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On the stability of third-grade fluids

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IT IS SHOWN that unlike the Navier-Stokes and second-grade fluids, fluids of grade three which are compatible with thermodynamics do not require the presence of the coefficient of viscosity μ in order that disturbances may die down asymptotically, to the null flow. Also, in marked contrast to the class of second-grade fluids, we find that the normal stress coefficient α_2 plays a less significant role that the normal stress coefficient α_1 in determining the characteristics of the stability of third-grade fluids.

Pokazano, że w przeciwieństwie do płynów Naviera-Stokesa i płynów rzędu drugiego, płyny trzeciego rzędu, które są niesprzeczne z zasadami termodynamiki, nie wymagają istnienia współczynnika lepkości μ do tego by zaburzenia mogły w nich zanikać asymptotycznie do zera. Podobnie, w odróżnieniu od płynów drugiego rzędu, stwierdzono, że współczynnik α_2 naprężenia normalnego odgrywa mniej istotną rolę niż współczynnik naprężenia normalnego α_1 w określaniu charakterystyki stateczności płynów trzeciego rzędu.

Показано, что в противовес к жидкостям Навье-Стокса и жидкостям второго порядка, жидкости трерьего порядка, которые не противоречат принципам термодинамики, не требуют существования коэффициента вязкости μ для того, чтобы возмущения могли в них асимптотически затухать к нулю. Аналогично констатировано, в отличие от жидкостей второго порядка, что коэффициент α_2 нормального напряжения играет менее существенную роль чем коэффициент нормального напряжения α_1 в определении характеристики устойчивости жидкостей третьего порядка.

1. Introduction

THE CAUCHY stress **T** for an incompressible homogeneous third-grade fluid is of the following form, [1]:

$$T = -p\mathbf{I}+\mu(\theta)\mathbf{A}_1+\alpha_1(\theta)\mathbf{A}_2+\alpha_2(\theta)\mathbf{A}_1^2+\beta_1(\theta)\mathbf{A}_3 +\beta_2(\theta)[\mathbf{A}_1\mathbf{A}_2+\mathbf{A}_2\mathbf{A}_1]+\beta_3(\theta)(\mathrm{tr}\,\mathbf{A}^2)\mathbf{A}_1, \tag{1.1}$$

where μ is the viscosity, α_1 and α_2 the normal stress moduli, β_1, β_2 and β_3 material moduli which resemble shear dependent viscosity — all these moduli being functions of temperature θ. The term $-p\mathbf{I}$ indicates the spherical stress due to the constraint of incompressibility and the tensors $\mathbf{A}_1$, $\mathbf{A}_2$ and $\mathbf{A}_3$ are the first three Rivlin-Ericksen tensors defined through

$$\mathbf{A}_1 = \mathbf{L}+\mathbf{L}^T,$$

and

$$\mathbf{A}_n = \dot{\mathbf{A}}_{n-1}+\mathbf{L}^T\mathbf{A}_{n-1}+\mathbf{A}_{n-1}\mathbf{L},$$

where

$$\mathbf{L} = \mathrm{grad}\,\mathbf{v}, \tag{(1.2)$_{1-3}$}$$

where **v** is the velocity, and the dot denotes material time differentiation.

While the constitutive assumption (1.1) may be considered as a third-order approximation to the response functional of a simple fluid in the sense of retardation (COLEMAN and NOLL [2]; TRUESDELL and NOLL [1]), it may also be considered as an exact model for some fluid, as it is done, when all the coefficients except the coefficient of viscosity are set to be zero; namely the classical Navier-Stokes fluid. A detailed study of the thermodynamics of a fluid whose constitutive relation is represented by Eq. (1.1), in the sense of an exact model, has been carried out by FOSDICK and RAJAGOPAL [3]. They find that if a fluid modeled by Eq. (1.1) is to be compatible with thermodynamics, that is meet the restrictions imposed by the Clausius-Duhem inequality and the assumption that the specific Helmholtz free energy be a minimum when the fluid is locally at rest(1), the material coefficients have to meet the following restrictions, [3]:

$$\begin{aligned} \mu(\theta) &\geqslant 0, \\ \alpha_1(\theta) &\geqslant 0, \\ \beta_1(\theta) = \beta_2(\theta) &= 0, \\ \beta_3(\theta) &\geqslant 0, \end{aligned} \tag{1.3}$$

and

$$-\sqrt{24\mu(\theta)\beta_3(\theta)} \leqslant \alpha_1(\theta)+\alpha_2(\theta) \leqslant \sqrt{24\mu(\theta)\beta_3(\theta)}.$$

