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Impulsive impact of nonlinear elastic rods—similarity solutions

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SIMILARITY solutions are determined for the problem of impulsive imapct of semi-infinite nonlinear elastic rods. By invoking invariance under a group of transformations, the governing nonlinear partial differential equation and the auxiliary conditions are transformed to an ordinary differential equation with appropriate auxiliary conditions. The resulting similarity representation is solved for both characteristic and shockwave propagation. For the general similarity representation, numerical solutions are obtained using the method of collocation. A closed form solution is obtained for a special case.

Notations

- x coordinate along the axis of the rod (Lagrangian coordinate system), where x denotes the position of the particle in the initial unstrained state,
- t time,
- $u(x, t)$ displacement,
- $\sigma(x, t)$ nominal compressive stress; compressive stress is assumed to be positive,
- $e(x, t)$ nominal compressive strain,
- $v(x, t)$ particle velocity,
- ϱ mass density of the material in the intially unstrained state,
- $X(t)$ distance from origin of the moving boundary,
- $X_s(t)$ shock front distanced from origin,
- ζ similarity variable,
- ζ_c similarity variable at the wavefront for characteristic propagation,
- $\bar{\zeta}$ similarity variable at the wavefront for shock-propagation,
- μ, q material parameters in the constitutive relationship,
- I strength of the impulse.

1. Introduction

THE PROBLEM of the impact of rods with nonlinear constitutive relationships has been the topic of research for quite some time. A useful technique for analyzing nonlinear partial differential models pertaining to the problem of impact and wave-propagation is the similarity analysis [1]. Similarity analysis is essentially a method of determining transformations of variables in the original problem description that converts the partial differential system to an ordinary differential system. TAULBEE, COZZARELLI and DYM [2] used the separation of the variables technique to obtain similarity solutions for the impact problem of nonlinear elastic and viscous rods. SESHADRI and SINGH [3] developed a technique for obtaining a relationship between the characteristics and the similarity coordinate at the moving boundary. The so-called "$s-c$ relation ship" is the condition at the moving boundary necessary to solve the similarity representation. A classification of wave-propagation problems based on invariance of governing equations under a group of transformations was presented by SESHADRI and SINGH [4].

In this paper the problem of impulsive impact of long nonlinear elastic rods is considered. Similarity solutions have been obtained from both characteristic and shock- wave propagation. The general similarity representation is solved numerically using the method of collocation. For the linear constitutive relationship, the closed form solution is obtained.

2. Governing equations and auxiliary conditions for impulsive impact

Within the framework of the uniaxial theory of thin rods, the governing equations of motion for small deformation are

$$\frac{\partial\sigma}{\partial x} = -\varrho\frac{\partial v}{\partial t},$$

$$\frac{\partial e}{\partial t} = -\frac{\partial v}{\partial x}, \tag{1}$$

$$\sigma = \mu(e)^{\frac{1}{q}},$$

where x is the Lagrangian coordinate, t is time, and the strain $e = -\dfrac{\partial u}{\partial x}$. Nominal compressive stresses and strain are considered positive. Equation (1) can be combined to give

$$\frac{\mu}{\varrho q}\left(-\frac{\partial u}{\partial x}\right)^{\frac{1-q}{q}}\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = 0. \tag{2}$$

Equation (2) is quasi-linear hyperbolic. The auxiliary condition at the impacted end of the rod can be written as

$$\sigma(x=0,t) = \mu\left(-\frac{\partial u}{\partial x}\right)^{\frac{1}{q}} = -I\delta(t). \tag{3}$$

where $\delta(t)$ is the Dirac delta-function. Since the displacement immediately ahead of the wavefront is zero,

$$u\big(x = X(t); t\big) = 0. \tag{4}$$

The initial conditions are given by

$$u(x, t=0) = 0,$$
$$\sigma(x, t=0) = 0. \tag{5}$$

3. Similarity analysis

The dependent and independent variables appearing in the problem description are rendered nondimensional by introducing arbitrary reference quantities as follows [1]:

(6)
$$\bar{x} = \frac{x}{x_0}, \quad \bar{t} = \frac{t}{t_0} \quad \text{and} \quad \bar{u} = \frac{u}{u_0},$$

where x_0, t_0 and u_0 are reference quantities that are determined such that a "minimum parametric" description results.

