90} = \frac{4x^2}{9};$$

$$\therefore x^4 - y^4 = x^4 - \frac{16x^4}{81} = 65, \text{ or } x^4 = 81;$$

$$\therefore x = 3 \text{ and } y = 2.$$

(96). From two towns, which were 168 miles distant, two persons, *A* and *B*, set out to meet each other; *A* went 3 miles the first day, 5 the next, 7 the third, and so on; *B* went 4 miles the first day, 6 the next, and so on. In how many days did they meet?

If x = the number of days,

$$\text{then } A\text{'s distance} = \{6 + (x-1)\,2\}\,\frac{x}{2},$$

$$B\text{'s distance} = \{8 + (x-1)\,2\}\,\frac{x}{2},$$

$$\text{and } \{7 + 2(x-1)\}\,x = 168, \text{ whence } x = -\frac{5}{4} \pm \frac{37}{4} = 8 \text{ days.}$$

(97). Three persons divide a certain sum of money amongst them in this manner: *A* takes the nth part of the whole, and $\frac{a}{n}$ £; *B* takes the nth part of the remainder, and $\frac{a}{n}$ £, and *C* takes the nth part of what then remained, and $\frac{a}{n}$ £; and then nothing was left. Find the sum.

Let x = the sum of money,

$$\text{then } A\text{'s part} = \frac{x}{n} + \frac{a}{n}; \ \therefore \text{ remainder} = \frac{(n-1)\,x}{n} - \frac{a}{n},$$

$$B\text{'s part} = \frac{(n-1)\,x}{n^2} - \frac{a}{n^2} + \frac{a}{n};$$

$$\therefore \text{ remainder} = \frac{(n-1)^2\,x}{n^2} - \frac{(n-1)\,a}{n^2} - \frac{a}{n},$$

$$C\text{'s part} = \frac{(n-1)^2\,x}{n^3} - \frac{(n-1)\,a}{n^3} - \frac{a}{n^2} + \frac{a}{n};$$

$$\therefore \text{ remainder} = \frac{(n-1)^3\,x}{n^3} - \frac{(n-1)^2\,a}{n^3} - \frac{(n-1)\,a}{n^2} - \frac{a}{n} = 0;$$

$$\therefore x = \frac{3n^2 - 3n + 1}{(n-1)^3} \times a£.$$

(99). A mercer sold a piece of cloth for 24*l.*, and gained as much per cent. as the cloth cost him. What was the price of the cloth?

Let x = price of cloth, then $x : 100 :: 24 : (100 + x)$.

whence $x^2 + 100x = 2400$, and $x = -50 \pm 70 = 20l.$

(100). Two detachments of foot being ordered to a station 39 miles distant, the one by marching $\frac{1}{4}$ mile per hour quicker than the other arrives one hour sooner. What was their speed?

Let x = speed per hour of one, and $x + \frac{1}{4}$ of the other;

$$\therefore \frac{39}{x} \text{ and } \frac{156}{4x+1} \text{ are their hours of marching,}$$

$$\text{and } \frac{39}{x} - 1 = \frac{156}{4x+1}; \ \therefore x = -\frac{1}{8} \pm \frac{25}{8} = 3;$$

$\therefore$ their speeds are 3 and $3\frac{1}{4}$ miles per hour.

(101). What number is that which, being multiplied by the sum of its digits, equals 1012, and if 63 be subtracted from it, its digits will be inverted?

Let x and y be the digits,

then we have $(10x + y) \times (x + y) = 1012$,

$$\text{and } 10x + y - 63 = 10y + x, \text{ or } x - y = 7,$$

$$\text{wherefore } (11x - 7)(2x - 7) = 1012,$$

$$\text{and } x = \frac{91}{44} \pm \frac{305}{44} = 9, \text{ and } y = 2, \text{ and } 92 = \text{number}.$$

(102). What two numbers are those whose difference being multiplied by the difference of their squares produces 576, and whose sum multiplied by the sum of their squares is 2336?

Let x and y be the numbers, then $(x^2 - y^2)(x - y) = 576$,

$$\text{and } (x^2 + y^2)(x + y) = 2336;$$

$$\therefore \frac{x^2 + y^2}{(x-y)^2} = \frac{73}{18}, \text{ or } \frac{2xy}{(x-y)^2} = \frac{55}{18}, \text{ and } \left(\frac{x+y}{x-y}\right)^2 = \frac{64}{9};$$

$$\therefore 3(x + y) = 8(x - y), \text{ or } 5x = 11y,$$

$$\text{whence } x - y = \frac{6y}{5}, \text{ and } x^2 - y^2 = \frac{96y^2}{25};$$

$$\therefore 6y \times 96y^2 = 576 \times 125, \text{ or } y = 5 \text{ and } x = 11.$$

(103). Given the sum of five numbers in arithmetical progression equal to 20, and the sum of their squares 90. What are the numbers?

Let $x - 2y$, $x - y$, x, $x + y$, and $x + 2y$ be the numbers;

$$\therefore 5x = 20, \text{ and } x = 4,$$

$$\text{and } 5x^2 + 10y^2 = 90; \therefore 10y^2 = 10, \text{ and } y = 1;$$

$\therefore$ the series is 2, 3, 4, 5, 6.

(104). The sum of three numbers in a geometrical progression is 91, and the sum of their squares is 4459. What are the numbers?

Let xy^{-1}, x, and xy be the numbers,

$$\text{then } x(1 + y + y^2) = 91y, \text{ and } x^2(1 + y^2 + y^4) = 4459y^2;$$

$$\therefore x(1 - y + y^2) = 49y; \therefore \frac{1 + y + y^2}{1 - y + y^2} = \frac{13}{7},$$

$$\text{whence } y = \frac{5}{3} \pm \frac{4}{3} = 3 \text{ or } \frac{1}{3}; \therefore x = 21,$$

and the series is 7, 21, 63.

(105). The sum of two numbers is 11, and the sum of their third powers 407. Required the numbers.

Let x = one number; $\therefore$ $11 - x$ = the other;

$$\therefore x^3 + (11 - x)^3 = 407,$$

$$\text{or } x^3 + 1331 - 363x + 33x^2 - x^3 = 407,$$

$$x^2 - 11x + \frac{121}{4} = -28 + \frac{121}{4} = \frac{9}{4};$$

$$\therefore x = \frac{11}{2} \pm \frac{3}{2} = 4 \text{ or } 7;$$

$\therefore$ 4 and 7 are the numbers.

(106). A body of men are just sufficient to form a hollow equilateral wedge three deep, and if 597 be taken away, the remainder will form a hollow square four deep, the front of which contains one man more than the square root of the number contained in the side of the wedge. What is the number?

Let x be the number in the side of the wedge, then the number of men in a complete wedge would be

$$(x + 1)\,\frac{x}{2},$$

and the side of the triangle inside the external will be $(x - 3)$;

$\therefore$ the side of 4th internal triangle will be $(x - 9)$,

and the number of men in this wedge will be

$$(x - 9 + 1)\,.\,\frac{x - 9}{2};$$

$\therefore$ number of men in wedge

$$= (x + 1)\,.\,\frac{x}{2} - (x - 8)\,.\,\frac{(x - 9)}{2} = 9x - 36,$$

$$\text{side of square} = \sqrt{x} + 1;$$

$$\therefore \text{men in square when 4 deep} = (\sqrt{x} + 1)^2 - (\sqrt{x} - 7)^2;$$

$$\therefore 9x - 36 = 16\sqrt{x} - 48 + 597;$$

$$\therefore 9x - 16\sqrt{x} = 585;$$

$$\therefore \sqrt{x} = \frac{8}{9} \pm \frac{73}{9} = 9;$$

$\therefore$ 81 is a side of the wedge,

and the number of men $= 9x - 36 = 693$.

(107). A vessel which was leaky was furnished with two pumps, which, being worked by A and B, A took 3 strokes to 2 of B's; but 4 of B's threw out as much water as 5 of A's. Now, had they pumped together, they would have emptied the vessel in $3\frac{3}{4}$ hours; but B first worked for as long as it would have taken A to empty the hold alone, then A commenced alone, and cleared the hold in $13\frac{1}{3}$ hours from B's commencement; but he pumped 100 gallons less than he would have done if they had both pumped together. Required the quantity of water in the hold, and the hourly leakage.

Let x = water in hold, y = hourly influx;

$n = A$'s strokes per hour, $\frac{2n}{3} = B$'s strokes,

z = gallons per stroke of A, nz = gallons per hour by A,

and $\frac{2n}{3} \cdot \frac{5z}{4} = \frac{5nz}{6}$ = gallons per hour by B.

By problem $\therefore \frac{15nz}{4} + \frac{15.5nz}{4.6} = x + \frac{15y}{4}$;

$$\therefore nz(30 + 25) = 8x + 30y;$$

$$x = \frac{55nz - 30y}{8} \dots\dots\dots\dots\dots (1).$$

Let u = hours in which A would empty it alone;

then $\frac{5unz}{6}$ = gallons drawn out by B,

and $\frac{40}{3} - u = \frac{40 - 3u}{3}$ = hours by A to finish it;

$$\therefore \frac{5unz}{6} + \frac{40 - 3u}{3} . nz = x + \frac{40y}{3},$$

$$5unz + 80nz - 6unz = 6x + 80y;$$

$$\therefore\ u = \frac{80nz - 6x - 80y}{nz} \ \dots\dots\dots\dots (2)$$

= time in which A would have emptied it alone;

$$\text{but again } \frac{40 - 3u}{3} nz = \frac{15nz}{4} - 100,$$

$$(160 - 12u)\, nz = 45nz - 1200;$$

$$\therefore\ u = \frac{115nz + 1200}{12nz} \ \dots\dots\dots\dots (3);$$

$$\text{but } nzu = x + uy;$$

$$\therefore\ u = \frac{x}{nz - y} = \frac{115nz + 1200}{12nz} \text{ from } (3);$$

$$\therefore\ x = \frac{(115nz + 1200)(nz - y)}{12nz} \ \dots\dots (4),$$

$$\text{also } u = \frac{x}{nz - y} = \frac{80nz - 6x - 80y}{nz} \text{ from } (2),$$

$$\text{and } x = \frac{80(nz - y)^2}{7nz - 6y} \dots\dots\dots\dots (5);$$

$$\therefore \text{ from (5 and 4) } \frac{16(nz - y)}{7nz - 6y} = \frac{23nz + 240}{12nz},$$

$$\text{whence } y = \frac{1680nz - (31nz)^2}{1440 - 54nz} \dots\dots\dots (6),$$

$$\text{also } x = \text{from (1) } \frac{55nz - 30y}{8} = \frac{(115nz + 1200)(nz - y)}{12nz} \text{ from (4)},$$

$$\text{whence } y = \frac{(13nz)^2 + 480nz}{28nz + 480} \dots\dots\dots (7);$$

$$\therefore \text{ from (6 and 7) } \frac{1680nz - (31nz)^2}{1440 - 54nz} = \frac{(13nz)^2 - 480nz}{28nz + 480},$$

$$\text{and } 166(nz^2) - 240nz.164 = 2.(240)^2,$$

$$(nz)^2 - 240nz.\frac{82}{83} + \left(\frac{240.41}{83}\right)^2 = \frac{(240)^2.1764}{83};$$

$$\therefore\ nz = \frac{240.41}{83} \pm \frac{240.42}{83} = 240;$$

$$\therefore\ y = \frac{(13+2)\ 240^2}{(28+2)\ 240} = 120 \text{ gallons hourly influx by leakage,}$$

$$\text{and } x = \frac{55-15}{8} \times 240 = 1200 \text{ gallons of water in hold.}$$

(108). A dealer bought a number of bushels of wheat, expecting to sell it in 6 months, at a profit of 3*s.* per bushel; but prices having fallen 1*s.* per bushel, he found he should lose the price of five bushels. He then kept it 12 months and sold it at 2*s.* per bushel under prime cost, and found that his loss was 10*s.* less than his expected gain. Now if interest be allowed at 5 per cent., what was the quantity and price of the wheat bought?

Let x = bushels of wheat,

y = price per bushel, $\frac{xy}{40}$ = 6 months' interest;

$$\therefore\ x(y+3) - \frac{xy}{40} - xy = \text{his expected gain}$$

$$= 3x - \frac{xy}{40},$$

$$\text{and } xy + \frac{xy}{40} - x(y-1) = \text{loss in the first case}$$

$$= 5y,$$

$$\text{or } \frac{xy}{40} + x = 5y;\ \therefore\ x = \frac{(5y)\ 40}{40+y} \ \ldots\ldots\ (1),$$

$$\text{also } xy + \frac{xy}{20} - x(y-2) = \text{loss by its sale}$$

$$= \frac{xy}{20} + 2x;$$

$$\text{but by problem } \frac{xy}{20} + 2x = 3x - \frac{xy}{40} - 10;$$

$$\therefore\ x = \frac{400}{40-3y} \ \ldots\ldots\ldots\ldots\ldots\ (2);$$

$$\text{and } \frac{y}{y+40} = \frac{2}{40-3y} \text{ from (1 and 2),}$$

$$\text{and } 40y - 3y^2 = 2y + 80;$$

$$\therefore\ y^2 - \frac{38y}{3} + \left(\frac{19}{3}\right)^2 = -\frac{80}{3} + \frac{361}{9} = \frac{121}{9};$$

$$\therefore\ y = -\frac{19}{3} \pm \frac{11}{3} = 10s. \text{ per bushel},$$

$$\text{and } x = \frac{400}{40 - 3y} = 40 \text{ bushels of wheat.}$$

(109). Out of a vessel containing 24 gallons of brandy there were drawn, at three successive times, a number of gallons, forming an ascending arithmetic series, and the difference of the squares of the extremes was equal to 16 times the means; the cask having been filled up with water between each draw, it was found that the last draw was only one-sixth of its original strength. Required the number of gallons of pure brandy drawn each time.

Let $(x - y)$, x, and $(x + y)$ be the number of gallons of the mixture drawn each time,

$$(x + y)^2 - (x - y)^2 = 16x,$$

$$4xy = 16x;$$

$$\therefore\ y = 4;$$

therefore $x - 4$ is number of gallons of spirit drawn out first time, and also the quantity of water introduced, and x is the number of gallons of the mixture drawn out the next time,

$$24 - (x - 4) : 24 : \text{spirit} : x;$$

$$\therefore \text{ spirit drawn out second time} = \frac{(28 - x).x}{24},$$

$$\text{spirit left in} = 24 - (x - 4) - \frac{\{24 - (x - 4)\}}{24}x$$

$$= \frac{(24 - x)\{24 - (x - 4)\}}{24};$$

spirits drawn out third time

$$: x + 4 :: \frac{(24 - x)(28 - x)}{24} : 24 :: 1 : 6;$$

$$\therefore\ (24 - x)(28 - x) = 96,$$

$$\text{whence } x = 26 \pm 10 = 36 \text{ or } 16;$$

$\therefore$ 16 gallons is the proper result,
since 36 gallons is impossible,
and the draws are 12, 16, and 20 gallons,
and the spirits drawn are 12, 8, and $3\frac{1}{3}$ gallons.

(110). P, Q, R represent three candidates at an election. Q polled as many plumpers wanting one, as the split votes betwixt P and R exceeded those betwixt himself and R; and the number of split votes betwixt Q and R was one more than twice the number betwixt Q and P. If P had not voted for himself and R, but for R only, and if 5 others who split betwixt P and Q had voted for Q only, Q would have just beaten P, and would have been 48 below R. The number of voters was 1341, of which 565 gave plumpers. Required the number of plumpers for each candidate, and the final state of the poll.

Let x, y, and z be plumpers for P, Q, and R;

$$\therefore x + y + z = 565.$$

Let m be the votes split between P and Q,
n those between P and R, and r those between Q and R;

$$\therefore m + n + r = 776,$$

then by problem $y + 1 = n - r$, and $r = 2m + 1$;

$$\therefore 3m + n = 775, \text{ and } y = n - r - 1 = 773 - 5m,$$

again $y + m + r = x + n + m - 5$; $\therefore y = 769 + x - 5m$,

whence $x = 4$ the number of plumpers for P,

also $y + m + r = z + n + r - 48$, whence $y = z + 727 - 4m$,

but $y + z = 561$; $\therefore 5 = 561 - y$; $\therefore 2y = 1288 - 4m$,

also $2y = 1546 - 10m$; $\therefore 258 = 6m$, or $m = 43$;

$\therefore y = 773 - 215 = 558$ plumpers for Q,

$z = 561 - 558 = 3$ plumpers for R,

$r = 86 + 1 = 87$, and $n = 776 - 130 = 646$;

$\therefore x + m + n = 4 + 43 + 646 = 693$ votes for P,

and $y + m + r = 558 + 43 + 87 = 688$ votes for Q,

also $z + n + r = 3 + 646 + 87 = 736$ votes for R.

(111). There is a number consisting of three digits, of which the first is to the second as the second to the third; the number itself is to the sum of its digits as 124 to 7; and if 594 be added to it, the digits will be inverted. Required the number.

Let $100x + 10y + z$ be the number,

$$\text{then } x : y :: y : z; \;\therefore\; xz = y^2,$$

$$\text{and } 100x + 10y + z : x + y + z :: 124 : 7,$$

$$\text{or } 99x + 9y : x + y + z :: 117 : 7;$$

$$\therefore\; z = \frac{64x - 6y}{13},$$

$$\text{also } 100z + 10y + x = 100x + 10y + z + 594,$$

$$\text{whence } z - x = 6; \;\therefore\; x + 6 = \frac{64x - 6y}{13},$$

$$\text{and } 17x = 2y + 26, \text{ also } z = \frac{y^2}{x} = x + 6; \;\therefore\; y^2 = x^2 + 6x;$$

$$\therefore\; \left(\frac{17x - 26}{2}\right)^2 = x^2 + 6x, \text{ or } 285x^2 - 908x + 676 = 0,$$

$$\text{whence } x = 2, \; z = 8, \text{ and } y = 4,$$

and the number is $200 + 40 + 8 = 248$.

(112). Find two numbers whose product is equal to the difference of their squares, and the difference of their cubes equal to the sum of their squares.

Let x, and mx be the numbers, then $m^2x^2 - x^2 = mx^2$,

$$\text{or } m^2 - m - 1 = 0, \text{ or } m = \frac{1 \pm \sqrt{(5)}}{2},$$

$$\text{also } m^3x^3 - x^3 = m^2x^2 + x^2,$$

$$(m^3 - 1)\,x = m^2 + 1,$$

$$\because\; m^2 = m + 1; \;\therefore\; m^3 - 1 = m^2 + m - 1 = 2m;$$

$$\therefore\; x = \frac{m + 2}{2m} = \frac{5 \pm \sqrt{5}}{2(1 \pm \sqrt{5})} = \pm\frac{\sqrt{5}}{2},$$

$$\text{and } mx = \frac{\sqrt{5} \pm 5}{4}.$$

(113). The length of a rectangular field is to its breadth as 6 : 5; one-sixth part of the area being planted, there remains 625 square yards. What are the sides of the field?

Let $6x$ and $5x$ be the length and breadth; $\therefore$ area $= 30x^2$,

$$\text{then } \frac{5}{6}(30x^2) = 625, \text{ or } x = 5,$$

and the sides are 30 and 25.

(114). Find two numbers such that the product of the greater and the cube of the less may be to the product of the less and the cube of the greater as 4 : 9; and the sum of the cubes of the numbers may be 35.

Let x and y be the numbers, then $xy^3 : x^3y :: 4 : 9$;

$$\therefore y^2 : x^2 :: 4 : 9; \therefore y = \frac{2x}{3},$$

$$\text{and } x^3 + \frac{8x^3}{27} = 35; \therefore x = 3, \text{ and } y = 2.$$

(115). A ship with a crew of 175 men set sail with a store of water sufficient to last to the end of the voyage. But in 30 days the scurvy made its appearance, and carried off 3 men every day; and at the same time a storm arose, which protracted the voyage 3 weeks. They were, however, just enabled to arrive in port, without any diminution in each man's daily allowance of water. Required the time of the passage, and the number of men alive when the vessel reached the harbour.

Let x = expected number of days;

$\therefore 175x$ = number of rations of water;

$\therefore x + 21$ = actual number of days,

$3(x + 21 - 30) = (x - 9)\,3$ = number of deaths,

and $x - 9$ = number of days of scurvy,

and rations of water lost

$$= \{6 + (x - 10)\,3\}\frac{x-9}{2} = (3x - 24)\frac{1}{2}(x - 9);$$

$$\therefore\ 175x = 175 \times (x+21) - \frac{1}{2}(3x-24)(x-9),$$

$$\text{whence } x^2 - 17x = 2378,$$

$$\text{and } x = \frac{17}{2} \pm \frac{99}{2} = 58 \text{ days};$$

$$\therefore\ x + 21 = 79 \text{ days the whole voyage,}$$

$$\text{number of men left} = 175 - 49 \times 3 = 28.$$

(116). There are three numbers, the difference of whose differences is 5, their sum is 20, and their continual product 130. Required the numbers.

Let x, y, z be the numbers, and x the greatest,

then $x - y$, $y - z$ are differences; $\therefore\ x - y - y + z = 5$,

$$\text{or } x + z = 2y + 5,\ x + y + z = 20;$$

$$\therefore\ 2y + 5 = 20 - y, \text{ or } y = 5,$$

$$\text{also } 5xz = 130;\ \therefore\ xz = 26;$$

$$\therefore\ (x+z)^2 - 4xz = (x-z)^2 = 225 - 104 = 121;$$

$$\therefore\ x - z = 11,\ x + z = 15;$$

$$\therefore\ x = 13,\ y = 5,\ z = 2.$$

(117). Five persons undertake to reap a field of 87 acres. The five terms of an arithmetical progression, whose sum is 20, will express the times in which they can severally reap an acre; and they altogether can finish the undertaking in 60 days. In how many days can each separately reap an acre?

$$\text{Let } x - 2y + x - y + x + x + y + x + 2y = 20;$$

$$\therefore\ x = 4;$$

$$\therefore\ \frac{1}{x-2y} + \frac{1}{x-y} + \frac{1}{x} + \frac{1}{x+y} + \frac{1}{x+2y} = \text{daily work of the whole,}$$

$$\text{or } \frac{2}{4-y^2} + \frac{8}{16-y^2} + \frac{1}{4} = \frac{87}{60},$$

$$\text{whence } y^2 = \frac{35 \pm 29}{6} = 1 \text{ or } \frac{32}{3};$$

P 2

$$\therefore\ y = 1 \text{ or } \pm\sqrt{\left(\frac{32}{3}\right)};$$

$$\therefore\ x - 2y = 4 - 2 = 2,\ 4 - y = 3,$$

and the time of each in reaping an acre will be respectively

2, 3, 4, 5, and 6 days.

(118). The fore-wheel of a carriage makes 6 revolutions more than the hind-wheel in going 120 yards; but if the periphery of each wheel be increased one yard, it will make only 4 revolutions more than the hind-wheel in the same space. Required the circumference of each.

Let x = periphery of fore-wheel, y of hind-wheel;

$$\therefore\ \frac{120}{x} = \text{number of revolutions of fore-wheel,}$$

$$\text{and } \frac{120}{y} = \text{number of hind-wheel;}$$

$$\therefore\ \frac{120}{x} = \frac{120}{y} + 6, \text{ or } y = \frac{20x}{20 - x},$$

$$\text{but } \frac{120}{x+1} = \frac{120}{y+1} + 4, \text{ or } y = \frac{31x + 1}{29 - x},$$

$$\text{whence } x = \frac{39}{22} \pm \frac{49}{22} = 4 \text{ yards, and } y = 5 \text{ yards.}$$

(119). A number of persons purchased a field for 345*l.* The youngest contributed a certain sum, the next 5*l.* more, the third 5*l.* more than the second, and so on to the oldest. For the greater accommodation of the seniors, the field was divided into two parts, the younger half taking a portion proportional to the sum they had subscribed: and in order that each might have an equal share in this portion, they agreed to equalise their contributions, and each to pay 22*l.* Required the number of persons and the sums paid by each.

Let $2x$ = number of persons, and y = sum paid by the youngest,

$$\text{then } 345 = \{2y + (2x - 1) \times 5\}\, x,$$

$$\text{and } 22x = \{2y + (x - 1) \times 5\}\, \frac{x}{2};\ \therefore\ 44 = 2y + 5x - 5,$$

$$\text{or } x = \frac{49 - 2y}{5}, \text{ and } (2x - 1) \times 5 = 98 - 4y - 5,$$

$$\text{whence } 345 = (2y + 93 - 4y) \times \frac{49 - 2y}{5}, \text{ whence } y = 12,$$

$$\text{and } 2x = 2 \times 5 = 10 \text{ persons,}$$

and the sums paid by each respectively are 12, 17, ... 57*l*.

(120). Some bees were sitting on a tree; at one time the square root of half their number flew away. Again eight-ninth of the whole flew away the second time; two bees remained. How many were there?

Let $2x^2$ = number of bees; $\therefore$ x flew away,

$$2x^2 - x - \frac{8}{9} 2x^2 = 2,$$

$$\text{whence } x = \frac{9}{4} \pm \frac{15}{4} = 6, \text{ and } 2x^2 = 72 \text{ number of bees.}$$

(121). The number of deaths in a besieged garrison amounted to 6 daily; and, allowing for this diminution, their stock of provisions was sufficient to last for 8 days. But on the evening of the sixth day 100 men were killed in a sally, and afterwards the mortality increased to 10 daily. Supposing the stock of provisions unconsumed at the end of the sixth day to support 6 men for 61 days; it is required to find how long it would support the garrison, and the number of men alive when the provisions were exhausted.

Let y = number of men in garrison on the morning of the day,

x = number of rations;

$\therefore$ the issue of rations will be 6 less on the second day;

$$\therefore 8y - (12 + 6 \times 6) \frac{7}{2} = 8y - 168 = x,$$

rations left at the end of the sixth day

$$= x - 6y + (12 + 4 \times 6) \frac{5}{2}$$

$$= x - 6y + 90 = 366;$$

$$\therefore\ 8y - 168 = 6y + 276;$$

$$\therefore\ y = 222, \text{ and } x = 1608,$$

men left at the end of the sixth day = 222 − 136 = 86,

and if z be the number of days the provisions will last, then the rations for the first day are 86, and 10 less daily for the remaining $(z - 1)$ days;

$$\therefore\ 86z - \{20 + (z - 2) \times 10\} \frac{z-1}{2} = 86z - 5z(z-1) = 366,$$

$$\text{whence } z^2 - \frac{91z}{5} + \left(\frac{91}{10}\right)^2 = \frac{961}{100};$$

$$\therefore\ z = \frac{91}{10} \pm \frac{31}{10} = 6 \text{ days},$$

and men left alive = 86 − 60 = 26 men.

(122). D sets out from F towards G, and travels 8 miles a day; after he had gone 27 miles, E sets out from G towards F, and goes every day $\frac{1}{20}$ of the whole journey; and after he had travelled as many days as he goes miles in one day, he met D. What is the distance of the two places?

Let x = miles from F to G;

$$\therefore\ \frac{x}{8} = \text{days by } D : \frac{27}{8} = 3\tfrac{3}{8} \text{ days},$$

$$\frac{x}{20} = \text{miles per day by } E = \text{also days by } D;$$

$$\therefore\ \frac{x^2}{400} = \text{distance gone by } E \text{ on meeting } D,$$

$$\text{but distance by } D = \frac{x}{20} \times 8 + 27;$$

$$\therefore\ \frac{2x}{5} + 27 + \frac{x^2}{400} = \text{whole distance} = x;$$

$$\therefore\ x^2 - 240x + (120)^2 = 3600;$$

$$\therefore\ x = 120 \pm 60 = 180 \text{ miles}.$$

(123). There are three numbers, the difference of whose differences is 3, their sum is 21, and the sum of the squares of the greatest and least is 137. Required the numbers.

Let x, y, z be the numbers, of which x is the greatest;

$$\therefore\ x - y,\ y - z \text{ are the differences;}$$

$$\therefore\ x - y - (y - z) = 3, \text{ or } 2y = x + z - 3,$$

$$\text{and } x + y + z = 21;\ \therefore\ 3y + 3 = 21;\ \therefore\ y = 6;\ \therefore\ x + z = 15,$$

$$\text{and } x^2 + 2xz + z^2 = 225, \text{ also } x^2 + z^2 = 137;\ \therefore\ 2xz = 88;$$

$$\therefore\ x^2 - 2xz + z^2 = 49, \text{ and } x - z = \pm 7;$$

$$\therefore\ x = 11,\ y = 6,\ z = 4.$$

(124). Three persons A, B, and C, went into a gaming-house; the sums which they severally had were in a decreasing geometrical progression. Upon quitting it, they found that the sums which they then had were in a decreasing arithmetical progression; that what B had remaining was to what he had lost in proportion of the sum to the difference of what he and C had at first; and that C had neither won nor lost. If C had won what A lost, he would then have had 64*l.* more than A had remaining; also, the whole sum which they had remaining was to what they had lost as 6 : 7. Required the sums which they had at first.

Let xr, x, and xr^{-1} be their respective sums,

$$\text{then } x - B\text{'s loss} : B\text{'s loss} :: x + xr^{-1} : x - xr^{-1} :: r + 1 : r - 1;$$

$$\therefore\ x : B\text{'s loss} :: 2r : r - 1;$$

$$\therefore\ B\text{'s loss} = \frac{x \times (r - 1)}{2r},\ B\text{'s remainder} = \frac{x\,(r + 1)}{2r},$$

$$\text{also } xr - A\text{'s loss} = xr^{-1} + A\text{'s loss} - 64;$$

$$\therefore\ A\text{'s loss} = \frac{x\left(r - \frac{1}{r}\right)}{2} + 32 = \frac{x\,(r^2 - 1)}{2r} + 32,$$

$$A\text{'s remainder} = xr - \frac{x\,(r^2 - 1)}{2r} - 32 = \frac{x\,(r^2 + 1)}{2r} - 32,$$

then, by arithmetical progression,

$$2B\text{'s remainder} = \frac{x(r+1)}{r} = \frac{x(r^2+1)}{2r} - 32 + \frac{x}{r},$$

$$\text{or } x = \frac{64r}{(r-1)^2},$$

$$\text{whole sum remaining} = 32\left\{\frac{r^2+r+4}{(r-1)^2} - 1\right\},$$

$$\text{whole loss} = 32\left\{\frac{r^2+r-2}{(r-1)^2} + 1\right\};$$

$$\therefore\ r^2+r+4-(r-1)^2 : (r^2+r-2)+(r-1)^2 :: 6 : 7,$$

$$\text{or } r^2+r+1 : (r-1)^2 - 3 :: 13 : 1,$$

$$\text{or } 3(r+1) : r^2-2r-2 :: 12 : 1,$$

$$\text{whence } 4r^2-9r-9=0, \text{ and } r = 3 \text{ or } -\frac{3}{4};$$

$$\therefore\ x = \frac{64\times 3}{4} = 48l.;\ \therefore \text{ their sums were } 144l., 48l., 16l.$$

(125). A cistern can be filled by 3 different pipes; by the first in $1\frac{1}{3}$ hours, by the second in $3\frac{1}{3}$ hours, and by the third in 5 hours; in what time will this cistern be filled when all three pipes are open at once?

Part per hour by first pipe $= \frac{3}{4}$, by second $= \frac{3}{10}$, by third $= \frac{1}{5}$,

and if x = number of hours,

$$\text{then } x\left(\frac{3}{4} + \frac{3}{10} + \frac{1}{5}\right) = 1,$$

$$\text{or } x = \frac{40}{30+12+8} = \frac{4}{5} \text{ of an hour} = 48 \text{ minutes.}$$

(126). There is a number consisting of two digits, which, when divided by the sum of its digits, gives a quotient greater by 2 than the tens or second digit; but if the digits be inverted, and the resulting number be divided by a number greater by unity than the sum of the digits, the quotient is greater by 2 than the preceding quotient. Find the number.

Let $10x + y$ be the number,

$$\text{then } \frac{10x + y}{x + y} = x + 2; \therefore y = \frac{8x - x^2}{x + 1},$$

$$\text{also } \frac{10y + x}{x + y + 1} = x + 4; \therefore y = \frac{x^2 + 4x + 4}{6 - x},$$

$$\text{whence } 19x^2 - 40x + 4 = 0,$$

$$\text{or } x = \frac{20}{19} \pm \frac{18}{19} = 2, \text{ and } y = \frac{12}{3} = 4,$$

and the number is 24.

(127). Find four numbers which exceed one another by unity, such that their continued product may be 120.

Let $(x - 1)$, x, $(x + 1)$, $(x + 2)$ be the numbers,

$$\text{then } (x^2 - 1)(x^2 + 2x) = 120,$$

$$\text{or } x^4 + 2x^3 - x^2 - 2x = 120;$$

$$\therefore x^4 + 2x^3 + x^2 - 2(x^2 + x) + 1 = 121;$$

$$\therefore x^2 + x = 1 \pm 11 = 12 \text{ or } -10,$$

$$\text{whence } x = -\frac{1}{2} \pm \frac{7}{2} = 3, \text{ and the numbers are } 2, 3, 4, 5.$$

(128). Two boys set off in opposite directions from the right angle of a triangular field, and ran along the sides without varying their velocities, which were in the ratio of 13 : 11. They met in the middle of the opposite side, and afterwards 30 yards from the point where they started. Required the lengths of the sides of the field.

Let C be the right, angle and the swifter run along the base from C to A, and D the middle point of the hypothenuse; then, since their speeds are as 13 : 11,

$$13 : 11 :: AC + AD : BC + BD,$$

$$\text{or } 24 : 2 :: AC + BC + AB : AC - BC \; (A),$$

$$\text{also } 13 : 11 :: AC + BC + AB + 30 : AC + BC + AB - 30,$$

$$\text{or } 24 : 2 :: 2 \times \text{perimeter} : 60; \therefore \text{perimeter} = 360,$$

$$\text{from } (A)\; AC - BC = 30; \therefore AC = 30 + BC,$$

$$\text{and } AB = 360 - AC - BC = 330 - 2BC,$$

$$\text{and } AB^2 = AC^2 + BC^2; \therefore (330 - 2BC)^2 = (30 + BC)^2 + BC^2,$$

whence $BC^2 - 690BC = -54000$, and $BC = 345 \pm 255 = 90$;

$$\therefore AC = 120, \text{ and } AB = 150.$$

(129). On January 1st, 1799, a certain beggar received from A as many groats as A was years old, who repeated a similar donation every January for the seven following years, during the last of which A died, his alms to the poor man having in all amounted to 7*l.* 18*s.* 8*d.* Required in what year he was born, and his age at his death.