For the purposes of our analysis, we shall assume that all the material moduli are constants. Thus, the Cauchy stress $\mathbf{T}$ is given by the form

$$\mathbf{T} = -p+\mu\mathbf{A}_1+\alpha_1\mathbf{A}_2+\alpha_2\mathbf{A}_1^2+\beta_3(\mathrm{tr}\mathbf{A}_1^2)\mathbf{A}_1. \tag{1.4}$$

The presence of the coefficient of viscosity μ is necessary if disturbances are ever to subside in both the Navier-Stokes fluid and the second-grade fluid, [4]. However, in this analysis we find that there is a departure from this necessity in third-grade fluids, wherein disturbances die down asymptotically even when the coefficient of viscosity "μ" is absent.(2) This decay is due to the presence of a higher order material coefficient β_3 which bears a certain resemblance to the notion of shear dependent viscosity which, by Eq. $(1.3)_4$, is required to be nonnegative if the constitutive assumption of a third-grade fluid is to be consistent with the restrictions imposed by thermodynamics.

Also, in contrast to the class of second-grade fluids, we find that the sign of the normal stress coefficient α_2 plays a less significant role than the sign of the normal stress coefficient α_1 in determining the stability of third-grade fluids. Since in the second-grade fluids $\alpha_1 = -\alpha_2$ such fluids with the coefficient $\alpha_2 > 0$ (i.e., $\alpha_1 < 0$) exhibit unstable characteristics (cf. [4], [5]), the same conclusions cannot be drawn from the normal stress coefficient α_2 being positive in third-grade fluids. In fact, third-grade fluids are stable irrespective of the sign of α_2 as long as $\alpha_1 > 0$, $\mu > 0$, $\beta_3 > 0$ and $|(\alpha_1+\alpha_2)| \leqslant \sqrt{24\mu\beta_3}$, [4]. However, we find that the magnitude of the sum of the normal stress coefficients $(\alpha_1+\alpha_2)$, and hence

(1) See Sect. 2 for a definition.

(2) While third-grade fluids whose coefficient $\mu = 0$ decay asymptotically, it is easy to establish that no initial disturbance can ever decay away in finite time. This is similar to the results established for second-grade fluids [4], and those for the Navier-Stokes fluid under special regularity assumptions [6, 7] and third-grade fluids where $\mu \neq 0$ [3].

the magnitude of α_2, plays an important but secondary role in determining the nature of the stability.

It has been shown, [4], that a fluid whose stress constitutive relation is given by Eq. (1.1) is stable when mechanically isolated(3) in the sense that the averaged stretching and the kinetic energy decay asymptotically to the null flow whenever the material moduli are such that $\mu > 0, \alpha_1 > 0, |\alpha_1+\alpha_2| < \sqrt{24\mu\beta_3}$ and $\beta_3 > 0$. We shall first investigate the stability of fluids modeled by Eq. (1.4) when $\mu = 0$. We shall study a very general class of flows and show that in such flows disturbances decay asymptotically to the null flow(4).

2. Stability

Let Ω denote the interior of a bounded three-dimensional domain which is occupied by an incompressible homogeneous fluid whose constitutive relation is given by Eq. (1.1) subject to the restrictions(5)

$$\mu = 0; \quad \alpha_1 \geqslant 0, \quad \text{and} \quad \beta_3 > 0 \tag{2.1}$$

in addition to those of Eq. $(1.3)_{1-5}$. The boundary $\partial\Omega$ is rigid and fixed and the fluid is assumed to adhere to the boundary, i.e.

$$\mathbf{v}(\mathbf{x}, t) = 0 \quad \text{on} \quad \partial\Omega\times[0, \infty]. \tag{2.2}$$

