Arch. Mech., 42, 2, pp. 245—251, Warszawa 1990

BRIEF NOTES

On some self-similar solutions of Euler equations

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A CLASS of axisymmetric self-similar solutions $v_R = V_R(\theta) R_\alpha, v_\theta = V_\theta(\theta) R^\alpha$ is considered An ordinary nonlinear equation for the stream function is given. In the special case $\alpha = 2$ when the equation becomes linear, solutions of this equation are analysed. The obtained solutions correspond both to the rotational and potential flows.

1. Introduction

LANDAU [1] in 1944 and independently SQUIRE [2] in 1951 (see also [3, 4]) found a self-similar solution of the Navier–Stokes equations for a steady axisymmetric flow. The solution found in [1] and [2] describes a jet flow inside a fluid, the components of velocity being of the form

$$v_R = V_R(\theta) R^{-1}, \quad v_\theta = V_\theta(\theta) R^{-1}, \tag{1.1}$$

where R and θ are spherical coordinates. It follows from the dimensional analysis that the only possible power of R in Eqs. (1.1) is -1. However, if an ideal fluid is concerned, the viscous terms in the Navier–Stokes equations being omitted, the latest limitation is no longer valid. In the following we shall consider a self-similar flow of an ideal fluid, the components of velocity being proportional to R^α, and we shall find a solution for the special case $\alpha = 2$.

2. Equation for the self-similar flow

The equations describing a steady, axisymmetric flow of an ideal fluid, or the Euler equations, may be written in the spherical coordinates as follows:

$$\begin{gathered}
\frac{\partial v_R}{\partial R} \frac{1}{R} \frac{\partial v_\theta}{\partial \theta} + \frac{2v_R}{R} + \frac{v_\theta \operatorname{ctg}\theta}{R} = 0, \\
v_R \frac{\partial v_R}{\partial R} + \frac{v_\theta}{R} \frac{\partial v_R}{\partial \theta} - \frac{v_\theta^2}{R} = -\frac{1}{\varrho} \frac{\partial p}{\partial R}, \\
v_R \frac{\partial v_\theta}{\partial R} + \frac{v_\theta}{R} \frac{\partial v_\theta}{\partial \theta} + \frac{v_R v_\theta}{R} = -\frac{1}{\varrho R} \frac{\partial p}{\partial t},
\end{gathered} \tag{2.1}$$

where v_R, v_θ are the components of the flow velocity, ϱ is density, p-pressure. We shall look for the solution of the system (2.1) in the form

$$
\begin{aligned}
v_R &= R^\alpha f(\eta),\\
v_\theta &= R^\alpha g(\eta),\\
p &= R^{2\alpha}\varrho P(\eta),
\end{aligned}
\tag{2.2}
$$

where

$$\eta = \cos\theta. \tag{2.3}$$

Inserting the relations (2.2) into the systeml (2.) of partial differential equations, one obtains ordinary differential equations with an independent variable η. The solutions of type (2.2) are self-similar. The continuity equation $(2.1)_1$ is fulfilled identically if the stream function $F(\eta)$ is introduced conforming to Eqs. (2.4):

$$
\begin{aligned}
f(\eta) &= -\frac{1}{\alpha+2}\frac{dF(\eta)}{d\eta},\\
g(\eta) &= -\frac{F(\eta)}{\sqrt{1-\eta^2}}.
\end{aligned}
\tag{2.4}
$$

Inserting the relations (2.2) and (2.4) into the momentum equations $(2.1)_2$ and $(2.1)_3$, one finds 2 ordinary differential equations for the functions $F(\eta)$ and $P(\eta)$. Upon some rearrangements, one may eliminate the pressure, thus obtaining the nonlinear ordinary differential equation for the stream function $F(\eta)$

$$\left(F^{\frac{2-\alpha}{2+\alpha}}F''\right)' + 4(1+\alpha)F^{\frac{2-\alpha}{2+\alpha}}\left(\frac{F}{1-\eta^2}\right)' = 0. \tag{2.5}$$