Let x = years old of A = number of groats the beggar received, then $x+1$, $x+2$, ... $x+7$ are the succeeding donations;

$$\therefore 8x + 28 = 476 \text{ groats}; \therefore x = 56 \text{ years old};$$

$$\therefore 1799 - 56 = 1743 \text{ when he was one year old};$$

$$\therefore \text{he was born in 1742, and died at 63 years old.}$$

(130). A person bought 2 cubical stacks of clover for 41*l.*, each of which cost as many shillings per solid yard as there were yards in a side of the other, and the greater stood on 9 square yards more than the less. What was the price of each?

Let x yards = side of one, and $m + x$ yards = side of other,

$$\text{then } (m+x)^2 = x^2 + 9; \therefore m^2 + 2mx = 9, \text{ or } x = \frac{9 - m^2}{2m},$$

$$\text{also } x^3(m+x) + (m+x)^3 x = 820, \text{ and } m + x = \frac{9 + m^2}{2m};$$

$$\therefore x(m+x)\{x^2 + (m+x)^2\} = 820,$$

$$\text{or } \frac{81 - m^4}{4m^2}\left\{\frac{(9-m^2)^2}{4m^2} + \frac{(9+m^2)^2}{4m^2}\right\}$$

$$= (81 - m^4)(81 + m^4) = 8 \times 820m^4;$$

$$\therefore m^8 + 6560m^4 + (\ \ )^2 = 10764961,$$

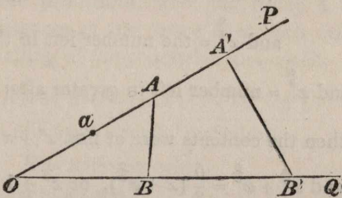
$$\text{and } m^4 = -3280 \pm 3281 = 1; \therefore m = 1,$$

$$\text{and } x = 4; \therefore \text{price of first} = \frac{64 \times 5}{20} l. = 16l.,$$

$$\text{and price of second} = \frac{125 \times 4}{20} l. = 25l.$$

(131). *A* and *B* were travelling along rectilinear roads, which intersected each other. *A* had proceeded *a* yards past the intersection of the roads, when *B* arrived at it; after an interval of *m* minutes, *A* was exactly on the left of *B*; and *n* minutes afterwards, *B* was on the right of *A*, and *b* yards distant from him. Supposing them never to have varied their speed, at what rate did each travel?

Let OP, OQ be the diverging roads,



$Oa = a$, A and B the first positions of A and B,

and $A'B'$ their second,

and x and y their speed respectively per minute;

$$\therefore\ OB = my,\ BB' = ny,\ Aa = mx,$$

$$\text{then } AO : OB :: OB' : OA',$$

$$\text{or } mx + a : my :: (m+n)\,y : (m+n)\,x + a,$$

$$\text{or } m(m+n)\,y^2 = \{(m+n)\,x + a\}\,(mx + a),$$

$$\text{also } A'B'^2 = b^2 = OB'^2 - OA'^2 = (m+n)^2 y^2 - \{(m+n)\,x + a\}^2,$$

$$\text{or } b^2 + \{(m+n)\,x + a\}^2 = \frac{\{(m+n)\,x + a)\,(mx + a)\,(m+n)}{m},$$

$$\text{whence } mb^2 - nax\,(m+n) + ma^2 = a^2\,(m+n),$$

$$\text{or } x = \frac{mb^2 - na^2}{na\,(m+n)}\ A\text{'s speed per minute,}$$

$$\text{and } y = \frac{b\,\sqrt{\{m^2b^2 + \sqrt{(n^2a^2)}\}}}{na\,(m+n)}.$$

(132). From each of two bags, containing a certain number of balls respectively, a person draws out a handful, and finds that the number remaining in the greater is exactly the cube of that remaining in the less, and exactly the square of one handful. He then draws out of the greater until he finds that the number remaining in it is exactly the square of that remaining in the less; and also, that if he now emptied the greater into the less, its original number will be increased by two-thirds. What was the number of the balls in each bag at first?

$$\text{Let } x = \text{balls in a handful;}$$

$$\therefore\ x^2 = \text{number left in the greater,}$$

$$\text{and } x^{\frac{2}{3}} = \text{the number left in the less,}$$

$$\text{and } x^{\frac{4}{3}} = \text{number left in greater after second draw,}$$

$$\text{then the contents were at first } x^2 + x, \text{ and } x + x^{\frac{2}{3}},$$

$$\text{and } x^{\frac{4}{3}} + x^{\frac{2}{3}} = \frac{5}{3}(x + x^{\frac{2}{3}}), \text{ or } x^{\frac{2}{3}} + 1 = \frac{5}{3}(x^{\frac{1}{3}} + 1),$$

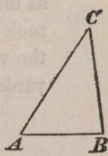
$$\text{whence } x^{\frac{1}{3}} = \frac{5}{6} \pm \frac{7}{6} = 2;\ \therefore\ x = 8,$$

and the contents of the larger were 72 balls, of the smaller 12 balls.

(133). A farmer laid up a stock of corn, expecting to sell it in 6 months at 3*s*. per bushel more than he gave for it. But the price of corn falling 1*s*. per bushel, he found that by selling it he should lose the cost price of 5 bushels; he therefore kept it to the end of the year, and selling it at 2*s*. per bushel under prime cost, found his loss to be 10*s*. less than his expected gain. Required the quantity of corn laid up, and the price per bushel, allowing 5 per cent. simple interest.

See Ex. 108.

(134). There are three towns, *A*, *B*, and *C*, the straight lines joining which form a right-angled triangle, *B* being situated at the right-angle, and the distance from *A* to *B* being the least of the three. A pedestrian making a circuit of them, at a uniform rate, finds that the time of his going from A to B, together with the time of going from B to C, exceeds the time of C to A by two hours and 40 minutes. A coach, which left A 4 hours after the pedestrian, to make the same circuit, overtakes him at the end of the 8th mile from B to C, the rate of the coach being three times that of the pedestrian; and after reaching A, and waiting there 6 hours and 40 minutes, it starts again to make the same circuit, and arrives again at A, exactly at the same time with the pedestrian, who had rested 4 hours at C. Find the distances of the towns from each other, and the rates of travelling of the pedestrian and coach.



Let $AB = x$, $BC = y$, $AC = \sqrt{(x^2 + y^2)}$,

z = miles per hour by traveller,

$3z$ = miles by coach,

$$\text{then } \frac{x+8}{z} = \frac{x+8}{3z} + 4; \therefore x = 6z - 8;$$

$$\therefore \frac{x+y}{z} = \frac{\sqrt{(x^2+y^2)}}{z} + \frac{8}{3}, \text{ or } \frac{x+y-\sqrt{(x^2+y^2)}}{z} = \frac{8}{3} \ (A),$$

$$\text{also } 4 + \frac{2\{x+y+\sqrt{(x^2+y^2)}\}}{3z} = \frac{x+y+\sqrt{(x^2+y^2)}}{z} + 4 - \frac{20}{3};$$

$$\therefore \frac{x+y+\sqrt{(x^2+y^2)}}{z} = 20 \ (B);$$

$$\therefore \text{ from } (A) \ \frac{x+y+\sqrt{(x^2+y^2)}}{x+y-\sqrt{(x^2+y^2)}} = \frac{15}{2}; \because y \text{ is} > x,$$

$$\text{whence } \frac{y+x}{y-x} = \frac{17}{7}, \text{ and } y = \frac{12x}{5};$$

$$\therefore \text{ from } (B) \ x + \frac{12x}{5} + \frac{13x}{5} = 20z, \text{ or } x = \frac{10z}{3}, \text{ also} = 6z - 8;$$

$$\therefore 10z = 18z - 24, \text{ and } z = 3; \therefore x = 10,$$

and $y = 24$, or distance $AB = 10$ miles, $BC = 24$ miles,

and $AC = \sqrt{(100 + 576)} = 26$,

and the rates of travelling are 3 and 9 miles.

(135). Three boats A, B, C, start in a race at the same instant; B being 20 yards behind A, and C the same distance behind B. A and B set off at an uniform rate, C advancing 1 yard less than A at every stroke. But B took 7 strokes to 6 of A's or C's, and increased its speed besides by 3 inches every stroke; so that by the time A had taken 42 strokes, B, though it had lost 16 yards by steering, was only 1 yard behind A. At this time B was observed to fall back, and its velocity decreased twice as fast as it had increased before; whilst C, quickening its stroke at the same instant in the ratio of 6 : 7, and gaining each stroke as much velocity as B lost, at the end of 28 strokes overtook B, which had lost 11 yards more by steering. Find the velocities with which they started, and the number of yards each made per stroke at first.

Let $y = A$'s stroke in yards,

$y - 1 = C$'s,

$x = B$'s first stroke,

$6 : 7 :: 42 : 49 =$ number of B's strokes corresponding to A's and C's 42;

$$\therefore \left[2x + 48\left(\frac{1}{12}\right)\right]\frac{49}{2} = (x + 2)\,49 = \text{space gone over by } B;$$

but it lost 16 yards by steering. Therefore the actual advance was $(x + 2)\,49 - 16 = 49x + 82$, and it has gained 19 yards on A; therefore

$$49x + 82 = 42y + 19,$$

$$\text{or } 7x = 6y - 9 \ldots\ldots\ldots\ldots\ldots\ldots(a).$$

During this time C has advanced $42(y - 1)$; therefore C is now behind B

$$20 + 42y + 19 - 42(y - 1) = 81 \text{ yards.}$$

Now C quickens its strokes in the ratio of 6 to 7; that is, takes them as rapidly as B and gains 6 inches every stroke; therefore in 28 strokes C will have advanced

$$\left[2(y - 1) + 27\left(\frac{1}{6}\right)\right]\frac{28}{2}, \text{ or } 28y + 35,$$

and B losing 6 inches each stroke will have gone over (the first stroke being equal to the last of the first 49; that is, $x+48\left(\frac{1}{12}\right)$, or $x+4$ yards)

$$\left[2(x+4)-27\left(\frac{1}{6}\right)\right]\frac{28}{2}, \text{ or } 28x+49,$$

but B loses 11 yards by steering; therefore actual advance of B is

$$28x+49-11, \text{ or } 28x+38 \text{ yards.}$$

Therefore per question

$$28y+35=28x+38+81,$$

$$y=x+3 \ldots\ldots\ldots\ldots(\beta);$$

$$\therefore (\alpha) \text{ becomes } 7x=6x+18-9,$$

$$\text{whence } x=9=B\text{'s stroke,}$$

$$y=12=A\text{'s,}$$

$$y-1=11=C\text{'s,}$$

and the velocities are proportional to the lengths of strokes multiplied by the rapidity with which they are made;

$$\therefore A\text{'s velocity} : B\text{'s velocity} : C\text{'s velocity}$$

$$:: 12\times 6 : 9\times 7 : 11\times 6$$

$$:: 72 : 63 : 66$$

$$:: 24 : 21 : 22.$$

(136). Two persons, A and B, on comparing the distances they had travelled, found that the square of the number of miles which A usually walked per hour exceeded the square of the number which B usually walked by 5; and that if to the square of the product of those numbers there be added the square of the sum of their fourth powers, augmented by the product of the square of the difference of their squares into the square of the product of the numbers themselves, the aggregate amount would be 10345. How many miles did each walk per hour?

$$\text{Let } \sqrt{(x)}=A\text{'s rate, } \sqrt{(y)}=B\text{'s rate,}$$

$$\text{then } x-y=5 \ldots\ldots\ldots\ldots(1),$$

$$\text{and } xy + (x^2 + y^2)^2 + (x - y)^2 \times xy = 10345 \ \ldots (2),$$

$$\text{from (1) } x^2 + y^2 = 25 + 2xy,$$

$$\text{by substitution } xy + (25 + 2xy)^2 + 25xy = 10345,$$

$$\text{or } 4x^2y^2 + 126xy = 9720,$$

$$\text{whence } 2xy = -\frac{63}{2} \pm \frac{207}{2} = 72,$$

$$\text{and } x^2 - 2xy + y^2 = 25;$$

$$\therefore\ x + y = 13, \text{ and } x - y = 5;$$

$$\therefore\ x = 9, \text{ and } \sqrt{(x)} = 3 \text{ miles } A\text{'s rate},$$

$$y = 4;\ \therefore\ \sqrt{(y)} = 2 \text{ miles } B\text{'s rate}.$$

(137). *A*, *B*, *C*, *D*, are four rough diamonds; the value of *C* in pounds is 52 less than the weight of *A* in carats, and the value of *C* and *D* in pounds is equal to the weight of *B* in carats; after being cut, each is found to have lost half its weight. The dust from *A* and *B* is worth 85*l.*; the value of *A* is to the value of *C* and *D* together with the dust from *A* and *B* as is the value of *B* to the value of *C* and *D*. A diamond weighing one carat when rough is worth 3*l.* when cut, and 2*l.* when uncut; the value is proportional to the square of the weight, and a carat of the dust is worth 1*l.* Find the value of *D* when cut.

Let w = value of *A*,
x = value of *B*,
y = value of *C*,
z = value of *D*,
m = dust from *A* after cutting,
n = ... *B* ....

Now, by hypothesis,

$$y + 52 = \text{carats in } A,$$

$$y + z = \text{carats in } B,$$

$$\text{also } \frac{1}{2}(y + 52) + \frac{1}{2}(y + z) = 85,$$

$$\text{or } 2y + z = 118;\ \therefore\ y = \frac{118 - z}{2}$$

Again, $w : y + z + m + n :: x : y + z$,

$$\text{or } \frac{w}{x} = \frac{y + z + m + n}{y + z},$$

and $m + n = 85$;

$$\therefore \frac{w}{x} = \frac{203 - y}{118 - y}.$$

Again, $w : x :: (y + 52)^2 : (y + z)^2$;

$$\therefore \frac{w}{x} = \frac{(y + 52)^2}{(y + z)^2};$$

$$\therefore \frac{203 - y}{118 - y} = \frac{(y + 52)^2}{(118 - y)^2},$$

whence $(203 - y)(118 - y) = (y + 52)^2$,

and $425y = 21250$;

$\therefore y = 50$;

$\therefore z = 118 - 2y = 18l.$;

$\therefore$ after cutting value of $z = 27l.$

(138). There are two sorts of metal, each being a mixture of gold and silver, but in different proportions. Two coins from these metals of the same weight are to each other in value as 11 to 17; but if to the same quantities of silver as before in each mixture double the former quantities of gold had been added, the values of two coins from them of equal weights would have been to each other as 7 to 11. Determine the proportion of gold to silver in each mixture, the values of equal weights of gold and silver being as 13 to 1.

Let x = units of gold,

y = units of silver;

$\therefore \frac{x + y}{p}$ may be one coin, and $\frac{mx + ny}{q}$ may be the other;

$\therefore \frac{p}{q} = \frac{x + y}{mx + ny}$ by hypothesis........(1).

Again, $\frac{x + y}{p} : \frac{mx + ny}{q} :: 11 : 17$, and if v be the value of x units of gold

$$x \text{ units of gold} : y \text{ units of gold} :: v : \frac{vy}{x};$$

$$\therefore \text{ value of } y \text{ units of silver} = \frac{vy}{13x};$$

$$\therefore \frac{v + \frac{vy}{13x}}{p} : \frac{mv + \frac{nvy}{13x}}{q} :: 11 : 17,$$

$$\text{or } \frac{p}{q} = \frac{221x + 17y}{143mx + 11ny} \text{ .... (2).}$$

Equating (1) and (2), we get, by reduction,

$$13mx^2 + 35nxy + ny^2 = 21mxy \text{ .... (3).}$$

Again, by question,

$$\frac{2x + y}{p'} = \frac{2mx + ny}{q'}, \text{ and } \therefore \frac{p'}{q'} = \frac{2x + y}{2mx + ny} \text{ .. (4),}$$

$$\text{and } \therefore \frac{26x + y}{p'} : \frac{26mx + ny}{q'} :: 7 : 11;$$

$$\therefore \frac{p'}{q'} = \frac{286x + 11y}{182mx + 7ny} \text{ .... (5).}$$

From (4) and (5), we get, by reduction,

$$52mx^2 + 68nxy + ny^2 = 40mxy \text{ ...... (6).}$$

From (6) and (3), we get, by subtraction and dividing off by x,

$$39mx + 33ny = 19my; \ \therefore n = \frac{19my - 39mx}{33y} \text{ .. (7).}$$

Substituting (7) in (3), we get

$$936x^2 + 67xy - 19y^2 = 0,$$

which being reduced to factors becomes

$$(104x + 19y)(9x - y) = 0;$$

$$\therefore 104x + 19y = 0,$$

$$\text{and } 9x - y = 0, \text{ or } \frac{x}{y} = \frac{1}{9}; \ \therefore x : y :: 1 : 9,$$

for the first mixture.

Again, since $\frac{x+y}{mx+ny} = \frac{221x+17y}{143mx+11ny}$;

$$\therefore \frac{\frac{x}{y}+1}{m\frac{x}{y}+n} = \frac{221\frac{x}{y}+17}{143m\frac{x}{y}+11n}.$$

But $\frac{x}{y} = \frac{1}{9}$ which we have just found;

$$\therefore \frac{10}{m+9n} = \frac{374}{143m+99n};$$

$$\therefore 1056m = 2376n,$$

$$\text{or } 4m = 9n;$$

$$\therefore \frac{m}{n} = \frac{9}{4}.$$

$$\text{But } \frac{x}{y} = \frac{1}{9};$$

$$\therefore \frac{mx}{ny} = \frac{1}{4},$$

$$\text{or } mx : ny :: 1 : 4;$$

therefore in the *first mixture* the proportion of gold to silver is as 1 : 9;

in the *second* mixture as 1 : 4.

Cor. But by the introduction of the other factor it appears that this question admits of two solutions, and that of the *second* is

$$x : y :: 19 : 104,$$

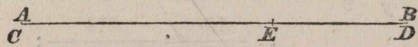
$$\text{and } mx : ny :: 36803 : 79976.$$

(139). From the towns C and D two travellers A and B, set out to meet each other, A beginning his journey 3 hours sooner than B. They meet at the distance of 20 miles from D, and A reaches D one hour before B arrives at C. The next day, B having begun to return at an early hour, meets A, who had then performed only $\frac{1}{7}$ of his journey back; and, notwithstanding a subsequent delay of 3 hours, arrives at D soon enough, were it necessary, to go 28 miles farther before A reaches C. Required the distance between the towns, and the rate at which each person travels.

Let distance $CD = x$,

rate of $A = r$,

rate of $B = r'$,



A is on the road $3r$ before B starts and they meet at 20 miles from D;

$$\therefore \frac{x - 3r - 20}{r} = \text{time before } A \text{ meets } B,$$

$$\frac{20}{r'} = \text{time before } B \text{ meets } A,$$

and as these times are equal

$$\frac{x - 3r - 20}{r} = \frac{20}{r'}, \text{ or } r'x - 3rr' - 20r' - 20r = 0 \,..(1),$$

and A reaches D one hour before B arrives at C, on the same grounds; therefore

$$\frac{20}{r} = \frac{x - 20}{r'} - 1, \text{ or } rr' + 20r' + 20r - rx = 0 \,..(2).$$

Similarly, A performs $\frac{6x}{7}$ of the journey while B performs $\frac{x}{7} + 28$ with a delay of 3 hours; hence

$$\frac{\frac{6x}{7}}{r} = \frac{\frac{x}{7} + 28}{r'} - 3, \text{ or } 6r'x + 21rr' - 196r - rx = 0 \,..(3).$$

From (1) and (3), we get

$$3rr' - 120r' + 76r + rx = 0.$$

From (2)

$$-rr' - 20r' - 20r + rx = 0;$$

$$\therefore r' = \frac{24r}{25 - r}.$$

Substituting this value in (1), we get

$$6x - 13r = 245 \quad \ldots\ldots\ldots\ldots(4).$$

Again, substituting this value of r' in (2), we get

$$x(25 - r) = 4r + 980 \quad \ldots\ldots\ldots\ldots(5).$$

Substituting in (5) the value of x from (4), we get, the quadratic in r,

$$13r^2 - 56r = 245;$$

$$\therefore\ r = 7 \text{ taking the positive sign};$$

$$\therefore\ r' = \frac{24r}{25 - r} = 9\tfrac{1}{3}.$$

But from (1) and (2),

$$x = \frac{2rr'}{r' - r} = \frac{2 \times 7 \times 9\frac{1}{3}}{9\frac{1}{3} - 7} = 56;$$

therefore, rate of A is 7 miles and of B is $9\frac{1}{3}$ miles per hour, and the distance from C to D is 56 miles.

(140). A person sets off to walk from Cambridge to London, at the rate of $3\frac{1}{2}$ miles an hour. In $2\frac{1}{2}$ hours he is overtaken by the Times, and at 10 minutes before 10 o'clock by the Fly; after resting $2\frac{1}{2}$ hours on the road, he starts again, and meets the Times on its return from London, and half a mile farther, the Fly, at 20 minutes past 5. Supposing the Times and Fly to have started from Cambridge at 6 and half-past 7 o'clock respectively, and from London at 3, determine the distance from Cambridge to London, and the rates at which the coaches travelled.

Let x = velocity of Times,

y = ... Fly.

When the Times overtakes him he has been on the road $2\frac{1}{2}$ hours, and has therefore walked $(2\frac{1}{2})(3\frac{1}{2}) = \frac{35}{4}$ miles. Hence the Times has been $\frac{35}{4x}$ hours on the road, and as it started at 6, it must then be $6 + \frac{35}{4x}$, from which deducting $2\frac{1}{2}$ hours leaves $\frac{7}{2}+\frac{35}{4x}$ for the time at which he left Cambridge. Therefore at $9\frac{5}{6}$ hours he has walked

$$\left[9\tfrac{5}{6}-\left(\frac{7}{2}+\frac{35}{4x}\right)\right]3\tfrac{1}{2}=\frac{133}{6}-\frac{245}{8x}\text{ miles.}$$

But the Fly which started at $7\frac{1}{2}$ hours will have travelled $(9\frac{5}{6}-7\frac{1}{2})\,y$ or $\frac{7y}{3}$ miles, and the two distances are equal by the question;

$$\therefore\ \frac{7y}{3}=\frac{133}{6}-\frac{245}{8x},$$

$$\text{or } 8xy=76x-105\ \ldots\ldots\ldots\ldots(a).$$

At 20 minutes past 5, or 20 minutes past 17 hours, in order to reckon the hours all from the same epoch, he will have been walking

$$17\tfrac{1}{3}-2\tfrac{1}{2}-\left(\frac{7}{2}+\frac{35}{4x}\right)=\left(\frac{34}{3}-\frac{35}{4x}\right)\text{ hours,}$$

and therefore will have gone over $\left(\frac{34}{3}-\frac{35}{4x}\right)\frac{7}{2}$ miles, which added to the distance gone by the Fly in $2\frac{1}{3}$ hours will give the whole distance from Cambridge to London,

$$\left(\frac{34}{3}-\frac{35}{4x}\right)\frac{7}{2}+\frac{7}{3}y\ \text{ or }=\frac{119}{3}-\frac{245}{8x}+\frac{7y}{3}.$$

Now the previous $\frac{1}{2}$ mile will have taken him $\dfrac{\frac{1}{2}}{3\frac{1}{2}}=\frac{1}{7}$ hours to walk; therefore, when he meets the Times it will have been on the road from London $2\frac{1}{3}-\frac{1}{7}=2\frac{4}{21}$ hours, during which it will have gone over $2\frac{4}{21}x$ miles;

$$\therefore\ \frac{7y}{3}=2\tfrac{4}{21}x-\frac{1}{2},\ \text{whence } 98y=92x-21\ \ldots(\beta),$$

$$y=\frac{92x-21}{98}.$$

Substitute this in (a), then

$$\frac{4x(92x-21)}{49}=76x-105, \text{ whence } x=8\tfrac{3}{4}=\text{velocity of Times},$$

$$y=\frac{92x-21}{98}=8=\text{velocity of Fly},$$

$$\frac{119}{3}-\frac{245}{70}+\frac{56}{3}=54\tfrac{5}{6}=\text{distance from Cambridge to London.}$$

(141). A stable-keeper bought 2 horses for 50*l.*, and at the end of the year sold one of them for double, and the other for half what he gave for it. The former being well fed and lightly worked, produced for its hire only the half of what it cost him, and consumed in keep as much per cent. on its price as the hire of the other produced on its price, the latter being kept for $\frac{5}{8}$ of as many guineas as it sold for pounds. The keep of the two amounted to 33*l.*, and the whole sum that he made by the horses was 9 times his profit on the sale. What did each horse cost?

Let £x = cost of first, $50-x$ of second, y = keep of first.

$$\text{He sold them for } 2x+\frac{50-x}{2}=\frac{3x+50}{2}.$$

$$\text{Profit on sale}=\frac{3x+50}{2}-50=\frac{3x-50}{2},$$

$$\text{also } x:y::50-x:\frac{(50-x)y}{x}=\text{hire of second},$$

$$\text{and } \frac{x}{2}=\text{hire of first.}$$

$$\text{Again, } y+\frac{5}{8}\,\frac{50-x}{2}\times\frac{21}{20}=33 \text{ by problem},$$

$$\text{or } 64y-21x=1062\ldots\ldots\ldots\ldots(1);$$

$$\therefore \text{ whole profit}=\frac{3x+50}{2}+\frac{x}{2}+\frac{50-x}{1}\times\frac{y}{x}-50-33;$$

$$\therefore \frac{2x^2-58x+50y-xy}{x}=9\times\frac{3x-50}{2}\ \ldots.(2);$$

$$\therefore\ 4x^2 - 116x + 100y - 2xy = 27x^2 - 450x,$$

$$\text{from (1) } 23x^2 - 334x = (50 - x)\frac{(1062 + 21x)}{32},$$

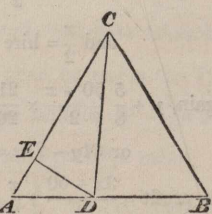
$$\text{whence } 757x^2 - 10676x = 53100,$$

$$\text{and } x = 18l. \text{ cost of one;}$$

$$\therefore\ 50 - x = 32l. \text{ cost of the other.}$$

(142). From the middle of a town two streets branched off and crossed a river that ran in a straight course, by two bridges, *A* and *B*. From their junction a sewer, equally inclined to both streets led to a point in the river at the distance of 6 chains from the bridge *A*, and a distance from *B* less by 11 chains than the length of the sewer; the expense of making it amounted to as many pounds per chain, as there were chains in the street leading to *A*. The sewer, however, being insufficient to carry off the water, an additional drain was made from a point in this street distant 4 chains from the bridge *A*, which entered the river at the same point with the sewer, and was equally inclined to the river and sewer. Now it was found that a drain down the middle of each street, at the rate of 9*l.* per chain, would have cost only 54*l.* more than the expense of the sewer. Required the lengths of the streets and the sewer, and the distance of its mouth from the bridge *B*.

A and *B* the bridges, *AC*, *CB* the streets,



CD the sewer $= x$, $AC = y$, $BD = x - 11$,

$AD = 6$ chains, $AE = 4$ chains, expense of sewer $= xy$,

$$\text{then } \frac{AC}{CB} = \frac{AD}{DB} = \frac{6}{x-11}; \therefore BC = \frac{y}{6}(x-11),$$

$$\text{and } \frac{CD}{AD} = \frac{CE}{AE} = \frac{y-4}{4};$$

$$\therefore x = \frac{6}{4}(y-4) = \frac{3y}{2} - 6,$$

$$\text{then } \frac{3y}{2}(x-11) + 9y = xy + 54,$$

whence $xy - 15y = 108$, and $y^2 - 14y = 72$,

and $AC = y = 7 \pm 11 = 18$ chains,

and $DC = x = 27 - 6 = 21$ chains, and $DB = 10$ chains,

also $BC = 3 \times 10 = 30$ chains.

(143). *A*, *B*, and *C*, were three architects; *A* and *B* built 4 warehouses, with flat roofs, each a large one and each a small one; the width of the two large ones being the same, and also that of the two small ones. *A* built his as long and as high as they were broad: but *B* made the length and height of his small one equal to the breadth of his large one, in such a manner that the difference between the solid content of those built by *A* and those built by *B* was 73728 feet. *C* also built a warehouse upon a square plot of ground which was equal to the difference between the ground plots occupied by those which *A* built, and found that it would have just stood upon 2688 feet square, if he had added 8 times as many square feet to the ground plot as there were feet in its width. How many feet wide were the several buildings erected by *A*, *B*, and *C*?

Let x = height of *A*'s large one and width of *B*'s large one,

y = height of *B*'s large one and length and height of *A*'s small one,

then x^3 and y^3 are the solid contents of *A*'s, x^2y and xy^2 of *B*'s,

$$\text{then } x^3 + y^3 - x^2y - xy^2 = 73728 \text{ .... (1)}.$$

Difference of ground plots of $A = x^2 - y^2 = C$'s ground plot,

$$\text{then } x^2 - y^2 + 8\sqrt{(x^2 - y^2)} + 16 = 2688 + 16 = 2704;$$

$$\therefore \sqrt{(x^2 - y^2)} = 48, \text{ and } x^2 - y^2 = 2304;$$

$$\therefore \text{ from (1) } \frac{(x^2 - y^2)(x - y)}{x^2 - y^2} = \frac{73728}{2304}, \text{ or } x - y = 32;$$

$$\therefore \frac{x^2 - y^2}{x - y} = \frac{2304}{72}, \text{ or } x + y = 72;$$

$\therefore$ width of A's and B's large one = 52 feet,
width of A's small one = 20 feet,
width of C's = $\sqrt{(2304)}$ = 48 feet.

(144). Into a cubical cistern, eight feet deep, and having an unknown leak, water is poured from two pumps worked by two men, A and B. They pump together till the vessel is half filled, when B falls asleep; A continues pumping till it is three-fourths filled, and then goes away; B afterwards waking, finds the cistern still half full, and after pumping till it is again three-fourths filled departs also, and meeting with A charges him with leaving his work unfinished. They return together and find the water $1\frac{1}{2}$ inch lower than when B left. The leak is now discovered and stopped, and by their joint efforts the vessel is filled in half the time in which they had worked together at first. They remark also that $10\frac{1}{3}$ hours had elapsed since they first began pumping, and that B had worked alone twice as long as A had. Supposing that a cubic foot contains $15\frac{5}{8}$ gallons, find the quantity of water thrown in by each pump.

Let x = gallons per minute by A, y ditto by B, z by leak, then time of filling one half the cistern at first

$$= \frac{512}{2} \times \frac{125}{8} \times \frac{1}{x + y - z} = \frac{32 \times 125}{x + y - z},$$

then time of filling one quarter by $A = \dfrac{16 \times 125}{x - z}$,

.... $B = \dfrac{16 \times 125}{y - z}$,

...... of emptying $\dfrac{1}{4}$ by leak = $\dfrac{16 \times 125}{z}$,

$$\text{then time of } \left(1\tfrac{1}{2}\text{ inch} = \frac{1}{8}\text{ foot}\right) = \frac{64}{8} \times \frac{125}{8} \times \frac{1}{z} = \frac{125}{z},$$

time by both in second case

$$= \frac{64}{1} \times \frac{17}{8} \times \frac{125}{8}\ \frac{1}{x+y} = \frac{17 \times 125}{x+y};$$

$$\therefore \frac{32}{x+y-z} = \frac{34}{x+y};$$

$$\therefore z = \frac{x+y}{17},\ x - z = \frac{16x-y}{17},\ y - z = \frac{16y-x}{17},$$

$$\text{but } \frac{1}{x-z} : \frac{1}{y-z} :: 1 : 2;\ \therefore \text{ whence } z = 2y - x = \frac{x+y}{17};$$

$$\therefore x = \frac{11y}{6},\ x - z = \frac{5y}{3},\ y - z = \frac{5y}{6},$$

$$\text{and } x + y - z = \frac{8y}{3};$$

$$\therefore \text{whole time} = \frac{125}{1}\left\{\frac{12}{y} + \frac{48}{5y} + \frac{96}{5y} + \frac{17 \times 6}{y} + \frac{6}{y}\right\} = 31 \times 20';$$

$$\therefore y = \frac{25 \times 744}{20 \times 31} = \frac{5}{1} \times \frac{186}{31} = 30 \text{ gallons by } B;$$

$$\therefore x = \frac{11}{6} \times \frac{30}{1} \text{ gallons} = 55 \text{ gallons by } A.$$

(145). A steamboat sets out from London 3 miles behind a wherry; and when at the same distance a-head of it, overtakes a barge floating down the stream, and reaches Gravesend $1\frac{1}{2}$ hours afterwards. Having waited, in order to land the passengers, $\frac{1}{5}$ of the time of coming down, it starts to return, and meets the wherry in $\frac{3}{4}$ hour, the barge being then $5\frac{1}{4}$ miles a-head of the steamboat; and occupies the same time in returning to London that the wherry did in coming down from thence. Required the distance between London and Gravesend, and the rate of each vessel, the tide being supposed to run out uniformly the whole time.

Let x = rate of boat, y = rate of wherry,
m = rate of barge = rate of tide,
since time of boat returning = time of wherry going down

$$x - m = y + m, \text{ or } m = \frac{x-y}{2};$$

$$\therefore \text{ speed of boat down} = x + \frac{x-y}{2} = \frac{3x-y}{2} \quad \cdots (1),$$

$$\text{speed up} = \frac{x+y}{2},$$

$$\text{time of boat down} = \frac{6}{x-y} + \frac{3}{2},$$

$$\text{time till it takes barge} = \frac{6}{x-y}.$$

Time between boat passing barge and meeting wherry on return

$$\frac{1}{5}\left(\frac{6}{x-y} + \frac{3}{2}\right) + \frac{3}{2} + \frac{3}{4} = \frac{6}{5(x-y)} + \frac{51}{20} \quad \ldots.(2).$$

Distance moved by barge in this time

$$= \frac{x-y}{2}\left\{\frac{6}{5(x-y)} + \frac{51}{20}\right\} \text{ miles},$$

and the boat has come up the river $\frac{3}{4} \times \frac{x+y}{2}$ miles;

therefore distance from Gravesend when boat passed barge down the river

$$= \frac{51(x-y)}{40} + \frac{3}{5} + \frac{3(x+y)}{8} + \frac{21}{4} = \frac{3}{2} \times \frac{3x-y}{2} \text{ from (1)},$$

$$\text{whence } 4x + y = 39 \quad \ldots\ldots\ldots\ldots(3).$$

Time of wherry going $(3 + 5\frac{1}{4})$ miles

$$= \frac{33}{4y} = \frac{6}{5(x-y)} + \frac{51}{20} \text{ from (2)},$$

$$\text{whence from (3) } \frac{11}{4(39-4x)} = \frac{2}{5(5x-39)} + \frac{17}{20},$$

whence $x = 9$ and $y = 3$; $\therefore m = 3$ miles,
and speed of boat with tide = 12 miles per hour,
and of wherry = 6,

and distance from London to Gravesend

$$= (1 + 1\tfrac{1}{2})\ 12 \text{ miles} = 30 \text{ miles}.$$

(146). A regiment in which there are between 10 and 100 officers and twice as many serjeants, in clearing the streets during a revolution, loses 2 officers; and after storming a barricade, in which 3 more fall and one accidentally joins, is obliged to retreat, and loses other 3. Whilst engaged in clearing the streets, the liability of an officer to fall is half that of a serjeant or private; but at the barricades as 4 : 3, and in the retreat as 3 : 4. Also, on leaving their barracks, the number, whose left-hand digits are the serjeants and right the officers, is 20 more than 10 times the number of privates; but in coming back (including the officer who joined) it is only 13 more, the number of officers being still greater than 10. Required the state of the regiment at first.