We shall assume that the body force $\mathbf{b}$ is derivable from a potential function. Then, it follows that the mechanical working(6)

$$\int_{\partial\Omega} \mathbf{T}\mathbf{n}\cdot\mathbf{v}da+\int_{\Omega} \varrho\mathbf{b}\cdot\mathbf{v}dv = 0 \tag{2.3}$$

for all $t \in [0, \infty)$, where $\mathbf{n}$ is the unit outer normal to $\partial\Omega$ and where ϱ is the mass density. Thus the power theorem has the form

$$\frac{d}{dt}\int_{\Omega}\frac{1}{2}\varrho|\mathbf{v}|^2dv+\int_{\Omega}\mathbf{T}\cdot\mathbf{L}dv = 0, \quad t \in [0, \infty). \tag{2.4}$$

We first observe that Eqs. $(1.3)_5$ and (2.1) imply that

$$(\alpha_1+\alpha_2) = 0. \tag{2.5}$$

(3) See Sect. 2 for a definition of mechanical isolation.

(4) The asymptotic decay of disturbances in a third-grade fluid whose coefficient of viscosity $\mu = 0$ is interesting in the light of the decay characteristics exhibited by third-grade fluids whose coefficient of viscosity $\mu \neq 0$, but whose normal stress coefficients α_1 and α_2 are such that $|\alpha_1+\alpha_2| = \sqrt{24\mu\beta_3}$. In this regard, see the remarks following Theorem 2.

(5) If $\beta_3 = 0$, then the fluid belongs to the class of second-grade fluids and the following analysis not valid.

(6) Thermodynamic processes which are such that the mechanical working vanishes over any period of time are said to be mechanically isolated during that period of time [8, 9]. We are thus interested in a body which has been subjected to arbitrary motions for all times $t \in (-\infty, 0)$ but which is mechanically isolated for all times $t \in [0, \infty)$.

Next, with the aid of Eqs. (1.4), (2.1) and (2.5), we can rewrite Eq. (2.4) in the form

$$\frac{d}{dt}\int_{\Omega} \varrho |\mathbf{v}|^2 dv + \frac{d}{dt}\frac{\alpha_1}{2}\int_{\Omega} |\mathbf{A}_1|^2 dv = -\beta_3 \int_{\Omega} |\mathbf{A}_1|^4 dv. \tag{2.6}$$

The above equality (2.6) can be rewritten in the form

$$\dot{E}(t) = -\frac{\beta_3}{\varrho}\int_{\Omega} |\mathbf{A}_1|^4 dv, \tag{2.7}$$

where

$$E(t) \equiv \int_{\Omega} |\mathbf{v}|^2 dv + \frac{\alpha_1}{2\varrho}\int_{\Omega} |\mathbf{A}_1|^2 dv. \tag{2.8}$$

$E(t)$ is a measure of the kinetic energy and the energy due to stretching of the fluid, $E(t)$ being zero if and only if $\mathbf{v} \equiv 0$ in Ω.

We are now in a position to prove the following:

THEOREM 1. *Let a third-grade fluid whose material coefficients meet* Eqs. $(1.3)_{1-5}$ *and* (2.1) *be mechanically isolated for all times* $t \in [0, \infty)$. *Then* $\dot{E}(t) < 0$ *for all times* $t \in [0, \infty) \ni : E(t) \neq 0$.

Proof. Since Eq. (2.2) is a sufficient condition for the validity of the Poincaré inequality, we obtain the following:

$$\int_{\Omega} |\mathbf{v}|^2 dv \leqslant C_p \int_{\Omega} |\operatorname{grad} \mathbf{v}|^2 dv = \frac{C_p}{2}\int_{\Omega} |\mathbf{A}_1|^2 dv, \tag{2.9}$$

where C_p is the domain-dependent Poincaré constant; the last equality $(2.9)_2$ being true by virtue of Eq. (2.2) and the divergence theorem.

