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A note on the instability of a vortex sheet leaving a semi-infinite plate

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THE CONTROVERSIAL results pertaining to the inviscid instability of a vortex sheet leaving a semi-infinite plate derived previously by Orszag and Crow and by Bechert and Michel have been re-examined. While the former found a solution of the problem, which, however, does not satisfy the full Kutta condition, the latter stated that without excitation by sources, only the trivial solution exists. It is found that both different methods agree completely, but that the no-Kutta-condition solution of Orszag and Crow, in the sense of Bechert and Michel, reflects the existence of a vortex sheet excited by a special dipole sheet along the plate. Furthermore it is shown that for vortex sheet problems the use of the pressure instead of the velocity potential is more convenient. Finally, it is proved that the Wiener–Hopf equation cf the present problem corresponds to the differential equation for the vortex sheet displacement derived by Bechert and Michel.

Przeanalizowano powtórnie kontrowersyjne wyniki dotyczące nieściśliwej niestateczności warstwy wirowej opuszczającej płytę półnieskończoną uzyskane przez Orszaga i Crowa oraz Becherta i Michela. Pierwsze z wymienionych rozwiązań nie spelniają pełnych warunków Kutty, w drugim zaś przypadku stwierdzono, iż bez wzbudzeń źródłowych istnieją wyłącznie rozwiązania trywialne. Ustalono pełną zgodność obu metod z tym, że rozwiązanie Orszaga–Crowa niespełniające warunków Kutty w sensie Becherta–Michela odpowiada istnieniu warstwy wirowej wzbudzonej przez specjalną warstwę dipolową rozłożoną wzdłuż płyty. Pokazano ponadto, że przy rozważaniu zagadnień warstwy wirowej wygodniej jest korzystać raczej z ciśnienia niż z potencjału prędkości. Wykazano wreszcie, że równanie Wienera–Hopfa omawianego problemu odpowiada równaniu różniczkowemu przemieszczeń warstwy wirowej wyprowadzonemu przez Becherta i Michela.

Повторно проанализированы спорные результаты, касающиеся несжимаемой неустойчивости вихревого слоя опускающего полубесконечную плиту, полученные Оршагом и Кроуом, а также Бехертом и Михелем. Первые из перечисленных решений не удовлетворяет полным условиям Кутта, во втором же случае констатировано, что без источников возбуждений существуют исключительно тривиальные решения. Установлено полное совпадение обоих этих методов, с тем, что решение Оршага–Кроуа, неудовлетворяющее условиям Кутта в смысле Бехерта–Михеля, отвечает существованию вихревого слоя, возбужденного специальным дипольным слоем, распределенным вдоль плиты. Кроме этого показано, что при рассмотрении задач вихревого слоя выгоднее использовать давление, чем потенциал скорости. Наконец, показано, что уравнение Винера–Хопфа обсуждаемой задачи, отвечает дифференциальному уравнению перемещений вихревого слоя, выведенному Бехертом и Михелем.

1. Introduction

HELMHOLTZ [1] studied the hydrodynamic instability of an infinitely extended vortex sheet as a simple model of a free shear layer in an inviscid, incompressible fluid and found instability for all frequencies. ORSZAG and CROW [2], in the following abbreviated by O and C, extended the theory to a vortex sheet leaving an infinitely thin and semi-infinitely long plate in an inviscid, incompressible fluid by application of the Wiener–Hopf technique. They found a solution which had, however, some unphysical properties. For instance, the vertical displacement $h(x, t)$ of the vortex sheet for periodic time dependence behaved close to the trailing edge ($x = 0$) like $h = O(x^{1/2})$. This "no-Kutta" condition had led to many discussions about the importance and relevance of the "full-Kutta" condition $\partial h/\partial x \to \to 0$ for $x \to 0$ for this problem. O and C discussed also modified solutions leading to a full and a rectified Kutta condition and considered the latter one most reasonable. Using a completely different method, BECHERT and MICHEL [3], in the following abbreviated by B and M, obtained the controversial result which states that a time-periodic solution to the problem does not exist except in the case when an external source excites the vortex sheet. However, the reason for the discrepancy with the Wiener–Hopf solution of O and C remained open, although B and M noted that the O and C-solution corresponds to a vortex sheet excited by a dipole at the trailing edge of the plate. Unfortunately, this important result has been widely ignored.

More complicated problems have been solved on the basis of the O and C-method, for instance, the compressible analogue with excitation by a simple point source by CRIGHTON and LEPPINGTON [4] and the axisymmetric jet problem by CRIGHTON [5], to mention only a few.

In the meantime, BECHERT [6] extended the theory with respect to certain flow parameters and to flow-field calculations. Later BECHERT and STAHL [7] were able to verify excellently the theoretical results for the excited vortex sheet by experiments. Hence, there is no doubt that the results of B and M are correct, but the question remains open what relation exists between both results of O and C and B and M.

The aim of the present note is to compare both methods and to find out the reasons for the (apparent) discrepancies of their results. It will become clear that both methods are correct and lead to identical results, but that the O and C-solution can be interpreted as a forced solution with the vortex sheet being excited by a special dipole sheet along the plate. Furthermore it will be mentioned that the use of the velocity potential for the disturbance velocity in both regions outside the vortex sheet can be misleading. As opposed to this, the pressure disturbance is found to be a more convenient variable. Finally, the comparison of both methods will show quite clearly the nonuniqueness of the Wiener–Hopf technique and the equivalence of the Wiener–Hopf equation of O and C with the differential equation derived by B and C.

In § 2 the governing equations of the inviscid, incompressible problem are derived with emphasis laid on the fact that the vortex sheet has to be considered as the limit case of a continuous, finitely thick shear layer. Following the method of B and M in Sect. 3, the pressure-displacement equation is solved by means of Poisson's integral. In Sect. 4 the problem is, following the method of O and C, solved by means of the Wiener–Hopf technique applied to the pressure field. Finally, in Sect. 5, both methods and their results are compared. In all cases, it is found that in the strict absence of any "source", a nontrivial solution of the problem does not exist.

2. Governing equations

Since we want to treat the vortex sheet as the limit of a shear layer of finite, but vanishing thickness, we consider a parallel basic shear flow with an x-velocity component $\overline{U}(y)$ in an inviscid, incompressible fluid. For small disturbance velocity u', v' and pressure p'

the linearized Euler and continuity equation in the presence of incompressible sources become

$$\varrho\left[\frac{\partial}{\partial t}+\bar{U}\frac{\partial}{\partial x}\right]u'+\varrho v'\frac{d\bar{U}}{dy} = -\frac{\partial p'}{\partial x}+f_x, \tag{2.1}$$

$$\varrho\left[\frac{\partial}{\partial t}+\bar{U}\frac{\partial}{\partial x}\right]v' = -\frac{\partial p'}{\partial y}+f_y, \tag{2.2}$$

$$\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y} = q, \tag{2.3}$$

where ϱ is the constant density, f_x and f_y are dipole-type source distributions and q is a simple source distribution. Furthermore the vertical particle displacement $h_p(x, y, t)$ is introduced, which is related to the v'-velocity by

$$v' = \left[\frac{\partial}{\partial t}+\bar{U}\frac{\partial}{\partial x}\right]h_p. \tag{2.4}$$