Let x = officers, $2x$ = serjeants, y = soldiers,

then $\frac{2}{x}$ = liability of each officer to fall in the street;

$\therefore \frac{4}{x}$ = ... serjeant and private;

$\therefore 2x\left(\frac{4}{x}\right) = 8$ = number of serjeants who fell,

and $y\left(\frac{4}{x}\right) = \frac{4y}{x}$ = ... privates ....

At the barricade,

$x - 2$ is the number of officers, of whom 3 fell;

$\therefore \frac{3}{x-2}$ = liability of each officer to fall,

and $\frac{3}{4}\left(\frac{3}{x-2}\right) = \frac{9}{4(x-2)}$ = ... serjeant and private,

hence $\frac{9}{4(x-2)}(2x-8) = \frac{9(x-4)}{2(x-2)}$ = serjeants who fell,

$\frac{9}{4(x-2)}\left(y - \frac{4y}{x}\right) = \frac{9y(x-4)}{4x(x-2)}$ = privates ....

In the retreat one officer has joined; therefore as 5 had fallen there are now $x-4$ and 3 fallen, hence

$$\frac{3}{x-4} = \text{liability of officers to fall;}$$

$$\therefore \frac{4}{3}\left(\frac{3}{x-4}\right) = \frac{4}{x-4} = \ldots\ldots\ldots \text{ serjeants and privates,}$$

$$\text{hence } \frac{4}{x-4}\left\{2x-8-\frac{9(x-4)}{2(x-2)}\right\} = \frac{8x-34}{x-2}$$

$$= \text{number of serjeants who fell,}$$

$$\frac{4}{x-4}\left\{y-\frac{4y}{x}-\frac{9y(x-4)}{4x(x-2)}\right\} = \frac{y}{x}\left(\frac{4x-17}{x-2}\right)$$

$$= \text{number of privates who fell.}$$

$$x+1-8 = x-7 = \text{number of officers remaining,}$$

$$2x-8-\frac{9x-4}{2x-2}-\frac{8x-34}{x-2} = \frac{4x^2-49x+136}{2(x-2)}$$

$$= \text{number of serjeants remaining,}$$

$$y-\frac{4y}{x}-\frac{9y}{4x}\cdot\frac{x-y}{x-2}-\frac{y}{x}\cdot\frac{(4x-17)}{(x-2)} = \frac{y}{x}\left\{\frac{4x^2-49x+136}{4(x-2)}\right\}$$

$$= \text{number of privates remaining;}$$

therefore by condition of question

$$100(2x)+x = 10y+20 \ldots\ldots\ldots\ldots(A),$$

$$100\left\{\frac{4x^2-49x+136}{2(x-2)}\right\}+x-7$$

$$= 10\frac{y}{x}\left\{\frac{4x^2-49x+136}{4(x-2)}\right\}+13\ldots\ldots(B),$$

from (A) we get

$$201x-10y = 20,$$

from (B) we get

$$(201x-10y)(4x^2-49x+136) = 39x^2-24x;$$

$$\therefore 20(4x^2-49x+136) = 39x^2-24x,$$

which gives $x = 20$ = officers,

$2x = 40$ = serjeants,

$y = 400$ = privates.

(147). In the first eruption of the Thames into the Tunnel, the water rose in the vertical shaft 8 times as fast as in the horizontal levels in the second eruption. It was observed also, that if the levels at the second influx had been 110 feet longer, the velocities of the water ascending in them in the first and second eruptions and when thus in creased would have formed an arithmetical progression, the common difference of which was $\frac{1}{9}$ of the difference of the velocities with which the water rose in the shaft in the two eruptions (that in the second being the greater); and had the levels been of the same length at the first as at the second eruption, the whole time of filling would have been half as long again. The tunnel consists of two equal levels, terminated by a vertical shaft of twice the breadth of either of them; the sections of the shaft and levels are supposed to be squares, and the height of the shaft above the upper surface of the levels to equal twice its breadth; the time of filling in the first eruption being 10 minutes less than in the second. Find the time in the second, and the dimensions of the tunnel.

Let x = length of level first time,

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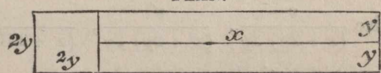
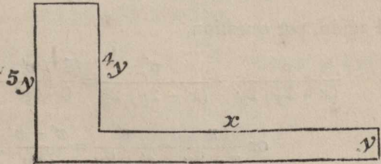


y = width = height of level;

$\therefore$ $2y$ = width = length of shaft,

$4y$ = height of shaft above top of level;

ELEVATION.



therefore content of levels including lower portion of shaft is

$$(x + 2y)\ (2y)\ (y).$$

Let x' = length of level second time; therefore content of levels with part of shaft as before will be

$$(x' + 2y)\ (2y)\ (y).$$

Let v and v' = volume of water poured in per minute at first and second eruption respectively, then the contents of shaft (upper portion) being $16y^3$, we have

$$\frac{(x + 2y)\ 2y^2 + 16y^3}{v} = \text{whole time of filling at first eruption,}$$

$$\frac{(x' + 2y')\ 2y^2 + 16y^3}{v'} = \ldots\ldots\ldots\ldots\ldots\ldots\ \text{second}\ \ldots\ldots;$$

$$\therefore\ \frac{(x + 2y)\ 2y^2 + 16y^3}{v} = \frac{(x' + 2y)\ 2y^2 + 16y^3}{v'} - 10\ ..(A).$$

Now the horizontal section of shaft = $(2y)^2 = 4y^2$,

and the horizontal section of levels and shaft together $= (x' + 2y)\ 2y$ at the second eruption; therefore, per question,

$$\frac{v}{4y^2} = 8\ \frac{v'}{(x' + 2y)\ 2y},\ \text{or}\ \frac{v}{2y} = \frac{8v'}{x' + 2y}\ \ldots.(B),$$

$$\text{also}\ \frac{v}{(x + 2y)\ 2y},\ \frac{v'}{(x' + 2y)\ 2y},\ \frac{v'}{(x' + 110 + 2y)\ 2y}$$

form an arithmetical progression;

$$\therefore\ \frac{v}{(x + 2y)\ 2y} + \frac{v'}{(x' + 110 + 2y)\ 2y} = \frac{2v'}{(x' + 2y)\ 2y},$$

$$\text{or}\ \frac{v}{x + 2y} + \frac{v'}{x' + 110 + 2y} = \frac{2v'}{x' + 2y}\ \ldots.\ (C).$$

And again, per question,

$$\frac{v}{(x + 2y)\ 2y} - \frac{v'}{(x' + 2y)\ 2y} = \frac{1}{9}\left(\frac{v'}{4y^2} - \frac{v}{4y^2}\right),$$

$$\text{or}\ \frac{v}{x + 2y} - \frac{v'}{x' + 2y} = \frac{v' - v}{18y}\ \ldots\ldots..(D).$$

Finally, if the tunnel had been as long at the first as at the second eruption, the time of filling would have been

$$\frac{(x'+2y)\,2y^2+16y^3}{v};$$

$$\therefore \frac{(x'+2y)\,2y^2+16y^3}{v}=\frac{3}{2}\left\{\frac{(x+2y)\,2y^2+16y^3}{v}\right\},$$

$$\text{or } x'+10y=\frac{3}{2}(x+10y),$$

$$\text{or } 2x'=3x+10y \text{ ....} (E).$$

From (D) we get

$$v'\left(\frac{1}{x'+2y}+\frac{1}{18y}\right)=v\left(\frac{1}{x+2y}+\frac{1}{18y}\right) \text{ .... } (\alpha),$$

from (B) we get

$$v(x'+2y)=v'(16y) \text{ ....} (\beta).$$

Multiply and cancel vv', then

$$1+\frac{x'+2y}{18y}=\frac{16y}{x+2y}+\frac{8}{9}, \text{ or } \frac{16y}{x+2y}=\frac{1}{9}+\frac{x'+2y}{18y} \text{ ..} (\gamma).$$

From (C) we have

$$v'\left(\frac{2}{x'+2y}-\frac{1}{x'+110+2y}\right)=v\left(\frac{1}{x+2y}\right).$$

Multiply by (B), and therefore

$$2-\frac{x'+2y}{x'+110+2y}=\frac{16y}{x+2y} \text{ .... } (\delta).$$

From (E) and (γ)

$$\frac{16y}{x+2y}=\frac{1}{9}+\frac{2x'+4y}{36y}=\frac{1}{9}+\frac{3x+14y}{36y};$$

$$\therefore 576y^2=4xy+8y^2+3x^2+28y^2+20xy,$$

$$\text{or } x^2+8xy=180y^2;$$

$$\therefore x=10y,$$

$$\text{and by } (E)\ x'=20y,$$

$$\text{by } (\delta)\ 2-\frac{22y}{22y+110}=\frac{16y}{12y}, \text{ whence } y=10;$$

$$\therefore x=10y=100,$$

$$x'=20y=200,$$

$$\text{by } (B)\ \frac{v}{20} = \frac{8v'}{220}, \text{ or } v = \frac{8}{11}v',$$

which value put in (*A*) gives

$$\frac{(120)(200) + 16000}{\frac{8}{11}v'} = \frac{(220)(200) + 16000}{v'} - 10,$$

$$\text{whence } 10v' = 5000,$$

$$v' = 500;$$

therefore time of filling second time

$$= \frac{(x' + 2y)\,2y^2 + 16y^3}{v'} = \frac{60000}{500} = 120 \text{ minutes} = 2 \text{ hours}.$$

(148). There are two vessels, *P* and *Q*, containing quantities of fluid in the ratio of 4 : 21, which consist of mixtures of wine and spirits in different proportions: *A* pumps a certain quantity out of *P* into *Q*, and then *B* pumps out into *Q* $\frac{3}{4}$ of what was left; the strength of the mixture *Q* is then found to be $\frac{12}{13}$ of its original strength. Now, if when *A* stopped, *B* had pumped the same quantity as before out of *Q* into *P*, instead of *P* into *Q*, the strength of the mixture *P* would have been exactly a mean proportional between the original strengths of *P* and *Q*; and *B* would have pumped out the same quantity of wine that he did before of spirits. Find the proportions of the wine and spirits in each of the vessels at first; and compare the quantities pumped out by *A* and *B*, the strength of spirits being supposed 3 times that of wine.

Let x and y = wine and spirits in *P* originally,

$x' \ldots y' = \ldots\ldots\ldots\ldots\ldots\ldots$ *Q* ....,

z = quantity pumped by *A*,

$z' = \ldots\ldots\ldots\ldots\ldots\ldots$ *B*.

Then, per question,

$$\frac{3}{4}(x + y - z) = z',$$

$$\text{or } x + y = \frac{4z' + 3z}{3} \text{ ....(1)},$$

$$\text{also } x' + y' = \frac{21}{4}(x + y);$$

$$\therefore x' + y' = \frac{28z' + 21z}{4} \text{ .... (2)},$$

then A pumps out of P

$$\frac{zx}{x+y} \text{ wine and } \frac{xy}{x+y} \text{ spirits,}$$

and B pumps out of P

$$\frac{z'x}{x+y} \text{ .... } \frac{z'y}{x+y} \text{ ......;}$$

therefore Q finally contains

$$x' + \frac{(z+z')\,x}{x+y} \text{ wine and } y' + \frac{(z+z')\,y}{x+y} \text{ spirits;}$$

if therefore the strength of wine be taken as units, we have

$$\text{original strength of } Q = \frac{x' + 3y'}{x' + y'},$$

$$\text{final .... } = \frac{x' + \frac{(z+z')\,x}{x+y} + 3\left\{y' + \frac{(z+z')\,y}{x+y}\right\}}{x' + y' + z + z'}.$$

Hence, by question,

$$\frac{x' + \frac{(z+z')\,x}{x+y} + 3\left\{y' + \frac{(z+z')\,y}{x+y}\right\}}{x' + y' + z + z'} = \frac{12}{13}\,\frac{x' + 3y'}{x' + y'},$$

whence, and by (1) and (2),

$$\frac{x + 3y}{x' + 3y'} = \frac{20z' + 27z}{273\,(z + z')} \text{ .... (3).}$$

But if B had pumped z' out of Q, he would have poured into P

$$\frac{\left(x' + \frac{zx}{x+y}\right)z'}{x' + y' + z} \text{ wine and } \frac{\left(y' + \frac{zy}{x+y}\right)z'}{x' + y' + z} \text{ spirits.}$$

The strength of P would now have been

$$\frac{x+3y-\dfrac{(x+3y)z}{x+y}+\dfrac{(x'+3y')z'}{x'+y'+z}+\dfrac{zz'(x+3y)}{(x+y)(x'+y'+z)}}{x+y-z+z'},$$

which is to be a mean proportional between P and Q's original strengths. Therefore

$$\left(\frac{x+3y}{x+y}\right)\left(\frac{x'+3y'}{x'+y'}\right)$$

$$=\left[\frac{(x+3y)(x+y-z)(x'+y'+z)+(x'+3y')(x+y)z'+(x+3y)zz'}{(x+y)(x'+y'+z)(x+y-z+z')}\right]^2$$

$$\ldots\ldots\ldots\ldots(4),$$

and since B would have pumped out the same quantity of wine as he had before of spirits

$$\frac{\left(x'+\dfrac{zx}{x+y}\right)z'}{x'+y'+z}=\frac{z'y}{x+y},$$

$$\text{or } \frac{x}{y}=\frac{y'+z}{x'+z}\ldots\ldots\ldots\ldots\ldots\ldots(5).$$

Now (1) and (4) give

$$\frac{x+3y}{x+y}\cdot\frac{x'+3y'}{x'+y'}=\left[\frac{(x+3y)(4x'+4y'+7z)+3(x'+3y')(x+y)}{7(x+y)(x'+y'+z)}\right]^2,$$

$$\text{or } \frac{x+3y}{x'+3y'}\cdot\frac{x+y}{x'+y'}=\left[\frac{\dfrac{x+3y}{x'+3y'}\cdot\{4.(x'+y')+7z\}+3(x+y)}{7(x'+y'+z)}\right]^2,$$

which by (1), (2), and (3) becomes

$$\frac{20z'+27z}{273(z+z')}\cdot\frac{4}{21}=\left(\frac{\dfrac{20z'+27z}{273(z+z')}\,28(z+z')+4z'+3z}{\dfrac{7}{4}(28z'+25z)}\right)^2$$

$$=16\left[\frac{80z'+108z+156z'+117z}{273(28z'+25z)}\right]^2,$$

$$\text{whence } \frac{260z'+351z}{z+z'}=\frac{4(55696z'^2+50625z^2+106200zz')}{784z'^2+625z^2+1400zz'},$$

which can be put under the form

$$(4z' - 5z)(4736z'^2 + 8020zz' + 3375z^2) = 0.$$

This last equation is verified by making $4z' - 5z = 0$, whence

$$z : z' :: 4 : 5;$$

therefore substituting for z' in the foregoing equations, they become

$$(1) \qquad x + y = \frac{8z}{3} \ldots\ldots\ldots\ldots\ldots\ldots \text{(i)},$$

$$(2) \qquad x' + y' = 14z \ldots\ldots\ldots\ldots\ldots \text{(ii)},$$

$$(3) \qquad \frac{x + 3y}{x' + 3y'} = \frac{16}{189} \ldots\ldots\ldots\ldots\ldots \text{(iii)},$$

$$(5) \qquad \frac{x}{y} = \frac{y' + z}{x' + z} \ldots\ldots\ldots\ldots \text{(iv)},$$

from (iv) we get

$$\frac{x + y}{y} = \frac{x' + y' + 2z}{x' + z} = \frac{x' + y' + 2z}{x' + y' + z - y'};$$

therefore by (i) and (ii)

$$\frac{8z}{3y} = \frac{16z}{15z - y'}, \text{ or } 6y = 15z - y' \ldots\ldots\ldots \text{(v)},$$

from (iii) we get

$$\frac{x + y + 2y}{x' + y' + 2y'} = \frac{16}{189}, \text{ or } \frac{\frac{8z}{3} + 2y}{14z + 2y'} = \frac{16}{189},$$

$$\text{or } 48y' - 567y = 420z.$$

Now (v) gives $48y' + 288y = 720z$;

$$\therefore \qquad 855y = 300z,$$

$$\text{or } y = \frac{20}{57}z,$$

$$\text{by (i) } x = \frac{8z}{3} - y = \frac{8z}{3} - \frac{20z}{57}, \text{ or } x = \frac{132}{57}z,$$

$$\text{by (v) } y' = 15z - 6y; \qquad \therefore y' = \frac{735}{57}z,$$

$$\text{by (ii) } x' = 14z - y'; \qquad \therefore x' = \frac{63}{57}z;$$

$$\therefore\ x : y :: 132 : 20 :: 33 : 5,$$

$$\text{or } x' : y' :: 63 : 735 :: 3 : 35;$$

$\therefore$ wine : spirits in P :: 33 : 5,

wine : spirits in Q :: 3 : 35,

quantity pumped by A : quantity by B :: 4 : 5.

INDETERMINATE EQUATIONS AND PROBLEMS.

XII. Find the integral values of the unknown quantities which satisfy the following equations:—

(1). $7x - 5y = 9$.

$$\text{Here } x = \frac{5y + 9}{7} = y + 1 - \frac{2y - 2}{7},$$

and $\frac{2y - 2}{7}$ must be a whole number; let then $y = 1$;

$$\therefore\ \frac{2y - 2}{7} = 0, \text{ and } x = 2.$$

If $y = 8$, then $x = 8 + 1 - 2 = 7$,

and the values of y will be 1, 8, 15, 22, &c.,

. . . . x . . . . . . 2, 7, 12, 17, &c.

(2). $5x - 7y = 21$.

$$\therefore\ x = y + 4 + \frac{2y + 1}{5}, \text{ and if } y = 2,\ x = 7,$$

if $y = 7$, 12, &c., $x = 14$, 21, &c.

(3). $11x + 35y = 500$.

$$\therefore\ x = 45 - 3y - \frac{2y - 5}{11}, \text{ and if } y = 8,\ x = 20,$$

if $y = 19$, $x = -15$;

therefore the first are the only positive integral values of x and y.

(4). $14x - 5y = 7$.

$$\therefore\ y = 3x - 1 - \frac{x + 2}{5}, \text{ if } x = 3,\ y = 7,$$

if $x = 8$, 13, &c., $y = 21$, 35, &c.

(5). $27x + 16y = 1600$.

$$\therefore\ y = 100 - \frac{27x}{16};\ \therefore \text{ if } x = 16,\ y = 73,$$

$$\text{if } x = 32, 48,\ y = 46, 19.$$

(6). $80x - 17y = 39$.

$$\therefore\ y = 5x - 2 - \frac{5x + 5}{17};\ \therefore \text{ if } x = 16,\ y = 73,$$

$$\text{if } x = 33, 46, \&c.,\ y = 153, 233.$$

(7). $3x + 5y = 26$.

$$\therefore\ x = 8 - 2y + \frac{y + 2}{3},\ \text{if } y = 1,\ x = 7,$$

$$y = 4,\ x = 2,\ \text{and if } y = 7,\ x = -3.$$

(8). $19x - 117y = 11$.

$$\therefore\ x = 6y + \frac{3y + 11}{19},\ \text{if } y = 9,\ x = 56,$$

$$\text{if } y = 28, 47, \&c.,\ x = 173, 290, \&c.$$

(9). $11x + 13y = 190$.

$$\therefore\ x = 17 - y - \frac{2y - 3}{11},\ \text{if } y = 7,\ x = 9,$$

if $y = 18$, $x = -4$, so that 7 and 9 are the only integral values.

(10). $7x + 13y = 71$.

$$\therefore\ x = 10 - 2y + \frac{y + 1}{7},\ \text{if } y = 6,\ x = -1,$$

$$\text{if } y = 13,\ x = -14,$$

and there are no positive integral values of x and y.

(11). $13x + 16y = 97$.

$$\therefore\ y = -x + 6 + \frac{3x + 1}{16},\ \text{if } x = 5,\ y = 2,$$

$$\text{if } x = 21,\ y = -11.$$

(12). $11x + 7y = 108$.

$$\therefore\ y = 15 - x - \frac{4x - 3}{7},\ \text{if } x = 6,\ y = 6,$$

$$\text{if } x = 13,\ y = -5.$$

(13). $49x - 15y = 11.$

$$\therefore\ y = 3x - 1 + \frac{4(x+1)}{15};\ \therefore \text{ if } x = 14,\ y = 45,$$

$$\text{if } x = 29, 44, \&c.,\ y = 94, 143, \&c.$$

(14). $10x + 9y = 1000.$

$$\therefore\ x = 100 - \frac{9y}{10}, \text{ if } y = 10,\ x = 91,$$

$$\text{if } y = 20, 30, 40, 50, \&c.,\ x = 82, 73, 64, 55, \&c.$$

(15). $20x - 21y = 38.$

$$\therefore\ x = y + 2 + \frac{y-2}{20}, \text{ if } y = 22,\ x = 25,$$

$$\text{if } y = 42, 62, \&c.,\ x = 25, 46, 67, \&c.$$

(16). $19x + 11 = 14y.$

$$\therefore\ y = x + 1 + \frac{5x-3}{14}, \text{ if } x = 9,\ y = 13,$$

$$\text{if } x = 23, 37, \&c.,\ y = 32, 51, \&c.$$

(17). $11x - 18y = 63.$

$$\therefore\ x = y + 6 + \frac{7y-3}{11}, \text{ if } y = 2,\ x = 9,$$

$$\text{if } y = 13, 24, 35, \&c.,\ x = 27, 45, 63, \&c.$$

(18). $5x + 7y = 29.$

$$\therefore\ x = 6 - y - \frac{2y+1}{5}, \text{ if } y = 2,\ x = 3,$$

$$\text{if } y = 7,\ x = -4.$$

(19). $2x + 3y = 35.$

$$x = 17 - y - \frac{y-1}{2}, \text{ if } y = 1,\ x = 16,$$

$$\text{if } y = 3, 5, 7, 9, 11,\ x = 13, 10, 7, 4, 1.$$

(20). $8x + 13y = 159.$

$$\therefore\ x = 20 - y - \frac{5y+1}{8}, \text{ if } y = 3,\ x = 15,$$

$$\text{if } y = 11,\ x = 2.$$

(21). $17x - 49y + 8 = 0.$

$$\therefore\ x = 3y - \frac{2y+8}{17}, \text{ if } y = 13,\ x = 37,$$

$$\text{if } y = 30, 47, \&c.,\ x = 86, 135, \&c.$$

(22). $5x + 3y = 78$.

$$\therefore\ y = 26 - \frac{5x}{3},\ \text{if } x = 3,\ y = 21,$$

$$x = 6, 9, 12, 15,\ y = 16, 11, 6, 1.$$

(23). $4x - 5z + 10 = 0$.

$$4x - 5y = 10;\ \therefore\ x = y + 2 + \frac{y+2}{4},\ \text{if } y = 2,\ x = 5,$$

$$\text{if } y = 6, 10, 14, \text{\&c.},\ x = 10, 15, 20, \text{\&c.}$$

(24). $20x - 21y = 38$, and $3y + 4z = 34$.

$$y = \frac{20x - 38}{21} = \frac{34 - 4z}{3};$$

$$\therefore\ 20x - 38 = 238 - 28z,$$

$$\text{or } x = 14 - z - \frac{2z+1}{5},\ \text{and if } z = 2,\ x = 11,\ \text{and } y = 8\tfrac{2}{3},$$

$$z = 7,\ x = 4,\ \text{and } y = 2.$$

(25). $5x + 4y + z = 272$, and $8x + 9y + 3z = 656$.

$$z = 272 - 4y - 5x = \frac{656 - 8x - 9y}{3};$$

$$\therefore\ 3y + 7x = 160,\ \text{and } y = 53 - 2x - \frac{x-1}{3},$$

$$\text{if } x = 1,\ y = 51,\ z = 63;\ \text{if } x = 4,\ y = 44,\ z = 76,$$

$$\text{if } x = 7,\ y = 37,\ z = 89;\ \text{if } x = 10,\ y = 30,\ z = 102, \text{\&c.}$$

(26). $x + 2y + 3z = 20$, and $4x + 5y + 6z = 47$.

$$x = 20 - 2y - 3z = \frac{47 - 5y - 6z}{4};$$

$$\therefore\ 3y + 6z = 33;\ \therefore\ y = 11 - 2z,\ \text{and if } z = 1,\ y = 9,\ x = -1,$$

$$\text{if } z = 2,\ y = 7,\ x = 0;\ \text{if } z = 3,\ y = 5,\ x = 1, \text{\&c.}$$

(27). $2x + 14y - 7z = 341$, and $10x + 4y + 9z = 473$.

$$x = \frac{341 + 7z - 14y}{2} = \frac{473 - 9z - 4y}{10},$$

$$\text{or } 2z - 3y = -56;\ \therefore\ z = -28 + \frac{3y}{2},\ \text{if } y = 2,\ z = -25,$$

$x = 169$; if $y = 4$, z is still negative, take $y = 20$, then $z = 2$, and $x = 37.5$, if $y = 22$, $z = 5$, and $x = 34$, &c.

(28). $2x + 5y + 3z = 108$, and $3x - 2y + 7z = 95$.

$$x = \frac{108 - 3z - 5y}{2} = \frac{95 - 7z + 2y}{3};$$

$$\therefore 19y - 5z = 134,$$

and $y = 7 + \frac{5z + 1}{19}$, and if $z = 15$, $y = 11$, and $x = 4$,

which are the only positive values.

(29). See Ex. (26).

(30). $6x + 7y + 4z = 122$, and $11x + 8y - 6z = 145$.

$$x = \frac{122 - 7y - 4z}{6} = \frac{145 - 8y + 6z}{11};$$

$$\therefore 29y + 80z = 472; \quad \therefore y = 16 - 3z + \frac{7z + 8}{29},$$

if $z = 3$, $y = 8$, $x = 9$; if $z = 32$, $y = -72$, $x = 498$.

(31). See Ex. (24).

(32). $7xy - 3y - 5x = 39$.

then $x = \frac{39 + 3y}{7y - 5}$, and if $y = 1$, $x = 21$,

if $y = 2$, $x = 5$, if $y = 3$, $x = 3$, if $y = 11$, $x = 1$.

(33). $3xy - 7x - 7y = 5$.

$x = \frac{7y + 5}{3y - 7}$, if $y = 3$, $x = 13$, if $y = 5$, $x = 5$.

(34). $10x + 9y + 7z = 58$.

$y = 6 - z - x + \frac{2z + 4 - x}{9}$, if $x = 1$, $z = 3$, and $y = 3$,

if $x = 2$, $z = 8$, and $y = -2$.

(35). $5xy = 3x + 24$.

$x = \frac{24}{5y - 3}$, and if $y = 1$, $x = 12$, if $y = 3$, $x = 2$,

which are the only integral values.

(36). $xy + 2x + 3y = 42$.

$y = \dfrac{42 - 2x}{x + 3}$, and if $x = 1$, $y = 10$, if $x = 3$, $y = 6$, if $x = 5$, $y = 4$.

(37). Find a number which, being divided by 39 and 56, leaves remainders 16 and 27 respectively.

Let N be the number, and x and y the quotients,

$$\text{then } N = 39x + 16 = 56y + 27\,;\ \therefore\ x = y + \frac{17y + 11}{39}.$$

$$\text{Let } 17y + 11 = 39m\,;\ \therefore\ y = 2m - 1 + \frac{5m + 6}{17}.$$

Let $5m + 6 = 17n$; $\therefore$ $m = 3n - 2 + \dfrac{2n + 4}{5}$; if $\therefore$ $n = 3$, $m = 9$, $y = 20$, and $x = 29$; $\therefore$ the number $= 20 \times 56 + 27 = 1147$, &c.

(38). Find the number of solutions of $11x + 15y - 1031$ in positive integers.

General Investigation. In turning $\dfrac{a}{b}$ into a continued fraction, let $\dfrac{n}{m}$ be the convergent immediately preceding $\dfrac{a}{b}$, then

$$am - bn = \pm 1 \ \ldots\ldots\ldots\ldots\ldots (1);$$

$$\therefore\ a(cm - bt) + b(at - cn) = 0 \ \ldots\ldots\ldots (2).$$

Hence, the *general* solution of

$$ax + by = 0 \ \ldots\ldots\ldots\ldots\ldots\ldots (3),$$

is seen, by comparing (2) with (3), to be

$$x = cm - bt,\ y = at - cn \ \ldots\ldots\ldots\ldots (4);$$

t being an *indeterminate*, and a taken for that coefficient which is positive when the 2nd side of (1) is + 1.

And since, for positive integral values of x, y, we must have

$$\frac{cm}{b} > t < \frac{cn}{a},$$

the *number* of solutions (N) of (3) in *positive integers* will be

$$N = \left\{\frac{cm}{b}\right\} - \left\{\frac{cn}{a}\right\} \ \ldots\ldots\ldots\ldots\ldots (5),$$

where $\left\{\frac{cm}{b}\right\}$ is put for the *integral part* of $\frac{cm}{b}$, and, if $\frac{cm}{b}$ is an *integer*, it must be diminished by 1, or we must consider $\frac{b}{b}$ as a *fraction*, and *reject it*, if zero values be excluded.

In the following solutions:

$$\text{In (38)}, \frac{a}{b} = \frac{15}{11}, \frac{n}{m} = \frac{4}{3}, c = 1031;$$

$$\therefore N = \{281\tfrac{2}{11}\} - \{274\tfrac{4}{15}\} = 7.$$

$$\text{In (41)}, \frac{a}{b} = \frac{2}{3}, \frac{n}{m} = \frac{1}{2}, c = 35;$$

$$\therefore N = \{23\tfrac{1}{3}\} - \{17\tfrac{1}{2}\} = 6.$$

$$\text{In (44)}, \frac{a}{b} = \frac{21}{5}, \frac{n}{m} = \frac{4}{1}, c = 800;$$

$$\therefore N = \{159\tfrac{5}{5}\} - \{152\tfrac{8}{21}\} = 7.$$

$$\text{In (45)}, \frac{a}{b} = \frac{21}{5}, \frac{n}{m} = \frac{4}{1}, c = 2800;$$

$$\therefore N = \{559\tfrac{5}{5}\} - \{533\tfrac{1}{3}\} = 26.$$

$$\text{In (51)}, \frac{a}{b} = \frac{9}{13}, \frac{n}{m} = \frac{2}{3}, c = 2000;$$

$$\therefore N = \{461\tfrac{2}{13}\} - \{444\tfrac{2}{9}\} = 17.$$

$$\text{In (55)}, \frac{a}{b} = \frac{4}{3}, \frac{n}{m} = \frac{1}{1}, c = 39;$$

$$\therefore N = \{12\tfrac{3}{3}\} - \{9\tfrac{3}{4}\} = 3.$$

$$\text{In (72)}, \frac{a}{b} = \frac{21}{5}, \frac{n}{m} = \frac{4}{1}, c = 240;$$

$$\therefore N = \{47\tfrac{8}{8}\} - \{45\tfrac{5}{7}\} = 2.$$

$$\text{In (73)}, \frac{a}{b} = \frac{21}{20}, \frac{n}{m} = \frac{1}{1}, c = 1600;$$

$$\therefore N = \{79\tfrac{20}{20}\} - \{76\tfrac{4}{13}\} = 3.$$

$$\text{In (74)}, \frac{a}{b} = \frac{21}{5}, \frac{n}{m} = \frac{4}{1}, c = 20000;$$

$$\therefore N = \{3999\tfrac{5}{5}\} - \{3809\tfrac{11}{21}\} = 190.$$

$$\text{In (75)}, \frac{a}{b} = \frac{100}{21}, \ \frac{n}{m} = \frac{19}{4}, \ c = 10000;$$

$$\therefore N = \{1904\tfrac{16}{21}\} - \{1900\} = 4.$$

Or find the least and greatest values of x or y, and we have

$$y = 69 - x + \frac{4x - 4}{15}, \text{ and if } x = 1, y = 68;$$

$\therefore$ the least value of $x = 1$, and the greatest value of $y = 68$, again,

$$x = 94 - y - \frac{4y + 3}{11}, \text{ and if } y = 2, x = 91;$$

$\therefore$ the least value of y is 2, and greatest value of x is 91, also, if

$$y = 13, \ x = 76,$$

and the series would be $y = 2, 13, 24 \ldots\ldots\ldots\ldots 68$,

and $x = 91, 76 \ldots\ldots\ldots\ldots\ldots 1$,

by an arithmetical progression, we have

$$68 = 2 + (n - 1)\ 11,$$

$$\text{or } n - 1 = 6; \ \therefore n = 7.$$

(39). If $x = 3$, $y = 5$, find the possible number of integral solutions of the equations $3x + 4y = 29$, and $7x - 2y = 11$.

$$x = 10 - y - \frac{y + 1}{3}, \text{ and if } y = 2, x = 7,$$

$$\text{also } y = 7 - x + \frac{x + 1}{4}, \text{ and if } x = 3, y = 5;$$

$$\therefore 5 = 2 + (n - 1)\,3; \ \therefore (n - 1) = 1, \text{ and } n = 2,$$

which is easily seen, for if we take $y = 5$, then $x = 3$,

from $7x - 2y = 11$,

$$\text{we have } y = 3x - 4 + \frac{x - 1}{2}, \text{ and if } x = 3, 5, 7, 9, \&c.$$

$$y = 6, 13, 20, \&c.,$$

and the number of solutions is unlimited.