From the definition of $E(t)$, Eq. (2.8), and Eq. (2.9) it follows that

$$E(t) \leqslant \left(\frac{\alpha_1 + \varrho C_p}{2\varrho}\right)\int_{\Omega} |\mathbf{A}_1|^2 dv. \tag{2.10}$$

Since ϱ and β_3 are strictly positive, the theorem follows from Eqs. (2.7) and (2.10).□

Next, we would like to investigate whether the disturbances in the fluid die down asymptotically. On rewriting Eq. (2.7) in the form

$$\dot{E}(t) + \frac{\beta_3}{\varrho}\int_{\Omega} |\mathbf{A}_1|^4 dv = 0, \tag{2.11}$$

and applying Holder's inequality, we obtain

$$\dot{E}(t) + \frac{\beta_3}{\varrho}\frac{1}{V(\Omega)}\left\{\int_{\Omega} |\mathbf{A}_1|^2 dv\right\}^2 \leqslant 0, \tag{2.12}$$

where $V(\Omega)$ is the volume measure of Ω. Thus, by virtue of Eq. (2.10), Eq. (2.12) can be rewritten as

$$\dot{E}(t) + \frac{\beta_3}{\varrho}\left(\frac{1}{V(\Omega)}\right)\left(\frac{2\varrho}{\alpha_1 + \varrho C_p}\right)^2 E^2(t) \leqslant 0. \tag{2.13}$$

Hence the following:

THEOREM 2 [7]. *Let a third-grade fluid whose coefficients satisfy the restriction* (1.3) *and* (2.1) *occupy the region within a fixed rigid container. Further, suppose that the motions are such that* Eq. (2.2) *is met. Then there exists a* $\lambda > 0$ *such that*

$$E(t) \leqslant E(0)\left\{\frac{1}{1+E(0)\lambda t}\right\}, \quad t \in [0, \infty),$$

where

$$\lambda \equiv \frac{\beta_3}{\varrho}\left(\frac{1}{V(\Omega)}\right)\left(\frac{2\varrho}{\alpha_1+\varrho C_p}\right)^2. \tag{2.14}$$

P r o o f. Follows trivially from integrating the inequality (2.13) and the definition of λ given in Eq. $(2.14)_2$. □

We now make the following remark on the roles played by the normal stress coefficients α_1 and α_2 in determining the stability of the class of third-grade fluids. It can be easily verified (cf. Appendix) that for a third-grade fluid whose coefficient of viscosity $\mu \neq 0$ and whose normal stress coefficients α_1 and α_2 are such that $|\alpha_1+\alpha_2| = \sqrt{24\mu\beta_3}$, one could pick initial data such that $\dot{E}(t)+\lambda E^2(t) > 0$ for all $t \in [0, \delta)$ where δ is some positive number and λ is given by Eq. (2.14). Thus, if two third-grade fluids which have the same material moduli β_3 and α_1, one of whose coefficient of viscosity μ is zero while the other whose coefficient of viscosity μ is *arbitrarily large* together with normal stress moduli α_2 such that $\alpha_1+\alpha_2 = \sqrt{24\mu\beta_3}$, are given identical initial disturbances, there is a future interval of time wherein the third-grade fluid with the arbitrarily large coefficient of viscosity is more disturbed than the one whose coefficient of viscosity is zero! However, this is not to say that such is the case for all future times. Even so, we see that the sum of the normal stress moduli $|\alpha_1+\alpha_2|$ determine to a certain extent the rate of decay of disturbances in the class of fluids under consideration.

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(7) The sign of the coefficient β_3 is crucial to our theorem. The following interesting result is a consequence of Eq. (2.13) if we assume that $\beta_3 < 0$. Firstly, we note by virtue of Eq. (2.11) that

$$\dot{E}(t) > 0, \quad t \in [0, \infty) \quad \text{provided } E(t) \neq 0.$$

Furthermore, by virtue of Holder's inequality, Eq. (2.11) can be rewritten as

$$\dot{E}(t) \geqslant \frac{|\beta_3|}{\varrho}\left(\frac{1}{V(\Omega)}\right)\left\{\int_{\Omega_t} |\mathbf{A}_1|^2 dv\right\}.$$

Thus, by Eq. (2.10), we obtain

$$\dot{E}(t) \geqslant \lambda E^2(t), \tag{A_1}$$

where

$$\lambda \equiv \frac{|\beta_3|}{\varrho}\frac{1}{(V(\Omega))}\left(\frac{2\varrho}{\alpha_1+\varrho C_p}\right)^2. \tag{A_2}$$

It can be easily verified that (A_1) implies that $E(t)$ does not exist for time $t > \dfrac{1}{\lambda E(0)}$ and that at $t = \dfrac{1}{\lambda E(0)}$, $E(t)$ becomes unbounded. Thus by Eqs. (2.8) and (2.10), both $\int_{\Omega_t} |\mathbf{v}|^2 dv$ and $\int_{\Omega_t} |\mathbf{A}_1|^2 dv$ become unbounded in finite time, a property which would be undesirable in any fluid which is to be modeled by Eq. (1.1).