If we take the x-derivative of Eq. (2.1) and the y-derivative of Eq. (2.2), we get with Eq. (2.3)

$$\frac{\partial^2 p'}{\partial x^2}+\frac{\partial^2 p'}{\partial y^2} = -2\varrho\frac{d\bar{U}}{dy}\frac{\partial v'}{\partial x}-\varrho\left[\frac{\partial}{\partial t}+\bar{U}\frac{\partial}{\partial x}\right]q+\frac{\partial f_x}{\partial x}+\frac{\partial f_y}{\partial y}. \tag{2.5}$$

Here v' can be eliminated by Eq. (2.4) yielding from Eqs. (2.5) and (2.2)

$$\frac{\partial^2 p'}{\partial x^2}+\frac{\partial^2 p'}{\partial y^2} = -\varrho\left[\frac{\partial}{\partial t}+\bar{U}\frac{\partial}{\partial x}\right]\left[2\frac{d\bar{U}}{dy}\frac{\partial h_p}{\partial x}+q\right]+\frac{\partial f_x}{\partial x}+\frac{\partial f_y}{\partial y}, \tag{2.6}$$

$$\varrho\left[\frac{\partial}{\partial t}+\bar{U}\frac{\partial}{\partial x}\right]^2 h_p = -\frac{\partial p'}{\partial y}+f_y. \tag{2.7}$$

Equations (2.6) and (2.7) constitute a system for the unknowns p' and h_p for given basic velocity $\bar{U}(y)$ and sources q, f_x, f_y.

In the following only the case of a discontinuous shear layer is treated by assuming

$$\bar{U}(y) = UH(y), \quad \frac{d\bar{U}}{dy} = U\frac{dH}{dy} = U\delta(y), \tag{2.8}$$

where $H(y)$ is the Heaviside unit-step function with $H(y>0) = 1$ and $H(y<0) = 0$. $\delta(y) = dH/dy$ is the Dirac delta function with $\delta(y \neq 0) = 0$ and $\delta(y = 0) = \infty$. For those who are not familiar with the functions $H(y)$ and $\delta(y)$, some relations which are used in the following are given in the Appendix 1.

Equation (2.8) implies that at $y = 0$ there is a vortex sheet with the vorticity $\Omega = -d\bar{U}/dy = -U\delta(y)$ in the fluid. A semi-infinitely long and infinitely thin, rigid plate is assumed at $y = 0$, $x \leqslant 0$. With Eq. (2.8), the pressure-displacement equations (2.6) and (2.7) become

$$\frac{\partial^2 p'}{\partial x^2}+\frac{\partial^2 p'}{\partial y^2} = -\varrho\left[\frac{\partial}{\partial t}+UH(y)\frac{\partial}{\partial x}\right]\left[2U\delta(y)\frac{\partial h_p}{\partial x}+q\right]+\frac{\partial f_x}{\partial x}+\frac{\partial f_y}{\partial y}, \tag{2.9}$$

$$\varrho\left[\frac{\partial}{\partial t}+UH(y)\frac{\partial}{\partial x}\right]^2 h_p = -\frac{\partial p'}{\partial y}+f_y. \tag{2.10}$$

As boundary condition we have to require that $\partial p'/\partial x$ and $\partial p'/\partial y \to 0$ for $|y| \to \infty$ and for $x \to -\infty$ far upstream save for the case when sources are there. Furthermore, the particle displacement $h_p(x, y, t)$ has to be continuous across the vortex sheet at $y = 0$ for physical reasons. Then $h(x, t) = h_p(x, 0, t)$ is the displacement of the vortex sheet. Along the plate ($y = 0, x \leqslant 0$) the normal velocity component v' and hence the displacement must be zero: $h(x, t) = 0$ for $x \leqslant 0$. In addition, if there is no source along the vortex sheet at $y = 0$, then the pressure must be continuous across $y = 0$. One can derive a patching condition from Eqs. (2.9) and (2.10) which requires that Eq. (2.10) be satisfied for $y \to \pm 0$. Finally, from physical reasons it follows that the displacement h has to be continuous along the vortex sheet and therefore $h \to 0$ for $x \to 0$.

It should be mentioned that without mean flow ($U = 0$), a nontrivial solution of Eqs. (2.9) and (2.10) satisfying the boundary conditions exists only in the presence of sources. Furthermore, if we assume, e.g., a simple point source $q = q_0(t)\,\delta(x-x_s)\,\delta(y-y_s)$ at x_s, y_s, the boundary condition along the plate requires additionally a dipole distribution $f_y = F_q(x, t)\,\delta(y)$ for $x \leqslant 0$. As a consequence of Eq. (2.10) with $h_p = 0$ for $y = 0$ and $x \leqslant 0$, we have a pressure difference across the plate:

(2.11) $$\Delta p = \lim_{\varepsilon \to 0}[p'(x, +\varepsilon, t) - p'(x, -\varepsilon, t)] = F_q(x, t) \quad \text{for} \quad x \leqslant 0.$$

This can be obtained by integration of Eq. (2.10) over $-\varepsilon \leqslant y \leqslant \varepsilon$ for $\varepsilon \to 0$. Some examples of solutions for $U = 0$ are given in Appendix 2 by means of conformal mapping.

In order to solve Eqs. (2.9) and (2.10), different approaches have been used by O & C and by B & M. Instead of the pressure p', O & C used the potential function Φ which is related to the pressure p' by

(2.12) $$p' = -\varrho\left[\frac{\partial}{\partial t} + UH(y)\frac{\partial}{\partial x}\right]\Phi.$$

For $y \neq 0$, Φ is the velocity potential. Since p' is continuous along the free vortex sheet, it follows that Φ must have a jump at $y = 0$ and that $\partial\Phi/\partial y$ is not bounded but will contain a $\delta(y)$-term. This is very inconvenient as compared with the pressure p', especially if the limits of $\partial\Phi/\partial y$ for $y \to \pm 0$ are needed. These limits make sense only if they are interpreted as $y \to \pm\varepsilon$ with $0 < \varepsilon \ll 1$. Notwithstanding, O & C applied the Wiener–Hopf technique to the problem to derive the solution for Φ. Instead, B & M derived a differential equation for the vortex sheet displacement $h(x, t)$ which has been derived from Eqs. (2.9) and (2.10) by means of symmetry conditions for the pressure p'.

In both methods the Helmholtz solution p_0, h_0 for the infinitely extended vortex sheet without plate has been separated by assuming

(2.13) $$p' = p_0 + p_c, \qquad h = h_0 + h_c.$$

For periodic time dependence, p_0 and h_0 are the Helmholtz solution, growing exponentially in the x-direction, of Eqs. (2.9) and (2.10) which satisfies the boundary conditions at $|y| \to \infty$ and the patching conditions at $y = 0$. The vortex sheet displacement h_0 is then

(2.14) $$h_0(x, t) = C_0 \exp(-i\mu_1 x - i\omega t),$$

where C_0 is an arbitrary constant amplitude, ω is the real cyclic frequency and μ_1 is the complex eigenvalue of the Helmholtz problem:

(2.15) $$\mu_1 = -(1-i)\omega/U.$$

There is an additional eigenvalue μ_2 which is the conjugate complex value of μ_1. This corresponds to a disturbance decaying exponentially for $x \to \infty$. An analysis of the Helmholtz solution for the disturbed free vortex sheet shows that, as required, the displacement of the vortex sheet and the pressure is continuous across the vortex sheet, but that the y-velocity component v has a jump and, as a consequence of the continuity equation, the x-velocity component u has a Dirac contribution. Since $h_0 \neq 0$ for $x \leqslant 0$ along the plate, a compensating field p_c, h_c is necessary to satisfy $h = h_0 + h_c = 0$ for $x \leqslant 0$. h_c and the gradients of p_c are assumed to vanish for $|x| \to \infty$.