Invoking the invariance of Eq. (2) and Eq. (3) under transformations, Eq. (6), the following parameters can be extracted:

(7)
$$\pi_e = \frac{\mu}{\varrho q} \cdot \frac{(u_0)^{\frac{1-q}{q}} t_0^2}{x_0^{(1+q)/q}},$$
$$\pi_b = \frac{\mu}{I} \cdot \left(\frac{u_0}{x_0}\right)^{\frac{1}{q}} t_0.$$

The subscripts e and b indicate that the parameters are obtained from equation and boundary conditions, respectively. Other boundary conditions do not contribute any additional parameters. The mathematical description of the problem can now be expressed as

(8)
$$\bar{u}(\bar{x}, \bar{t}) = F\left(\frac{x}{x_0}, \frac{t}{t_0}; \pi_e, \pi_b\right).$$

Minimum parametric description leads to the similarity transformation for the boundary value problem, Eqs. (2) to (5). This is achieved by setting $\pi_e = 1$ and $\pi_b = 1$. Recognizing that x_0, t_0 and u_0 do not appear in the original description, the similarity transformation can be obtained as [1]

(9)
$$u(x, t) = \beta t^n F(\zeta),$$

where

$$\zeta = \frac{Kx}{t^m},$$
$$k = \left(\frac{\mu}{\varrho q}\right)^{-\frac{1}{2}} \left(\frac{I}{\mu}\right)^{\frac{q-1}{2}},$$
$$\beta = \sqrt{\frac{\mu}{\varrho q}} \left(\frac{I}{\mu}\right)^{\frac{q+1}{2}},$$
$$m = \frac{1+q}{2}$$

and

$$n = \frac{1-q}{2}.$$

Substituting Eq. (9) into Eq. (2) and simplifying, the resulting nonlinear ordinary differential equation can be written as

(10)
$$[(-F')^{\frac{1-q}{2}} - m^2\zeta^2]F'' - m(m-2n+1)\zeta F' - n(n-1)F = 0.$$

4. Characteristic propagation

For this case, the discontinuity or the wave propagates along the characteristics. The auxiliary condition, Eq. (3), can be written in the similarity coordinate as

$$F'(0) = 0 \tag{11}$$

since $t\delta'(t) = 0$ for $t > 0$. At the moving boundary $x = X(t)$, Eq. (4) can be transformed as

$$F(\zeta_c) = 0, \tag{12}$$

where ζ_c satisfies the "similarity-characteristic" relationship [1],

$$\zeta_c = \frac{(-F')^{\frac{1-q}{2q}}}{m}. \tag{13}$$

The similarity solutions for the problem of characteristic propagation can be obtained by solving Eq. (10) in conjunction with Eqs. (11) to (13). Once the variation of F with ζ is obtained, the particle displacement can be determined using Eq. (9).

5. Shock-wave propagation

When the magnitude of the impulsive impact is large or the constitutive relationship is of a specific form, shock-wave formation could occur [5]. The moving boundary now satisfies the so-called "jump conditions" instead of the similarity- characteristic relationship which can be written as

$$\begin{aligned} \langle v\rangle &= c\langle e\rangle, \\ \langle \sigma\rangle &= \varrho c\langle v\rangle. \end{aligned} \tag{14}$$

The symbol $\langle\ \rangle$ means the difference between variables on either side of the shock-front. We now use the variable $G(\zeta)$ instead of $F(\zeta)$ for shock propagation. Substituting Eq. (9) into the first part of Eq. (14) and recognizing that $e = -\frac{\partial u}{\partial x}$ and $v = \frac{\partial u}{\partial t}$,

$$G(\bar{\zeta}) = 0. \tag{15}$$

(Note that $u(x, t) = \beta t^n G(\zeta)$ for shock propagation.)