It has been shown in [3] that if $\alpha_1 \leqslant 0$, and the other material moduli meet the restrictions of Eq. (1.3), disturbances continue to grow in mechanically isolated flows of fluids of grade three. However, it is also shown that the disturbances decay asymptotically to the null flow if $\alpha_1 \geqslant 0$ irrespective of the nature of α_2, provided $|\alpha_1+\alpha_2| < \sqrt{24\mu\beta_3}$. Our analysis shows that the magnitude of $|\alpha_1+\alpha_2|$, and thus the magnitude of α_2 does to some extent determine the character of the decay of the disturbances for the class of the third-grade fluids considered, the material moduli α_2, however, playing a much less significant role than the moduli α_1.

We conclude our analysis by showing that while the disturbances die down asymptotically for third-grade fluids with $\mu = 0$, there is always, however, some disturbance at $t \in [0, \infty)$. For any number $\gamma(t) \geqslant 0$, by Eqs. (2.7) and (2.8):

$$\dot{E}(t)+\gamma(t)E(t) = -\frac{\beta_3}{\varrho}\int_\Omega |\mathbf{A}_1|^4 dv+\frac{\gamma(t)\alpha_1}{2\varrho}\int_\Omega |\mathbf{A}_1|^2 dv+\gamma(t)\int_\Omega |\mathbf{v}|^2 dv.$$

Since

$$\int_\Omega |\mathbf{v}|^2 dv \geqslant 0$$

and

$$\gamma(t) \geqslant 0,$$

$$\dot{E}(t)+\gamma(t)E(t) \geqslant -\frac{\beta_3}{\varrho}\int_\Omega |\mathbf{A}_1|^4 dv+\frac{\gamma(t)\alpha_1}{2\varrho}\int_\Omega |\mathbf{A}_1|^2 dv.$$

Let us define

$$\gamma(t) \equiv \frac{\beta_3}{2\alpha_1}\frac{\int_\Omega |\mathbf{A}_1|^4 dv}{\int_\Omega |\mathbf{A}_1|^2 dv}.$$

Then,

$$\dot{E}(t)+\gamma(t)E(t) \geqslant 0.$$

The fact that $\gamma(t) \geqslant 0$, equality holding if and only if $E(t) = 0$, follows from the mean value theorem for integrals (Fleming 1977, p. 190). Thus we have established

THEOREM 3. *Let a third-grade fluid whose material coefficients meet* Eqs. $(1.3)_{1-5}$ *and* (2.1) *be mechanically isolated for all time* $t \in [0, \infty)$. *Then there exists a* $\gamma(t) > 0$ *such that*

$$\dot{E}(t)+\gamma(t)E(t) \geqslant 0.$$

Appendix

In this section, we show that for a third-grade fluid whose material moduli meet $(\alpha_1 + \alpha_2) = \sqrt{24\mu\beta_3}$, one could pick initial conditions governing the flow in a manner that it has $\dot{E}(t)+\lambda E^2(t) > 0$ for all $t \in [0, \delta)$ for some positive δ.

Consider initial conditions such that the first Rivlin-Ericksen tensor $\mathbf{A}_1(\mathbf{x}, t)$ is given through

(A.1)
$$\mathbf{A}_1(\mathbf{x}, 0) = \begin{pmatrix} \frac{1}{\sqrt{6}}\sqrt{\mu/\beta_3}+g(\mathbf{x}) & 0 & 0 \\ 0 & \frac{1}{\sqrt{6}}\sqrt{\mu/\beta_3}-g(\mathbf{x}) & 0 \\ 0 & 0 & -\frac{2}{\sqrt{6}}\sqrt{\mu/\beta_3} \end{pmatrix},$$

where g is a sufficiently differentiable real-valued function defined on the body, such that

$$0 \leqslant g(\mathbf{x}) < \varepsilon, \quad \forall x \in \Omega,$$

where ε is some positive number.