3. The solution for $\alpha = 2$

It is difficult to solve analytically Eq. (2.5) for an arbitrary value of α. In the following, we shall limit ourselves to the particular case $\alpha = 2$. Then Eq. (2.5) becomes linear:

$$F''' + 12\left(\frac{F}{1-\eta^2}\right)' = 0 \tag{3.1}$$

and can be integrated analytically. It is convenient for the following considerations to represent the first integral of Eq. (3.1) in the form

$$F'' + \frac{12F}{1-\eta^2} = 10(c_1-1), \tag{3.2}$$

where c_1 is an arbitrary constant. We shall consider also a homogeneous equation (3.3) corresponding to a nonhomogeneous equation (3.2):

$$F'' + \frac{12F}{1-\eta^2} = 0. \tag{3.3}$$

It may be easily checked that Eqs. (3.3) and (3.2) are satisfied respectively by the particular solutions $\varphi_1(\eta)$ and $\varphi_2(\eta)$

$$\varphi_1(\eta) = (1-\eta^2)(1-5\eta^2), \tag{3.4}$$

$$\varphi_2(\eta) = (1-\eta^2)(c_1+1-5\eta^2). \tag{3.5}$$

A standard method (see [5]) may be used to eliminate an unknown function from the homogeneous linear equation if a particular solution is known. As a result of a substitution

$$F(\eta) = \varphi_1(\eta)y(\eta) \tag{3.6}$$

applied to Eq. (3.3), we obtain the following equation for $y(\eta)$

$$(1-\eta^2)(1-5\eta^2)y''+(40\eta^3-24\eta)y' = 0 \tag{3.7}$$

which does not contain the unknown y in explicit form. The last equation being of the first order with regard to y can be integrated by the standard way. The general solution of Eq. (3.7) is the following:

$$y = \left[\frac{\frac{26}{3}\eta-10\eta^3}{(1-\eta^2)(1-5\eta^2)}+\ln\frac{1+\eta}{1-\eta}\right]c_2+c_3-1, \tag{3.8}$$

where c_2 and c_3 are arbitrary constants. The general solution of the homogeneous equation (3.3) is to be obtained by substitution of the solution (3.8) for y to (3.6). The general solution of the nonhomogeneous equation (3.2) can be represented as the sum of the last solution and the particular solution $(3.5)_2$ of the nonhomogeneous equation. Thus the general solution of Eq. (3.2) or the original equation (3.1) becomes

$$F(\eta) = c_1F_1(\eta)+c_2F_2(\eta)+c_3F_3(\eta), \tag{3.9}$$

where

$$\begin{aligned} F_1(\eta) &= 1-\eta^2,\\ F_2(\eta) &= (1-\eta^2)(1-5\eta^2),\\ F_3(\eta) &= \frac{26}{3}\eta-10\eta^3+(1-\eta^2)(1-5\eta^2)\ln\frac{1+\eta}{1-\eta}, \end{aligned} \tag{3.10}$$

c_1, c_2, c_3 being arbitrary constants. The functions $F_1(\eta)$, $F_2(\eta)$, $F_3(\eta)$ for $0 \leqslant \eta \leqslant 1$ are shown in Fig. 1.

We shall demonstrate that for the flows determined by the system (2.2) the equations of stream lines can be represented explicitly by the stream function. The differential equation of a stream line is

$$\frac{1}{R}\frac{dR}{d\theta} = \frac{v_R}{v_\theta}. \tag{3.11}$$

The velocity components v_R and v_θ can be expressed by the stream function from Eqs. (2.2) and (2.4)

$$v_R = -\frac{1}{4}\frac{dF}{d\eta}R^2, \quad v_\theta = -\frac{F}{\sqrt{1-\eta^2}}R^2.$$

Upon substitution of the last expressions into Eq. (3.11), and taking into account that following Eq. (2.3)

$$\frac{d\eta}{d\theta} = -\sqrt{1-\eta^2},$$

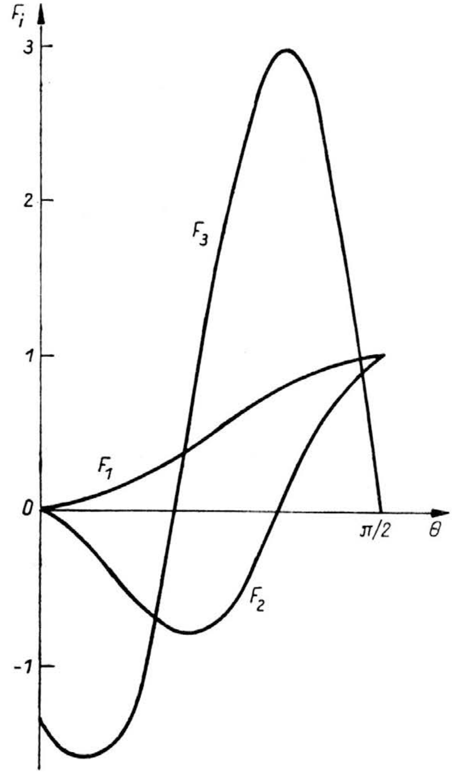

FIG. 1. Particular solutions $F_1(\eta)$, $F_2(\eta)$, $F_3(\eta)$.