In order to compare the different methods of O & C and B & M and their results, both methods will be re-examined in the following sections. To simplify the analysis, we restrict ourselves to a simple point source at (x_s, y_s) with

$$q(x, y, t) = q_0(t)\,\delta(x-x_s)\,\delta(y-y_s) \tag{2.16}$$

for $y_s < 0$, i.e., the source is located in the fluid at rest. Furthermore we assume $f_x \equiv 0$ and restrict the dipole distribution f_y to a dipole sheet along the plate with a separate contribution $F_q(x, t)\,\delta(y)$ belonging to the simple source, as mentioned above. Hence we have

$$f_y = [F_q(x, t) + F_d(x, t)]\,\delta(y) \tag{2.17}$$

which vanishes for $x > 0$.

Furthermore, since $h_p(x, y, t)$ has to be continuous at $y = 0$, we put the displacement of the vortex sheet $h_p(x, 0, t) = h(x, t)$. Then, according to Appendix 1, we have $\delta(y)h_p(x, y, t) = \delta(y)h(x, t)$. Finally, the Helmholtz solution p_0, h_0 satisfies Eqs. (2.9) and (2.10) for $q = f_x = f_y = 0$. Hence we obtain from Eq. (2.9) an equation for p_c:

$$\frac{\partial^2 p_c}{\partial x^2} + \frac{\partial^2 p_c}{\partial y^2} = -2\varrho U\delta(y)\left[\frac{\partial}{\partial t} + UH(y)\frac{\partial}{\partial x}\right]\frac{\partial h_c}{\partial x} - \varrho\frac{dq_0}{dt}\,\delta(x-x_s)\,\delta(y-y_s) + (F_q+F_d)\,\delta'(y). \tag{2.18}$$

The first patching condition for $y \to \pm 0$ and all x is due to Eq. (2.10):

$$\lim_{y\to 0}\left[\frac{\partial p_c}{\partial y} - (F_q+F_d)\,\delta(y)\right] = -\varrho\left[\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right]^2 h_c, \tag{2.19}$$

$$\lim_{y\to -0}\left[\frac{\partial p_c}{\partial y} - (F_q+F_d)\,\delta(y)\right] = -\varrho\frac{\partial^2 h_c}{\partial t^2}. \tag{2.20}$$

The limits on the left-hand sides of Eqs. (2.19) and (2.20) remain bounded since in Eq. (2.10) the left-hand side is bounded because of the continuity of h_p and the finite jump by $H(y)$. Thus, in the presence of a dipole sheet ($F_q + F_d \neq 0$) along the plate $x < 0$, $\partial p_c/\partial y$ must contain a $\delta(y)$-term which cancels the corresponding Dirac function in Eqs. (2.19) and (2.20). By adding both Eqs. (2.19) and (2.20), we obtain a differential equation for h_c:

$$\varrho\left\{\left[\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right]^2 + \frac{\partial^2}{\partial t^2}\right\}h_c = -\left[\frac{\partial p_c}{\partial y} - (F_q+F_d)\,\delta(y)\right]_{y\to+0} - \left[\frac{\partial p_c}{\partial y} - (F_q+F_d)\,\delta(y)\right]_{y\to-0}. \tag{2.21}$$

The second patching condition is given by the pressure continuity across the free vortex sheet for $x > 0$:

$$p_c(x, +0, t) = p_c(x, -0, t) \tag{2.22}$$

while the condition $h = 0$ along the rigid plate for $x < 0$ requires

$$h_c(x, t) = -h_0(x, t). \tag{2.23}$$

In the next section the solution of the problem is derived by a method similar to that of B & M.

3. Solution of the pressure-displacement equation by means of the Poisson integral

B & M considered only time-periodic solutions of Eq. (2.18) and split p_c into an instability field p_i and a source field p_s:

$$p_c = p_i + p_s. \tag{3.1}$$

For the moment being we retain arbitrary time dependence. The instability field p_i has to comply with

$$\frac{\partial^2 p_i}{\partial x^2} + \frac{\partial^2 p_i}{\partial y^2} = -2\varrho U \delta(y)\left[\frac{\partial}{\partial t} + UH(y)\frac{\partial}{\partial x}\right]\frac{\partial h_c}{\partial x} = L(x, y, t) \tag{3.2}$$

while the source field is defined with $U \equiv 0$ and determined by

$$\frac{\partial^2 p_s}{\partial x^2} + \frac{\partial^2 p_s}{\partial y^2} = -\varrho \frac{dq_0}{dt}\delta(x-x_s)\delta(y-y_s) + (F_q + F_d)\delta'(y). \tag{3.3}$$

For special cases the solution to this equation can be obtained by conformal mapping as shown in Appendix 2. The solution p_s is related to its y-velocity component v_s by Eq. (2.2) which yields with $\bar{U} \equiv 0$

$$\varrho \frac{\partial v_s}{\partial t} = -\frac{\partial p_s}{\partial y} + (F_q + F_d)\delta(y). \tag{3.4}$$

Here $v_s(x, 0, t) = 0$ for $x < 0$, and v_s is continuous at $y = 0, x > 0$.

B & M concluded from Eq. (3.2) and from the boundary conditions for $y \to \pm\infty$ that p_i has to be a symmetric function with respect to y. We shall show this more strictly by using the Poisson integral. If we treat Eq. (3.2) as a Poisson equation for p_i and assume that h_c vanishes for $x \to \pm\infty$, then the Poisson integral yields the solution of Eq. (3.2):

$$p_i(x, y, t) = \frac{1}{4\pi}\int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\eta\, L(\xi, \eta, t)\ln[(x-\xi)^2 + (y-\eta)^2], \tag{3.5}$$

where $L(x, y, t)$ is defined by Eq. (3.2). The integration with respect to η can be performed if we take the properties of the Dirac function $\delta(\eta)$ into acount (see Appendix 1). Then we get with Eq. (3.2)

$$p_i(x, y, t) = -\frac{\varrho}{4\pi}\int_{-\infty}^{\infty} d\xi \left[U^2 \frac{\partial^2 h_c}{\partial \xi^2} + 2U \frac{\partial^2 h_c}{\partial \xi\, \partial t}\right]\ln[(x-\xi)^2 + y^2]. \tag{3.6}$$

It is obvious from Eq. (3.6) that p_i is, in fact, a symmetric function of y. Hence the condition (2.22) is satisfied for all x. For $y \neq 0$ we can interchange the differentiation with respect to x or y with the integration over ξ. Then it follows that $\partial p_i/\partial x$ and $\partial p_i/\partial y$ vanish for $|y| \to \infty$. Especially, we obtain

(3.7)
$$\frac{\partial p_i}{\partial y} = -\frac{\varrho}{2}\int\limits_{-\infty}^{\infty} d\xi\left[U^2\frac{\partial^2 h_c}{\partial \xi^2}+2U\frac{\partial^2 h_c}{\partial\xi\,\partial t}\right]\frac{1}{\pi}\,\frac{y}{(x-\xi)^2+y^2}.$$

The last factor in the integrand of Eq. (3.7) tends for $y \to \pm 0$ to a Dirac function (see Appendix 1). Thus we find