At the shock-front, $\bar{\zeta}$, the stresses and strains are discontinuous whereas the displacement, while zero, is continuous. The second part of Eq. (14) can be expressed as

$$\bar{\zeta} = \frac{2}{(1+q)}\sqrt{\frac{\mu}{\varrho}}\,\beta^{\frac{1-q}{2q}}k^{\frac{1+q}{2q}}(-G')^{\frac{1-q}{2q}}. \tag{16}$$

Equation (16) can be further simplified to give

$$\bar{\zeta} = \frac{2\sqrt{q}}{(1+q)}(-G')^{\frac{1-q}{2q}}. \tag{17}$$

In terms of the distance to the shock-front

$$X_s(t) = \frac{2C_0}{(1+q)} \left(\frac{I}{\mu}\right)^{\frac{1-q}{2}} (-G')^{\frac{1-q}{2q}} t^{\frac{1+q}{2}}, \tag{18}$$

where C_0 is the elastic wave-speed.

In order to obtain similarity solutions for the shock-wave propagation problem, Eq. (10) is solved along with Eqs. (11), (15) and (18).

6. Analytical and numerical solutions

When $q = 1$ the constitutive relationship corresponds to the linear elastic material. The solution can be written as

$$F(\zeta) = G(\zeta) = H\left(t - \frac{x}{C_0}\right), \tag{19}$$

where $H()$ is the Heaviside function. In terms of the original variables,

$$u(x, t) = \frac{C_0 I}{E} H\left(t - \frac{x}{C_0}\right). \tag{20}$$

The solution compares with that reported elsewhere [5]. The stress-wave can therefore be written as

$$\sigma(x, t) = -I\delta\left(t - \frac{x}{C_0}\right). \tag{21}$$

For the linear case, the characteristics and the shock-front would coincide.

For the general nonlinear problem, Eq. (10) together with Eqs. (11), (12) and (13) are used for characteristic propagation, while Eq. (10) is used in conjunction with Eqs. (11), (15) and (18) for shock-wave propagation. Numerical solutions are obtained using the method of collocation. The method of collocation, as it applies to characteristic propagation, is discussed here. However, the procedure is similar for the propagation problem involving shock-waves.

It is assumed that at $\zeta = 0$, $F(0) = F_0$, where F_0 is as yet to be determined. The initial value problem, Eq. (10) along with Eqs. (11) and (21), is now considered. Furthermore, the solution is assumed to be

$$F(\zeta) = F_0 e^{\zeta^2} + \alpha_1 \zeta^2 + \alpha_2 \zeta^3 + \ldots, \tag{22}$$

where $F_0, \alpha_1, \alpha_2, \ldots$ are unknowns. $F(\zeta)$ in Eq. (21) satisfies the conditions Eq. (11) and $F(0) = F_0$. Substitution of Eq. (21) into Eq. (10) wouid give a remainder term which can be written as

$$R(F_0, \alpha_1, \alpha_2, \ldots; \zeta) \neq 0 \tag{23}$$

since the assumed form, Eq. (21), is not the solution. Collocation can be performed by stipulating that the remainder vanishes at certain values of ζ. This would lead to several equations which can be solved to determine the unknowns. We assume here that $\alpha_2 = \alpha_3 = \ldots = 0$, so that there are only two unknowns, i.e., α_1 and F_0. If the remainder is allowed to vanish at $\zeta = \zeta^* = 0.5$, then

(24) $$R(F_0, \alpha_1; \zeta^*) = 0.$$

Using Eq. (12) in conjunction with Eq. (21),

(25) $$F_0 e^{\zeta_C^2} + \alpha_1 \zeta_C = 0.$$

Additionally, at the wave-front, the similarity-characteristic relationship, Eq. (13) needs to be satisfied. Therefore,

(26) $$m\zeta_C = [-2\zeta_C e^{\zeta_C^2} F_0 - 2\alpha_1 \zeta_C]^{\frac{1-q}{2q}}.$$

Equations (24), (25) and (26) contain three unknowns F_0, α_1 and ζ_C, which can be solved using the Newton-Raphson approximations.

The procedure is similar for the numerical solution of the shock-wave propagation problem.