(40). Find two fractions, with 7 and 9 for their denominators, whose sum is $\frac{19}{21}$; and three fractions, having denominators, 3, 4, 5, whose sum is $\frac{47}{60}$.

$$\text{By problem, } \frac{x}{7} + \frac{y}{9} = \frac{19}{21}, \text{ or } 9x + 7y = 57;$$

$$\therefore y = 8 - x - \frac{2x - 1}{7}, \text{ and if } x = 4, y = 3,$$

and the fractions are

$$\frac{4}{7} + \frac{1}{3} = \frac{19}{21}.$$

$$\text{Again, let } \frac{x}{3} + \frac{y}{4} + \frac{z}{5} = \frac{47}{60},$$

$$\text{then } x = \frac{47}{20} - \frac{3(5y + 4z)}{20} = 2 - \frac{15y + 12z - 7}{20},$$

and if $y = 1$, $z = 1$, and $x = 1$;

$$\therefore \frac{1}{3} + \frac{1}{4} + \frac{1}{5} = \frac{47}{60}.$$

(41). Find the number of integral solutions of $2x + 3y = 35$.

$$x = 18 - y - \frac{y + 1}{2}, \text{ and if } y = 1, x = 16,$$

$$\text{also } y = 12 - x + \frac{x - 1}{3}, \text{ and if } x = 1, y = 11;$$

$$\therefore 11 = 1 + (n - 1)2; \ \therefore n - 1 = 5, \text{ and } n = 6.$$

(42). Find the number of integral solutions of $20x + 15y + 6z = 171$.

$$x = 8 - y + \frac{5y - 6z + 11}{20}, \text{ if } z = 1, y = 3, x = 6,$$

also $y = 7$, $x = 3$; if $y = 1$, $z = 6$ or 16,
and $x = 6$ or 3, if $z = 11$, $y = 3$, $x = 3$, if $z = 6$, $y = 5$, $x = 3$.
and there are six positive integral solutions.

(43). See Ex. (37).

(44). In how many different ways can 20*l.* be paid in half-guineas and half-crowns?

Take the sums in sixpences, and let x be the number of half-guineas, and y the number of half-crowns;

$$\therefore 21x + 5y = 800;$$

$$\therefore y = 160 - 4x - \frac{x}{5}, \text{ and if } x = 5,\ y = 139,$$

$$\text{also } x = 38 - \frac{5y-2}{21}, \text{ and if } y = 13,\ x = 35;$$

$$\therefore 35 = 5 + (n-1)\,5, \text{ and } n - 1 = 6; \therefore n = 7.$$

(45). In how many ways can 140*l.* be paid in guineas and five-shilling pieces only?

Let x = number of guineas, y = number of crowns;

$$\therefore 21x + 5y = 2800,$$

$$\text{and } y = 560 - \frac{21x}{5}, \text{ and if } x = 5,\ y = 539,$$

$$\text{also } x = 133 - \frac{5y-7}{21}, \text{ and if } y = 14,\ x = 130;$$

$$\therefore 130 = 5 + (n-1)\,5, \text{ and } n - 1 = 25; \therefore n = 26.$$

(46). A number of men and women contributed to a charity 50*l.*; each man gave 19*s.*, each woman 11*s.* How many were there of each?

Let x = number of men, y = number of women,

$$\text{then } 19x + 11y = 50 \times 20, \text{ and } y = 90 - 2x + \frac{3x+10}{11},$$

and if $x = 4$, $y = 84$; if $x = 15$, $y = 65$,

if $x = 26$, $y = 46$, and if $x = 37$, $y = 27$,

and if $x = 48$, $y = 8$.

(47). A person bought sheep and lambs for 8 guineas; the sheep cost 26*s.* each, the lambs 15*s.* How many of each did he buy?

Let x = number of sheep, and y = number of lambs,

$$\text{then } 26x + 15y = 168,$$

$$\text{then } y = 11 - 2x + \frac{4x+3}{15}, \text{ if } x = 3,\ y = 6,$$

which is the only solution.

(48). A farmer selling live stock consisting of oxen and horses, sold each ox for 8*l.*, each horse for 27*l.*, and all the oxen for 97*l.* more than all the horses. How many oxen and horses did he sell?

Let x = number of oxen, y = number of horses,

$$\text{then } 8x = 27y + 97;$$

$$\therefore x = 12 + 3y + \frac{3y + 1}{8}, \text{ and if } y = 5,\ x = 29,$$

$$\text{if } y = 13,\ x = 56, \text{ \&c., \&c.}$$

(49). Required to pay 50*l.* in guineas and three-shilling pieces only.

Let x = number of guineas, y = number of three-shilling pieces,

$$\text{then } 21x + 3y = 50 \times 20, \text{ and } y = 333 + 7x + \frac{1}{3},$$

and the problem is therefore impossible.

(50). It is required to find a number such that if it is divided by 11 the remainder is 3, and if by 17 the remainder is 10.

Let N = the number, and x and y the quotients,

$$\text{then } N = 11x + 3 = 17y + 10;\ \therefore x = 2y - \frac{5y - 7}{11}, \text{ if } y = 8,\ x = 13,$$

$$\text{if } y = 19,\ x = 30, \text{ and the number of solutions is infinite,}$$

$$\text{and } N = 146 = 333 = \text{\&c.}$$

(51). Find the number of solutions that $9x + 13y = 2000$ will admit of in positive integers.

$$x = 222 - y - \frac{4y - 2}{9}, \text{ if } y = 5,\ x = 215,$$

$$\text{also } y = 154 - x + \frac{4x - 2}{13}, \text{ and if } x = 7,\ y = 149;$$

$$\therefore 149 = 5 + (n - 1)\,9;\ \therefore n - 1 = 16, \text{ and } n = 17.$$

(52). Find a number which, being divided by 39, leaves a remainder 16, and by 56 a remainder 24.

Let N = the number, and x and y the quotients,

$$\text{then } 39x + 16 = 56y + 24, \text{ then } x = y + \frac{17y + 8}{39}.$$

Let $17y + 8 = 39m$; $\therefore\ y = 2m + \dfrac{5m - 8}{17}$, let $5m - 8 = 17n$;

$$\therefore\ m = 3n + 1 + \frac{2n + 3}{5},\ \text{and if } n = 1,\ m = 5,$$

$$y = 11,\ \text{and } x = 16;$$

$$\therefore\ \text{the number} = 39 \times 16 + 16 = 640,\ \text{and so on,}$$

$$\text{if } n = 6,\ m = 22,\ y = 50,\ x = 72.$$

(53). Show that the solution of $ax + by = c$ in positive integers is always possible if a be prime to b, and $c > ab - (a + b)$.

Let $c - (ab - a - b) = f$, and $x + 1 = x'$, then $y = a - 1 - y'$,

$$\text{where } y' = \frac{ax' - f}{b},\ \text{or } ax' - by' = f\ \ldots\ldots(1).$$

Hence $ax + by = 0$ is solvable in positive integers, if such values of y' can be found from (1), that $y = (a - 1) - y' =$ a positive integer or zero; therefore $y' \leqq (a - 1)$. Now, as in the solution of (38), we find, for the *general solution* of (1),

$$x' = bt \pm mf,\ \ y' = at \pm nf \ldots\ldots\ldots\ldots (2),$$

where t is an indeterminate, and $\dfrac{n}{m}$ the convergent preceding $\dfrac{a}{b}$. But it is obvious that one or more of the values of y' in (2) must be $< a$; therefore $y = (a - 1) - y' =$ zero or a positive integer; and therefore $ax + by = c$ is always solvable in positive integers, if a is prime to b, and $c > ab - a - b$.

It must, however, be understood that zero values are not excluded here: thus, if $3x + 2y = 4$, where $4 > 3 \times 2 - (3 + 2)$, the only solution in positive integers is $x = 0$, $y = 2$.

(54). Find two numbers such that their sum shall be equal to the sum of their squares.

Let x and y be the numbers, and $y = mx$;

$$\therefore\ x(m + 1) = (m^2 + 1)\,x^2;$$

$$\therefore\ x = \frac{m + 1}{m^2 + 1},\ \text{and if } m = 1,\ x = 1,\ \text{if } m = 2,\ x = \frac{3}{5},\ m = 3,\ x = \frac{2}{5}:$$

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$$\therefore \text{ if } x = 1,\ y = 1,\ x = \frac{3}{5},\ y = \frac{6}{5},\ \&c. = \&c.,$$

and the numbers are 1, 1; $\frac{3}{5}, \frac{6}{5}$; $\frac{2}{5}, \frac{6}{5}$; $\frac{5}{17}, \frac{20}{17}$, and so on.

(55). How many fractions are there having 3 and 4 as denominators, whose sum is $3\frac{1}{4}$?

$$\text{Let the fractions be } \frac{x}{3} + \frac{y}{4} = \frac{13}{4};\ \therefore\ 4x + 3y = 39,$$

$$\text{and } y = 13 - \frac{4x}{3},\ \text{and if } x = 3,\ y = 9,$$

$$\text{also } x = 10 - y + \frac{y-1}{4},\ \text{and if } y = 1,\ x = 9;$$

$$\therefore\ 9 = 3 + (n-1)\,3;\ \therefore\ n - 1 = 2,\ \text{and } n = 3.$$

(56). Find a number which, being divided by 3, 4, 5, successively, leaves the remainders 2, 3, 4.

Let N = the number, and x, y, z the quotients;

$$\therefore\ N = 3x + 2 = 4y + 3 = 5z + 4;$$

$$\therefore\ x = y + \frac{y+1}{3},\ \text{and if } y = 2,\ x = 3,\ \text{if } y = 14,\ x = 19,$$

$$\text{also } z = y - \frac{y+1}{5}\ (\text{if } y = 14) = 11;\ \therefore\ N = 59,\ 119,\ \&c.$$

(57). Find three square numbers which form an arithmetic progression.

Let the numbers be x^2, m^2x^2, and n^2x^2,

$$\text{then } m^2 - 1 = n^2 - m^2;\ \therefore\ m^2 = \frac{n^2+1}{2},$$

and if $n = 7$, $m = 5$, and x may be any number, and the three terms are 1, 25, 49.

Or let the three terms be x^2, y^2, z^2,

$$\text{then } x^2 - y^2 = y^2 - z^2,$$

$$\text{or } (x-y)(x+y) = (y-z)(y+z).$$

$$\text{Let } \frac{m}{n} = \frac{y+z}{x+y} = \frac{x-y}{y-z},$$

$$\text{whence } y = \frac{nz - mx}{m - n} = \frac{nx + mz}{m + n},$$

$$\text{and } (m^2 - n^2 + 2mn)\, x = (2mn - m^2 + n^2)\, z,$$

$$\text{and if } x^2 = (m^2 - n^2 - 2mn)^2,\ z^2 = (m^2 - n^2 + 2mn)^2,$$

$$\text{and } y^2 = \frac{x^2 + z^2}{2} = (n^2 + m^2)^2,$$

and the common difference $= 2mn\,(n^2 - m^2)$.

(58). Find a perfect number, or one which is equal to the sum of all the numbers which divide it without remainder.

Let $x^n y$ be the required number,

and its divisors are $1, x, x^2 \ldots x^n, y\,(1 + x + x^2 \ldots x^{n-1})$,

$$\text{which equal } \frac{x^{n+1} - 1}{x - 1} + y\,\frac{x^n - 1}{x - 1} = x^n y \text{ by problem;}$$

$$\therefore\ x^{n+1} - 1 + y\,(x^n - 1) = y\,(x^{n+1} - x^n);$$

$$\therefore\ y = \frac{x^{n+1} - 1}{x^{n+1} - 2x^n + 1},$$

and that y may be a whole number, let

$x^{n+1} - 2x^n = 0$, or $x = 2$; $\therefore\ y = 2^{n+1} - 1$, and if $n = 0$, $y = 1$, $n = 1$, $y = 3$, then $xy = 6 = 1 + 2 + 3$, the sum of its divisors, $n = 2$, $y = 7$, then $x^2 y = 28 = 14 + 7 + 4 + 2 + 1$, and so on.

(59). Find the least number which, being divided by 17 and 26, shall leave remainders 7 and 13 respectively.

Let N be the number, and x and y the quotients,

$$\text{then } N = 17x + 7 = 26y + 13, \text{ or } x = y + \frac{3\,(3y + 2)}{17},$$

$$\text{and } y = 5,\ x = 8,$$

and the number $= 17 \times 8 + 7 = 143 = 26 \times 5 + 13$.

(60). Required two square numbers whose difference shall be equal to a given square number (b^2).

$$\text{Let } x^2 - y^2 = \text{a square} = (x - my)^2,$$

$$\text{then } x^2 - y^2 = x^2 - 2mxy + m^2 y^2;$$

$$\therefore\ 2mx = (m^2 + 1)\, y, \text{ and if } y = 2m,\ x = m^2 + 1;$$

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$$\therefore\ (m^2+1)^2 - 4m^2 = (m^2-1)^2, \text{ and if } m = 1,\ y = 2,\ x = 3,$$

$$\text{if } m = 2,\ y = 4, \text{ and } x = 5, \text{ and } 5^2 - 4^2 = 9 = 3^2,$$

$$\text{and if } m = 3,\ y = 6,\ x = 10, \text{ and so on,}$$

$$\text{if } x^2 - y^2 = \frac{(m^2+1)^2 y^2 - 4m^2y^2}{4m^2} = b^2,$$

$$\text{then } y^2 = \frac{4m^2b^2}{(m^2-1)^2},\ x^2 = \frac{(m^2+1)^2}{4m^2} \times \frac{4m^2b^2}{(m^2-1)^2} = \frac{(m^2+1)^2 b^2}{(m^2-1)^2},$$

$$\text{if } m = 2,\ y^2 = \frac{16b^2}{9},\ x^2 = \frac{25b^2}{9},$$

$$\text{and if } b^2 = 64,\ x^2 - y^2 = \frac{64(25-16)}{9} = 64.$$

(61). Find the least whole number which, being divided by 28, 19, and 15, shall leave remainders respectively 19, 15, 11.

Let N be the number, and x, y, z the quotients,

$$\text{then } N = 28x + 19 = 19y + 15 = 15z + 11;$$

$$\therefore\ z = y + \frac{4(y+1)}{15},$$

$$\text{and if } y = 14,\ z = 18,\ y = 29,\ z = 37,\ y = 44,\ z = 56,$$

$$\text{but } x = y - \frac{9y+4}{28}, \text{ and if } y = 40,\ x = 27,\ y = 68,\ x = 46,$$

and now we have to find when the values of y in the two series become equal, or

$$14 + (n-1)\,15 = 40 + (m-1)\,28,$$

$$\text{whence } 15n = 28m + 13,$$

$$\text{and } n = 2m + 1 - \frac{2(m+1)}{15},\ m = 14,\ n = 27;$$

$$\therefore\ 14 + 26 \times 15 = 404 = y;$$

$$\therefore\ x = 404 - \frac{3640}{28} = 274, \text{ and } z = 404 + \frac{4(405)}{15} = 512,$$

$$\text{and } N = 28 \times 274 + 19 = 19 \times 404 + 15 = 7691.$$

(62). Find the numbers which, when divided by 2 and 3, leave the remainders 1 and 2 successively.

Let N be the number, and x and y the quotients,

$$\text{then } N = 2x + 1 = 3y + 2; \;\therefore\; x = y + \frac{y+1}{2},$$

$$\text{and if } y = 1,\; x = 2,\; y = 3,\; x = 5;$$

$$\therefore\; N = 5,\ 11,\ \&c.$$

(63). Find the least number which, being divided by 3, 5, 7, and 2, shall leave for remainders 2, 4, 6, and 0, respectively.

Let N be the number, and x, y, z, ω the quotients,

$$\text{then } N = 3x + 2 = 5y + 4 = 7z + 6 = 2\omega;$$

$$\therefore\; \omega = 3z + 3 + \frac{z}{2},\ \text{and if } z = 2,\; \omega = 10,\; z = 4,\; \omega = 17,$$

$$(A)\; y = z + \frac{2(z+1)}{5},\ \text{and if } z = 4,\; y = 6,\; z = 9,\; y = 13,$$

and the equal values of z in these two series are 4;

$$\therefore\; 2\omega = 7 \times 4 + 6 = 34,\ \text{also } 5y + 4 = 30 + 4 = 34,$$

$$x = 2y + 1 - \frac{y+1}{3},\ \text{and if } y = 2,\; x = 4,\; y = 5,\; x = 9,$$

and to find the corresponding values of y in this equation and (A), we have

$$6 + (n-1)\,7 = 2 + (m-1)\,3,\ \text{or } 3m = 7n;$$

$$\therefore\; m = \frac{7n}{3},\ \text{and if } n = 3,\; m = 7;$$

$$\therefore\; y = 20,\ \text{and } x = 34;\; \therefore\; 5y + 4 = 104,\ \text{and } 3x + 2 = 104;$$

$\therefore$ the least number is 104.

(64). Find two numbers, such that if the first be multiplied by 17, and the second by 26, the first product shall exceed the second by 7.

Let x and y be the numbers, then $17x - 26y = 7$;

$$\therefore\; x = y + 1 + \frac{9y-10}{17};\ \text{let } 9y - 10 = 17m;\; \therefore\; y = 2m + 1 - \frac{m-1}{9},$$

$$\text{and if } m = 10,\; y = 20,\ \text{if } m = 1,\; y = 3,$$

$$\text{and } x = 3 + 1 + 1 = 5,\ \text{and the numbers are } 85 - 78 = 7,$$

$$\text{if } y = 20,\; x = 21 + 10 = 31,$$

$$\text{and the numbers are } 17 \times 31 - 26 \times 20 = 7.$$

(65). In a foundry two kinds of cannon are cast; each of the first sort weighs 16 cwt., and each of the second 25 cwt.; and for the second kind there are used 1 cwt. of metal less than for the first. How many are there of each kind?

Let x and y be the numbers,

$$\text{then } 16x - 25y = 1; \quad \therefore\ x = y + \frac{9y+1}{16}, \text{ let } 9y + 1 = 16m;$$

$$\therefore\ y = 2m - \frac{2m+1}{9}, \text{ and if } m = 4,\ y = 7, \text{ and } x = 11,$$

$$m = 13,\ y = 23,\ x = 36.$$

(66). Divide the number 1591 into two parts, such that the one may be divisible by 23 and the other by 34.

$$\text{Let } 23x + 34y = 1591;$$

$$\therefore\ x = 69 - y - \frac{11y-4}{23}.$$

$$\text{Let } \frac{11y-4}{23} = m; \quad \therefore\ y = 2m + \frac{m+4}{11},$$

$$\text{and if } m = 7,\ y = 15, \text{ and } x = 47,$$

$$\text{or if } m = 18,\ y = 38, \text{ and } x = 13;$$

$$\therefore\ 23 \times 47 + 34 \times 15 = 1591, \text{ also } 23 \times 13 + 38 \times 34 = 1591.$$

(67). Required three numbers, such that if the first is multiplied by 7, the second by 9, and the third by 11, the first product shall be 1 less than the second, and 2 greater than the third.

Let x, y, z be the numbers,

$$\text{then } 7x = 9y - 1, \text{ also } 7x = 11z + 2;$$

$$\therefore\ 7x = 9y - 1 = 11z + 2, \text{ and } y = z + \frac{2z+3}{9}, \text{ and if } z = 3,\ y = 4,$$

$$\text{and } x = \frac{36-1}{7} = 5.$$

(68). Required three numbers, such that the sum of their products by the numbers 3, 5, 7, respectively, may be 560, and the sum of their products by the numbers 9, 25, 49, respectively, may be 2920.

Let x, y, z be the numbers,

then $3x + 5y + 7z = 560$, and $9x + 25y + 49z = 2920$;

$\therefore\ 9x + 15y + 21z = 1680$,

whence $10y + 28z = 1240$, or $5y + 14z = 620$;

$\therefore\ y = 124 - \dfrac{14z}{5}$, and if $z = 5$, $y = 110$, and x is negative.

Let then $z = 15$, then $y = 82$, and $x = 15$.

Let $z = 30$, then $y = 40$, and $x = 50$.

(69). Find a number, N, which, being divided by 11, gives the remainder 3; divided by 19, gives the remainder 5; and divided by 29, gives the remainder 10.

Let $N = 11x + 3 = 19y + 5 = 29z + 10$;

$\therefore\ x = 2y - \dfrac{3y - 2}{11}$, and if $y = 8$, $x = 14$, $y = 19$, $x = 33$,

also $x = 3z + 1 - \dfrac{4(z+1)}{11}$, and if $z = 10$, $x = 27$, $z = 21$, $x = 56$.

Let then $14 + (n - 1)\,19 = 27 + (m - 1)\,29$,

or $n = m + \dfrac{10m + 3}{19}$, and if $m = 13$, $n = 20$,

or $x = 14 + 19 \times 19 = 27 + 12 \times 29 = 375$;

$\therefore$ the corresponding values of $z = 10 + 12 \times 11 = 142$,

or .... $y = 8 + 19 \times 11 = 217$;

$\therefore\ N = 11x + 3 = 19y + 5 = 29z + 10$

$= 11 \times 375 + 3 = 19 \times 217 + 5 = 142 \times 29 + 10 = 4128$.

(70). Find two square numbers whose sum is a square.

Let $x^2 + y^2 = \text{a square} = (x - my)^2 = x^2 - 2mxy + m^2y^2$..(1);

$\therefore\ x = \dfrac{(m^2 - 1)\,y}{2m}$, and if $y = 2m$, $x = m^2 - 1$,

and from (1) $(m^2 - 1)^2 + 4m^2 = (m^2 + 1)^2$,

and if $m = 2$, $(m^2 + 1)^2 = 25$,

and $x^2 = 9$, and $y^2 = 16$, or if $m = 4$, $(m^2 + 1) = 17^2$,

and so on, if $m = 4$, $x^2 = 15^2$, $y^2 = 8^2$, and $x^2 + y^2 = 289 = 17^2$.

(71). Find two square numbers whose difference is a square.

$$\text{Let } x^2 - y^2 = (x - my)^2 = \text{a square};$$

$$\therefore x = \frac{(m^2+1)\,y}{2m}, \text{ and if } y = 2m,\ x = m^2 + 1,$$

and $x^2 - y^2 = (m^2+1)^2 - 4m = (m^2-1)^2$, and as above in (60),

if $m = 2$, $x^2 = 25$, $y^2 = 16$, and $x^2 - y^2 = 9$, and so on.

(72). In how many ways can a person pay a bill of 12*l.* in crowns and guineas?

Let x = number of guineas, and y = number of crowns;

$$\therefore 21x + 5y = 240,$$

$$\text{and } y = 48 - \frac{21x}{5}, \text{ and if } x = 5,\ y = 27,$$

$$\text{also } x = 11 - \frac{5y-9}{21}, \text{ and if } y = 6,\ x = 10;$$

$$\therefore 10 = 5 + (n-1)\,5, \text{ and } 5n = 10;\ \therefore n = 2.$$

(73). In how many ways can 80*l.* be paid in guineas and sovereigns?

Let x = number of guineas, y = number of sovereigns,

$$21x + 20y = 1600,$$

$$\text{and } y = 80 - \frac{21x}{20}, \text{ and if } x = 20,\ y = 59,$$

$$\text{also } x = 76 - y + \frac{y+4}{21}, \text{ if } y = 17,\ x = 60;$$

$$\therefore 60 = 20 + (n-1)\,20;\ \therefore n - 1 = 2, \text{ and } n = 3.$$

(74). In how many ways can 1000*l.* be paid in crowns and guineas?

If x be the number of crowns, y the number of guineas,

$$\text{then } 5x + 21y = 20000;$$

$$\therefore x = 4000 - \frac{21y}{5}, \text{ and if } y = 5,\ x = 3979,$$

$$\text{also } y = 952 - \frac{5x-8}{21}, \text{ and if } x = 10,\ y = 950;$$

$$\therefore 5 + (n-1)\,5 = 950, \text{ and } n = 190.$$

(75). In how many ways can $500l.$ be paid in guineas and $5l.$ notes?

Let x = number of guineas, y = number of $5l.$ notes,

$$21x + 100y = 10000;$$

$$\therefore\ y = 100 - \frac{21x}{100}, \text{ and if } x = 100,\ y = 79,$$

$$\text{also } x = 476 - 5y + \frac{5y+4}{21}, \text{ and if } y = 16,\ x = 400;$$

$$\therefore\ 79 = 16 + (n-1)\,21;\ \therefore\ n - 1 = 3, \text{ and } n = 4.$$

(76). A regiment of foot (less than 1000), when put in column with 13 men in front, wanted 9 men to complete the last rank; with 15 men in front, then 14 were wanting; but with 17 men in front, the ranks were complete. What was the strength of the regiment?

Let N be the number of men, x, y, z, the number of ranks,

$$\text{then } N = 13x - 9 = 15y - 14 = 17z;$$

$$\therefore\ x = y + \frac{2y-5}{13}, \text{ and if } y = 9,\quad x = 10,\ N = 121,$$

$$y = 22,\ x = 25,\ N = 316,$$

$$y = 61,\ x = 70,\ N = 901 = 17 \times 53;$$

$\therefore\ z = 53$, and 901 is the first number that answers the conditions.

(77). Is it possible to pay $100l.$ in guineas and moidores (worth $27s.$) only?

$$\text{Let } 21x + 27y = 2000, \text{ then } x = 95 - y - \frac{6y-5}{21}.$$

$$\text{Let } 6y - 5 = 21m;\ \therefore\ y = 3m + 1 + \frac{3m-1}{6}.$$

$$\text{Let } \frac{3m-1}{6} = n;\ \therefore\ m = 2n + \frac{1}{3},$$

and no integral value of n can make m also integral; therefore the problem is impossible.

(78). I owe a person 1*s*., and have nothing about me but sovereigns, and he nothing but dollars, worth 4*s*. 3*d*. each; how must I discharge the debt?

Let x = number of sovereigns, y = number of dollars;

$$\therefore\ 20x - \frac{17y}{4} = 1,\ \text{or } 17y = 80x - 4;$$

$$\therefore\ y = 5x - \frac{5x+4}{17},\ \text{and if } x = 6,\ y = 28,$$

$$\text{or } 6 \times 20 - 7 \times 17 = 1s.$$

(79). A man bought 20 birds for 20*s*., consisting at geese of 4*s*., quails at 6*d*., and snipes at 3*d*. each. How many had he of each?

Let x, y, z be the number of each,

$$\text{then } 4x + \frac{y}{2} + \frac{z}{4} = 20,\ \text{or } 16x + 2y + z = 80;$$

$$\text{also } x + y + z = 20;$$

$$\therefore\ 15x + y = 60;\ \therefore\ y = 60 - 15x,\ \text{and if } x = 1,\ y = 45,\ z = -26,$$

$$\text{but if } x = 2,\ y = 30,\ z = -12,$$

$$\text{or if } x = 3,\ y = 15,\ z = 2 \text{ are the possible solutions.}$$

(80). A jeweller wishes to mix gold of 14, 11, and 9 carats fine, so as to form a composition of 30 oz. in weight of 12 carats fine. Find all the ways in which this can be done in whole numbers.

Let x, y, z be the ounces respectively of each of the three,

$$\text{then } 14x + 11y + 9z = 360,$$

$$\text{also } 9x + 9y + 9z = 270;$$

$$\therefore\ 5x + 2y = 90,\ \text{and } y = 45 - \frac{5x}{2},$$

$$\text{and if } x = 2,\ y = 40,\ \text{and } z = -12,\ \text{which is impossible,}$$

$$\text{if } x = 12,\ y = 15,\ z = 3,\ \text{if } x = 14,\ y = 10,\ z = 6,$$

$$\text{if } x = 16,\ y = 5,\ z = 9;\ \therefore\ \text{there are three possible answers.}$$

(81). A person buys 100 head of cattle for 100*l.*, viz. oxen at 10*l.* each, cows at 5*l.* each, calves at 2*l.* each, and sheep at 10*s.* each. How many of each did he buy?

Let x, y, z, and w be the numbers respectively of oxen, cows, calves, and sheep,

$$\text{then } x + y + z + w = 100, \text{ also } 10x + 5y + 2z + \frac{w}{2} = 100;$$

$$\therefore 20x + 10y + 4z + w = 200, \text{ and } 19x + 9y + 3z = 100;$$

$$\therefore z = 33 - 3y - 6x - \frac{x-1}{3}, \text{ and if } x = 4, \ z = 8 - 3y,$$

$$\text{and if } y = 1, \ z = 5;$$

$\therefore$ 4 oxen + 1 cow + 5 calves + 90 sheep = 100,

or 1 ox + 1 cow + 24 calves + 74 sheep = 100.

(82). A farmer buys 120 head of cattle, pigs, goats, and sheep, for 400*l.* Each pig costs 4*l.* 10*s.*, each goat 3*l.* 5*s.*; and each sheep 1*l.* 5*s.* How many are there of each kind?

Let x, y, z be the numbers respectively of pigs, goats, and sheep;

$$\therefore x + y + z = 120; \ \therefore \frac{9x}{2} + \frac{13}{4}y + \frac{5}{4}z = 400,$$

$$\text{or } 18x + 13y + 5z = 1600, \text{ also } 5x + 5y + 5z = 600;$$

$$\therefore 13x + 8y = 1000; \ \therefore y = 125 - \frac{13x}{8},$$

$$\text{and if } x = 8, \ y = 112, \text{ and } z = 0,$$

$$\text{but if } x = 16, \ y = 99, \text{ and } z = 5.$$

(83). A person wishes to buy 20 animals for 20*l.*, sheep at 31*s.*, pigs at 11*s.*, and rabbits at 1*s.* each. In how many ways can he do so?

Let x, y, z be the numbers respectively of sheep, pigs, and rabbits;

$$\therefore x + y + z = 20, \text{ and } 31x + 11y + z = 400;$$

$$\therefore 30x + 10y = 380, \text{ or } 3x + y = 38;$$

$$\therefore y = 38 - 3x, \text{ and if } x = 7, \ y = 17, \text{ and } z = -4,$$

$$\begin{aligned} \text{but if } x &= 8, \quad y = 14, \text{ and } z = -2, \\ x &= 9, \quad y = 11, \qquad z = 0, \\ x &= 10, \; y = 8, \qquad z = 2, \\ x &= 11, \; y = 5, \qquad z = 4, \\ x &= 12, \; y = 2, \qquad z = 6, \end{aligned}$$

and there are only 3 possible ways of making the purchase.

(84). A farmer went to market with 100*l.* to buy 100 head of geese, sheep, and oxen, at 1*s.* each for the geese, 1*l.* for the sheep, and 5*l.* for the oxen. How many of each did he buy?

Let x, y, z be the whole numbers respectively of geese, sheep, and oxen,

$$\text{then } x + 20y + 100z = 2000, \text{ also } x + y + z = 100;$$

$$\therefore\ 19y + 99z = 1900,$$

$$\text{or } y = 100 - \frac{99z}{19}, \text{ and if } z = 19,\ y = 1, \text{ and } x = 80,$$

which are the only positive values.

(85). A farmer buys horses and oxen, and gives 31 crowns for each horse, and 20 crowns for each ox, and he finds that the oxen cost him 7 crowns more than the horses. How many of each did he buy?

Let x = number of horses, and y = number of oxen,

$$\text{then } 31x = 20y - 7;\ \therefore\ y = x + \frac{11x + 7}{20}, \text{ and if } x = 3,\ y = 5,$$

$$x = 23, 43, 63, \&c.,\ y = 36, 67, 98, \&c.$$

LOGARITHMIC AND EXPONENTIAL EQUATIONS.

XIII. Find the value of x and y in the following equations:—

(1). $12^x = 168$.

$$\therefore\ x \log 12 = \log 168,$$

$$\text{and } x = \frac{2\cdot225309}{1\cdot079181} = 2\cdot08.$$

(2). $10^x = 1250$.

$$\therefore\ x = \log 1250 = 3{\cdot}09691.$$

(3). $(15)^x = 1250,\ (16)^x = 1000$.

$$15^x = 1250;\ \therefore\ x = \frac{\log 1250}{\log 15} = \frac{3{\cdot}09691}{1{\cdot}176091} = 2{\cdot}6332,$$

$$16^x = 1000;\ \therefore\ x = \frac{3}{\log 16} = \frac{3}{1{\cdot}204120} = 2{\cdot}49045.$$

(4). $14^x = 63y,\ 17^x = 87y$.

$$x\log 14 = \log 63 + \log y,$$

$$x\log 17 = \log 87 + \log y;\ \therefore\ x = \frac{\log 87 - \log 63}{\log 17 - \log 14} = \frac{{\cdot}140178}{{\cdot}084321} = 1{\cdot}662,$$

$$\log y = 1{\cdot}66\,(1{\cdot}146128) - 1{\cdot}799341 = {\cdot}103231;$$

$$\therefore\ y = 1{\cdot}2745.$$

(5). $2^x \times 3^y = 560,\ 5x = 7y$.