we obtain for the stream line the equation

$$\frac{dR}{R} = -\frac{1}{4}\frac{dF}{F}$$

which can be integrated independently of the form of the function $F(\eta)$. The result is

$$R = \frac{c}{\sqrt[4]{|F(\eta)|}}. \tag{3.12}$$

It follows from Eq. (3.12) that for such flows the stream lines are geometrically similar, the constant c being positive.

It is easy to show that the solution $F_1(\eta)$ corresponds to the rotational flow while the solutions $F_2(\eta)$ and $F_3(\eta)$ are potential.

4. Analysis of solutions

The analysis will be performed for some particular combinations of the constants c_1, c_2, c_3. First, consider the case $c_3 = 0$. The stream function is then of the form

$$F(\eta) = (1-\eta^2)[(c_1+c_2)-5c_2\eta^2]. \tag{4.1}$$

Because $F(\eta)$ is an even function, the stream lines are symmetric with respect to the plane $\theta = \pi/2$. The function $F(\eta)$ defined by Eq. (4.1) has at the most 2 positive roots: $\eta = \eta_1 = 1$ independent of the values of constants c_1 and c_2, and $\eta = \eta_2 = \sqrt{\dfrac{c_1+c_2}{5c_2}}$ determined for $c_2 \neq 0$, and real for $c_1/c_2 \geqslant -1$. Consequently, there are two regions on the plane c_1, c_2, characterized by different flow patterns (Fig. 2).

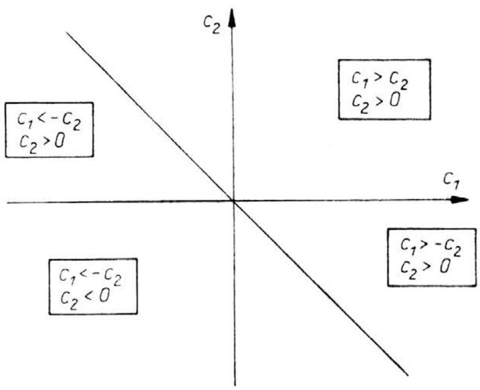

FIG. 2. Division of the plane c_1, c_2 into regions of different flow patterns ($c_3 = 0$).

In the region 1, the root η_2 is imaginary, therefore the streamlines have only one, asymptote corresponding to the axis of symmetry. On the plane $\theta = \pi/2$, sources are placed. An example of such streamline is shown in Fig. 3a in the meridional plane.

In the region 2, η_2 is real and not equal to 0. In this case two kinds of streamlines exist (Fig. 3b). Inside the cone $\theta = \theta_2$ (where $\theta_2 = \arccos \eta_2$) the streamlines have 2 asymptotes: a generating line and the axis of the cone. The streamlines outside the cone have 1 asymptote — a generating line of the cone; in the points where streamlines intersect the plane sources are located.

In the intermediate case $\eta_2 = 0$, i.e., $c_1 = -c_2$, streamlines with 2 perpendicular asymptotes are obtained (Fig. 3c).

In the case $c_2 = 0$ streamlines have an analogous form to that in the region 1.

Next, let us consider the case $c_1 = c_2 = 0$, $c_3 \neq 0$. This time the stream function $F(\eta) = c_3 F_3(\eta)$ has 2 roots: $\eta = \eta_1 = 0$ and $\eta = \eta_2$ where $0 < \eta_2 < 1$. Consequently the plane $\theta = \pi/2$ is an asymptotic plane, sources being located at the axis $\theta = 0$ (Fig. 3d).