(3.8)
$$\lim_{y\to\pm 0}\frac{\partial p_i}{\partial y} = -\frac{\varrho}{2}\int\limits_{-\infty}^{\infty} d\xi\left[U^2\frac{\partial^2 h_c}{\partial \xi^2}+2U\frac{\partial^2 h_c}{\partial\xi\,\partial t}\right][\pm\delta(x-\xi)] = \mp I(x).$$

The integral $I(x)$ cannot be evaluated unless the term in the square brackets is continuous for all ξ. Nevertheless, by introducing Eqs. (3.8) and (3.4) into Eq. (2.21), we note that $I(x)$ drops out. Since the Helmholtz solution h_0 satisfies the homogeneous part of Eq. (2.21), we get for the vortex sheet displacement $h = h_0+h_c$:

(3.9)
$$\left\{\left[\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right]^2+\frac{\partial^2}{\partial t^2}\right\}h = \frac{\partial v_s(x,\,+0,\,t)}{\partial t}+\frac{\partial v_s(x,\,-0,\,t)}{\partial t} = 2\,\frac{\partial v_s(x,0,t)}{\partial t}.$$

The required boundary condition is $h(x,t) = 0$ for $x \leqslant 0$. The exciting velocity v_s of the sources is unequal to zero only for $x \geqslant 0$.

The homogeneous equation (3.9) has already been given by HOWE [8], while the inhomogeneous equation (3.9) and its solution has been discussed by MICHALKE [9]. Here we shall restrict ourselves to the case of periodic time dependence of v_s and h proportional to $\exp(-i\omega t)$ where ω is the real frequency. Without changing the symbol for the vortex sheet displacement ($h(x,t) := h(x)\exp(-i\omega t)$) and putting $v_s(x,0,t) = V_s(x)H(x) \times \exp(-i\omega t)$, we obtain from Eq. (3.9)

(3.10)
$$U^2\frac{d^2h}{dx^2}-2i\omega U\frac{dh}{dx}-2\omega^2 h = -2i\omega V_s(x)H(x).$$

Eq. (3.10) corresponds to the equation derived by B & M. Its solution satisfying $h = 0$ for $x < 0$ is easily found to be:

(3.11)
$$h(x) = \frac{-2i\omega}{U^2(\mu_1+\mu_2)}\int\limits_{-\infty}^{x} d\xi\, V_s(\xi)H(\xi)\big[\exp[i\mu_2(\xi-x)]-\exp[i\mu_1(\xi-x)]\big].$$

Here μ_1 is the Helmholtz eigenvalue (2.15) and μ_2 its conjugate complex value. The solution (3.11) yields $h(x) = 0$ for $x \leqslant 0$ and tends for $x \to \infty$ to the exponentially growing Helmholtz solution with the amplitude being completely determined by the exciting velocity V_s.

From this equation (3.11), B & M concluded that a nontrival solution does not exist without excitation ($V_s \equiv 0$). B & M calculated the displacement h for excitation by a point source $q_0 \neq 0$ leading to $V_s = O(x^{-1/2})$. The solution (3.11) then satisfied the full Kutta condition $h = O(x^{3/2})$. This can easily be seen by considering the expansion of Eq. (3.11) close to the trailing edge of the plate for $x \to +0$:

$$h(x) = \frac{2\omega(\mu_1-\mu_2)}{U^2(\mu_1+\mu_2)} \int_{-\infty}^{x} d\xi V_s(\xi) H(\xi)(x-\xi) + \dots \tag{3.12}$$

Excellent agreement of theoretical results with experimental ones has been found by BECHERT and STAHL [7] in the range of validity of the vortex sheet approximation.

For the physical understanding it is necessary to emphasize that, by excitation ($q_0 \neq 0$), also a dipole distribution $F_q \neq 0$ along the plate exists which, due to Eq. (2.11), corresponds to a pressure difference Δp across the plate (see Appendix 2). Due to Eq. (3.3) a pressure difference Δp can also be generated for $q_0 = F_q = 0$ by a dipole sheet F_d alone along the plate for $x < 0$. In this particular case the solution to Eq. (3.3) with the Poisson integral for periodic time dependence is given by

$$p_s(x, y) = \frac{\partial}{\partial y}\left[\frac{1}{4\pi}\int_{-\infty}^{0} d\xi F_d(\xi)\ln[(x-\xi)^2+y^2]\right] = \frac{\partial W}{\partial y}. \tag{3.13}$$

Furthermore, since the function W satisfies the equation

$$\frac{\partial^2 W}{\partial x^2}+\frac{\partial^2 W}{\partial y^2} = F_d(x)\,\delta(y), \tag{3.14}$$

we get from Eqs. (3.4), (3.13) and (3.14) with $q = F_q = 0$:

$$-i\omega\varrho v_s = -\frac{\partial p_s}{\partial y}+F_d\,\delta(y) = \frac{\partial^2 W}{\partial x^2}. \tag{3.15}$$

Hence the inhomogeneous part of the differential equation (3.10) becomes

$$-2i\omega V_s(x) H(x) = \frac{2}{\varrho}\lim_{y\to 0}\frac{\partial^2 W}{\partial x^2}. \tag{3.16}$$

With Eq. (3.13) and with $\xi = -\zeta$ we find

$$\frac{\partial^2 W}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{1}{2\pi}\int_0^{\infty} d\zeta F_d(-\zeta)\frac{\zeta+x}{(\zeta+x)^2+y^2}\right] = -\frac{1}{2\pi}\mathscr{R}\left[\int_0^{\infty} d\zeta\frac{F_d(-\zeta)}{(\zeta+z)^2}\right], \tag{3.17}$$

where $z = x+iy$ and $\mathscr{R}$ means "real part of".

Since the left-hand side of Eq. (3.16) vanishes identically for $x < 0$, the question is whether dipole distributions F_d of this type do exist. Let us try:

$$F_d(x) = G(-x)^{\nu-1}, \quad x < 0, \quad G = \text{constant}. \tag{3.18}$$

Then the integral (3.17) can be evaluated (see GRADSHTEYN and RYZHIK [10], formula 3.194.6):

$$\int_0^{\infty} d\zeta\frac{\zeta^{\nu-1}}{(\zeta+z)^2} = \frac{(1-\nu)\pi z^{\nu-2}}{\sin(\nu\pi)}, \quad 0 < \nu < 2. \tag{3.19}$$

The branch cut of $z^{\nu-2}$ is located in the negative real axis. It follows from Eqs. (3.15) to (3.19) with $z = x+iy = r\exp(i\varphi)$:

$$(3.20) \qquad -i\omega\varrho v_s(x,y) = -\frac{(1-\nu)G}{2\sin(\nu\pi)}\mathscr{R}[z^{\nu-2}] = -\frac{(1-\nu)G}{2\sin(\nu\pi)}\frac{\cos((\nu-2)\varphi)}{r^{2-\nu}}.$$

For $\varphi \to \pm\pi$, v_s has to vanish for $y \to \pm 0$, $x < 0$. Hence admissible values of ν are either $\nu = 1/2$ leading to

$$(3.21) \qquad \varrho v_s(x,y) = \frac{G}{4i\omega}\mathscr{R}[z^{-3/2}]; \qquad \varrho V_s(x>0) = \frac{G}{4i\omega}x^{-3/2}$$

or $\nu = 3/2$ leading to

$$(3.22) \qquad \varrho v_s(x,y) = \frac{G}{4i\omega}\mathscr{R}[z^{-1/2}], \qquad \varrho V_s(x>0) = \frac{G}{4i\omega}x^{-1/2}.$$

It follows with Eq. (3.12) from Eqs. (3.22) and (3.18) that a dipole sheet with $F_d \sim (-x)^{1/2}$ leads to the full-Kutta-condition solution with $h = O(x^{3/2})$, while $F_d \sim (-x)^{-1/2}$ with Eqs. (3.21) yields the no-Kutta-condition solution $h = O(x^{1/2})$ derived by O & C. In the latter case it is necessary in evaluating Eq. (3.12) to replace $\xi^{-3/2}H(\xi)$ due to Eqs. (3.21) by $\mathscr{R}[(\xi+iy)^{-3/2}]$ and taking the limit $y \to +0$ after the integration. Hence it is proved that the no-Kutta-condition solution of O & C can be obtained by the method of B & M, if only a dipole source distribution along the plate is assumed with $F_d \propto (-x)^{-1/2}$ leading to $V_s \propto x^{-3/2}$.