We have $x\log 2 + y\log 3 = \log 560$, and $5x = 7y$;

$$\therefore\ 7y\log 2 + 5y\log 3 = 5\log 560;$$

$$\therefore\ y = \frac{5\log 560}{7\log 2 + 5\log 3} = \frac{13{\cdot}74094}{4{\cdot}492815} = 3{\cdot}0583;$$

$$\therefore\ x = \frac{7y}{5} = 4{\cdot}28162.$$

(6). $20^x = 100$.

$$\therefore\ x = \frac{2}{1{\cdot}301030} = 1{\cdot}537.$$

(7). $a^{bx+d} = c$.

$$a^{bx} \times a^d = c;\ \therefore\ bx\log a = \log c - d\log a, \text{ or } x = \frac{\log c - d\log a}{b\log a}.$$

(8). $2^x = 769$.

$$\therefore\ x = \frac{\log 769}{\log 2} = \frac{2{\cdot}885926}{{\cdot}301030} = 9{\cdot}586.$$

(9). $a^{mx} \times b^{nx} = c$.

$$\therefore\ (a^m \times b^n)^x = c;\ \therefore\ x = \frac{\log c}{m\log a + n\log b}.$$

(10). $a^{\sqrt{(x)}} = b$.

$$\therefore\ \sqrt{x} = \frac{\log b}{\log a};\ \therefore\ x = \left(\frac{\log b}{\log a}\right)^2.$$

(11). $\log x = \frac{1}{2}\log a - \frac{1}{4}\log b$.

$$\log x = \frac{1}{2}\log a - \frac{1}{4}\log b = \log\frac{a^{\frac{1}{2}}}{b^{\frac{1}{4}}}; \ \therefore \ x = a^{\frac{1}{2}}b^{-\frac{1}{4}}.$$

(12). $2\log x + 2\log y = 5, \ 2\log x - 2\log y = 1$.

$$\therefore \ \log x = 1{\cdot}5, \text{ and } x = 31{\cdot}625,$$
$$\text{also } \log y = 1; \ \therefore \ y = 10.$$

(13). $8^x \times 9^x = 4{\cdot}9$.

$$\therefore \ x = \frac{\log 4{\cdot}9}{\log 72} = \frac{{\cdot}690196}{1{\cdot}857333} = {\cdot}371606.$$

(14). $\log x = 2n\log a + 2m\log b - 2p\log c$.

$$\log x = \log\left(\frac{a^{2n} \times b^{2m}}{c^{2p}}\right); \ \therefore \ x = a^{2n}.b^{2m}.c^{-2p}.$$

(15). $2^{x+1} + 4^x = 80$.

$$2^{2x} + 2 \times 2x = 80; \ \therefore \ 2^{2n} + 2 \times 2^x + 1 = 81,$$
$$\text{and } 2^x = -1 \pm 9 = 8;$$
$$\therefore \ x = \frac{\log 8}{\log 2} = \frac{{\cdot}903090}{{\cdot}301030} = 3.$$

(16). $7^{\frac{x}{2}+\frac{y}{3}} = 2401, \ 6^{\frac{x}{4}+\frac{y}{2}} = 1296$.

$$7^{\frac{x}{2}+\frac{y}{3}} = 7^4; \ \therefore \ \frac{x}{2} + \frac{y}{3} = 4,$$

$$\text{also } 6^{\frac{x}{4}+\frac{y}{2}} = 6^4; \ \therefore \ \frac{x}{4} + \frac{y}{2} = 4; \ \therefore \ x = 4, \ y = 6.$$

(17). $(7 \times 9^{-\frac{2}{3}})^x = (864)^{\frac{1}{6}}$.

$$(7 \times 9^{-\frac{2}{3}})^x = \left\{\frac{7}{\sqrt[3]{(81)}}\right\}^x = (864)^{\frac{1}{6}};$$

$$\therefore \ x \log\frac{7}{\sqrt[3]{(81)}} = \frac{1}{6}\log 864, \text{ or } x = \frac{{\cdot}489419}{{\cdot}845098 - {\cdot}6361616} = 2{\cdot}342.$$

(18). $\log x = \log n + \log y, \ ax + by = c$.

$$\log x = \log(ny); \ \therefore \ x = ny; \ \therefore \ any + by = c,$$

$$\text{or } y = \frac{c}{an+b}, \ x = \frac{nc}{an+b}.$$

(19). $(a+b)^{2x} \times (a^4 - 2a^2b^2 + b^4)^{x-1} = (a-b)^{2x}$.

$$(a+b)^{2x} \times (a^2 - b^2)^{2x-2} = (a-b)^{2x};$$

therefore by extracting the square root

$$\frac{(a+b)^x . (a^2-b^2)^x}{a^2-b^2} = (a-b)^x;$$

$$\therefore (a-b)^x = 0, \text{ or } a = b, \text{ also } (a+b)^{2x} = a^2 - b^2;$$

$$\therefore (a+b)^x = \sqrt{(a^2-b^2)}, \text{ and } x = \frac{\log \sqrt{(a^2-b^2)}}{\log(a+b)} = \frac{1}{2}\,\frac{\log(a^2-b^2)}{\log(a+b)},$$

$$\text{if } (a+b)^2\,(a^2-b^2)^{2x-2} = (a-b)^{2x},$$

$$\text{then } (a^2-b^2)^x = (a-b)^x \times \frac{(a^2-b^2)}{(a+b)}, \text{ or } (a+b)^x = a-b;$$

$$\therefore x = \frac{\log(a-b)}{\log(a+b)}.$$

(20). $\log_{10} x = 3 \log_{10} a - 2$.

$$\log_{10} x = 3 \log_{10} \frac{a}{100} = \log \frac{a^3}{100}; \;\therefore x = \frac{a^3}{100}.$$

(21). $bc^{mx} = ab^{nx}$.

$$\therefore (b^n \times c^{-m})^x = \frac{b}{a}; \;\therefore x = \frac{\log b - \log a}{n \log b - m \log c}.$$

(22). $a^{2x} + 1 = a^{4x}$.

$$a^{4x} - a^{2x} + \frac{1}{4} = 1 + \frac{1}{4}; \;\therefore a^{2x} = \frac{1 \pm \sqrt{5}}{2}; \;\therefore x = \frac{\log \frac{1}{2}(1 \pm \sqrt{5})}{2 \log a}.$$

(23). $a^{2x} - b = a^x$.

$$a^{2x} - a^x + \frac{1}{4} = b + \frac{1}{4}; \;\therefore a^x = \frac{1}{2} \pm \frac{\sqrt{(4b+1)}}{2};$$

$$\therefore x = \frac{\log \frac{1}{2}\{1 \pm \sqrt{(4b+1)}\}}{\log a}.$$

(24). $x^{\frac{y}{x}} = y, \; x^{\frac{p}{q}} = y$.

$$\text{If } x^{\frac{y}{x}} = y, \text{ and } x^{\frac{p}{q}} = y; \;\therefore \frac{y}{x} = \frac{p}{q};$$

$$\therefore y = \frac{px}{q}, \text{ and } \frac{x^{\frac{p}{q}}}{x} = \frac{p}{q}; \;\therefore x^{\frac{p-q}{q}} = \frac{p}{q};$$

$$\therefore x = \left(\frac{p}{q}\right)^{\frac{q}{p-q}}, \text{ and } y = \frac{p}{q}\left(\frac{p}{q}\right)^{\frac{q}{p-q}} = \left(\frac{p}{q}\right)^{\frac{p}{p-q}}.$$

(25). $(a^x)^x \times (b^y)^y = c,\ nx = my.$

$$\text{Let } a^x = z,\ b^y = \omega;\ \therefore\ z^x \times \omega^y = c;$$

$$\therefore\ x \log z + y \log \omega = \log c, \text{ but } y = \frac{nx}{m};$$

$$\therefore\ mx \log z + nx \log \omega = m \log c;$$

$$\therefore\ x = \frac{m \log c}{m \log z + n \log \omega},$$

$$\text{but } \log \omega = y \log b = \frac{n}{m} x \log b, \text{ and } \log z = x \log a;$$

$$\therefore\ x^2 = \frac{m^2 \log c}{m^2 \log a + n^2 \log b};\ \therefore\ x = \frac{m \sqrt{(\log c)}}{\sqrt{(m^2 \log a + n^2 \log b)}},$$

$$\text{and } y = \frac{n \sqrt{(\log c)}}{\sqrt{(m^2 \log a + n^2 \log b)}}.$$

(26). $x^{\frac{y}{x}} = y,\ x^3 = y^2.$

$$x^{\frac{y}{x}} = y, \text{ and } x^{\frac{3}{2}} = y;\ \therefore\ 2y = 3x, \text{ and } \left(\frac{2y}{3}\right)^{\frac{3}{2}} = y;\ \therefore\ y = \frac{27}{8},$$

$$\text{and } x = \frac{2}{3} \times \frac{27}{8} = \frac{9}{4}.$$

(27). $3^{x+y} = 20 \times 2^x,\ 2x = 5y.$

$$\text{If } 3^{x+y} = 20 \times 2^x, \text{ and } x = \frac{5y}{2};$$

$$\therefore\ 3^{\frac{7y}{2}} = 20 \times 2^{\frac{5y}{2}}, \text{ or } 3^{7y} \times 2^{-5y} = 400,$$

$$\text{or } y = \frac{\log 400}{7 \log 3 - 5 \log 2} = 1.419,\ x = 3{\cdot}5475.$$

(28). $3^{2x} \times 5^{3x-4} = 7^{x-1} \times 11^{2-x}.$

$$\therefore\ 3^{2x} \times 5^{3x} \times 11^x = \frac{11^2 \times 5^4}{7} \times 7^x;$$

$$\therefore\ \left(\frac{3^2 \times 5^3 \times 11}{7}\right)^x = \frac{11^2 \times 5^4}{7};$$

$$\therefore\ x = \frac{2 \log 11 + 4 \log 5 - \log 7}{2 \log 3 + 3 \log 5 + \log 11 - \log 7}$$

$$= \frac{4{\cdot}033567}{3{\cdot}247447} = 1{\cdot}242.$$

(29). $3^{x^2-4x+5} = 1200.$

$$\therefore\ 3^{x^2-4x+4} = 400 \text{ (by dividing by 3)} = 20^2;$$

$$\therefore\ (x-2)^2 \log 3 = 2 \log 20,$$

$$\text{or } x - 2 = \sqrt{\left(\frac{2\log 20}{\log 3}\right)} = \pm\, 2{\cdot}33;$$

$$\therefore\ x = 4{\cdot}33, \text{ or } -0{\cdot}33.$$

(30). $x^{x+y} = y^{4a},\ y^{x+y} = x^a.$

$$\therefore\ y^{\frac{4a}{x+y}} = x = y^{\frac{x+y}{a}};\ \therefore\ (x+y)^2 = 4a^2, \text{ and } x + y = \pm\, 2a,$$

$$\text{whence } y^{2a} = x^a;\ \therefore\ x = y^2;$$

$$\therefore\ y^{y^2+y} = y^{2a};\ \therefore\ y^2 + y + \frac{1}{4} = 2a + \frac{1}{4}, \text{ and } y = -\frac{1}{2} \pm \frac{1}{2}\sqrt{(8a+1)},$$

$$\text{and } x = \frac{1}{2}(4a+1) \mp \frac{1}{2}\sqrt{(8a+1)}.$$

(31). $2a^{4x} + a^{2x} = a^{6x}.$

$$\therefore\ a^{4x} - 2a^{2x} + 1 = 2;\ \therefore\ a^{2x} = 1 \pm \sqrt{2};\ \therefore\ x = \frac{\log(1 \pm \sqrt{2})}{2\log a}.$$

(32). $\{\sqrt{(x)}\}^{\sqrt[4]{(x)}+\sqrt[4]{(y)}} = \{\sqrt{(y)}\}^{\frac{8}{3}},\ \{\sqrt{(y)}\}^{\sqrt[4]{(x)}+\sqrt[4]{(y)}} = \{\sqrt{(x)}\}^{\frac{2}{3}}.$

$$\text{If } \{\sqrt{(x)}\}^{\sqrt[4]{(x)}+\sqrt[4]{(y)}} = \{\sqrt{(y)}\}^{\frac{8}{3}} = y^{\frac{4}{3}};\ \therefore\ x^{\frac{3}{8}(\sqrt[4]{x}+\sqrt[4]{y})} = y,$$

$$\text{also } y = x^{\frac{2}{3}\frac{1}{\sqrt[4]{(x)}+\sqrt[4]{(y)}}};\ \therefore\ (\sqrt[4]{x} + \sqrt[4]{y})^2 = \frac{2}{3} \times \frac{8}{3} = \frac{16}{9},$$

$$\text{and } \sqrt[4]{x} + \sqrt[4]{y} = \pm\frac{4}{3};\ \therefore\ x = y^{\frac{8}{3}\times\frac{3}{4}} = y^2,$$

$$\text{and } \sqrt{y} + \sqrt[4]{y} + \frac{1}{4} = \frac{4}{3} + \frac{1}{4};\ \therefore\ y^{\frac{1}{4}} = -\frac{1}{2} \pm \frac{1}{2}\sqrt{\left(\frac{19}{3}\right)},$$

$$\text{or } y = \left\{-\frac{1}{2} \pm \frac{1}{6}\sqrt{(57)}\right\}^4, \text{ and } x = y^2.$$

(33). $a^x - a^{-x} = 2c.$

$$a^{2x} - 2ca^x + c^2 = 1 + c^2;\ \therefore\ a^x = c \pm \sqrt{(1+c^2)};$$

$$\therefore\ x = \frac{\log\{c \pm \sqrt{(1+c^2)}\}}{\log a}.$$

(34). $x^x - x^{-x} = 3(1 + x^{-x})$.

$$x^{2x} - 1 = 3(x^x + 1); \therefore x^{2x} - 3x^x + \frac{9}{4} = 4 + \frac{9}{4} = \frac{25}{4};$$

$$\therefore x^x = \frac{3}{2} \pm \frac{5}{2} = 4, \text{ or } -1 = 2^2; \therefore x = 2.$$

(35). $a^x \times b^y = k$, $nx = my$.

If $a^x \times b^y = k$, and $nx = my$, then $a^x \times b^{\frac{nx}{m}} = k$;

$$\therefore x = \frac{m \log k}{m \log a + n \log b}, \text{ and } y = \frac{n \log k}{m \log a + n \log b}.$$

(36). $a^{x^2} \times b^{\frac{1}{2}y^2} = r$, $nx = my$.

If $a^{x^2} \times b^{\frac{1}{2}y^2} = r$, and $x = \frac{my}{n}$,

$$\text{then } a^{\frac{m^2y^2}{n^2}} \times b^{\frac{1}{2}y^2} = r; \therefore \frac{m^2}{n^2} y^2 \log a + \frac{1}{2} y^2 \log b = \log r;$$

$$\therefore y = \frac{n\sqrt{(2 \log r)}}{\sqrt{(2m^2 \log a + n^2 \log b)}}, \text{ and } x = \frac{my}{n}.$$

(37). $y^x = x^y$, $x^a = y^b$.

See Ex. (24).

RATIO, PROPORTION, AND VARIATION.

XIV. (1). Which is the greater of the two ratios, 15 : 16, or 16 : 17; 3 : 5, or 5 : 8, and 19 : 20, or 20 : 21?

$$\frac{15}{16} >< \frac{16}{17}, \text{ or } 255 >< 256, \text{ but } 256 > 255;$$

$\therefore$ the ratio of 16 : 17 > 15 : 16,

$$\frac{3}{5} >< \frac{5}{8}; \therefore 24 >< 25, \text{ but } 25 > 24; \therefore 5 : 8 > 3 : 5,$$

in the same way 20 : 21 >: 19 : 20.

(2). Compare the ratios 7 : 8 and 10 : 11; 19 : 25 and 56 : 74.

$$\text{Here } \frac{7}{8} : \frac{10}{11} :: 77 : 80, \text{ and } \frac{19}{25} : \frac{56}{74} :: 1406 : 1440.$$

(3). When $x : y :: 2 : 1$, which is the greater, $2ax : 3by$ or $3a : 2b$?

$$\text{If } x = 2y,\ 2ax : 3by :: 4ay : 3by :: 2a : 3b;$$

$$\therefore\ 3a : 2b > 2a : 3b.$$

(4). Show that $a : b > ax : bx + y$; but $< ax : bx - y$.

$$\frac{a}{b} >< \frac{ax}{bx \pm y},\ 1 >< \frac{bx}{bx + y};\ \therefore\ \frac{1}{0} >< \frac{bx}{y},\ \text{but } \frac{1}{0} > \frac{bx}{y};$$

$$\therefore\ \frac{a}{b} > \frac{ax}{bx + y},\ \text{in second case } \frac{1}{0} < \frac{bx}{-y};\ \therefore\ \frac{a}{b} < \frac{ax}{bx - y}.$$

(5). Prove that $a^3 + b^3 : a^2 + b^2 > a^2 + b^2 : a + b$.

$$\frac{a^3 + b^3}{a^2 + b^2} >< \frac{a^2 + b^2}{a + b},\ \text{or } a^4 + a^3b + ab^3 + b^4 >< a^4 + 2a^2b^2 + b^4,$$

or $ab\,(a^2 + b^2) >< 2a^2b^2$, i.e. $(a - b)^2 > 0$; the first > latter.

(6). Which is the greater of the ratios, $a + x : a - x$ and $a^2 + x^2 : a^2 - x^2$, a being $> x$?

$$\frac{a + x}{a - x} >< \frac{a^2 + x^2}{a^2 - x^2},\ \text{or } (a + x)^2 > a^2 + x^2;$$

$$\therefore\ a + x : a - x \text{ is the greater.}$$

(7). Which is the >, $\frac{a + x}{a}$ or $\frac{4x}{a + x}$; $\frac{a^2 - x^2}{a^3 - x^3}$ or $\frac{a - x}{a^2 - x^2}$?

$$\frac{a + x}{a} >< \frac{4x}{a + x},\ \text{or } (a + x)^2 >< 4ax,\ \text{or } (a - x)^2 > 0;$$

$$\therefore\ \frac{a + x}{a} \text{ is the greater;}$$

$$\frac{a^2 - x^2}{a^3 - x^3} >< \frac{a - x}{a^2 - x^2};\ \therefore\ \frac{a + x}{a^2 + ax + x^2} >< \frac{1}{a + x},\ \text{or } (a + x)^2 > a^2 + ax + x^2;$$

$$\therefore\ \frac{a^2 - x^2}{a^3 - x^3} \text{ is the greater.}$$

(8). If x be small compared with a, show that the ratios of $(a \pm x)^3$, $(a \pm x)^4$ are nearly equal respectively to $a \pm 3x$ and $a \pm 4x$; and that of

$$(a \pm x)^{\frac{1}{3}},\ (a \pm x)^{\frac{1}{4}} \text{ to } a \pm \frac{x}{3} \text{ and } a \pm \frac{x}{4}.$$

then $(a \pm x)^3 = a \pm 3x$ nearly,

and $(a \pm x)^4 = a \pm 4x$ nearly, by binomial theorem,

so also $(a \pm x)^{\frac{1}{3}} = a \pm \frac{x}{3}$ nearly, and $(a \pm x)^{\frac{1}{4}} = a \pm \frac{x}{4}$ nearly.

(9). Find the ratio compounded of $a : x$, $x : y$, and $y : b$; also of $x + a : x + b$, and $a(x + b) : b(x + a)$.

Compound ratio $= \frac{a}{x} \times \frac{x}{y} \times \frac{y}{b} = \frac{a}{b}$, also $\frac{a(x+a)(x+b)}{b(x+b)(x+a)} = \frac{a}{b}$.

(10). Find the ratio compounded of $a + x : a - x$, $a^2 + x^2 : (a + x)^2$, and $(a^2 - x^2)^2 : a^4 - x^4$.

$$\text{The compound ratio} = \frac{a+x}{a-x} \times \frac{a^2+x^2}{(a+x)^2} \frac{a^2-x^2}{a^2+x^2} = 1.$$

(11). If $a : b > c : d$, show that $a + c : b + d < a : b$, but $> c : d$.

$$\text{If } \frac{a}{b} > \frac{c}{d}, \text{ then } \frac{a}{c} > \frac{b}{d}; \ \therefore\ 1 + \frac{a}{c} > 1 + \frac{b}{d};$$

$$\therefore\ a + c : b + d > c : d,$$

$$\text{also } \frac{c}{a} < \frac{d}{b}, \text{ or } 1 + \frac{c}{a} < 1 + \frac{d}{b}; \ \therefore\ a + c : b + d < a : b.$$

(12). If a be the greatest of the four proportionals a, b, c, d; show that $a + d > b + c$; and that $a^n + d^n > b^n + c^n$.

If a, b, c, d be proportionals, and a the greatest;

$$\therefore\ c \text{ is} > d,$$

$$\text{and } \frac{a}{d} > \frac{b}{c}; \ \therefore\ \frac{a}{d} + 1 > \frac{b}{c} + 1, \text{ or } \frac{a+d}{d} > \frac{b+c}{c},$$

much more then is $a + d > b + c$, since $c > d$,

so also $a^n + d^n > b^n + c^n$.

(13). Prove that $a - x : a + x$ is $>$ or $< a^2 - x^2 : a^2 + x^2$, according as $a : x$ is a ratio of less or greater inequality.

$$\frac{a-x}{a+x} > \text{ or } < \frac{a^2-x^2}{a^2+x^2} \text{ is } a^2 + x^2 > \text{ or } < (a+x)^2,$$

$$\text{or } 0 < 2ax; \ \therefore\ \text{in all cases } \frac{a-x}{a+x} < \frac{a^2-x^2}{a^2+x^2}.$$

(14). If $\frac{a}{b} = \frac{c}{d}$, show that $\frac{a+b}{a-b} = \frac{c+d}{c-d}$.

$$\text{If } \frac{a}{b} = \frac{c}{d}, \text{ then } \frac{a}{b} + 1 = \frac{c}{d} + 1, \text{ or } \frac{a+b}{b} = \frac{c+d}{d},$$

$$\text{also } \frac{a}{b} - 1 = \frac{c}{d} - 1; \ \therefore \ \frac{a-b}{b} = \frac{c-d}{d}, \text{ and } \frac{a+b}{a-b} = \frac{c+d}{c-d}.$$

(15). If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f}$, prove that $a : b :: a + c + e : b + d + f$.

$$\therefore \ ad = bc, \text{ so also } af = be;$$

$$\therefore \ ab + ad + af + \&c. = b\,(a + c + e + \&c.);$$

$$\therefore \ \frac{a}{b} = \frac{a + c + e + \&c.}{b + d + f + \&c.}.$$

(16). If $\frac{a}{b} = \frac{c}{d}$, prove that $\frac{ma \pm nb}{pa \pm qb} = \frac{mc \pm nd}{pc \pm qd}$.

$$\text{If } \frac{a}{b} = \frac{c}{d}, \text{ then } \frac{ma}{b} = \frac{mc}{d}, \text{ and } \frac{ma}{b} \pm n = \frac{mc}{d} \pm n,$$

$$\text{or } \frac{ma \pm nb}{b} = \frac{mc \pm nd}{a}, \text{ so also } \frac{pa \pm qb}{b} = \frac{pc \pm qd}{d};$$

$$\therefore \ \frac{ma \pm nb}{pa \pm qb} = \frac{mc \pm nd}{pc \pm qd}.$$

(17). If $\frac{a}{b} = \frac{b}{c} = \frac{c}{d}$, then $a : c :: a^2 : b^2$, and $a : d :: a^3 : b^3$.

$$\text{If } \frac{a}{b} = \frac{b}{c}, \text{ then } \frac{a}{b} \times \frac{b}{c} = \frac{b}{c} \times \frac{b}{c} = \frac{a}{b} \times \frac{a}{b}; \ \therefore \ a : c :: a^2 : b^2,$$

$$\text{and if } \frac{a}{b} = \frac{b}{c} = \frac{c}{d},$$

$$\text{then } \frac{a}{b} \times \frac{b}{c} \times \frac{c}{d} = \frac{a}{b} \times \frac{a}{b} \times \frac{a}{b},$$

$$\text{or } a : d :: a^3 : b^3.$$

(18). Show that if $(a + b)^2 : (a - b)^2 :: b + c : b - c$,

$$\text{then } a : b :: \sqrt{(2a - c)} : \sqrt{(c)};$$

$$\text{and if } a : b : c : d,$$

$$\text{then } \sqrt{(a - b)} : \sqrt{(c - d)} :: \sqrt{(a)} - \sqrt{(b)} : \sqrt{(c)} - \sqrt{(d)}.$$

$$\text{If } (a+b)^2 : (a-b)^2 :: b+c : b-c,$$

$$\text{then } (a+b)^2 : 4ab :: b+c : 2c,$$

$$\text{or } a^2c + 2abc + cb^2 = 2ab^2 + 2abc;$$

$$\therefore\ a^2c = 2ab^2 - b^2c, \text{ or } \frac{a^2}{b^2} = \frac{2a-c}{c},$$

$$\text{or } a : b :: \sqrt{(2a-c)} : \sqrt{(c)},$$

$$\text{if } \frac{a}{b} = \frac{c}{d};\ \therefore\ \frac{a-b}{b} = \frac{c-d}{d}, \text{ or } \frac{\sqrt{(a-b)}}{\sqrt{(b)}} = \frac{\sqrt{(c-d)}}{\sqrt{d}},$$

$$\text{also } \frac{\sqrt{a}}{\sqrt{b}} = \frac{\sqrt{c}}{\sqrt{d}};\ \therefore\ \frac{\sqrt{a}-\sqrt{b}}{\sqrt{b}} = \frac{\sqrt{c}-\sqrt{d}}{\sqrt{d}};$$

$$\therefore\ \frac{\sqrt{(a-b)}}{\sqrt{a}-\sqrt{b}} = \frac{\sqrt{(c-d)}}{\sqrt{c}-\sqrt{d}},$$

$$\text{or } \sqrt{(a-b)} : \sqrt{(c-d)} :: \sqrt{a}-\sqrt{b} : \sqrt{c}-\sqrt{d}.$$

(19). If $(a+b+c+d)(a-b-c+d) = (a-b+c-d)(a+b-c-d)$, show that $a : b :: c : d$.

$$\text{If } \{(a+d)+(b+c)\} \times \{(a+d)-(b+c)\}$$

$$= \{(a-d)-(b-c)\}\{(a-d)+(b-c)\},$$

$$\text{then } (a+d)^2 - (b+c)^2 = (a-d)^2 - (b-c)^2,$$

$$\text{or } 4ad = 4bc, \text{ or } a : b :: c : d.$$

(20). If $x : y :: a^3 : b^3$, and $a : b :: \sqrt[3]{(c+x)} : \sqrt[3]{(d+y)}$, show that $cy = dx$.

$$\text{If } \frac{x}{y} = \frac{a^3}{b^3}, \text{ and } \frac{a^3}{b^3} = \frac{c+x}{d+y}, \text{ then } \frac{x}{y} = \frac{c+x}{d+y},$$

$$\text{or } \frac{c+x}{x} = \frac{d+y}{y};\ \therefore\ \frac{c}{x} = \frac{d}{y}, \text{ or } cy = dx.$$

(21). Form into proportions the equations

$$ab = a^2 - x^2, \text{ and } x^2 + y^2 = 2ax.$$

$$\text{If } ab = a^2 - x^2 = (a-x)(a+x);\ \therefore\ a : a-x :: a+x : b,$$

$$\text{also if } y^2 = 2ax - x^2 = x(2a-x),$$

$$\text{or } \frac{y}{x} = \frac{2a-x}{y}, \text{ or } y : x :: 2a-x : y.$$

(22). If x be to y in the duplicate ratio of a to b, and a be to b in the sub-duplicate ratio of $a+x$ to $a-y$, then

$$a-y : y :: x+y : x-y.$$

$$\text{If } x^2 : y^2 :: a : b, \text{ and } a^{\frac{1}{2}} : b^{\frac{1}{2}} :: a+x : a-y;$$

$$\therefore x^2 : y^2 :: (a+x)^2 : (a-y)^2, \text{ or } x : y : a+x : a-y \ (A),$$

$$\text{also } x = \frac{ay}{a-2y}; \ \therefore a+x = \frac{a^2-ay}{a-2y};$$

$$\therefore \text{ from } (A)\ \frac{x}{y} = \frac{(a-y)\,a}{(a-2y)\,(a-y)} = \frac{a}{a-2y};$$

$$\therefore x+y : x-y :: a-y : y.$$

(23). Find two numbers in the ratio of 3 to 5, such that the difference of their squares : difference of their cubes :: 8 : 147.

Let $3x$ and $5x$ be the numbers;

$$\therefore (5x)^2-(3x)^2 : (5x)^3-(3x)^3 :: 8 : 147,$$

$$\text{or } 16 : 98x :: 8 : 147;$$

$$\therefore x = \frac{2\times 147}{98} = \frac{147}{49} = 3; \ \therefore \text{ the numbers are 9 and 15.}$$

(24). Find two numbers in the ratio of 9 : 16, such that when each number is increased by 15, they shall be in the ratio of 2 to 3.

Let $9x$ and $16x$ be the numbers,

$$\text{then } 9x+15 : 16x+15 :: 2 : 3;$$

$$\therefore 9x+15 : 7x :: 2 : 1; \ \therefore 14x = 9x+15, \text{ and } x = 3;$$

$\therefore$ the numbers are 27 and 48.

(25). Find two numbers which are to each other as 3 : 4, and their sum : sum of their squares :: 7 : 50.

Let $3x$ and $4x$ be the numbers, then $7x : 9x^2+16x^2 :: 7 : 50$;

$\therefore 25x = 50$, and $x = 2$, and the numbers are 6 and 8.

(26). If $\dfrac{\sqrt[n]{(x)} - m\sqrt[n]{(y)}}{\sqrt[n]{(x)} + m\sqrt[n]{(y)}} = \dfrac{\sqrt[2n]{(x)} - m\sqrt[2n]{(x-y)}}{\sqrt[2n]{(x)} + m\sqrt[2n]{(x-y)}}$,

$$\text{then } \frac{x}{y} = \frac{1\pm\sqrt{(5)}}{2}.$$

$$\text{If } \frac{\sqrt[n]{(x)} - m\sqrt[n]{(y)}}{\sqrt[n]{(x)} + m\sqrt[n]{(y)}} = \frac{\sqrt[2n]{(x)} - m\,\sqrt[2n]{(x-y)}}{\sqrt[2n]{(x)} + m\sqrt[2n]{(x-y)}},$$

$$\text{then } \frac{\sqrt[n]{(x)}}{m\sqrt[n]{(y)}} = \frac{\sqrt[2n]{(x)}}{m\sqrt[2n]{(x-y)}}, \text{ or } \frac{x^2}{y^2} = \frac{x}{x-y},$$

$$\text{or } x^2 - xy = y^2;\ \therefore\ \frac{x^2}{y^2} - \frac{x}{y} + \frac{1}{4} = 1 + \frac{1}{4} = \frac{5}{4};$$

$$\therefore\ \frac{x}{y} = \frac{1 \pm \sqrt{5}}{2}.$$

(27). If x, the first of three magnitudes, $x, y, z, \propto yz$, and $y^2 \propto xz$, then $x \propto z^3$.

$$\text{Let } x = myz, \text{ and } y^2 = pxz;$$

$$\therefore\ y^2 = \frac{x^2}{m^2z^2} = pxz;\ \therefore\ x = m^2pz^3 \propto z^3.$$

(28). If $s \propto t^2$ when f is constant, and $s \propto f$ when t is constant, also $2s = f$ when $t = 1$; find the equation between f, s, t.

Let $s \propto ft^2$; $\therefore\ s = mft^2$, and by problem when $t = 1$,

$$\frac{f}{2} = m \times f;\ \therefore\ m = \frac{1}{2}, \text{ and } s = \frac{1}{2}ft^2.$$

(29). A passenger in a railway train observes that another train on a parallel line moving in an opposite direction occupies 2″ in passing him; but when in the same direction, it occupies 30″. Compare the speed of the trains.

Let x miles = speed of passenger's train,

y miles of the other train;

$\therefore\ x + y$ miles relative speed in opposite direction,

$y - x$ in same direction;

$\therefore\ y - x : y + x :: 2 : 30;$

$\therefore\ x : y :: 28 : 32 :: 7 : 8.$

(30). If $y \propto x$, and when $x = 3$, $y = 21$; find the equation between x and y.

Let $y = mx$, and when $x = 3$, $y = 21$; $\therefore\ 3m = 21$, and $m = 7$;

$$\therefore\ y = 7x.$$

(31). If $xy \propto x^2 + y^2$, and when $x = 3$, $y = 4$; find the equation between x and y.

$$\text{Let } xy = m(x^2 + y^2), \text{ and when } x = 3,\ y = 4;$$

$$\therefore 12 = m \times 25, \text{ or } m = \frac{12}{25},$$

$$\text{and } 25xy = 12(x^2 + y^2), \text{ whence } 3x = 4y,\ 4x = 3y.$$

(32). If $y^2 \propto a^2 - x^2$, and when $x^2 = a^2 - b^2$, $a^2y^2 = b^4$; find the equation between x and y.

$$\text{Let } y^2 = m(a^2 - x^2);\ \therefore b^4a^{-2} = mb^2, \text{ or } m = b^2a^{-2};$$

$$\therefore a^2y^2 = b^2(a^2 - x^2), \text{ and } a^2y^2 + b^2x^2 = a^2b^2.$$

(33). If $y^2 \propto x^2 - a^2$, and when $x = (a^2 + b^2)^{\frac{1}{2}}$, $y = \frac{b^2}{a}$; required the equation between x and y.

$$\text{Let } y^2 = m(x^2 - a^2);\ \therefore b^4a^{-2} = m(b^2);\ \therefore m = b^2a^{-2},$$

$$\text{and } a^2y^2 = b^2x^2 - a^2b^2, \text{ or } b^2x^2 - a^2y^2 = a^2b^2.$$

(34). If $y^2 \propto x$, and when $x = a$, $y = \pm 2a$; find the equation between x and y.

$$\text{Let } y^2 = ma;\ \therefore 4a^2 = ma, \text{ or } m = 4a;$$

$$\therefore y^2 = 4ax.$$

(35). If $ax + by = cx + dy$, show that $x \propto y$.

$$\text{Since } (a - c)x = (d - b)y;\ \therefore x = \frac{d - b}{a - c}y,$$

and a, b, c, d are constants; $\therefore x \propto y$.

(36). If $x \propto y$ and $y \propto z$, show that

$$(ax + by + cz) \propto [h(xy)^{\frac{1}{2}} + k(xz)^{\frac{1}{2}} + l(yz)^{\frac{1}{2}}].$$

$$\text{Let } x = my,\ y = nz, \text{ then } my^2 = xy,\ nz^2 = zy;$$

$$\therefore my^2 \times nz^2 = xzy^2, \text{ or } mnz^2 = xz,$$

$$\text{then } ax = may = a\sqrt{(m)}\sqrt{(xy)},$$

$$by = nbz = b\sqrt{(n)}\sqrt{(zy)},$$

$$cz = \frac{c}{\sqrt{(mn)}}\sqrt{(xz)};$$

$$\therefore\ ax + by + cz = a\sqrt{(m)}\sqrt{(xy)} + b\sqrt{(n)}\sqrt{(zy)}$$

$$+ \frac{c}{\sqrt{(mn)}}\sqrt{(xz)} \propto h\sqrt{(xy)} + k\sqrt{(xz)} + l\sqrt{(yz)},$$

$$\text{where } h = a\sqrt{(m)},\ l = b\sqrt{(n)},\ k = c\,(mn)^{-\frac{1}{2}}.$$

(37). If $a + b \propto a - b$, show that $a^2 + b^2 \propto ab$; and if $a \propto b$, show that $a^2 - b^2 \propto ab$.

$$\text{If } (a + b) \propto (a - b), \text{ then } (a + b) = m\,(a - b);$$

$$\therefore\ a^2 + 2ab + b^2 = m^2(a^2 + b^2) - 2m^2ab,$$

$$\text{or } (m^2 - 1)(a^2 + b^2) = 2\,(m^2 + 1)\,ab,$$

$$\text{or } a^2 + b^2 = \frac{2\,(m^2 + 1)\,ab}{m^2 - 1};\ \therefore\ a^2 + b^2 \propto ab,$$

$$\text{if } a = mb,\ \frac{a^2 - b^2}{ab} = \frac{m^2 - 1}{m};\ \therefore\ a^2 - b^2 \propto ab.$$

(38). There are two vessels, A and B, each containing a mixture of water and wine, A in the ratio of $2 : 3$, B in the ratio of $3 : 7$. What quantity must be taken from each in order to form a third mixture which shall contain 5 gallons of water and 11 of wine?