As yet, attention has not been paid to the case where c_3 and, at least, one of the constants c_1 or c_2, are different from 0. Then $F(\eta)$ is neither an even nor an odd function. A detailed analysis of particlar flow patterns and ranges of c_1, c_2, c_3 corresponding to

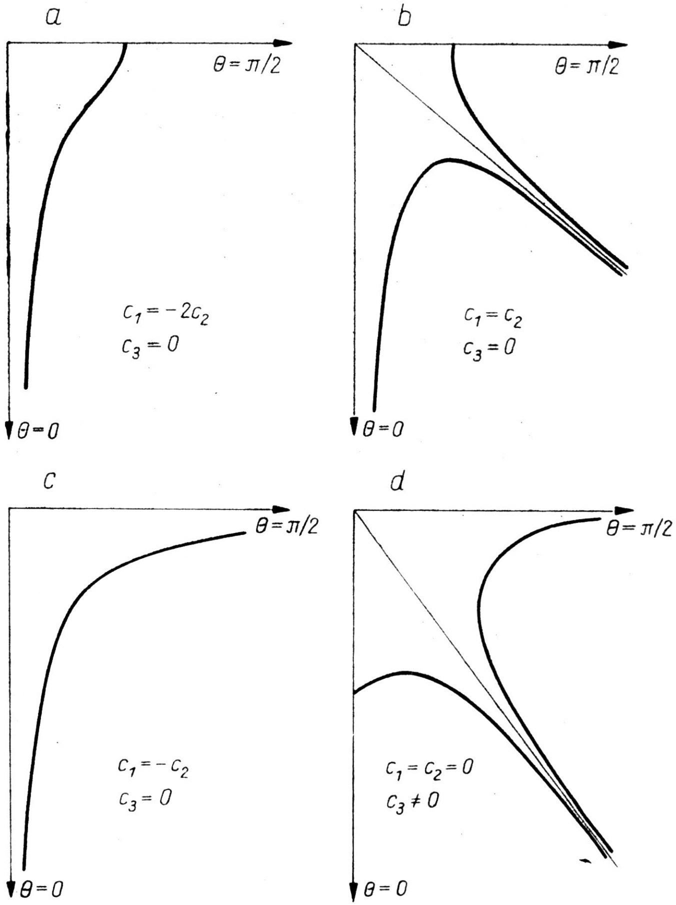

FIG. 3. Stream lines for $c_3 = 0$ (a, b, c) and $c_1 = c_2 = 0$ (d).

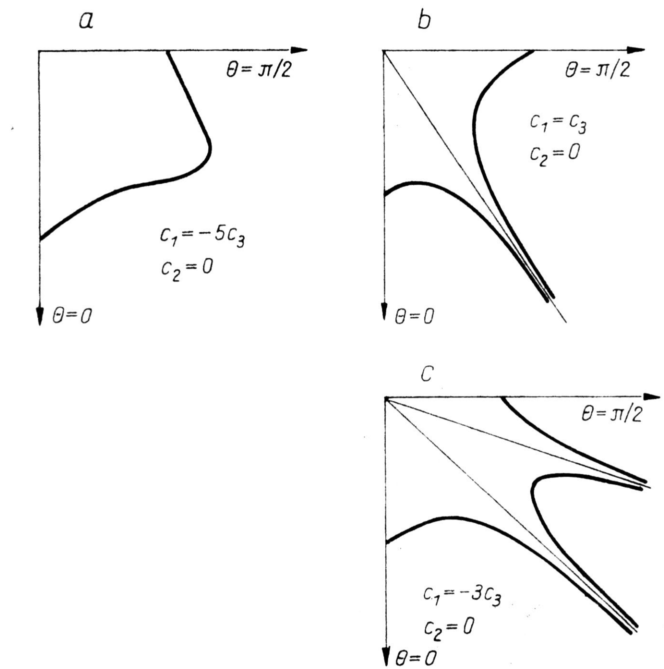

FIG. 4. Stream lines for $c_3 \neq 0$, $|c_1|+|c_2| \neq 0$.

them is too complicated to be presented here. It is worthy of mention, however, that in this case sources are distributed both on the axis of symmetry and on the plane $\theta = \pi/2$. For some values of c_1, c_2, c_3 the function $F(\eta)$ has 2 roots inside the interval (0, 1). Some flow patterns, corresponding to the last case are represented in Fig. 4.

In all the cases considered, the velocity tends to infinity for $R \to \infty$. Therefore the only reasonable flows of this class are flows in a bounded region with very specific boundary conditions.

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Received November 13, 1989.