The solutions have been obtained for both characteristic and shock-wave propagation for several values of the nonlinear exponent, q. Figure 1 is a plot of $F(\zeta)$ and $G(\zeta)$ versus ζ. Figure 2 is a plot of similarity variable at the wavefront as a function of the nonlinear exponent, q. It can be seen that for $q < 1$, similarity solutions of the "first kind" [1] are obtained, i.e, the shock-front lies behind the leading discontinuity or the characteristics.

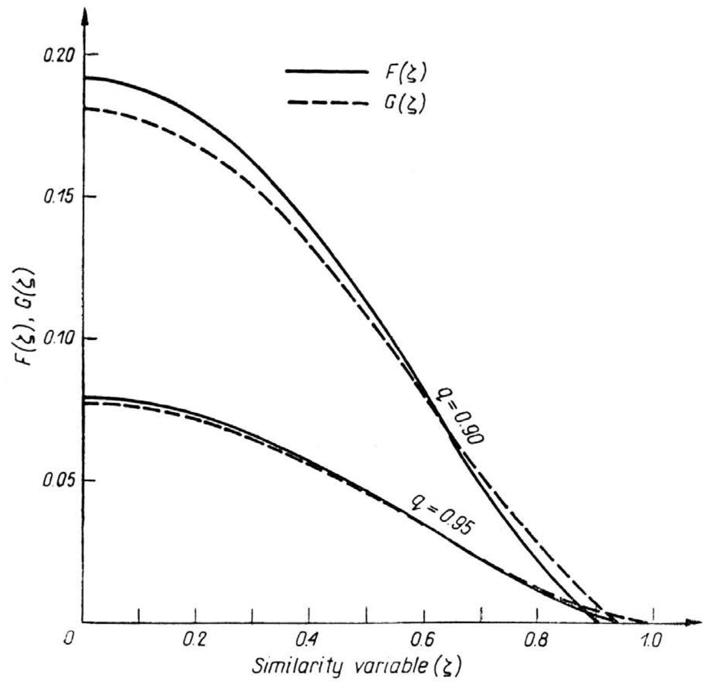

FIG. 1. $F(\zeta)$ and $G(\zeta)$ versus similarity variable, ζ.

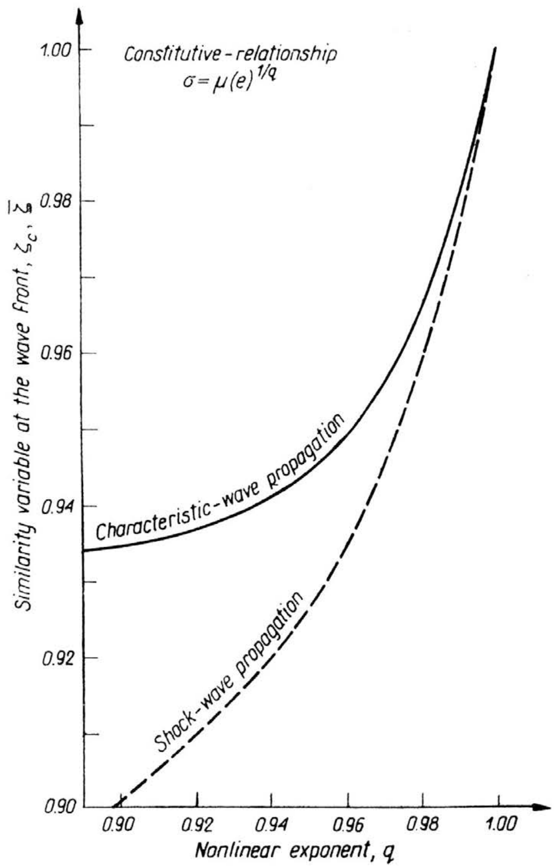

FIG. 2. Similarity variable at the wavefront versus q.

7. Conclusions

Similarity solutions for the problem of the impulsive impact of nonlinear elastic rods have been obtained. Similarity transformations are derived using the concept of invariance of the governing equations and auxiliary conditions under a group of transformations. The resulting ordinary differential equation is subjected to auxiliary conditions including some specific conditions at the wavefront. These conditions have been obtained using the "similarity-characteristic relationship" and the "jump conditions". The technique presented in the paper can be extended to propagation problems in acoustics and continuum mechanics.

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