Before discussing both dipole sheet solutions in more detail, let us re-examine the method of O & C in the next section.

4. Solution by means of the Wiener–Hopf technique

As already mentioned in Sect. 2 O & C used the velocity potential Φ in their calculation. For the reasons already mentioned we prefer to use the pressure p_c for the solution by means of the Wiener–Hopf technique. Analogously to O & C, we denote

$$(4.1) \qquad p_c = \begin{cases} P_{c1}(x,y)\exp(-i\omega t) & y>0,\\ p_{c2}(x,y)\exp(-i\omega t) & y<0.\end{cases}$$

From Eq. (2.18) we see that for $y \neq 0$ and $q_0 \equiv 0$ both functions are solutions of the Laplace equation. In order to avoid problems with the asymptotic behaviour of the functions for $|y| \to \infty$, the Laplace equation is replaced by

$$(4.2) \qquad \frac{\partial^2 p_{ci}}{\partial x^2} + \frac{\partial^2 p_{ci}}{\partial y^2} = \varepsilon^2 p_{ci} \qquad i = 1, 2.$$

Here ε is a real positive constant which is set zero later. The boundary conditions require $p_{c1} \to 0$ for $y \to +\infty$, and $p_{c2} \to 0$ for $y \to -\infty$. The first patching condition at $y = 0$ valid for all x is given by Eqs. (2.19) and (2.20). These yield for the "bounded" part of $\partial p_{ci}/\partial y$

$$(4.3) \qquad \left.\frac{\partial p_{c1}}{\partial y}\right|_{y\to+0} = -\varrho\left[U\frac{\partial}{\partial x} - i\omega\right]^2 h_c; \qquad \left.\frac{\partial p_{c2}}{\partial y}\right|_{y\to-0} = \varrho\omega^2 h_c.$$

Finally, the second patching condition along the free vortex sheet and along the rigid plate are given by Eqs. (2.22) and (2.23).

In order to solve the problem, a Fourier transform with respect to x is introduced:

$$P_1(y, \lambda) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} dx p_{c1}(x, y) \exp(i\lambda x) \tag{4.4}$$

and corresponding expressions for $P_2(y, \lambda)$ and $C(\lambda)$ as Fourier transforms of p_{c2} and h_c, respectively.

The solutions of Eq. (4.2) satisfying the boundary conditions for $|y| \to \infty$ lead to

$$\begin{aligned} P_1(y, \lambda) &= A(\lambda)\exp[-(\lambda^2+\varepsilon^2)^{1/2}y], \\ P_2(y, \lambda) &= B(\lambda)\exp[+(\lambda^2+\varepsilon^2)^{1/2}y], \end{aligned} \tag{4.5}$$

where the branch cuts for $(\lambda^2+\varepsilon^2)^{1/2}$ as function of λ are chosen in a way that the real part is positive for all complex λ (see O & C). Then in the strip $-\varepsilon < \mathscr{I}(\lambda) < \varepsilon$ both functions are analytic. The Fourier transform of the first patching condition (4.3) yields

$$P_1'(+0, \lambda) = \varrho[U\lambda+\omega]^2 C(\lambda), \qquad P_2'(-0, \lambda) = \varrho\omega^2 C(\lambda). \tag{4.6}$$

With Eqs. (4.5) it follows

$$A(\lambda) = -\varrho(U\lambda+\omega)^2(\lambda^2+\varepsilon^2)^{-1/2}C(\lambda), \qquad B(\lambda) = \varrho\omega^2(\lambda^2+\varepsilon^2)^{-1/2}C(\lambda). \tag{4.7}$$

To satisfy the second set of the patching conditions (2.22) and (2.23), half-range Fourier transforms will be used as defined by

$$C_- = (2\pi)^{-1/2} \int_{-\infty}^{0} dx h_c(x) e^{i\lambda x}, \qquad C_+ = (2\pi)^{-1/2} \int_{0}^{\infty} dx h_c(x) e^{i\lambda x}. \tag{4.8}$$

It is obvious that $C = C_+ + C_-$. Analogous relations hold for P_1 and P_2. Then Eqs. (2.22) and (2.23) yield with the half-range Fourier transform of Eq. (2.14)

$$A_+ = B_+, \qquad C_- = iC_0(2\pi)^{-1/2}(\lambda-\mu_1)^{-1}. \tag{4.9}$$

Furthermore, from Eq. (4.7), it follows

$$\begin{aligned} A_+ + A_- &= -\varrho(U\lambda+\omega)^2(\lambda^2+\varepsilon^2)^{-1/2}(C_+ + C_-), \\ B_+ + B_- &= \varrho\omega^2(\lambda^2+\varepsilon^2)^{-1/2}(C_+ + C_-). \end{aligned} \tag{4.10}$$

With Eqs. (4.9) we find from Eqs. (4.10)

$$(B_- - A_-)/\varrho = \frac{[(U\lambda+\omega)^2+\omega^2]}{(\lambda^2+\varepsilon^2)^{1/2}} [C_+ + iC_0(2\pi)^{-1/2}(\lambda-\mu_1)^{-1}]. \tag{4.11}$$

By means of the Helmholtz eigenvalues μ_1 and μ_2 given by Eq. (2.15) it follows that

$$(U\lambda+\omega)^2+\omega^2 = U^2\lambda^2+2\omega U\lambda+2\omega^2 = U^2(\lambda-\mu_1)(\lambda-\mu_2). \tag{4.12}$$

Then we can split Eq. (4.11) in a function F_- defined by

$$F_- = -(\lambda-i\varepsilon)^{1/2}(A_- - B_-)/\varrho U^2 \tag{4.13}$$

and a function F_+ defined by

$$F_+ = [(\lambda-\mu_1)(\lambda-\mu_2)C_+ + iC_0(2\pi)^{-1/2}(\lambda-\mu_2)](\lambda+i\varepsilon)^{-1/2}, \tag{4.14}$$

F_- is analytic in $-\infty < \mathcal{S}(\lambda) < \varepsilon$, F_+ is analytic in $-\varepsilon < \mathcal{S}(\lambda) < +\infty$. Both functions are analytic in the common strip of overlap $-\varepsilon < \mathcal{S}(\lambda) < \varepsilon$ and there Eq. (4.11) requires that $F_+ = F_-$.