Let x be the gallons taken from the first and y from the second,

$$\text{then } \frac{2x}{5} + \frac{3y}{10} : \frac{3x}{5} + \frac{7y}{10} :: 5 : 11;$$

$$\therefore\ 44x + 33y = 30x + 35y, \text{ or } y = 7x,$$

$$\text{also } x + y = 16;\ \therefore\ x + 7x = 16, \text{ and } x = 2 \text{ gallons},$$

$$y = 14 \text{ gallons}.$$

(39). If two globes of gold, whose radii are a and a', are melted and formed into one solid globe, what is its radius?

Let x = the required radius,

and since the volume of the globe $\propto$ (rad)^3,

let its volume $= mx^3 = ma^3 + ma'^3$;

$$\therefore\ x = \sqrt[3]{(a^3 + a'^3)}.$$

(40). The value of diamonds $\propto$ as the square of their weights, and the square of the value of rubies $\propto$ as the cube of their weights; a diamond of a carats is worth m times a ruby of b carats, and both together are worth £c. Find the values of a diamond and ruby, each weighing x carats.

Since the value of a diamond $\propto$ (weight)^2,

the value of a diamond of n carats $\propto n^2 = pn^2$,

.... ruby .... $\propto n^{\frac{3}{2}} = qn^{\frac{3}{2}}$,

where p and q are constant quantities;

$\therefore$ the value of a diamond of a carats $= pa^2$,

of a ruby of b carats $= qb^{\frac{3}{2}}$,

and we have by problem $pa^2 = mqb^{\frac{3}{2}}$, also $pa^2 + qb^{\frac{3}{2}} = c$;

$$\therefore mqb^{\frac{3}{2}} + qb^{\frac{3}{2}} = c, \text{ or } q = \frac{c}{(m+1)\,b^{\frac{3}{2}}};$$

$$\therefore mpa^2 + pa^2 = mc; \ \therefore p = \frac{mc}{(m+1)\,a^2};$$

$$\therefore \text{value of diamond} = \frac{mcn^2}{(m+1)\,a^2}, \text{ of ruby} = \frac{cn^{\frac{3}{2}}}{(m+1)\,b^{\frac{3}{2}}}.$$

ARITHMETIC, GEOMETRIC, AND HARMONIC PROGRESSION.

XV. In an *Arithmetic Progression*, if a be the first term, d the common difference, n the number of terms, l the last term, and S the sum of n terms; then

$$l = a + (n-1)\,d, \ S = [2a + (n-1)\,d]\,\frac{n}{2}.$$

Find the sum of the following Arithmetic Series:—

(1). $2 + 4 + 6 +$ &c. to 16 terms.

$$\text{Sum to 16 terms} = (4 + 15 \times 2)\,\frac{16}{2} = 34 \times 8 = 272.$$

(2). $1 + 3 + 5 +$ &c. to 100 terms and n terms.

$$\text{Sum to 100 terms} = (2 + 99 \times 2)\,\frac{100}{2} = 100 \times 100 = 100^2,$$

$$\text{and to } n \text{ terms} = \{2 + (n - 1)\,2\}\,\frac{n}{2} = n^2.$$

(3). $\frac{1}{2} + \frac{1}{3} + \frac{1}{6} +$ &c. to 7 terms and 14 terms.

$$\text{The common difference is } \frac{1}{3} - \frac{1}{2} = -\frac{1}{6};$$

$$\therefore \text{ sum to 7 terms} = \left(1 - 6 \times \frac{1}{6}\right)\frac{7}{2} = 0,$$

$$\text{and to 14 terms} = \left(1 - 13 \times \frac{1}{6}\right) 7 = -\frac{49}{6} = -8\tfrac{1}{6}.$$

(4). $-5 - 3 - 1 +$ &c. to 8 terms.

$$\text{Sum to 8 terms} = (-10 + 7 \times 2)\,4 = 16.$$

(5). $\frac{2}{3} + \frac{7}{15} + \frac{4}{15} +$ &c. to 6 terms.

$$\text{The common difference} = \frac{7}{15} - \frac{10}{15} = -\frac{3}{15} = -\frac{1}{5};$$

$$\therefore \text{ sum to 6 terms} = \left(\frac{4}{3} - 5 \times \frac{1}{5}\right) \times 3 = 1.$$

(6). $\frac{n-1}{n} + \frac{n-2}{n} + \frac{n-3}{n} +$ &c. to n terms.

$$\text{The common difference is } -\frac{1}{n};$$

$$\therefore \text{ sum to } n \text{ terms} = \left\{2\left(\frac{n-1}{n}\right) - \frac{n-1}{n}\right\}\frac{n}{2} = \frac{n-1}{2}.$$

(7). $6 + \frac{11}{2} + 5 +$ &c. to 25 terms.

$$\text{Sum to 25 terms} = \left(12 - 24 \times \frac{1}{2}\right)\frac{25}{2} = (6 - 6)\,25 = 0.$$

(8). $116 + 108 + 100 +$ &c. to 10 terms.

$$\text{Sum to 10 terms} = (232 - 9 \times 8)\,5 = 800.$$

(9). $\frac{1}{2} - \frac{2}{3} - \frac{11}{6} -$ &c. to 13 terms.

The common difference $= -\frac{2}{3} - \frac{1}{2} = -\frac{7}{6}$;

$\therefore$ sum to 13 terms $= \left(1 - 12 \times \frac{7}{6}\right)\frac{13}{2} = -\frac{169}{2} = -84\frac{1}{2}$.

(10). $\frac{1}{2} - 1 - \frac{5}{2} -$ &c. to 30 terms.

The common difference is $-\frac{3}{2}$;

$\therefore$ sum to 30 terms $= \left(1 - 29 \times \frac{3}{2}\right) 15 = -\frac{85 \times 15}{2} = -637\frac{1}{2}$.

(11). $\frac{1}{3}, \frac{1}{4}, \frac{1}{6}$, &c. to 8 terms and 16 terms.

The common difference $= \frac{1}{4} - \frac{1}{3} = \frac{3}{12} - \frac{4}{12} = -\frac{1}{12}$;

$\therefore$ sum to 8 terms $= \left(\frac{2}{3} - \frac{7}{12}\right) \times 4 = \frac{1}{3}$,

and to 16 terms $= \left(\frac{2}{3} - \frac{15}{12}\right) \times 8 = -\frac{7}{12} \times \frac{8}{1} = -4\frac{2}{3}$.

(12). $\frac{1}{2} + \frac{3}{8} + \frac{2}{8} +$ &c. to 8 terms and 20 terms.

Common difference $= \frac{3}{8} - \frac{1}{2} = \frac{3}{8} - \frac{4}{8} = -\frac{1}{8}$;

$\therefore$ sum to 8 terms $= \left(1 - \frac{7}{8}\right) \times 4 = \frac{1}{2}$;

and to 20 terms $= \left(1 - \frac{19}{8}\right) \times 10 = -\frac{55}{4} = -13\frac{3}{4}$.

(13). $\frac{5}{7}$, $1\frac{2}{7}$ + &c. to 10 terms and n terms.

Sum to 10 terms $= \left(\frac{10}{7} + 9 \times \frac{4}{7}\right) 5 = \frac{46 \times 5}{7} = 32\frac{6}{7}$,

and to n terms $= \left\{\frac{10}{7} + (n-1) \times \frac{4}{7}\right\} \frac{n}{2} = \left(\frac{2n+3}{7}\right) n$, (14). $\frac{a-b}{a+b}+\frac{3a-2b}{a+b}+\frac{5a-3b}{a+b}+$ &c. to n terms.

$$\text{Common difference} = \frac{3a-2b}{a+b}-\frac{a-b}{a+b}=\frac{2a-b}{a+b};$$

$\therefore$ sum to n terms

$$=\left\{\frac{2(a-b)}{a+b}+\frac{(n-1)(2a-b)}{a+b}\right\}\times\frac{n}{2}=\frac{n}{2(a+b)}\{2na-(n+1)b\}.$$

(15). $\frac{n-1}{n}+\frac{n-3}{n}+\frac{n-5}{n}+$ &c. to n terms.

$$\text{Sum to } n \text{ terms} = \left\{\frac{2(n-1)}{n}-\frac{(n-1)\times 2}{n}\right\}\frac{n}{2}=0.$$

(16). $(a+x)^2+(a^2+x^2)+(a-x)^2+$ &c. to n terms.

$$\text{The common difference} = (a^2+x^2)-(a+x)^2=-2ax;$$

$\therefore$ sum to n terms

$$=\{2(a+x)^2-2ax(n-1)\}\frac{n}{2}=\{(a+x)^2-ax(n-1)\}n.$$

(17). $\left(\frac{1}{a}-\frac{n}{x}\right)+\left(\frac{1}{a}-\frac{n-1}{x}\right)+\left(\frac{1}{a}-\frac{n-2}{x}\right)+$ &c. to n terms.

$$\text{Common difference} = \frac{1}{x};$$

$$\therefore \text{sum to } n \text{ terms} = \left\{2\left(\frac{1}{a}-\frac{n}{x}\right)+\frac{n-1}{x}\right\}\frac{n}{2}=\frac{n}{a}-\frac{n(n+1)}{2x}.$$

(18). The sum of an arithmetic series being 1455, the first term 5, and the number of terms 30; find the common difference.

By substitution in the formula $s=\{2a+(n-1)d\}\frac{n}{2}$,

we have $(10+29\times d)\,15=1455$;

$$\therefore 29\times d=97-10;\quad \therefore d=\frac{87}{29}=3.$$

(19). The sum of an arithmetic series is 91, the common difference 2, and the last term 19. Find the number of terms.

Here we have the last term $=a+(n-1)\,2=19$; $\therefore a+2n=21$,

also $\{2a+(n-1)\,2\}\frac{n}{2}=91$; $\therefore a+n-1=\frac{91}{n}$,

or $n^2-20n+100=-91+100=9$; $\therefore n=10\pm 3=13$ or 7.

(20). The first term of an arithmetic series is $3\frac{1}{3}$, the common difference $1\frac{4}{9}$, and the sum 22. Find the number of terms.

$$\text{Here } \left\{\frac{20}{3} + (n-1)\frac{13}{9}\right\}\frac{n}{2} = 22,$$

$$\text{whence } n^2 + \frac{47n}{13} + \left(\frac{47}{26}\right)^2 = \frac{396}{13} + \left(\frac{47}{26}\right)^2 = \frac{22801}{676};$$

$$\therefore\ n = -\frac{47}{26} \pm \frac{151}{26} = 4.$$

(21). What number of terms of the series 54, 51, 48, &c. must be taken to make 513?

$$\text{Here } \{108 - (n-1)\,3\}\frac{n}{2} = 513,$$

$$\text{whence } n^2 - \frac{111n}{3} + \left(\frac{111}{6}\right)^2 = -\frac{1026}{3} + \left(\frac{111}{6}\right)^2 = \frac{9}{36};$$

$$\therefore\ n = \frac{111}{6} \pm \frac{3}{6} = 19 \text{ or } 18.$$

(22). Required the number of terms, when the first term is 7, the common difference 2, and the sum 40.

$$\text{Here } \{14 + (n-1)\,2\}\frac{n}{2} = 40;$$

$$\therefore\ n^2 + 6n + 9 = 49;\ \therefore\ n = -3 \pm 7 = 4.$$

(23). Required the number of terms when the sum = 2·748, the first term ·034, and the common difference ·0004.

$$\text{Here we have } \{\cdot068 + (n-1)(\cdot0004)\}\frac{n}{2} = 2\cdot748,$$

$$\text{or } \{\cdot034 + (n-1)(\cdot0002)\}\,n = 2\cdot748,$$

and by moving the decimal place

$$\{340 + (n-1)\times 2\}\,n = 27480;$$

$$\therefore\ n^2 + 169n + \left(\frac{169}{2}\right)^2 = 13740 + \left(\frac{169}{2}\right)^2 = \frac{83521}{4};$$

$$\therefore\ n = -\frac{169}{2} \pm \frac{289}{2} = 60.$$

(24). Find 3 numbers in arithmetic progression whose product $= 120$, and sum 15.

Let $x - y$, x, and $x + y$ be the numbers;

$\therefore$ their sum $= 3x = 15$; $\therefore$ $x = 5$,

and $(5 - y) \times 5 \times (5 + y) = 120$; $\therefore$ $25 - y^2 = 24$; $\therefore$ $y = \pm 1$,

and the series is 4, 5, 6.

(25). Write down the second and seventh terms of the arithmetic series, whose fifth and ninth terms are 1 and 9.

The fifth term $= a + 4d = 1$,

ninth term $= a + 8d = 9$; $\therefore$ $d = 2$, and $a = -7$;

$\therefore$ the second term is -5, and the seventh term $= -7 + 12 = 5$.

(26). If the sum of three numbers in arithmetic progression be 15, and the sum of the squares 93; what are the numbers?

Let $x - y$, x, and $x + y$ be the numbers; $\therefore$ $3x = 15$, and $x = 5$,

then $(5 - y)^2 + 5^2 + (5 + y)^2 = 93$,

or $2y^2 = 93 - 75 = 18$; $\therefore$ $y = \pm 3$,

and the series 2, 5, 8.

(27). Find three numbers in arithmetic progression whose sum is 24, and their product 480.

Let $x - y$, x, and $x + y$ be the numbers; $\therefore$ $x = 8$,

and $(8 - y) \times 8 \times (8 + y) = 480$; $\therefore$ $64 - y^2 = 60$; $\therefore$ $y = \pm 2$,

and the series is 6, 8, 10.

(28). The first two terms of an arithmetic series being together $= 18$, and the next term being 12, how many terms beginning with the first must be taken to make 78?

Let $2a + d = 18$, and $a + 2d = 12$, whence $a = 8$, and $d = 2$,

$$\text{then } \{16 + (n - 1)\, 2\} \frac{n}{2} = 78,$$

$$\text{or } n^2 + 7n + \frac{49}{4} = 78 + \frac{49}{4} = \frac{361}{4};$$

$$\therefore\ n = -\frac{7}{2} \pm \frac{19}{2} = 6.$$

(29). The first term is $n^2 - n + 1$, the common difference 2; find the sum of n terms.

$$\text{Sum of } n \text{ terms} = \{2n^2 - 2n + 2 + (n-1)\,2\}\frac{n}{2} = n^2 \times n = n^3.$$

(30). Insert 7 arithmetic means between 1 and $-\frac{1}{2}$.

If there be 7 means, there are 9 terms;

$$\therefore \text{ ninth term} = 1 + 8d = -\frac{1}{2}; \quad \therefore d = -\frac{3}{16},$$

$$\text{and the series is } 1, \frac{13}{16}, \frac{10}{16}, \frac{7}{16}, \frac{4}{16}, \frac{1}{16}, -\frac{2}{16}, -\frac{5}{16}, -\frac{1}{2}.$$

(31). Insert 4 arithmetic means between 2 and -18.

The number of terms is 6; $\therefore$ sixth term $= 2 + 5d = -18$;

$\therefore d = -4$, and the series is $2, -2, -6, -10, -14, -18$.

(32). The fifth and ninth terms of an arithmetic series are 13 and 25; find the seventh term.

The fifth term $= a + 4d = 13$, ninth term $= a + 8d = 25$;

$\therefore 4d = 12$, or $d = 3$; $\therefore a = 1$;

and the seventh term $= 1 + 18 = 19$.

(33). If the sum of three numbers in arithmetic progression be 30, and the sum of their squares 308, what are the numbers?

Let $x - y$, x, and $x + y$ be the numbers; $\therefore x = 10$,

and $(10 - y)^2 + 10^2 + (10 + y)^2 = 308$;

$\therefore 2y^2 = 8$, and $y = \pm 2$;

$\therefore$ the series is 8, 10, 12.

(34). If the sum of n arithmetic means between 1 and 19 is to the sum of the first $(n - 2)$ of them :: 5 : 3; find the series.

$$\text{The } (n+2)^{\text{th}} \text{ term} = 1 + (n+1)\,d = 19; \quad \therefore d = \frac{18}{n+1},$$

$$\text{and the first mean is } 1 + \frac{18}{n+1} = \frac{n+19}{n+1};$$

$$\therefore \left\{\frac{2(n+19)}{n+1} + (n-1)\frac{18}{n+1}\right\}\frac{n}{2} : \left\{\frac{2(n+19)}{n+1} + \frac{(n-3)\times 18}{n+1}\right\}\frac{n-2}{2} :: 5 : 3,$$

$$\text{or } (20n + 20)\, n : (20n - 16) \times (n - 2) :: 5 : 3,$$

$$\text{or } (n + 1)\, n : (5n - 4) \times (n - 2) :: 1 : 3,$$

$$\text{whence } 2n^2 - 17n + 8 = 0;$$

$$\therefore\ n^2 - \frac{17n}{2} + \left(\frac{17}{4}\right)^2 = -\frac{8}{2} + \frac{289}{16} = \frac{225}{16};$$

$$\therefore\ n = \frac{17}{4} \pm \frac{15}{4} = 8, \text{ and } d = \frac{18}{9} = 2;$$

$\therefore$ the series is 1, 3, 5, 7, 9, 11, 13, 15, 17, 19.

(35). Divide $\frac{n}{7}(n + 4)$ into n parts, so that each term shall exceed the preceding one by a common difference.

$\frac{n}{7}(n+4)$ must be the sum of an arithmetic series of n terms;

$$\therefore\ \frac{n}{7}(n + 4) = \frac{n}{2}\left(\frac{2n + 8}{7}\right) = \left\{\frac{10}{7} + (n - 1)\frac{2}{7}\right\}\frac{n}{2};$$

$\therefore\ \frac{5}{7}$ is the first term, $\frac{2}{7}$ the common difference,

and n the number of terms,

if $n = 2$, the sum of 2 terms $= \frac{12}{7}$;

if $n = 4$, the sum of 4 terms $= \frac{32}{7}$,

and the series is $\frac{5}{7}$, 1, $1\frac{2}{7}$, $1\frac{4}{7}$, &c.

(36). Find four numbers in arithmetic progression, such that the product of the extremes shall be 27, and the product of the means 35.

Let $x - 3y,\ x - y,\ x + y,\ x + 3y$ be the series,

then $x^2 - 9y^2 = 27$, and $x^2 - y^2 = 35$; $\therefore\ 8y^2 = 8$; $\therefore\ y = \pm 1$,

and $x^2 = \pm 6$; $\therefore$ the series is 3, 5, 7, 9.

(37). If four numbers be in arithmetic progression, whose sum is 24 and product 945, what are the numbers?

Let $x - 3y,\ x - y,\ x + y$, and $x + 3y$ be the series, then $4x = 24$; $\therefore\ x = 6$, and $(36 - 9y^2)(36 - y^2) = 945$;

$$\therefore\ 9y^4 - 360y^2 + 1296 = 945,$$

$$\text{whence } y^2 = \frac{60}{3} \pm \frac{57}{3} = 1 \text{ or } 39;\ \therefore\ y = \pm 1,$$

and the series is 3, 5, 7, 9.

(38). If the n^{th} and m^{th} terms of an arithmetic progression be m and n respectively, find the number of terms whose sum is $\frac{1}{2}.(m+n).(m+n-1)$, and the last term of the series.

The n^{th} term $= a + (n-1)d = m$, the m^{th} term $= a + (m-1)d = n$;

$$\therefore\ (n-m)d = m - n;\ \therefore\ d = -1, \text{ and } a = m + n - 1,$$

and if x = number of terms, then sum

$$= \{2(m+n-1) - (x-1)\}\frac{x}{2} = \frac{(m+n)(m+n-1)}{2};$$

$$\therefore\ x^2 - (2m+2n-1)x + (\ )^2 = -(m+n)(m+n-1) + \left(m+n-\frac{1}{2}\right)^2 = \frac{1}{4};$$

$$\therefore\ x = m + n - \frac{1}{2} \pm \frac{1}{2} = m + n \text{ or } m + n - 1,$$

$$\text{the last term} = m + n - 1 - (m + n - 1) = 0,$$

$$\text{or} = m + n - 1 - (m + n - 2) = 1.$$

(39). The sum of the first n terms of a series in arithmetic progression is $\left(na - \frac{n+1}{2} \times b\right)\frac{n}{a+b}$. Find the series.

$$\left\{na - \frac{(n+1)}{2}b\right\}\frac{n}{a+b} = \text{the sum of a series to } n \text{ terms}$$

$$= \left\{\frac{2na - (n+1)b}{a+b}\right\}\frac{n}{2} = \left\{\frac{2(a-b)}{a+b} + \frac{(n-1)(2a-b)}{a+b}\right\}\frac{n}{2};$$

$\therefore$ the first term is $\frac{a-b}{a+b}$, the common difference $= \frac{2a-b}{a+b}$,

$$\text{and the series is } \frac{a-b}{a+b} + \frac{3a-2b}{a+b} + \&c.,$$

otherwise, let $n = 1$,

$$\text{then the expression} = \frac{2a-2b}{2} \cdot \frac{1}{a+b} = \frac{a-b}{a+b},$$

$$\text{and if } n=2, \text{ it} = \frac{4a-3b}{2} \times \frac{2}{a+b} = \frac{4a-3b}{a+b};$$

$$\therefore \text{ common difference} = \frac{2a-b}{a+b}.$$

(40). If a steam engine is observed to pass over 4 feet in the first second, and 88 feet in the sixtieth second of its motion, how far will it travel in the first minute, supposing its motion to be increased each second by a constant quantity?

$$\text{Sixtieth term} = 4 + 59 \times d = 88; \;\therefore\; d = \frac{84}{59} \text{ feet};$$

$$\therefore \text{ distance in } 60'' = \left(8 + 59 \times \frac{84}{59}\right) \times 30 = 2760 \text{ feet} = 920 \text{ yards}.$$

(41). The $(n+1)^{\text{th}}$ term of an arithmetic series is $\frac{ma-nb}{a-b}$. Required the sum of the series to $(2n+1)$ terms.

$$\text{The } (n+1)^{\text{th}} \text{ term} = \frac{ma-nb}{a-b} = \frac{ma}{a-b} - \frac{nb}{a-b};$$

$\therefore$ the first term $= \frac{ma}{a-b}$, and the common difference $= \frac{b}{a-b}$;

$\therefore$ the sum of $(2n+1)$ terms

$$= \left(\frac{2ma}{a-b} - \frac{2n \times b}{a-b}\right) \times \frac{2n+1}{2} = \frac{ma-nb}{a-b} \times (2n+1).$$

(42). Insert 3 arithmetic means between 1 and 11.

The number of terms will be 5;

$$\therefore \text{ the fifth term} = 1 + 4d = 11; \;\therefore\; d = \frac{5}{2},$$

$$\text{and the series is } 1, \frac{7}{2}, 6, \frac{17}{2}, 11.$$

(43). Insert 9 arithmetic means between 1 and – 1.

$$\text{The eleventh term} = 1 + 10 \times d = -1; \;\therefore\; d = -\frac{1}{5},$$

$$\text{and the series is } 1, \frac{4}{5}, \frac{3}{5}, \frac{2}{5}, \text{\&c.} -1.$$

(44). The sums of n terms of two arithmetical progressions are as $13 - 7n : 1 + 3n$; show that their first terms are as $3 : 2$, and their second terms as $-4 : 5$.

The number of terms are the same in each, but the first terms and common difference are unknown. Let the series be

$$a,\ a + x,\ a + 2x + \ldots,$$
$$b,\ b + y,\ b + 2y + \ldots;$$
$$\therefore\ 2a + (n-1)\,x : 2b + (n-1)\,y :: 13 - 7n : 1 + 3n,$$
$$\text{and if } n = 1,\ 2a : 2b :: 6 : 4,\ \text{ or } a : b :: 3 : 2,$$
$$\text{but if } n = 3,\ 2a + 2x : 2b + 2y :: -8 : 10,$$
$$a + x : b + y :: -4 : 5.$$

(45). The sum of the first two terms of an arithmetical progression is 4, and the fifth term is 9; find the series.

$2a + d = 4$, $a + 4d = 9$; $\therefore\ 2a + 8d = 18$; $\therefore\ d = 2$, and $a = 1$;

$\therefore$ the series is 1, 3, 5, &c.

(46). The first two terms of an arithmetical progression being together 18, and the next three terms 12; how many terms must be taken to make 28?

$$2a + b = 18,\ \text{the next three terms} = 3a + 9b = 12,$$
$$\text{whence } b = -2,\ \text{and } a = 10;$$
$$\therefore\ \{20 - (n-1)\,2\}\,\frac{n}{2} = 28,\ \text{whence } n^2 - 11n + \frac{121}{4} = \frac{9}{4};$$
$$\therefore\ n = \frac{11}{2} \pm \frac{3}{2} = 7 \text{ or } 4.$$

(47). The latter half of $2n$ terms of an arithmetical series is equal to one-third of the sum of $3n$ terms of the same series.

The latter half of $2n$ terms

$$= \{2a + (2n-1)\,d\}\,n - \{2a + (n-1)\,d\}\,\frac{n}{2}$$
$$= \{2a + (3n-1)\,d\}\,\frac{n}{2};$$
$$\therefore\ \{2a + (3n-1)\,d\}\,\frac{n}{2} = \frac{1}{3}\{2a + (3n-1)\,d\}\,\frac{3n}{2} = \{2a + (3n-1)\,d\}\,\frac{n}{2}.$$

Q.E.D.

(48). The difference between the sums of m and n terms of an arithmetical progression : the sum of $m+n$ terms $:: m-n : m+n$.

By problem

$$\{2a+(m-1)d\}\frac{m}{2}-\{2a+(n-1)d\}\frac{n}{2} : \{2a+(m+n-1)d\}\frac{m+n}{2}$$

$$:: a(m-n)+\{m^2-n^2-(m-n)\}\frac{d}{2} : \left\{a+(m+n-1)\frac{d}{2}\right\}(m+n)$$

$$:: a+(m+n-1)\frac{d}{2}(m-n) : \left\{a+(m+n-1)\frac{d}{2}\right\}(m+n)$$

$:: m-n : m+n$. Q.E.D.

(49). Determine the relation between a, b, and c, that they may be respectively the pth, qth, and rth terms of an arithmetic series.

The pth term $= x+(p-1)d = a$, the qth term $= x+(q-1)d = b$, the rth $= x+(r-1)d = c$;

$$\therefore (p-q)d = a-b, \text{ also } (q-r)d = b-c;$$

$$\therefore \frac{a-b}{p-q} = \frac{b-c}{q-r}, \text{ whence } (q-r)a+(r-p)b+(p-q)c = 0.$$

(50). If S, S', S'' be the sums of three arithmetic series, 1 be the first term of each, and the respective differences be 1, 2, 3; prove that $S+S'' = 2S'$.

$$S = \{2+(n-1)\}\frac{n}{2},\ S' = \{2+(n-1)2\}\frac{n}{2},\ S'' = \{2+(n-1)3\}\frac{n}{2},$$

$$\text{then } S+S'' = \{4+4(n-1)\}\frac{n}{2} = \{2+2(n-1)3\}n = 2S'.$$

(51). If there be p arithmetical progressions, each beginning from unity, whose common differences are $1, 2, 3 \ldots p$; shew that the sum of their nth terms is

$$= \frac{1}{2}\cdot[(n-1).p^2+(n+1).p].$$

The nth term of the first series $= 1+n-1 = n = n$,

. . . . second series $= 1+(n-1)2 = 2n-1 = n+n-1$,

. . . . third series $= 1+(n-1)3 = 3n-2 = n+2(n-1)$,

. . . . fourth series $= 1+(n-1)4 = 4n-3 = n+3(n-1)$,

. . . . pth series $= 1+(n-1)p = 1+pn-p = n+(p-1)(n-1)$;

∴ the sum of the nth terms

$$= \{2n + (p-1)(n-1)\}\frac{p}{2} = \frac{1}{2}\{(n-1)p^2 + (n+1)p\}.$$

(52). If a and b are respectively the first term and common difference of an arithmetic series, S_n the sum of n terms, S_{n+1} the sum of $(n+1)$ terms, &c.; prove that

$$S_n + S_{n+1} + S_{n+2} + \text{\&c. to } n \text{ terms}$$

$$= (3n-1).n.\frac{a}{2} + (7n-2).(n-1).n.\frac{b}{6}.$$

$$\begin{aligned} 2S_n &= \{2a + (n-1)b\}n &&= 2na + n.(n-1)b, \\ 2S_{n+1} &= (2a+nb)(n+1) &&= 2(n+1)a + n(n+1)b, \\ 2S_{n+2} &= \{2a + (n+1)b\}(n+2) &&= 2(n+2)a + (n+1)(n+2)b, \\ \text{\&c.} &= \text{\&c.} &&= \text{\&c.}; \end{aligned}$$

∴ sum to n terms $= \{n + (n+1) + (n+2) + \text{\&c. to } n \text{ terms}\}\,2a$
$+ \{(n-1)n + n(n+1) + (n+1)(n+2) + \text{\&c. to } n \text{ terms}\}\,b$

$$= (3n-1)na + \left\{\begin{matrix} n^2 + (n+1)^2 + (n+2)^2 + \text{\&c. to } n \text{ terms} \\ -n-(n+1)-(n+2) - \text{\&c. to } n \text{ terms}\end{matrix}\right\} \times b \ldots (A),$$

but $1^2 + 2^2 + 3^2 + \text{\&c. to } n \text{ terms} = \frac{1}{6}\{n(n+1)(2n+1)\}$;

∴ $1^2 + 2^2 + 3^2 + \text{\&c. to } (n-1) \text{ terms} = \frac{1}{6}(n-1)(n)(2n-1)$;

∴ $1^2 + 2^2 + 3^2 + 4^2 + \text{\&c.} + n^2 + (n+1)^2 + \text{\&c. to } (2n-1) \text{ terms}$

$$= \frac{1}{6}(2n-1)(2n)(4n-1);$$

$$\therefore (A) = \frac{n}{6}\{(4n-2)(4n-1) - (n-1)(2n-1)\}b - (3n-1)\frac{nb}{2}$$

$$= \frac{nb}{3}(7n^2 - 9n + 2) = \frac{nb}{3}(7n-2)(n-1);$$

$$\therefore \text{sum of series} = (3n-1)\frac{na}{2} + (n-1)(7n-2)\frac{nb}{6}.$$

(53). If $S_1, S_2, S_3 \ldots S_p$ be the sums of p arithmetic progressions continued to n terms, and their first terms be 1, 2, 3, 4

&c., and their common differences 1, 3, 5, 7, &c.; shew that $S_1 + S_2 + S_3 + \ldots + S_p = \frac{1}{2} \cdot (np + 1)\, np$.

Here $S_1 = \{2 + (n-1)\}\frac{n}{2} = (n+1)\frac{n}{2}$: $S_2 = \{4 + (n-1)\,3\}\frac{n}{3} = (3n+1)\frac{n}{2}$,

$S_3 = \{6 + (n-1)\,5\}\frac{n}{2} = (5n+1)\frac{n}{2}$: $S_5 = \{8 + (n-1)\,7\}\frac{n}{2} = (7n+1)\frac{n}{2}$,

and $S_p = \{2p + (n-1)(2p-1)\}\frac{n}{2} = \{(2p-1)\,n + 1\}\frac{n}{2}$;

$$\therefore\ S_1 + S_2 + S_3 + \&c.\ S_p = \frac{np}{2} + n\,\{1 + 3 + 5 + \&c.\ (2p-1)\}\frac{n}{2}$$

$$= \frac{np}{2} + \{2 + (p-1)\,2\}\frac{p}{2} \times \frac{n^2}{2} = (np+1)\frac{np}{2}.$$

(54). In an arithmetic series if the $(p + q)$th term be $= m$, and the $(p - q)$th $= n$, show that the pth term $= \frac{m+n}{2}$, and the qth term $= m - (m - n)\frac{p}{2q}$.

If a be the first term, and b the common difference,

then $(p + q)$th term $= a + (p + q - 1)\,b = m \ldots .(1)$,

and $(p - q)$th term $= a + (p - q - 1)\,b = n$,

and $\left(\frac{m+n}{2}\right) = a + (p - 1)\,b = p$th term;

$\therefore$ from (1) $bq = m - \frac{m+n}{2} = \frac{m-n}{2}$, and $a + (q-1)\,b = m - bp$;

$$\therefore\ q\text{th term} = a + (q-1)\,b = m - p.\frac{m-n}{2q}.$$

(55). $S_1, S_2, S_3 \ldots S_m$ are the sum of m arithmetic progressions to n terms, and their first terms are respectively 1, 3, 5, &c., and their common differences 1, 3, 5, 7; shew that

$$S_1 + S_2 + S_3 \ldots + S_m = \frac{n^3}{2}(n + 1).$$

$S_1 = \{2 + (n-1)\}\frac{n}{2} = (n+1)\frac{n}{2}$: $S_2 = \{6 + (n-1)\,3\}\frac{n}{2} = 3(n+1)\frac{n}{2}$,

$$S_3 = \{10 + (n-1)\,5\}\,\frac{n}{2} = 5\,(n+1)\,\frac{n}{2}, \text{ \&c.,}$$

$$\text{and } S_m = \{2\,(2m-1) + (n-1)\,(2m-1)\}\,\frac{n}{2} = (2m-1)\,(n+1)\,\frac{n}{2};$$

$$\therefore \text{ the sum} = \frac{n\,(n+1)}{2}\,\{1 + 3 + 5 + \ldots (2m-1)\}$$

$$= \frac{n\,(n+1)\,m^2}{2}.$$

XVI. In a *Geometric Progression*, if a be the first term, r the common ratio, n the number of terms, and S the sum of n terms; then

$$S = a \times \frac{r^n - 1}{r - 1}, \text{ and the } n\text{th term} = ar^{n-1};$$

but if r be a proper fraction, and n = infinity, then $S = \dfrac{a}{1-r}$.