The functions F_+ and F_- agree with those derived by O & C by means of the velocity potential Φ. From the asymptotic behaviour of the Wiener–Hopf equation $F_+ = F_-$ for $|\lambda| \to \infty$, O & C concluded that $F_+ = F_- = E =$ constant. Then:

(4.15) $$C_+ = [(\lambda-\mu_1)(\lambda-\mu_2)]^{-1}[(\lambda+i\varepsilon)^{1/2}E - iC_0(2\pi)^{-1/2}(\lambda-\mu_2)].$$

Since C_+ has to be regular at $\lambda = \mu_1$ (μ_1 is located in the upper half-plane), it follows:

(4.16) $$(\mu_1+i\varepsilon)^{1/2}E = iC_0(2\pi)^{-1/2}(\mu_1-\mu_2).$$

This equation relates both constants E and C_0. Furthermore we have

(4.17) $$A_- - B_- = -\varrho U^2 E(\lambda-i\varepsilon)^{-1/2}.$$

With Eqs. (4.9) the results follows:

(4.18) $$C(\lambda) = C_+ + C_- = [(\lambda-\mu_1)(\lambda-\mu_2)]^{-1}E(\lambda+i\varepsilon)^{1/2}$$

and

(4.19) $$A(\lambda) - B(\lambda) = -\varrho U^2 E(\lambda-i\varepsilon)^{-1/2}$$

which, together with Eqs. (4.7) and (4.5), determines P_1 and P_2.

We note from Eqs. (4.5) and (4.19) that $A-B$ is the Fourier transform of the pressure difference $\Delta p(x) = p_c(x, +0) - p_c(x, -0)$ across the plate. Hence $E \neq 0$ implies that according to Eq. (2.11) a dipole sheet is present along the plate. The inverse Fourier transform of Eq. (4.19) yields (see LIGHTHILL [11]):

(4.20) $$\Delta p(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\lambda [A(\lambda) - B(\lambda)] e^{-i\lambda x} = -\varrho U^2 E(2i/-x)^{1/2} H(-x) e^{\varepsilon x}.$$

We see that Eq. (4.20) indicates the presence of a dipole sheet with $F_d = \Delta p$ corresponding for $\varepsilon = 0$ to that of Eq. (3.18) with $\nu = 1/2$. The inverse Fourier transform of $C(\lambda)$ given by Eq. (4.18) yields h_c. For $\varepsilon \to 0$, O & C found the no-Kutta-condition solution $h = h_0 + h_c = O(x^{1/2})$. However, if there is no dipole sheet along the plate ($E = 0$), only the trivial solution $h \equiv 0$ would exist.

Finally, it should be mentioned that O & C noted that "by adding a periodic, irrotational surging around the edge of the plate" a full-Kutta condition solution with $h = O(x^{3/2})$ can be obtained. This "added" flow corresponds to the "folded" parallel flow discussed in Appendix 2 and yields a dipole sheet like that of Eq. (3.18) with $\nu = 3/2$.

5. Comparison and discussion of both methods

We shall now compare the results of O & C with that of the method used by B & M. It is obvious that the no-Kutta-condition solution of O & C corresponds to that which is obtained by the method of B & M, when a dipole sheet (3.18) with $\nu = 1/2$ along the plate is assumed. This is equivalent to a pressure difference across the plate. The corresponding flow field is generated by a "folded" dipole, as can be seen by comparing Eqs. (3.21) and (A2.11).

The full-Kutta-condition solution also mentioned by O & C requires a dipole sheet along the plate, too. Its distribution F_d is given by Eq. (3.18) with $\nu = 3/2$. The corresponding flow field is generated by a "folded" parallel flow, as follows from the comparison of Eqs. (3.22) and (A2.17). Hence it is obvious that the type of the Kutta condition which is found depends on the exciting source type applied, or more precisely, on the transverse velocity $v_s(x, 0, t)$ exciting the free vortex sheet, as can be seen from Eq. (3.9) or from Eqs. (3.11) and (3.12). In the absence of any source generating a nonzero $v_s(x, 0, t)$ and, equivalently, a pressure difference Δp across the plate, then no instability wave can exist. This confirms the statement of B & M. The conclusion can also be drawn from the results of O & C (case $E = 0$).

It should, however, be mentioned that the expressions "source" and "excitation" may be a little bit ambiguous. It may be a question of philosophy to denote the necessary pressure difference across the plate as the cause for the instability wave to develop ("excitation by a dipole source sheet") or to mention the instability wave to be the cause for the pressure difference across the plate. In the latter case of interpretation it is, however, difficult to explain why such quite different pressure distributions across the plate should be generated by the instability wave of the vortex sheet having always the same asymptotic behaviour. Furthermore, note from Eq. (A2.9) that the pressure of the "folded" dipole has an infinitely large pressure jump at the trailing edge between $x \to -0$ and $x \to +0$, which is physically quite unrealistic if the presence of a source is excluded.

A dipole sheet along the plate corresponds to the inhomogeneous term $f_y = F_d(x)\,\delta(y)$ of the linearized Euler equation which has to be a given quantity. Its existence is a necessary condition to generate a fluctuating flow around the trailing edge of the plate in absence of any other source. Since only in this way an instability wave is generated, it seems to be reasonable to denote the inhomogeneous term in the Euler equation as a "source". As already mentioned, a simple point source placed in the stagnant fluid requires also a dipole sheet along the plate, as was shown in Appendix 2. BECHERT [6] discussed the effect of the source position $z_s = x_s + iy_s$ on the excitation of the vortex sheet. He found that the excitation become ineffective if the simple source is placed downstream of the trailing edge of the plate right at the vortex sheet. In this case the dipole sheet along the plate vanishes, as can be seen from Eq. (A2.3) for a source position $z_s = x_s > 0$. Hence we can conclude that the existence of a dipole sheet along the plate inducing a flow around the trailing edge of the plate is the necessary condition for the existence of an amplified instability wave past the trailing edge of the plate. This seems to be reasonable since for $U \equiv 0$ only the inhomogeneous equations have a nontrivial solution satisfying all boundary conditions, i.e., if sources are present (at least, $F_d \neq 0$). The same is obviously true even for $U \neq 0$.

It can be shown by means of Eq. (3.9) that even a sound wave coming from the upstream stagnant fluid will generate a $v_s(x, 0, t)$-distribution along the free vortex sheet which leads to an amplified instability wave. The same is true for flow disturbances in the potential flow. Assume a potential vortex upstream of the trailing edge convected in the potential flow parallel to the plate. If the vortex approaches and passes the trailing edge of the plate, a transient instability wave is excited, as can be derived from Eq. (3.9). Again in this case, the vortex requires a dipole sheet along the plate in order to admit only a tangential velocity component at the plate. In both cases the full Kutta condition is satisfied.

Finally, it remains to be shown that the results of the Wiener–Hopf technique applied by O & C are identical with those of B & M. The Wiener–Hopf equation $F_+ = F_-$ yields with Eqs. (4.9), (4.13) and (4.14)

$$(\lambda-\mu_1)(\lambda-\mu_2)C = -(\lambda^2+\varepsilon^2)^{1/2}(A-B)/\varrho U^2. \tag{5.1}$$

Here C is the Fourier transform of h_c and

$$A-B = \lim_{y\to+0} P_1(y,\lambda) - \lim_{y\to-0} P_2(y,\lambda) = \Delta P(\lambda) \tag{5.2}$$

is the Fourier transform of the pressure difference Δp across the plate corresponding to a dipole sheet with strength $F_d = \Delta p$, as already mentioned. The left-hand side of Eq. (5.1) is the Fourier transform of a differential equation for h_c. With Eq. (4.12) the inverse Fourier transform of Eq. (5.1) yields

$$U^2\frac{d^2h_c}{dx^2} - 2i\omega U\frac{dh_c}{dx} - 2\omega^2 h_c = R(x)/\varrho, \tag{5.3}$$

where

$$R(x) = (2\pi)^{-1/2}\int\limits_{-\infty}^{\infty} d\lambda(\lambda^2+\varepsilon^2)^{1/2}\Delta P(\lambda)\exp(-i\lambda x). \tag{5.4}$$

It is obvious that the left-hand side of Eq. (5.3) is identical with that of Eq. (3.10) derived by B & M if we take into account the fact that the Helmholtz solution h_0 satisfies the homogeneous equation (5.3). Hence we should have $R(x)/\varrho = -2i\omega V_s(x)H(x)$ if both methods are to be equivalent. Moreover, we know from Eqs. (3.10) and (3.11) that a solution $h = h_0+h_c$ complying with the condition $h = 0$ for $x \leqslant 0$ requires $R(x) = 0$ for $x \leqslant 0$.

To interpret the function $R(x)$ of Eq. (5.4), we note that we obtain with Eqs. (5.2) and (4.5)

$$(\lambda^2+\varepsilon^2)^{1/2}(A-B) \equiv (\lambda^2+\varepsilon^2)^{1/2}\Delta P = -\lim_{y\to+0}\frac{\partial P_1}{\partial y} - \lim_{y\to-0}\frac{\partial P_2}{\partial y}. \tag{5.5}$$

From this and Eq. (5.4) we find that

$$R(x) = -\lim_{y\to+0}\frac{\partial p_c}{\partial y} - \lim_{y\to-0}\frac{\partial p_c}{\partial y}. \tag{5.6}$$

This corresponds to the right-hand side of Eq. (2.21) if, in the presence of dipole sheets, only the "bounded" contribution of $\partial p_c/\partial y$ is taken. Hence $R(x) \neq 0$ for $x > 0$ requires that p_c has an antisymmetric contribution at $y = 0$.

In the case that we admit only a dipole sheet along the plate at $x < 0$ as source, it follows from Eqs. (3.10), (3.16) and (5.3) that we must have the identity

$$R(x) = 2\lim_{y\to0}\frac{\partial^2 W}{\partial x^2}, \tag{5.7}$$

where W is defined by Eq. (3.13). To show this Eq. (5.4) is re-written as

$$(5.8)\qquad R(x) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} d\lambda(\lambda^2+\varepsilon^2)\Delta P(\lambda)[\lambda^2+\varepsilon^2]^{-1/2}\exp(-i\lambda x)$$

$$= \lim_{y\to 0}\left[\varepsilon^2 - \frac{\partial^2}{\partial x^2}\right]\left\{(2\pi)^{-1/2}\int_{-\infty}^{\infty} d\lambda\Delta P(\lambda)(\lambda^2+\varepsilon^2)^{-1/2}\exp[-i\lambda x-(\lambda^2+\varepsilon^2)^{1/2}y]\right\}.$$

On the other hand, we have with Eqs. (2.11) and (2.17)

$$(5.9)\qquad \Delta P(\lambda) = (2\pi)^{-1/2}\int_{-\infty}^{\infty} d\xi\Delta p(\xi)e^{i\lambda\xi} = (2\pi)^{-1/2}\int_{-\infty}^{0} d\xi F_d(\xi)e^{i\lambda\xi}.$$

With Eq. (5.9) we obtain from Eq. (5.8)

$$(5.10)\qquad R(x) =$$

$$= \lim_{y\to 0}\left[\varepsilon^2 - \frac{\partial^2}{\partial x^2}\right]\left\{\frac{1}{2\pi}\int_{-\infty}^{0} d\xi F_d(\xi)\int_{-\infty}^{\infty} d\lambda(\lambda^2+\varepsilon^2)^{-1/2}\exp[i\lambda(\xi-x)-(\lambda^2+\varepsilon^2)^{1/2}y]\right\}.$$

The inner integral of Eq. (5.10) can be expressed by twice the modified Bessel function $K_0(z_1)$ where

$$(5.11)\qquad z_1 = \varepsilon[(\xi-x)^2+y^2]^{1/2}$$

(see GRADSHTEYN and RYZHIK [10], formula 3.961.2). Hence Eq. (5.10) becomes

$$(5.12)\qquad R(x) = \lim_{y\to 0}\left[\varepsilon^2 - \frac{\partial^2}{\partial x^2}\right]\left\{\frac{1}{\pi}\int_{-\infty}^{0} d\xi F_d(\xi)K_0\left(\varepsilon[(\xi-x)^2+y^2]^{1/2}\right)\right\}.$$

Since for $\varepsilon \to +0$ $K_0(z_1) \to -\ln(z_1/2)$, Eq. (5.12) becomes in the limit $\varepsilon \to +0$ with Eqs. (3.13) and (3.14)

$$(5.13)\qquad R(x) = 2\lim_{y\to 0}\frac{\partial^2}{\partial x^2}\left[\frac{1}{4\pi}\int_{-\infty}^{0} d\xi F_d(\xi)\ln[(\xi-x)^2+y^2]\right] = 2\lim_{y\to 0}\frac{\partial^2 W}{\partial x^2}.$$

Hence Eq. (5.7) is proven. This means that the Wiener–Hopf equation $F_+ = F_-$ of O & C is equivalent to the differential equation (3.10) of B & M. The result indicates the well-known fact that the Wiener–Hopf technique cannot yield a unique solution. In the present case, $F_+ = F_-$ only constitutes a differential equation which relates the vortex sheet displacement to the dipole sheet along the plate or, more precisely, to the normal velocity distribution along the free vortex sheet induced by that dipole sheet.

As a consequence of these results, the equivalence of both methods of O & C and B & M is verified.

6. Conclusion

The comparison of the different methods applied to the instability problem of a vortex sheet past a semi-infinite plate has shown that both methods are completely equivalent. The Wiener–Hopf equation of O & C is essentially the Fourier transform of the differential equation derived by B & M relating the vortex sheet displacement to the exciting transverse velocity along the free vortex sheet. It is proved that a nontrivial solution of the problem only exists if any kind of sources is present which leads to inhomogeneous terms in the original differential equations. These sources require, in the presence of the half-infinite long plate, an additional dipole sheet along the plate which generates a pressure difference across the plate and, as a consequence, a flow around the trailing edge of the plate. Sound waves or flow disturbances in the potential flow can also excite the vortex sheet. In the case of Orszag and Crow, only a special dipole sheet along the plate is the exciting source.

Appendix 1. Some properties of the Heaviside and Dirac functions

We consider the Heaviside function $H(x)$ and the Dirac function $\delta(x) = dH/dx$ as limits of continuous functions, e.g.:

(A1.1)
$$H(x) = \lim_{\varepsilon\to\pm 0}\left[\frac{1}{2}\pm\frac{1}{\pi}\arctan(x/\varepsilon)\right] = \begin{bmatrix} 0, & x<0, \\ 1/2, & x=0, \\ 1, & x>0, \end{bmatrix}$$

(A1.2)
$$\delta(x) = \lim_{\varepsilon\to\pm 0}\frac{1}{\pi}\frac{(\pm\varepsilon)}{x^2+\varepsilon^2} = \begin{bmatrix} 0, & x\neq 0, \\ \infty, & x=0. \end{bmatrix}$$

The properties of the Dirac function lead to the following result:

(A1.3)
$$\int_{-\infty}^{x} d\xi\, F(\xi)\,\delta(\xi) = H(x)F(0)$$

provided $F(x)$ is bounded and continuous at $x = 0$. Especially, for $F(x) = 1$ we have

(A1.4)
$$\int_{-\infty}^{x} d\xi\,\delta(\xi) = H(x)$$

which is equal to unity for $x > 0$. Hence we have equivalently $F(x)\,\delta(x) = F(0)\,\delta(x)$.