Find the sum of the following series:—

(1). $\frac{1}{3}, \frac{1}{6}, \frac{1}{12}$, &c. to 8 terms.

$$\frac{1}{6} \div \frac{1}{3} = \frac{1}{6} \times \frac{3}{1} = \frac{1}{2} \text{ the ratio;}$$

$$\therefore\ S = \frac{1}{3} \times \frac{\left(\frac{1}{2}\right)^8 - 1}{\frac{1}{2} - 1} = \frac{1}{3}\,\frac{2^8 - 1}{128} = \frac{1}{3}\,\frac{256 - 1}{128} = \frac{255}{3 \times 128} = \frac{85}{128}.$$

(2). $\frac{1}{2}, \frac{1}{3}, \frac{2}{9}$, &c. to 6 terms and infinity.

$$\frac{1}{3} \div \frac{1}{2} = \frac{2}{3};$$

$$\therefore \text{ sum to 6 terms} = \frac{1}{2}\,\frac{\left(\frac{2}{3}\right)^6 - 1}{\frac{2}{3} - 1} = \frac{1}{2}\,\frac{3^6 - 2^6}{3^5} = \frac{729 - 64}{2 \times 243} = \frac{665}{486},$$

$$\text{and to infinity} = \frac{\frac{1}{2}}{1 - \frac{2}{3}} = \frac{3}{2}.$$

(3). $4 - 2 + 1 - \&c.$ to 12 terms.

$$\therefore -2 \div 4 = -\frac{2}{4} = -\frac{1}{2};$$

$$\therefore \text{ sum to 12 terms} = 4\,\frac{\left(-\frac{1}{2}\right)^{12} - 1}{-\frac{1}{2} - 1} = \frac{2^{12} - 1}{3 \times 512} = \frac{1365}{512}.$$

(4). $\frac{1}{3} + \frac{1}{4} + \frac{3}{16} + \&c.$ to 10 terms and infinity.

$$\frac{1}{4} \div \frac{1}{3} = \frac{3}{4} \text{ the ratio};$$

$$\therefore \text{ sum to 10 terms} = \frac{1}{3}\,\frac{\left(\frac{3}{4}\right)^{10} - 1}{\frac{3}{4} - 1} = \frac{1}{3}\,\frac{4^{10} - 3^{10}}{4^9} = \frac{989527}{786432},$$

$$\text{to infinity} = \frac{\frac{1}{3}}{1 - \frac{3}{4}} = \frac{4}{3}.$$

(5). $3 - 1 + \frac{1}{3} - \&c.$ to 8 terms and infinity.

$$\text{The ratio} = -1 \div 3 = -\frac{1}{3};$$

$$\therefore \text{ sum to 8 terms} = 3\,\frac{\left(-\frac{1}{3}\right)^{8} - 1}{-\frac{1}{3} - 1} = \frac{3^8 - 1}{3^6 \times 4} = \frac{1640}{729},$$

$$\text{to infinity} = \frac{3}{1 + \frac{1}{3}} = \frac{9}{4}.$$

(6). $\frac{1}{5} + \frac{1}{15} + \frac{1}{45} + \&c.$ to 10 terms and infinity.

$$\text{The ratio is } \frac{1}{15} \div \frac{1}{5} = \frac{1}{3};$$

$\therefore$ sum to 10 terms

$$= \frac{1}{5}\,\frac{\left(\frac{1}{3}\right)^{10} - 1}{\frac{1}{3} - 1} = \frac{1}{5}\,\frac{3^{10} - 1}{3^9 \times 2} = \frac{59048}{10\,(19683)} = \frac{29524}{98415},$$

$$\text{to infinity} = \frac{\frac{1}{5}}{1 - \frac{1}{3}} = \frac{3}{10}.$$

(7). $1 - 2x + 2x^2 - 2x^3 + \&c.$ to infinity.

Sum to infinity $= 1 - 2x\,(1 - x + x^2 - \&c.$ to infinity)

$$= 1 - 2x\,\frac{1}{1+x} = \frac{1-x}{1+x}.$$

(8). $\sqrt{\left(\frac{2}{5}\right)} - \sqrt{\left(\frac{1}{3}\right)} + \frac{1}{3}\sqrt{\left(\frac{5}{2}\right)} - \&c.$ to 10 terms and infinity.

$$\text{The ratio} = -\sqrt{\left(\frac{1}{3}\right)} \div \sqrt{\left(\frac{2}{5}\right)} = -\sqrt{\left(\frac{5}{6}\right)};$$

$$\therefore \text{ sum to 10 terms} = \sqrt{\frac{2}{5}}\,\frac{\left(-\frac{5}{6}\right)^5 - 1}{-\sqrt{\left(\frac{5}{6}\right)} - 1}$$

$$= \sqrt{\frac{2}{5}}\,\frac{6^5 - 5^5}{\{\sqrt{(5)} + \sqrt{6}\} \times 6^{\frac{9}{2}}} = \sqrt{\frac{3}{5}} \times \frac{4651}{3888\,(\sqrt{6} + \sqrt{5})},$$

$$\text{to infinity} = \frac{\sqrt{\frac{2}{5}}}{1 + \sqrt{\frac{5}{6}}} = \frac{2\sqrt{3}}{\sqrt{(30)} + 5} = \frac{2\sqrt{3}\,(\sqrt{6} - \sqrt{5})}{\sqrt{5}}.$$

(9). $\frac{2}{5} - \sqrt{\left(\frac{2}{5}\right)} + 1 - \&c.$ to 8 terms.

$$\text{The ratio} = -\sqrt{\frac{2}{5}} \div \frac{2}{5} = -\sqrt{\frac{5}{2}};$$

$$\therefore \text{sum to 8 terms} = \frac{2}{5}\,\frac{\left(\frac{5}{2}\right)^4 - 1}{-\sqrt{\frac{5}{2}} - 1} = -\frac{2\sqrt{2} \times 609}{5(\sqrt{5} + \sqrt{2})16} = -\frac{\sqrt{2} \times 609}{40(\sqrt{5} + \sqrt{2})},$$

$$\text{the same to infinity} = \frac{\frac{2}{5}}{1+\sqrt{\frac{5}{2}}} = \frac{2\sqrt{2}}{5(\sqrt{5}+\sqrt{2})}.$$

(10). $2 + \sqrt[4]{(8)} + \sqrt{(2)} + \&c.$ to 12 terms.

$$\text{The ratio} = 2^{\frac{3}{4}} \div 2 = 2^{-\frac{1}{4}};$$

$$\therefore \text{ sum to 12 terms} = 2 \times \frac{\left(\frac{1}{2}\right)^3 - 1}{\frac{1}{\sqrt[4]{2}} - 1} = \frac{2(8-1)}{(\sqrt[4]{2}-1)\times 2^{1\frac{1}{4}}} = \frac{7}{4 - 2\sqrt[4]{(8)}},$$

$$\text{and the same to infinity} = \frac{2}{1 - \sqrt[4]{\frac{1}{2}}} = \frac{2\sqrt[4]{(2)}}{\sqrt[4]{2} - 1}.$$

(11). $3 + 9^{\frac{1}{3}} + 3^{\frac{1}{3}} + \&c.$ to n terms and infinity.

$$\text{The ratio} = 9^{\frac{1}{3}} \div 3 = 3^{-\frac{1}{3}};$$

$$\therefore \text{ sum to } n \text{ terms} = \frac{3 \times (3^{-\frac{n}{3}} - 1)}{3^{-\frac{1}{3}} - 1} = \frac{(3^{\frac{n}{3}} - 1)}{3^{\frac{n-4}{3}}(\sqrt[3]{3} - 1)},$$

$$\text{and the same to infinity} = \frac{3}{1 - \sqrt[3]{\frac{1}{3}}} = \frac{3\sqrt[3]{3}}{\sqrt[3]{3} - 1}.$$

(12). $\frac{\sqrt{(2)} + 1}{\sqrt{(2)} - 1} + \frac{1}{2 - \sqrt{(2)}} + \frac{1}{2} + \&c.$ to infinity.

$$\text{The ratio} = \frac{1}{\sqrt{2}(\sqrt{2} - 1)} \times \frac{\sqrt{2} - 1}{\sqrt{2} + 1} = \frac{1}{2 + \sqrt{2}} = \frac{2 - \sqrt{2}}{2};$$

$\therefore$ sum of the series

$$= \frac{\sqrt{2} + 1}{\sqrt{2} - 1} \times \frac{1}{1 - \frac{(2 - \sqrt{2})}{2}} = \frac{\sqrt{2} + 1}{\sqrt{2} - 1} \times \frac{2}{\sqrt{2}} = 4 + 3\sqrt{2}.$$

(13). $1 - \frac{2}{3} + \frac{4}{9} -$ &c. to infinity.

$$\text{Sum to infinity} = \frac{1}{1 + \frac{2}{3}} = \frac{3}{5}.$$

(14). $1 + 2 - \frac{1}{4} + \frac{1}{32} -$ &c. to infinity.

$$\text{Sum to infinity} = 1 + \frac{2}{1 + \frac{1}{8}} = \frac{16}{9} + 1 = \frac{25}{9}.$$

(15). $\frac{a}{x}\sqrt{\left(\frac{3}{2}\right)} + \sqrt{\left(\frac{a}{x}\right)} + \frac{2}{3} +$ &c. to n terms.

$$\text{The ratio} = \sqrt{\left(\frac{a}{x} \times \frac{2}{3}\,\frac{x^2}{a^2}\right)} = \sqrt{\left(\frac{2x}{3a}\right)};$$

∴ the sum of the series to n terms

$$= \frac{a}{x}\sqrt{\frac{3}{2}}\,\frac{\left(\frac{2x}{3a}\right)^{\frac{n}{2}} - 1}{\sqrt{\left(\frac{2x}{3a}\right)} - 1} = \frac{a}{x}\sqrt{\frac{3}{2}}\,\frac{(2x)^{\frac{n}{2}} - (3a)^{\frac{n}{2}}}{(3a)^{\frac{n-1}{2}}\,(\sqrt{2x} - \sqrt{3a})}$$

$$= \frac{(2x)^{\frac{n}{2}} - (3a)^{\frac{n}{2}}}{x\sqrt{6}\,(3a)^{\frac{n-3}{2}}\,(\sqrt{2x} - \sqrt{3a})}.$$

(16). $\frac{a}{b} - \frac{a-b}{b^2}x + \frac{a-b}{b^3}x^2 -$ &c. to infinity.

$$\text{The series is } \frac{a}{b} - \frac{a-b}{1} \times \frac{x}{b^2}\left\{1 - \frac{x}{b} + \frac{x^2}{b^2} - \text{\&c.}\right\}$$

$$= \frac{a}{b} - \frac{a-b}{1} \times \frac{x}{b^2} \times \frac{1}{1 + \frac{x}{b}} = \frac{a}{b} - \frac{(a-b)\,x}{b\,(b+x)} = \frac{a+x}{b+x}.$$

(17). $\left(\frac{x}{y}\right)^{\frac{1}{2}} - \left(\frac{y}{x}\right)^{\frac{1}{2}} + \left(\frac{y}{x}\right)^{\frac{3}{2}} +$ &c. to infinity.

$$\text{The ratio} - \sqrt{\frac{y}{x}} \times \sqrt{\frac{y}{x}} = -\frac{y}{x};$$

$$\therefore \text{ the sum of the series to infinity} = \frac{\sqrt{\dfrac{x}{y}}}{1+\dfrac{y}{x}} = \frac{x\sqrt{(x)}}{\sqrt{y}\,(x+y)}.$$

(18). $x^p + x^{p+q} + x^{p+2q} +$ &c. to n terms.

$$\text{The ratio} = x^{p+q} \div x^p = x^q;$$

$$\therefore\ x^p + x^{p+q} + x^{p+2q} \text{ to } n \text{ terms} = x^p \times \frac{x^{nq}-1}{x^q-1}.$$

(19). $x - y + \frac{y^2}{x} - \frac{y^3}{x^2} +$ &c. to n terms.

$$\text{The ratio} = -\frac{y}{x};$$

$$\therefore\ x - y + \frac{y^2}{x} - \text{\&c. to } n \text{ terms} = x\,\frac{\left(-\dfrac{y}{x}\right)^n - 1}{\left(-\dfrac{y}{x}\right)-1} = \frac{1}{x^{n-2}}\,\frac{x^n-(-y)^n}{(x+y)}.$$

(20). $\frac{1}{\sqrt{(2)}\,\{1+\sqrt{(2)}\}} + \frac{1}{\{1+\sqrt{(2)}\}\,\{2+\sqrt{(2)}\}} + \frac{1}{\{2+\sqrt{(2)}\}\{3+2\sqrt{(2)}\}}$
+ &c. to infinity.

$$\text{The ratio} = \frac{1}{(1+\sqrt{2})\,(2+\sqrt{2})} \times \frac{2+\sqrt{2}}{1} = \frac{1}{1+\sqrt{2}} = \sqrt{2}-1;$$

$$\therefore \text{ sum of series to infinity} = \frac{1}{2+\sqrt{2}} \times \frac{1}{1-(\sqrt{2}-1)} = \frac{1}{4-2} = \frac{1}{2}.$$

(21). Find 3 geometric means between 2 and 32.

If the means be 3, the number of terms = 5;

$\therefore$ the fifth term $= 2r^4 = 32$; $\therefore\ r^4 = 16$, and $r = \pm 2$;

$\therefore$ the means are 4, 8, 16.

(22). Find 3 geometric means between $\frac{1}{2}$ and 128, and 9 and $\frac{1}{9}$.

As above the fifth term $= \frac{1}{2} \times r^4 = 128$;

$\therefore\ r^4 = 256$, $r^2 = 16$, and $r = \pm 4$;

$\therefore$ the means are 2, 8, 32.

Again, the fifth term $= \frac{1}{9} \times r^4 = 9$; $\therefore$ $r^4 = 81$, and $r = \pm 3$;

$\therefore$ the means are $\frac{1}{3}$, 1, 3.

(23). Find a mean proportional between 2 and 18, and ·05 and ·2.

If x be the mean proportional,

then $2 : x :: x : 18$; $\therefore$ $x^2 = 36$, and $x = 6$.

Again, $\cdot05 : x :: x : \cdot2$; $\therefore$ $x^2 = \cdot01$; $\therefore$ $x = \cdot1$.

(24). Find 3 terms in geometrical progression, whose sum is 14, and the sum of their squares 84.

Let the series be $a + ar + ar^2 = 14$; $\therefore$ $a^2 + a^2r^2 + a^2r^4 = 84$,

$$\text{and } \frac{a(1 + r^2 + r^4)}{1 + r + r^2} = a(1 - r + r^2) = 6;$$

$$\therefore \frac{1 + r + r^2}{1 - r + r^2} = \frac{7}{3}, \text{ and } \frac{1 + r^2}{2r} = \frac{10}{8} = \frac{5}{4};$$

$$\therefore r^2 - \frac{5r}{2} + \left(\frac{5}{4}\right)^2 = -1 + \frac{25}{16} = \frac{9}{16}; \therefore r = \frac{5}{4} \pm \frac{3}{4} = 2 \text{ or } \frac{1}{2},$$

$$\text{and } a = \frac{14}{1 + 2 + 4} = 2, \text{ and the series is } 2, 4, 8.$$

(25). Find the nth term and the sum of n terms of the series 1, 5, 13, 29, 61, &c., and 1, 3, 7, 15, 31, &c.

The series $= 1 + 5 + 13 +$ &c. to n terms

$= 4 + 8 + 16 +$ &c. to n terms

$- 3 - 3 - 3 -$ &c. to n terms

$= 4(2^n - 1) - 3n$,

and the nth term $= 4 . 2^{n-1} - 3 = 2^{n+1} - 3$,

the series $= 1, 3, 7, 15$ to n terms

$= 2 + 4 + 8 + 16$ to n terms

$- 1 - 1 - 1 - 1$ to n terms

$= 2(2^n - 1) - n$,

and the nth term $= 2 . 2^{n-1} - 1 = 2^n - 1$.

(26). If the difference of two numbers is 48, and the arithmetic mean exceed the geometric by 18; what are the numbers?

Let x and y be the numbers;

$$\therefore\ x - y = 48,\ \text{and the arithmetic mean} = \frac{x+y}{2} = y + 24,$$

$$\text{also the geometric mean } \sqrt{(xy)} = \sqrt{(y^2 + 48y)},$$

$$\text{and } y + 24 - \sqrt{(y^2 + 48y)} = 18;$$

$$\therefore\ y^2 + 12y + 36 = y^2 + 48y;\ \therefore\ y = 1;\ \therefore\ x = 49.$$

(27). If the second term of a geometric series be 4 and the fifth 256, find the series.

$$\text{Second term} = ar = 4,\ \text{the fifth term} = ar^4 = 256;$$

$$\therefore\ r^3 = 64,\ \text{and } r = 4;\ \therefore\ a = 1;$$

$$\therefore\ \text{the series is } 1, 4, 16.$$

(28). If the sum of 3 numbers in a geometric progression be 7, and the sum of their squares 21, what is the series?

$$\text{If } a + ar + ar^2 = 7,\ \text{and } a^2(1 + r^2 + r^4) = 21;$$

$$\therefore\ a\left(\frac{1 + r^2 + r^4}{1 + r + r^2}\right) = a(1 - r + r^2) = \frac{21}{7} = 3;$$

$$\therefore\ \frac{1 + r + r^2}{1 - r + r^2} = \frac{7}{3},\ \text{or } \frac{1 + r^2}{2r} = \frac{10}{8} = \frac{5}{4},$$

$$\text{or } r^2 - \frac{5r}{2} + \left(\frac{5}{4}\right)^2 = -1 + \frac{25}{16} = \frac{9}{16};$$

$$\therefore\ r = \frac{5}{4} \pm \frac{3}{4} = 2,\ \text{or } -\frac{1}{2};\ \therefore\ a = 1,$$

and the series is 1, 2, 4.

(29). If the product of four numbers in geometric progression be 64, and the sum of their products taken 2 together be 70, what is the series?

If a, ar, ar^2, ar^3 be the four numbers, $a^4r^6 = 64$; $\therefore\ a^2r^3 = 8$,

$$\text{sum of products } a^2r + a^2r^2 + a^2r^3 + a^2r^3 + a^2r^4 + a^2r^5 = 70,$$

$$\text{but } 2a^2r^3 = 16;$$

$$\therefore\ a^2r(1 + r + r^3 + r^4) = 54,\ \text{but } a^2r = \frac{8}{r^2};$$

$$\therefore\ 4(1 + r + r^3 + r^4) = 27r^2,$$

$$\text{or } r^4 + r^3 - \frac{27r^2}{4} + r + 1 = 0;$$

$$\therefore \left(r + \frac{1}{r}\right)^2 + \left(r + \frac{1}{r}\right) + \frac{1}{4} = \frac{27}{4} + 2 + \frac{1}{4} = \frac{36}{4},$$

$$\text{and } r + \frac{1}{r} = -\frac{1}{2} \pm \frac{6}{2} = \pm\frac{5}{2} \text{ or } -\frac{7}{2},$$

$$\text{and } r^2 - \frac{5r}{r} + \frac{25}{16} = \frac{9}{16};\ \therefore\ r = \frac{5}{4} \pm \frac{3}{4} = 2, \text{ and } a = 1,$$

and the series is 1, 2, 4, 8.

(30). Insert three geometric means between 1 and $\frac{1}{8}$, and between 100 and $\frac{64}{25}$.

Since the number of means is 3, the number of terms is 5;

$$\therefore \text{ fifth term} = r^4 = \frac{1}{8};\ \therefore\ r = \frac{1}{\sqrt[4]{(8)}};$$

$$\therefore \text{ the means are } \frac{1}{\sqrt[4]{(8)}},\ \frac{1}{\sqrt{(8)}},\ \frac{1}{\sqrt[4]{(8^3)}},$$

$$\text{also the fifth term} = 100 \times r^4 = \frac{64}{25};\ \therefore\ r^4 = \frac{16}{625};\ \therefore\ r = \frac{2}{5},$$

and the means are 40, 16, $\frac{32}{5}$.

(31). Find a geometric series such that the sum of the first two terms shall be $1\frac{1}{3}$ and of the next two 12.

Let a, ar, ar^2, &c. be the series,

$$\text{then } a(1 + r) = \frac{4}{3}, \text{ and } ar^2(1 + r) = 12;$$

$$\therefore \frac{ar^2}{a} = \frac{12}{1} \times \frac{3}{4}, \text{ or } r^2 = 9;\ \therefore\ r = 3, \text{ and } a = \frac{4}{3} \times \frac{1}{4} = \frac{1}{3};$$

$\therefore$ the series is $\frac{1}{3}$, 1, 3, 9, &c.

(32). In a geometrical progression, if the $(p + q)$th term $= m$, and the $(p - q)$th term $= n$; then will the pth term $= \sqrt{(mn)}$,

T

$$\text{and the } q\text{th term} = m\left(\frac{n}{m}\right)^{\frac{p}{2q}}.$$

Let the first term be a, and r the ratio,

then $(p+q)$th term $= ar^{p+q-1} = m$, and $(p-q)$th $= ar^{p-q-1} = n$,

then $mn = a^2r^{2(p-1)}$; $\therefore$ the pth term $= ar^{p-1} = \sqrt{(mn)}$,

$$\text{also } \frac{m}{n} = r^{2q}; \ \therefore\ r = \left(\frac{m}{n}\right)^{\frac{1}{2q}},$$

$$\text{and the } q\text{th term} = ar^{q-1} = \frac{m}{rp} = m\left(\frac{n}{m}\right)^{\frac{p}{2q}}.$$

(33). If the pth and qth terms of a geometrical progression be P and Q, then will the nth term $= \left(\dfrac{Q^{p-n}}{P^{q-n}}\right)^{\frac{1}{p-q}}$, and the sum of

$$n \text{ terms} = \left(\frac{Q^{p-n}}{P^{q-1}}\right)^{\frac{1}{p-q}} \left\{\frac{P^{\frac{n}{p-q}} - Q^{\frac{n}{p-q}}}{P^{\frac{1}{p-q}} - Q^{\frac{1}{p-q}}}\right\}.$$

The pth term $= ar^{p-1} = P$; qth $= ar^{q-1} = Q$;

$$\therefore\ r^{p-q} = \frac{P}{Q}, \text{ and } r = \left(\frac{P}{Q}\right)^{\frac{1}{p-q}}, \text{ and } a = \frac{P}{r^{p-1}} = P \times \left(\frac{Q}{P}\right)^{\frac{p-1}{p-q}},$$

$$\text{the } n\text{th term} = ar^{n-1} = P \times \left(\frac{Q}{P}\right)^{\frac{p-1}{p-q}} \times \left(\frac{P}{Q}\right)^{\frac{n-1}{p-q}} = \left(\frac{Q^{p-n}}{P^{q-n}}\right)^{\frac{1}{p-q}},$$

sum of n terms

$$= P\left(\frac{Q}{P}\right)^{\frac{p-1}{p-q}} \frac{\left\{\left(\frac{P}{Q}\right)^{\frac{n}{p-q}} - 1\right\}}{\left(\frac{P}{Q}\right)^{\frac{1}{p-q}} - 1} = \left(\frac{Q^{p-n}}{P^{p-1}}\right)^{\frac{1}{p-q}} \left\{\frac{P^{\frac{n}{p-q}} - Q^{\frac{n}{p-q}}}{P^{\frac{1}{p-q}} - Q^{\frac{1}{p-q}}}\right\}.$$

(34). If a, b, and c be the pth, qth, and rth terms of a geometrical progression, then will $a^{q-r}\, b^{r-p}\, c^{p-q} = 1$.

If m be the first term, and s the ratio,

the pth term $= ms^{p-1} = a$, qth $= ms^{q-1} = b$, rth $= ms^{r-1} = c$,

then $m = as^{1-p} = bs^{1-q} = cs^{1-r}$,

$$\text{and } s^{q-p} = ba^{-1}; \text{ also } s^{r-q} = cb^{-1};$$

$$\therefore (ba^{-1})^{\frac{1}{q-p}} = (cb^{-1})^{\frac{1}{r-q}}; \therefore (ba^{-1})^{r-q} = (cb^{-1})^{q-p};$$

$$\therefore \text{ or } a^{q-r} \times b^{r-p} \times c^{p-q} = 1.$$

(35). Insert three geometric means between 39 and 3159; also, between 37 and 2997.

Since the number of means is 3, the number of terms is 5,

and the fifth term $= 39r^4 = 3159$; $\therefore r^4 = \frac{3159}{39} = 81$; $\therefore r = 3$,

and the means are 117, 351, 1053,

also the fifth term $= 37 \times r^4 = 2997$; $\therefore r^4 = \frac{2997}{37} = 81$; $\therefore r = 3$,

and the means are 111, 333, 999.

(36). In the geometrical progression $(x - y) + \left(\frac{y^2}{x} - \frac{y^3}{x^2}\right) + \&c.$, show that the sum of n terms : sum $(n = \text{inf.}) :: x^{2n} - y^{2n} : x^{2n}$.

$$\text{The ratio} = \frac{y^2 (x - y)}{x^2 (x - y)} = \frac{y^2}{x^2};$$

$$\therefore \text{sum to infinity} = \frac{(x - y)\, x^2}{x^2 - y^2} = \frac{x^2}{x + y},$$

$$\text{sum to } n \text{ terms} = (x - y)\, \frac{y^{2n} - x^{2n}}{x^{2n-2}(y^2 - x^2)};$$

$$\therefore \text{by problem } \frac{-(y^{2n} - x^{2n})}{x^{2n-2}(x + y)} : \frac{x^2}{x + y} :: x^{2n} - y^{2n} : x^{2n}.$$

(37). Show that the nth term of $\frac{1}{\sqrt{(2)}} + \frac{1}{2} + \frac{1}{2\sqrt{(2)}} + \&c. = \frac{1}{\sqrt{(2^n)}}$

and the sum of n terms $= \frac{1}{\sqrt{(2^n)}}\left\{\frac{\sqrt{(2^n)} - 1}{\sqrt{(2)} - 1}\right\}$.

$$\text{The ratio} = \frac{1}{2} \times \frac{\sqrt{2}}{1} = \frac{1}{\sqrt{(2)}}; \therefore n\text{th term} = \frac{1}{\sqrt{2}} \times \left(\frac{1}{2}\right)^{\frac{1}{2}(n-1)} = \frac{1}{\sqrt{(2^n)}};$$

$$\therefore \text{sum of } n \text{ terms} = \frac{\frac{1}{\sqrt{(2)}}\left\{\left(\frac{1}{2}\right)^{\frac{1}{2}(n)} - 1\right\}}{\frac{1}{\sqrt{(2)}} - 1} = \frac{1}{\sqrt{(2^n)}}\,\frac{\sqrt{(2^n)} - 1}{\sqrt{(2)} - 1},$$

$$\text{sum to infinity} = \frac{\frac{1}{\sqrt{(2)}}}{1 - \frac{1}{\sqrt{(2)}}} = \frac{1}{\sqrt{(2)} - 1}.$$

(38). Show that $\frac{3}{2} - 1 + \frac{2}{3} - \&c.$ to inf. $= \frac{9}{10}$.

$$\therefore \text{ the ratio} = -1 \times \frac{2}{3} = -\frac{2}{3};$$

$$\therefore \text{ sum to infinity} = \frac{\frac{3}{2}}{1 + \frac{2}{3}} = \frac{9}{10}.$$

(39). Show that $\sqrt{\left(\frac{3}{2}\right)} + \sqrt{\left(\frac{2}{3}\right)} + \frac{2}{3}\sqrt{\left(\frac{2}{3}\right)} + \&c.$ to n terms $= \sqrt{\left(\frac{3}{2}\right)}\left(\frac{3^n - 2^n}{3^{n-1}}\right)$; and to inf. $= 3\sqrt{\left(\frac{3}{2}\right)}$.

$$\text{The ratio} = \sqrt{\left(\frac{2}{3}\right)} \times \sqrt{\left(\frac{2}{3}\right)} = \frac{2}{3};$$

$$\therefore \text{ sum to } n \text{ terms} = \frac{\sqrt{\frac{3}{2}}\left(\left(\frac{2}{3}\right)^n - 1\right)}{\frac{2}{3} - 1} = \sqrt{\frac{3}{2}}\left(\frac{3^n - 2^n}{3^{n-1}}\right),$$

$$\text{sum to infinity} = \frac{\sqrt{\frac{3}{2}}}{1 - \frac{2}{3}} = 3\sqrt{\frac{3}{2}}.$$

(40). Show that $\frac{1}{3} + \frac{1}{6\sqrt{(-1)}} - \frac{1}{12} - \&c.$ to inf. $= \frac{2}{15}\{2 - \sqrt{(-1)}\}$.

$$\text{The ratio} = \frac{1}{6\sqrt{(-1)}} \times \frac{3}{1} = \frac{1}{2\sqrt{(-1)}};$$

$\therefore$ sum to infinity

$$= \frac{1}{3} \times \frac{1}{1 - \frac{1}{2\sqrt{(-1)}}} = \frac{2\sqrt{(-1)}}{3}\,\frac{\{2\sqrt{(-1)} + 1\}}{(-4 - 1)} = \frac{2\{2 - \sqrt{(-1)}\}}{15}.$$

(41). Show that

$$\frac{a}{(1+x)^n} + \frac{ax}{(1+x)^{n+1}} + \frac{ax^2}{(1+x)^{n+2}} + \text{\&c. to inf.} = \frac{a}{(1+x)^{n-1}}.$$

$$\text{The ratio} = \frac{ax}{(1+x)^{n+1}} \times \frac{(1+x)^n}{a} = \frac{x}{(1+x)};$$

$$\therefore \text{ sum to infinity} = \frac{a}{(1+x)^n\left(1 - \dfrac{x}{1+x}\right)} = \frac{a}{(1+x)^{n-1}}.$$

(42). In a geometrical progression, show that

$$r = \frac{S-a}{S-l}.\ l(S-l)^{n-1} - a(S-a)^{n-1} = 0,$$

$$r^n - \frac{Sr^{n-1}}{S-l} + \frac{l}{S-l} = 0, \quad S = \frac{l^{\frac{n}{n-1}} - a^{\frac{n}{n-1}}}{l^{\frac{1}{n-1}} - a^{\frac{1}{n-1}}}.$$

$$\text{The last term } l = ar^{n-1}, \ S = \frac{a(r^n-1)}{r-1};$$

$$\therefore \frac{S-a}{S-l} = \frac{a(r^n-1) - a(r-1)}{a(r^n-1) - ar^{n-1}(r-1)} = \frac{r^n - r}{r^{n-1}-1} = \frac{r^{n-1}-1}{r^{n-1}-r} \times r = r,$$

$$\text{and } r = (la^{-1})^{\frac{1}{n-1}}; \ \therefore (S-a) = (S-l)(la^{-1})^{\frac{1}{n-1}};$$

$$\therefore (S-a)\,a^{\frac{1}{n-1}} - (S-l)\,l^{\frac{1}{n-1}} = 0,$$

$$\text{or } a(S-a)^{n-1} - l(S-l)^{n-1} = 0,$$

$$\text{also } \frac{S}{l} = \frac{r^n-1}{(r-1)r^{n-1}}; \ \therefore \frac{S}{S-l} = \frac{r^n-1}{r^n-1-r^n+r^{n-1}} = \frac{r^n-1}{r^{n-1}-1};$$

$$\therefore r^n - \frac{Sr^{n-1}-S}{S-l} - 1 = 0, \text{ or } r^n - \frac{Sr^{n-1}}{S-l} + \frac{l}{S-l} = 0,$$

$$\text{again } l = ar^{n-1}; \ \therefore r = (la^{-1})^{\frac{1}{n-1}}, \text{ and } S = \frac{a(r^n-1)}{(r-1)};$$

$$\therefore S = \frac{a\{(la^{-1})^{\frac{n}{n-1}} - 1\}}{(la^{-1})^{\frac{1}{n-1}} - 1} = \frac{l^{\frac{n}{n-1}} - a^{\frac{n}{n-1}}}{l^{\frac{1}{n-1}} - a^{\frac{1}{n-1}}}.$$

(43). Find the geometrical progression, when the sum of the first and second terms is 9, the sum of the first and third is 15; and show how many terms of $6 + 4 + 2\frac{2}{3} + \&c.$ amount to 18.

Let the first + second term $= a + ar = q$; $\therefore a + ar^2 = 15$;

$$\therefore \frac{r^2+1}{r+1} = \frac{5}{3}, \text{ whence } r = \frac{5}{6} \pm \frac{7}{6} = 2 \text{ or } -\frac{1}{3};$$

$\therefore a = 3$, and the series is 3, 6, 12, &c.,

$$\text{or } a = \frac{9}{1-\frac{1}{3}} = \frac{27}{2} = 13\tfrac{1}{2};$$

$\therefore$ the series is $13\frac{1}{2}, -4\frac{1}{2}, 1\frac{1}{2} -$ &c.

$$\text{Let } \frac{6\left\{\left(\frac{2}{3}\right)^n - 1\right\}}{\frac{2}{3} - 1} = 18; \ \therefore \left(\frac{2}{3}\right)^n - 1 = -1; \ \therefore \left(\frac{2}{3}\right)^n = 0,$$

and the problem is impossible in finite terms, but if

$$n = \infty, \text{ then } \frac{6}{1-\frac{2}{3}} = 18.$$

(44). In any geometrical progression, consisting of an odd number of terms, the sum of the squares of the terms is equal to the sum of all the terms multiplied by the difference of the odd and even terms.

Let $a + ax + ax^2 + \&c. + ax^{2n}$ be the series;

$\therefore$ number of terms $= 2n + 1$ where n is any integer,

$$\text{then sum of } 2n \text{ terms} = a\,\frac{x^{2n+1} - 1}{x - 1},$$

$$\text{sum of odd terms} = a\,\frac{(x^2)^{n+1} - 1}{x^2 - 1},$$

$$\text{sum of even terms} = ax\,\frac{(x^2)^{n-1} - 1}{x^2 - 1};$$

$$\therefore \text{difference of sums} = \frac{a}{x+1}(x^{2n+1} + 1) \text{ by reduction,}$$

$$\text{sum of squares} = \frac{a^2\{(x^2)^{2n+1} - 1\}}{x^2 - 1} = \frac{(x^{2n+1} - 1)}{x - 1} \times \frac{a}{x+1}(x^{2n+1} + 1)$$

= sum of series × difference of sum of odd and even terms.

(45). In any geometrical progression, the sum of the first and last terms is greater than the sum of any other two terms equidistant from the extremes.