From Eqs. (A1.2) and for $F(x) = x^n$, $n \geqslant 1$ it follows from Eq. (A1.3) that

(A1.5)
$$x^n\delta(x) \equiv 0.$$

Furthermore, if $G = G(H(x))$ is a function discontinuous at $x = 0$ according to its argument $H(x)$, then we have

(A1.6)
$$I = \int_{-\infty}^{\infty} dx\,F(x)G(H(x)\,\delta(x) = F(0)\int_{-\infty}^{\infty} dx\,G\big(H(x)\big)\,dH/dx = F(0)\int_{0}^{1} dH\,G(H).$$

Especially for $G = H(x)$, Eq. (A1.6) yields $I = F(0)/2$.

The function $2H(x)-1$ is identical with the function $\mathrm{sgn}(x)$. Hence we have

(A1.7)
$$\frac{d}{dx}[\mathrm{sgn}(x)] = \frac{d}{dx}[2H(x)-1] = 2\delta(x).$$

Appendix 2. Source fields in the presence of a plate

Solutions of Eq. (3.3) can be obtained by conformal mapping of simple flow fields.

We consider the complex z-plane ($z = x+iy$) with the half-infinite plate at $y = 0$, $x < 0$. This can be conformally mapped in the upper half-plane of the Z-plane ($Z = X+ +iY$) by

$$Z = iz^{1/2}. \tag{A2.1}$$

The whole X-axis then corresponds to the "up-folded" plate.

A *simple point source* at $Z_s = X_s+iY_s$, $Y_s > 0$ with its mirror image in the Z-plane has the complex potential:

$$w(Z) = [Q(t)/2\pi][\ln(Z-Z_s)+\ln(Z-\bar{Z}_s)], \tag{A2.2}$$

where $Q(t)$ is the source strength and the overbar indicates the complex conjugate. It is obvious that for $Z = X =$ real w is also real. Then the X-axis is a streamline. If we go back to the z-plane, Eq. (A2.2) can be written with (A2.1):

$$w(z) = [Q(t)/2\pi]\left[\ln(z-z_s)+\ln[(\sqrt{z}+\sqrt{\bar{z}_s})/(\sqrt{z}+\sqrt{z_s})]+\ln(-1)\right]. \tag{A2.3}$$

Here the first part is the source at $z = z_s$ in the z-plane, while the other part takes the effect of the half-infinite plate into account. The complex velocity is

$$u-iv = dw/dz \tag{A2.4}$$

and the pressure is due to the linearized Bernoulli equation

$$p = -\varrho\mathcal{R}[\partial w/\partial t], \tag{A2.5}$$

where "$\mathcal{R}$" means: "real part of". The expressions for these quantities will not be given here. They have been used previously by B & M. However, it can be found that along the plate at $y = 0$, $x < 0$ there is a u-velocity jump, which corresponds to a vortex sheet, and a presssure jump, which corresponds to a dipole sheet with F_q as defined by Eq. (2.11).

We now consider an *X-dipole at the origin of the Z-plane*. Its complex potential is given by

$$w(Z) = [D(t)/2\pi]Z^{-1} \tag{A.2.6}$$

which can also be interpreted as the limit of two counter-rotating potential vortices on the Y-axis of equal amount of circulation Γ placed at $Z_0 = \pm iY_0$, if $Y_0 \to 0$, while $D = 2Y_0\Gamma$ remains constant. In the z-plane we get a "*folded*" *dipole* whose complex potential becomes with Eq. (A2.1)

$$w(z) = [D(t)/2\pi i]z^{-1/2}. \tag{A2.7}$$

The complex velocity is with $z = x+iy = r\ \exp(i\varphi)$ and with Eq. (A2.4)

$$u-iv = [iD(t)/4\pi]z^{-3/2} = [D(t)/4\pi]r^{-3/2}[\sin(3\varphi/2)+i\cos(3\varphi/2)]. \tag{A2.8}$$

We see that for $\varphi \to \pm\pi$ along the plate $v = 0$ and $u = \pm D(t)/4\pi r^{-3/2}$. The pressure becomes, due to Eqs. (A2.5) and (A2.7)

(A2.9) $$p = \left(\varrho \frac{dD}{dt}/2\pi\right) \mathcal{R}[iz^{-1/2}] = \varrho \frac{dD}{dt} \sin(\varphi/2)/(2\pi \sqrt{r})$$
$$= \varrho \frac{dD}{dt} \operatorname{sgn}(y) [(r-x)/2]^{1/2}/(2\pi r).$$

Here also a pressure jump exists along the plate at $y = 0$, $x < 0$ which yields, due to Eq. (2.11), a dipole sheet with the strength

(A2.10) $$F_d = \varrho \frac{dD}{dt} \frac{1}{\pi} [(|x|-x)/2x^2]^{1/2} = \varrho \frac{dD}{dt} [H(-x)/|x|]^{1/2}/\pi.$$

Note that this discontinuity is associated with the branch cut of the function $z^{1/2}$ located in the negative real axis, i.e., at the location of the plate. Furthermore, the y-component of the velocity, v, is due to Eq. (A2.8):

(A2.11) $$v = -[D(t)/4\pi]\mathcal{R}([z^{-3/2}].$$

Finally, we consider a *parallel flow in the Z-plane* with

(A2.12) $$w(Z) = W(t)Z.$$

This yields a "*folded*" *parallel flow* in the z-plane with Eq. (A2.1) which has also been considered by B & M:

(A2.13) $$w(z) = W(t)iz^{1/2}.$$

The complex velocity is with $z = x+iy = r\exp(i\varphi)$ and with Eq. (A2.4):

(A2.14) $$u-iv = [iW(t)/2]z^{-1/2} = W(t)[\sin(\varphi/2)+i\cos(\varphi/2)]/(2r^{1/2}).$$

At $\varphi \to \pm\pi$ along the plate again we have $v = 0$ and $u = \pm W(t)/(2r^{1/2})$. The pressure is due to Eqs. (A2.5) and (A2.13):

(A2.15) $$p = -\varrho \frac{dW}{dt} \mathcal{R}[iz^{1/2}] = \varrho \frac{dW}{dt} r^{1/2} \sin(\varphi/2) = \varrho \frac{dW}{dt} \operatorname{sgn}(y)[(r-x)/2]^{1/2}.$$

Again we have a pressure jump across the plate for $y = 0$, $x < 0$ which yields a dipole sheet due to Eq. (2.11) with the strength

(A2.16) $$F_d = 2\varrho \frac{dW}{dt} [(|x|-x)/2]^{1/2} = 2\varrho \frac{dW}{dt} [|x| H(-x)]^{1/2}.$$

The v-velocity becomes, from Eq. (A2.14),

(A2.17) $$v = -(W(t)/2)\mathcal{R}[z^{-1/2}].$$

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