If a be the first term, and l the last term, and m and n any intermediate terms equidistant from the extremes, and r the ratio, then

$$m = ar^x, \ l = nr^x;$$

$$\therefore \ a + l = mr^{-x} + nr^x, \text{ and } \therefore > m + n.$$

(46). If S_1, S_2, S_3, &c., S_n, be the sums of n geometrical progressions, whose first terms are a, $2a$, $3a$, &c., na; then will

$$S_1 + S_2 + S_3 + \&c. + S_n = \frac{n(n+1)}{2}\left(\frac{r^n - 1}{r - 1}\right) a.$$

$$S_1 + S_2 + S_3 + \&c. \ a\frac{(r^n - 1)}{r - 1} + 2a\frac{r^n - 1}{r - 1} + \&c.$$

$$= (a + 2a + 3a + \&c.)\frac{r^n - 1}{r - 1} = \frac{n(n+1)}{2} \cdot \frac{r^n - 1}{r - 1} \times a.$$

(47). If a, b, c, d, &c., be n quantities in geometrical progression, then will $\frac{1}{a^2 - b^2}$, $\frac{1}{b^2 - c^2}$, $\frac{1}{c^2 - d^2}$, &c. be in geometrical progression, and the sum of n terms will be $\frac{1}{b^{2(n-1)}} \cdot \frac{a^{2n} - b^{2n}}{(a^2 - b^2)^2}$.

Let the ratio $= r$, then $b = ar$, $c = ar^2$, $d = ar^3$, and $r = ba^{-1}$;

$\therefore$ the sum to n terms

$$= \frac{1}{a^2(1 - r^2)} + \frac{1}{a^2r^2(1 - r^2)} + \frac{1}{a^2r^4(1 - r^2)} + \&c.$$

$$= \frac{1}{a^2(1 - r^2)}\left\{1 + \frac{1}{r^2} + \frac{1}{r^4} + \&c. \text{ to } n \text{ terms}\right\}$$

$$= \frac{1}{a^2\{1 - (ba^{-1})^2\}}\left[\frac{(ab^{-1})^{2n} - 1}{\{(ab^{-1})^2 - 1\}} = \frac{1}{b^{2n-2}} \times \frac{a^{2n} - b^{2n}}{(a^2 - b^2)^2}\right].$$

(48). If the arithmetic mean between a and b be twice as great as the geometric,

$$\text{then } \frac{a}{b} = \frac{2 + \sqrt{(3)}}{2 - \sqrt{(3)}}.$$

The arithmetic mean $= \frac{a + b}{2}$, the geometric $= \sqrt{(ab)}$, by problem $a + b = 4\sqrt{(ab)}$; $\therefore\ a - b = 2\sqrt{(3ab)}$,

$$\text{and } \frac{a+b}{a-b} = \frac{2\sqrt{(ab)}}{\sqrt{(3ab)}} = \frac{2}{\sqrt{3}};\ \therefore\ \frac{a}{b} = \frac{2+\sqrt{3}}{2-\sqrt{3}}.$$

(49). If a, b, c, d, e, be in *Harmonic Progression*, then

$$a : c :: a - b : b - c;$$
$$b : d :: b - c : c - d;$$
$$c : e :: c - d : d - e;$$

and their reciprocals are in arithmetic progression;

$$\text{i.e. } ab - ac = ac - bc;$$

$$\text{or, } \frac{1}{c} - \frac{1}{b} = \frac{1}{b} - \frac{1}{a}.$$

Then $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}, \frac{1}{d}$ form an arithmetic progression,

$$\text{and } \frac{1}{b} - \frac{1}{a} = \frac{1}{c} - \frac{1}{b}, \text{ or } \frac{a-b}{ab} = \frac{b-c}{bc};$$

$$\therefore \frac{a-b}{b-c} = \frac{a}{c}, \text{ so also } \frac{1}{c} - \frac{1}{b} = \frac{1}{d} - \frac{1}{c},$$

and $b : a :: b - c : c - d$, &c.

(50). If a and b be the first two terms of an harmonic series, find the nth term.

If a and b be the first two terms of an harmonic series,

then $\frac{1}{a}, \frac{1}{b}$ are the first of an arithmetic series,

and $\frac{1}{b} - \frac{1}{a} = \frac{a-b}{ab}$ is the common difference,

and the nth term of the arithmetic series

$$= \frac{1}{a} + (n-1) \times \frac{a-b}{ab} = \frac{b + (n-1)(a-b)}{ab};$$

$\therefore$ the nth term of the harmonic series

$$= \frac{ab}{b + (n-1)(a-b)} = \frac{ab}{(n-1)a - (n-2)b}.$$

(51). Insert two harmonic means between 6 and 24, and six between 3 and $\frac{6}{23}$.

The arithmetic means between $\frac{1}{6}$ and $\frac{1}{24}$ are two;

$\therefore$ number of terms is 4,

and the fourth term $= \frac{1}{6} + 3 \times d = \frac{1}{24}$; $\therefore d = -\frac{1}{24}$,

and the arithmetic series is $\frac{1}{6}, \frac{3}{24}, \frac{2}{24}, \frac{1}{24}$;

$\therefore$ the harmonic means are 8 and 12,

also if the arithmetic means be 6 between $\frac{1}{3}$ and $\frac{23}{6}$,

the number of terms is 8;

$\therefore$ the eighth term $= \frac{1}{3} + 7 \times d = \frac{23}{6}$; $\therefore d = \frac{1}{2}$,

and the arithmetic series is $\frac{1}{3}, \frac{5}{6}, \frac{8}{6}, \frac{11}{6}, \frac{14}{6}, \frac{17}{6}, \frac{20}{6}, \frac{23}{6}$,

and the harmonic means are $\frac{6}{5}, \frac{4}{3}, \frac{6}{11}, \frac{3}{7}, \frac{6}{17}, \frac{3}{10}$.

(52). The sum of three consecutive terms in harmonic progression is $1\frac{8}{15}$, and the first term is 1; find the series.

If $1 + 1 + b$, $1 + 2b$ be the arithmetic series,

$$1 + \frac{1}{1+b} + \frac{1}{1+2b} = \frac{23}{15} \text{ in the harmonic series;}$$

$$\therefore b^2 - \frac{216}{16} = 22, \text{ whence } b = \frac{21}{32} \pm \frac{43}{32} = 2;$$

$\therefore$ the arithmetic series is 1, 3, 5, &c.,

and the harmonic 1, $\frac{1}{3}, \frac{1}{5}, \frac{1}{7}$, &c.

(53). If the arithmetic mean between a and b be equal to m times the harmonic,

$$\text{then } \frac{a}{b} = \frac{\sqrt{(m)} + \sqrt{(m-1)}}{\sqrt{(m)} - \sqrt{(m-1)}},$$

and if the geometric be equal to m times the harmonic,

$$\text{then } \frac{a}{b} = \frac{m + \sqrt{(m^2-1)}}{m - \sqrt{(m^2-1)}}.$$

$$\text{The arithmetic mean} = \frac{a+b}{2}, \text{ the harmonic} = \frac{2ab}{a+b},$$

$$\text{by problem } (a+b)^2 = 4mab,$$

$$\text{or } (a+b) = 2\sqrt{(mab)},\ (a-b) = 2\sqrt{\{(m-1)\,ab\}};$$

$$\therefore \frac{a+b}{a-b} - \frac{2\sqrt{(mab)}}{2\sqrt{\{(m-1)\,ab\}}}, \text{ and } \frac{a}{b} = \frac{\sqrt{(m)}+\sqrt{(m-1)}}{\sqrt{(m)}-\sqrt{(m-1)}},$$

$$\text{also if } \sqrt{(ab)} = \frac{m\,(2ab)}{(a+b)},$$

$$\text{and } a+b = 2m\sqrt{(ab)},\ \text{also } (a-b) = 2\sqrt{\{(m^2-1)\,ab\}};$$

$$\therefore \frac{a+b}{a-b} = \frac{m}{\sqrt{(m^2-1)}}, \text{ and } \frac{a}{b} = \frac{m+\sqrt{(m^2-1)}}{m-\sqrt{(m^2-1)}}.$$

(54). Insert two harmonic means between 2 and 4.

$$\text{The arithmetic means between } \frac{1}{2} \text{ and } \frac{1}{4} \text{ are two};$$

$$\therefore \text{the fourth term} = \frac{1}{2} + 3 \times d = \frac{1}{4};\ \therefore d = -\frac{1}{12},$$

$$\text{and the arithmetic series is } \frac{1}{2},\ \frac{5}{12},\ \frac{4}{12},\ \frac{1}{4};$$

$$\therefore \text{the harmonic means are } \frac{12}{5} \text{ and } 3.$$

(55). Insert four harmonic means between 2 and 12.

$$\text{The arithmetic means between } \frac{1}{2} \text{ and } \frac{1}{12} \text{ are } 4;$$

$$\therefore \text{sixth term} = \frac{1}{2} + 5 \times d = \frac{1}{12};\ \therefore d = -\frac{1}{12},$$

$$\text{also the arithmetic series is } \frac{1}{2},\ \frac{5}{12},\ \frac{4}{12},\ \frac{3}{12},\ \frac{2}{12},\ \frac{1}{12};$$

$$\therefore \text{the harmonic means are } \frac{12}{5},\ 3,\ 4,\ 6.$$

(56). Insert n harmonic means between x and y.

$$\text{Since the arithmetic means between } \frac{1}{x} \text{ and } \frac{1}{y} \text{ are } n,$$

the number of terms is $n + 2$;

$\therefore$ the $(n+2)$th term $= \frac{1}{x} + (n+1)\,d = \frac{1}{y}$; $\therefore d = \frac{x-y}{xy\,(n+1)}$,

and $\frac{1}{x} + \frac{x-y}{xy\,(n+1)} = \frac{(n+1)\,y + x - y}{xy\,(n+1)} = \frac{ny+x}{xy\,(n+1)}$,

and the arithmetic series is

$$\frac{1}{x},\ \frac{ny+x}{(n+1)\,xy},\ \frac{(n-1)\,y+2n}{(n+1)\,xy},\ \ldots\ldots\ \frac{y+nx}{(n+1)\,xy},\ \frac{1}{y};$$

$\therefore$ the harmonic series is

$$x,\ \frac{(n+1)\,ny}{x+ny},\ \frac{(n+1)\,ny}{(n-1)\,y+2x},\ \ldots\ldots\ \frac{(n+1)\,ny}{y+nx},\ y.$$

(57). The sum of three terms of a harmonic series is $1\frac{1}{12}$, and the first term is $\frac{1}{2}$; find the series, and continue it both ways.

If 2, $2+b$, $2+2b$ be the arithmetic series,

then $\frac{1}{2} + \frac{1}{2+b} + \frac{1}{2+2b} = \frac{13}{12}$, whence $b^2 + \frac{3b}{7} + (\ \ )^2 = \frac{289}{196}$;

$$\therefore b = -\frac{3}{14} \pm \frac{17}{14} = 1 \text{ or } -\frac{10}{7};$$

$\therefore$ the arithmetic series is 2, 3, 4, 5,

and the harmonic series is 0, 1, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$.

(58). The first two terms of a harmonic series are a and b; it is required to continue the series.

See Ex. 50,

then the arithmetic series is $\frac{1}{a}$, $\frac{a+b}{ab}$, $\frac{1}{b}$, $\frac{2a-b}{ab}$, &c.,

and the harmonic series is a, $\frac{ab}{a+b}$, b, $\frac{ab}{2a-b}$, ... $\frac{ab}{3a-2b}$, &c.

(59). Compare the arithmetic, geometric, and harmonic means between a and b, and show that the geometric mean is a mean proportional between the arithmetic and harmonic.

$$\text{Let } \frac{a+b}{2} : \sqrt{(ab)} : x : \frac{2ab}{a+b};$$

$$\therefore x = \frac{ab}{\sqrt{(ab)}} = \sqrt{(ab)}, \text{ the geometric mean.}$$

(60). The arithmetic mean between two numbers exceeds the geometric by 13, and the geometric exceeds the harmonic by 12; what are the numbers?

The arithmetic mean between x and $y = \frac{x+y}{2}$,

the geometric $= \sqrt{(xy)}$, the harmonic $= \frac{2xy}{x+y}$,

by problem $\frac{x+y}{2} = \sqrt{(xy)} + 13$;

$\therefore\ x - 2\sqrt{(xy)} + y = 26$, or $(\sqrt{x} - \sqrt{y})^2 = 26 \ldots\ldots (A)$,

also $\frac{x+y}{2xy} + 12 = \sqrt{(xy)} = \frac{x+y}{2} - 13$,

whence $(x-y)^2 = 50\,(x+y) \ldots\ldots\ldots\ldots (B)$;

$\therefore$ from $\frac{B}{A}\ (\sqrt{x} + \sqrt{y})^2 = \frac{25\,(x+y)}{13}$, or $\frac{x + 2\sqrt{(xy)} + y}{(x+y)} = \frac{25}{13}$,

whence $\sqrt{x} = \frac{13\sqrt{y}}{12} \pm \frac{5\sqrt{y}}{12} = \frac{3\sqrt{y}}{2}$;

$\therefore$ from (A) $(3\sqrt{y} - 2\sqrt{y})^2 = 4 \times 26$; $\therefore\ y = 104$, and $x = 234$.

(61). If the geometric mean between two quantities x and y be to the harmonics as $1 : n$, show that

$$x : y :: 1 + \sqrt{(1 - n^2)} : 1 - \sqrt{(1 - n^2)}.$$

If $\frac{\sqrt{(xy)} \times (x+y)}{2xy} = \frac{1}{n} = \frac{x+y}{2\sqrt{(xy)}}$,

then $x + y = \frac{2\sqrt{(xy)}}{n}$, and $x - y = \frac{2\sqrt{\{(1-n^2)\,xy\}}}{n}$;

$\therefore\ \frac{x+y}{x-y} = \frac{1}{\sqrt{(1-n^2)}}$, and $\frac{x}{y} = \frac{1 + \sqrt{(1-n^2)}}{1 - \sqrt{(1-n^2)}}$.

(62). Find the relation which must subsist between a, b, and c, that they may be the pth, qth, and rth terms of an harmonic series.

In the arithmetic series the pth term $= x + (p-1)\,d = \frac{1}{a}$,

.... qth term $= x + (q-1)\,d = \frac{1}{b}$,

.... rth term $= x + (r-1)\,d = \frac{1}{c}$;

$$\therefore p\text{th} - q\text{th} = (p-q)\,d = \frac{1}{a} - \frac{1}{b},\ q\text{th} - r\text{th} = (q-r)\,d = \frac{1}{b} - \frac{1}{c};$$

$$\therefore \frac{b-a}{a\,(p-q)} = \frac{c-b}{c\,(q-r)},$$

whence $bc(q-r) + ac(r-p) + ab(p-q) = 0$.

(63). The geometric mean between x and y : arithmetic mean :: 4 : 5; find the value of xy^{-1}.

$$\text{By problem } \sqrt{(xy)} : \frac{x+y}{2} :: 4 : 5,$$

$$\text{or } x + y = \frac{5\sqrt{(xy)}}{2};\ \therefore x - \frac{5\sqrt{(xy)}}{2} + \frac{25y}{16} = \frac{9y}{16};$$

$$\therefore \sqrt{x} = \left(\frac{5}{4} \pm \frac{3}{4}\right)\sqrt{y} = 2\sqrt{y} \text{ or } \frac{1}{2}\sqrt{y};$$

$$\therefore xy^{-1} = 4 \text{ or } \frac{1}{4}.$$

(64). If S_1, S_2, S_3, &c. denote the sums of an infinite number of infinite decreasing geometric series, whose first terms are a, a^2, a^3, &c. and common ratios r, $2r$, $3r$; prove that

$$\frac{1}{S_1} + \frac{1}{S_2} + \frac{1}{S_3} + \&\text{c.} = \frac{a(1-r)-1}{(a-1)^2}.$$

$$S_1 = \frac{a}{1-r},\ S_2 = \frac{a^2}{1-2r},\ S_3 = \frac{a^3}{1-3r},\ \&\text{c.};$$

$$\therefore \frac{1}{S_1} + \frac{1}{S_2} + \frac{1}{S_3} + \&\text{c. to inf.} = \frac{1}{a} - \frac{r}{a} + \frac{1}{a^2} - \frac{2r}{a^2} + \frac{1}{a^3} - \frac{3r}{a^3} + \&\text{c}$$

$$= \frac{1}{a}\left(1 + \frac{1}{a} + \frac{1}{a^2} + \&\text{c.}\right) - \frac{r}{a}\left(1 + \frac{1}{a} + \frac{1}{a^2} + \&\text{c.}\right)$$

$$- \frac{r}{a^2}\left(1 + \frac{1}{a} + \frac{1}{a^2} + \&\text{c.}\right)$$

$$= \frac{1}{a}\left(\frac{1}{1-\frac{1}{a}}\right) - \frac{r}{a}\left(1 + \frac{1}{a} + \frac{1}{a^2} + \&\text{c.}\right)\left(1 + \frac{1}{a} + \frac{1}{a^2} + \&\text{c.}\right)$$

$$= \frac{1}{a-1} - \frac{r}{a-1} \times \frac{a}{a-1} = \frac{a(1-r)-1}{(a-1)^2}.$$

(65). If S_1, S_2, S_3, &c. S_n be the sums of n geometric series, whose first terms are a, $2a$, $3a \ldots na$, and r the common ratio in each; prove that

$$S_1 + S_2 + S_3 + \&c.\ S_n = \frac{n(n+1)}{2}\left(\frac{r^n - 1}{r - 1}\right) a.$$

$$S_1 = a\frac{r^n - 1}{r - 1},\quad S_2 = 2a\frac{r^n - 1}{r - 1},\quad S_3 = 3a\frac{r^n - 1}{r - 1}, \&c.;$$

$$\therefore\ S_1 + S_2 + S_3 + \&c. \text{ to } n \text{ terms}$$

$$= (a + 2a + 3a + \&c. \text{ to } n \text{ terms})\frac{r^n - 1}{r - 1} = \frac{n(n+1)}{2}\,\frac{r^n - 1}{r - 1} \times a.$$

(66). If S represent the sum of an infinite geometric series, whose first term is a, and common ratio r, S_2 the sum of the squares, S_3 the sum of the cubes &c. of the terms; prove that

$$\frac{1}{S_1} + \frac{1}{S_2} + \&c. \text{ to inf.} = \frac{1}{a-1} - \frac{r}{a-r}.$$

The series will be a, ar, ar^2, &c. $a^2(1 + r^2 + r^3 + \&c.)$;

$$\therefore\ S_1 = \frac{a}{1-r},\quad S_2 = \frac{a^2}{1-r^2},\quad S_3 = \frac{a^3}{1-r^3};$$

$$\therefore\ \frac{1}{S_1} + \frac{1}{S_2} + \frac{1}{S_3} + \&c. \text{ to inf.} = \frac{1}{a} - \frac{r}{a} + \frac{1}{a^2} - \frac{r^2}{a^2} + \frac{1}{a^3} - \frac{r^3}{a^3} + \&c.$$

$$= \frac{1}{a}\,\frac{1}{\left(1 - \frac{1}{a}\right)} - \frac{r}{a}\,\frac{1}{\left(1 - \frac{r}{a}\right)} = \frac{1}{a-1} - \frac{r}{a-r}.$$

PERMUTATIONS AND COMBINATIONS.

XVII. If n things be taken r together, the

$$\text{number of permutations} = n(n-1)(n-2)\ldots(n-r+1),$$

$$\text{number of combinations} = \frac{n(n-1)(n-2)\ldots(n-r+1)}{1.2.3\ldots r}.$$

If quantities recur, and they be taken altogether,

$$\text{number of permutations} = \frac{n(n-1)(n-2)\ldots 3.2.1}{(1.2\,3p) \times (1.2.3\ldots r)\ \&c.}.$$

(1). If ten letters, a, b, c, &c. be combined 5 together; how many combinations will they make?

$$\text{The numbers of combination 5 together} = \frac{10.9.8.7.6}{1.2.3.4.8} = 252.$$

(In this question there was intended to be added, and in how many of the combinations will neither a nor b occur?) Then if a and b be taken out, there will be left 8 things to combine five together, and the number $= \frac{8.7.6.5.4}{1.2.3.4.5} = 56$, and in these neither ($a$) nor ($b$) occur, and the difference (196) is the number of combinations in which a and b do occur. If the question were, in how many of the combinations do the two letters (a) and (b) occur together, then the ten things must be divided into two classes, one of eight and the other of two things; then the two things must enter into every combination, snd as five things are to be in each combination, the eight things must be combined three together, and the number ot combinations is $\frac{8.7.6}{1.2.3} = 56 = c_3^8$ and the number of combinations of 2 things 2 together $= \frac{2.1}{1.2} = 1 = c_2^2$, therefore the number of combinations in which a and b occur $= c_3^8 . c_2^2 = 56$.

(2). How many different sums may be formed with a guinea, a half-guinea, a crown, a half-crown, a shilling, and a sixpence?

Number of combinations	1 at once		= 6
....	2 together	$= \frac{6.5}{1.2}$	= 15
....	3	$= \frac{6.5.4}{1.2.3}$	= 20
....	4	$= \frac{6.5.4.3}{1.2.3.4}$	= 15
....	5	$= \frac{6.5.4.3.2}{1.2.3.4.5}$	= 6
	all together =		= 1
Whole number of combinations			= 63

(3). How many different sums may be formed with the following coins: a farthing, a penny, a sixpence, a shilling, a crown, a half-sovereign, a guinea, and a moidore?

The number of coins is 8;

$\therefore$ number of combinations 1 together $= 8$

.... 2 $= \dfrac{8.7}{1.2} = 28$

.... 3 $= \dfrac{8.7.6}{1.2.3} = 56$

.... 4 $= \dfrac{8.7.6.5}{1.2.3.4} = 70$

.... 5 $= \dfrac{8.7.6.5.4}{1.2.3.4.5} = 56$

.... 6 $= \dfrac{8.7.6.5.4.3}{1.2.3.4.5.6} = 28$

.... 7 $= \dfrac{8.7...2}{1.2...7} = 8$

.... 8 $= \quad = 1$

$\therefore$ whole number of combinations $= 255$

(4). At an election where every voter can vote for any number of candidates not greater than the number to be elected, there are 4 candidates and 3 members to be chosen. In how many ways can a man vote?

Number of combinations of 4 things 3 together $= \dfrac{4 \times 3 \times 2}{1 \times 2 \times 3} = 4$

.... 2 $= \dfrac{4 \times 3}{1 \times 2} = 6$

.... 1 $= \quad = 4$

$\therefore$ number of ways $= 14$

(5). How many days can 5 persons be placed in different positions about a table at dinner?

This is, how many permutations can be made out of 5 things all together?

and the number $= 5 \times 4 \times 3 \times 2 \times 1 = 120$.

(6). From a company of 50 men, 4 are draughted off every night to guard; on how many different nights can a different guard be posted, and on how many of these will any particular soldier be engaged?

The number of combinations of 50 things 4 together

$$= \frac{50 \times 49 \times 48 \times 47}{1 \times 2 \times 3 \times 4} = 230300,$$

and if one soldier be left out, we have the combination of 49 taken 4 together

$$= \frac{49 \times 48 \times 47 \times 46}{1 \times 2 \times 3 \times 4} = 211876;$$

∴ the number of nights on which any one soldier will be on guard and the difference of these numbers = 18424.

(7). Find all the permutations that can be formed out of the letters of the words (1) Baccalaureus, (2) Mississippi, (3) Hippopotamus. (4) Commencement.

Baccalaureus contains 2 *c*'s, 2 *u*'s, 3 *a*'s, and 12 letters, and the number of permutations

$$= \frac{12.11.10 \ldots 2.1}{(1.2)\ (1.2.3)\ (1.2)} = 19958400.$$

In Mississippi there are 4 *s*'s, 4 *i*'s, 2 *p*'s, and 11 letters; therefore number of permutations

$$= \frac{11.10.9.8 \ldots 2.1}{(1.2.3.4)\ (1.2.3.4)\ (1.2)} = 34650.$$

In Hippopotamus there are 3 *p*'s, 2 *o*'s, and 12 letters, and the number will be found to be 39916800.
In Commencement there are 3 *m*'s, 2 *n*'s, 2 *c*'s, 3 *e*'s, and 12 letters, and the number of permutations will be found to be 3326400.

(8). If the number of things : the number of variations, 3 together :: 1 : 20, then the number of things = 6.

$$\text{By problem } n : n(n-1)(n-2) :: 1 : 20;$$

$$\therefore n^2 - 3n + 2 = 20, \ n = \frac{3}{2} \pm \frac{9}{2} = 6.$$

(9). If the number of permutations of n things r together : number $(r-1)$ together :: 10 : 1, and the number of combinations r together : number $(r-1)$ together :: 5 : 3; find n and r.

By problem

$$n(n-1)\ldots(n-r+2)(n-r+1) : n(n-1)\ldots(n-r+2) :: 10 : 1;$$

$$\therefore n - r + 1 = 10,$$

also $\dfrac{n(n-1)\ldots(n-r+2)(n-r+1)}{1.2\ldots(n-1)\,r} : \dfrac{n(n-1)\ldots(n-r+2)}{1.2\ldots(r-1)} :: 5 : 3;$

$$\therefore \frac{n-r+1}{r} : 1 :: 5 : 3; \ \therefore 3n - 3r + 3 = 5r;$$

$$\therefore n = \frac{8r-3}{3} = 9 + r; \ \therefore r = 6, \text{ and } n = 15.$$

(10). If the number of variations of n things 3 together : the number of variations of $(n+2)$ things 3 together :: 5 : 12; then $n = 7$.

By problem $n(n-1)(n-2) : (n+2)(n+1)n :: 5 : 12$,

$$\text{whence } 7n^2 - 51n = -14; \ \therefore n = \frac{51}{14} \pm \frac{47}{14} = 7.$$

(11). If the number of variations of n things 4 together : the number of variations of $\frac{2n}{3}$ things 4 together :: 13 : 2; then $n = 15$.

By problem

$$n(n-1)(n-2)(n-3) : \frac{2n}{3}\left(\frac{2n}{3}-1\right)\left(\frac{2n}{3}-2\right)\left(\frac{2n}{3}-3\right) :: 13 : 2,$$

$$\text{or } 81(n-1)(n-2) : 2(2n-3)(2n-9) :: 13 : 1,$$

$$\text{whence } 23n^2 - 381n = -540, \text{ and } n = \frac{381}{46} \pm \frac{309}{46} = 15.$$

(12). If the number of variations of n things 3 together be 12 times as great as the number of variations of $\frac{n}{2}$ things 3 together; what is the number of variations of the same n things n together?

By problem $n(n-1)(n-2) = 12\left\{\frac{n}{2}, \left(\frac{n}{2}-1\right), \left(\frac{n}{2}-2\right)\right\}$,

$$\text{whence } n^2 - 12n + 36 = 16; \ \therefore n = 6 \pm 4 = 10,$$

and number of permutations of 10 things altogether

$$= 10 \times 9 \times 8 \times 7 \ldots 2 \times 1 = 3628800.$$

(13). If the number of combinations of n things taken 4 together is to the number taken 2 together :: 15 : 2; then $n = 12$.

$$\text{By problem } \frac{n(n-1)(n-2)(n-3)}{1.2.3.4} : \frac{n(n-1)}{1.2} :: 15 : 2,$$

$$\text{whence } n^2 - 5n = 84, \text{ and } n = \frac{5}{2} \pm \frac{19}{2} = 12.$$

(14). If the number of permutations of $(2n + 1)$ things $(n - 1)$ together : number of permutations of $(2n - 1)$ things n together :: 3 : 5; then $n = 4$.

By problem $(2n+1)\,2n\,(2n-1)\ldots(n+3) : (2n-1)(2n-2)\ldots$

$$(n+3)(n+2)(n+1)\,n :: 3 : 5,$$

$$\text{or } 2(2n+1) : (n+2)(n+1) :: 3 : 5,$$

$$\text{whence } 3n^2 - 11n = 4, \text{ and } n = \frac{11}{6} \pm \frac{13}{6} = 4.$$

(15). If the number of variations of n things taken 3 together = six times the number of combinations of n things taken 4 together; then $n = 7$.

$$\text{By problem } n(n-1)(n-2) = \frac{6n(n-1)(n-2)(n-3)}{1.2.3.4},$$

$$\text{whence } 4 = n - 3, \text{ or } n = 7.$$

(16). If the number of combinations of n things taken 5 together is to the number taken 3 together :: 18 : 5; then $n = 12$.

By problem

$$\frac{n(n-1)(n-2)(n-3)(n-4)}{1.2.3.4.5} : \frac{n(n-1)(n-2)}{1.2.3} :: 18 : 5,$$

$$\text{whence } n^2 - 7n = 60, \text{ and } n = \frac{7}{2} \pm \frac{17}{2} = 12.$$

(17). There is a certain number of things of which taken 8 together the variations are 80, and taken 10 together, 960; how many must be taken away from the original number that the combinations of the remaining things taken 2 together may be 15?

If n be the number,

then $n(n-1)\ldots(n-7) = 80$, and $n(n-1)\ldots(n-8)(n-9) = 960$;

$$\therefore (n-8)(n-9) = \frac{960}{80} = 12, \text{ and } n = \frac{17}{2} \pm \frac{7}{2} = 12,$$

and if n things be combined 2 together the number

$$= \frac{n(n-1)}{1.2} = 15;$$

$$\therefore n^2 - n = 30, \text{ and } n = \frac{1}{2} \pm \frac{11}{2} = 6;$$

$\therefore$ the number to be subtracted $= 12 - 6 = 6$.

(18). The number of permutations of n things taken r together : the number of permutations of n things taken $r-1$ together :: 42 : 1; and the corresponding combinations :: 1 : 1; find the values of n and r.

$$n(n-1)\ldots(n-r+2)(n-r+1) : n(n-1)\ldots(n-r+2) :: 42 : 1;$$

$$\therefore n - r + 1 = 42,$$

$$\text{also } \frac{n(n-1)\ldots(n-r+2)(n-r+1)}{1.2\ldots(r-1)\,r} : \frac{n(n-1)\ldots(n-r+2)}{1.2.3\ldots(r-1)} :: 1 : 1;$$

$$\therefore n - r + 1 = r; \ \therefore n = 2r - 1 = r + 41; \ \therefore r = 42, \text{ and } n = 83.$$

(19). How often might a common die be thrown, so as to *expose* five different faces?

A die has six faces, therefore the only possible number of changes are that each face should successively be to the board, which will be *six* times.

(20). How many combinations can be made in all of 6 things taken 1, 2, 3, 4, 5, 6 together?

This question is in reality the same as Ex. 2.

(21). If the number of combinations of $\frac{n}{3}$ things 2 together is 15; $n = 18$.

$$\text{By problem } \frac{n}{3}\left(\frac{n}{3} - 1\right) \times \frac{1}{1.2} = 15,$$

$$\text{whence } n^2 - 3n = 270, \text{ and } n = \frac{3}{2} \pm \frac{32}{2} = 18.$$

(22). If the number of combinations of $\frac{n}{2}$ things 4 together is $3\frac{3}{4}$ of the number of combinations of $\frac{n}{3}$ things 3 together; then $n = 12$.

$$\text{By problem } \frac{n}{2} \times \left(\frac{n}{2} - 1\right)\left(\frac{n}{2} - 2\right)\left(\frac{n}{2} - 3\right) \times \frac{1}{1.2.3.4}$$

$$= \frac{15}{4} \cdot \frac{n}{3}\left(\frac{n}{3} - 1\right)\left(\frac{n}{3} - 2\right) \times \frac{1}{1.2.3},$$

$$\text{whence } 9n^2 - 134n + 312 = 0, \text{ and } n = \frac{67}{9} \pm \frac{41}{9} = 12.$$

(23). If the number of combinations of $n + 1$ things 4 together is 9 times the number of combinations of n things 2 together; then $n = 11$.

$$\text{By problem } \frac{(n+1)\,n\,(n-1)\,(n-2)}{1.2.3.4} = 9\,\frac{n\,(n-1)}{1.2},$$

$$\text{whence } n^2 - n = 110, \text{ and } n = \frac{1}{2} \pm \frac{21}{2} = 11.$$

(24). If the number of combinations of n things 3 together is $\frac{5}{18}$ of the number 5 together; then $n = 12$.

$$\text{By problem } \frac{n(n-1)(n-2)}{1.2.3} = \frac{5}{18}\,\frac{n\,(n-1)(n-2)(n-3)\,(n-4)}{1.2.3.4.5},$$

$$\text{whence } n^2 - 7n = 60, \text{ and } n = \frac{7}{2} \pm \frac{17}{2} = 12.$$

(25). How many words of 6 letters may be made out of the 26 letters of the alphabet, with 2 out of the 5 vowels in every word?

Number of combinations of 5 vowels 2 together $= \frac{5.4}{1.2} = 10$,

number of variations of 21 consonants 4 together $= 21.20.19.18$, and the number of variations of each of these with two vowels added

$$= 6.5.4.3.2.1 = 720;$$

$\therefore$ whole number of combinations $= 21 \times 20 \times 19 \times 18 \times 7200$

$$= 1034208000.$$

(26). A person wishes to make up as many different dinner parties as he can out of an acquaintance of 24; how many should he invite at a time?

If r be the required number,

$$\text{number of combinations of 24 men together} = \frac{24.23\ldots(25-r)}{1.2.3.4\ldots r},$$

$$\text{and } 25 - r \text{ must be greater than } r \text{ or } = r + 1;$$

$$\therefore\ 2r = 24, \text{ and } r = 12.$$

(27). Show that the number of different combinations of n things, taken 1, 2, 3...n together, of which p are of one sort, q of another, r of another, &c. $= (p+1)(q+1)(r+1)\ldots - 1$.

First, let us consider p things of one sort; the number of ways in which they may be taken is p, viz. 1 thing, or 2 things, or 3 things, ... or p things; and if we include *no thing* as being one way, there will be $(p+1)$ ways. If now we introduce q things of another sort, in the same manner they may be taken $(q+1)$ ways, and each of these $(q+1)$ ways being combined with the $(p+1)$ ways of the former sort makes altogether

$$(p+1)(q+1) \text{ ways,}$$

including *no thing* as one way.

Now introducing r things of another kind we shall have for the total number of combinations

$$(p+1)(q+1)(r+1),$$

but if we reject "*no thing*" as one way, the number of combinations will be

$$(p+1)(q+1)(r+1)\ldots - 1.$$

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