Firstly, we observe that

(A.2)
$$\operatorname{tr}\mathbf{A}_1(\mathbf{x}, 0) = 0, \quad \forall x \in \Omega,$$

(A.3)
$$|\mathbf{A}_1(\mathbf{x}, 0)|^2 = \frac{\mu}{\beta_3}+2g^2(\mathbf{x}),$$

(A.4)
$$\operatorname{tr}\mathbf{A}_1^3(\mathbf{x}, 0) = -\frac{1}{\sqrt{6}}(\mu/\beta_3)^{3/2}+\sqrt{6\mu/\beta_3}\,g^2(\mathbf{x}),$$

and

(A.5)
$$|\mathbf{A}_1(\mathbf{x}, 0)|^4 = \left(\frac{\mu}{\beta_3}\right)^2+4\frac{\mu}{\beta_3}g^2(\mathbf{x})+4g^4(\mathbf{x}).$$

If $E(t)$ is as defined in Eq. (2.8), then it can be shown (cf. [3]) that in a mechanically isolated flow of an incompressible third-grade fluid

(A.6)
$$\dot{E}(t) = -\frac{1}{\varrho}\int_{\Omega}[\mu|\mathbf{A}_1(\mathbf{x}, t)|^2+(\alpha_1+\alpha_2)\operatorname{tr}\mathbf{A}_1^3(\mathbf{x}, 0)+\beta_3|\mathbf{A}_1(\mathbf{x}, t)|^4]dv\},$$

for all $t \in [0, \infty)$.

When the material moduli of the third-grade fluid are such that

$$(\alpha_1+\alpha_2) = \sqrt{24\mu\beta_3},$$

it follows from Eq. (A.2) through (A.6), that

(A.7)
$$\dot{E}(0) = -\frac{18\mu}{\varrho}\int_{\Omega}g^2(\mathbf{x})dv-\frac{4\beta_3}{\varrho}\int_{\Omega}g^4(\mathbf{x})dv.$$

We would like to show that, for the initial conditions under question, for the λ as defined in Eq. (2.14),

(A.8)
$$\dot{E}(0)+\lambda E^2(0) > 0.$$

If we then define a function $H(t)$ through

(A.9)
$$H(t) \equiv \dot{E}(t)+\lambda E^2(t),$$

we would like to show that

$$H(0) > 0,$$

and the continuity of the function H would assure us that

$$H(t) > 0, \quad \forall t \in [0, \delta),$$

where δ is some positive number, and hence would validate our claim.

From the definition of the function $E(t)$ (2.8), it follows that

$$E(0) \geqslant \frac{\alpha_1}{2\varrho} \int_{\Omega} |\mathbf{A}_1(\mathbf{x}, 0)|^2 dv, \tag{A.10}$$

since

$$\int_{\Omega} |\mathbf{v}(\mathbf{x}, 0)|^2 dv \geqslant 0.$$

Thus

$$\dot{E}(0) + \lambda E^2(0) \geqslant \dot{E}(0) + \lambda \left\{ \frac{\alpha_1}{2\varrho} \int_{\Omega} |\mathbf{A}_1(\mathbf{x}, 0)|^2 dv \right\}^2 \tag{A.11}$$

$$= -\frac{18\mu}{\varrho} \int_{\Omega} g^2(\mathbf{x}) dv - \frac{4\beta_3}{\varrho} \int_{\Omega} g^4(\mathbf{x}) dv + \lambda \left\{ \frac{\alpha_1}{2\varrho} \int_{\Omega} |\mathbf{A}_1(\mathbf{x}, 0)|^2 dv \right\}^2,$$

by virtue of Eq. (A.7) and (A.10).

It then follows from Eq. (A.2) and (A.3), and the fact that μ, α_1, β_3, ϱ and λ are non-negative, that the inequality (A.11) can be rewritten as

$$\dot{E}(0) + \lambda E^2(0) \geqslant -\frac{18\mu}{\varrho} \varepsilon^2 V(\Omega) - \frac{4\beta^3}{\varrho} \varepsilon^4 V(\Omega) + \frac{\lambda \alpha_1}{2\varrho} \frac{\mu^2}{\beta_3^2} V^2(\Omega). \tag{A.12}$$

In obtaining the above inequality, we have also made use of the fact that the function g is non-negative. It can be very easily verified that an $\varepsilon > 0$ such that

$$\varepsilon \equiv \left[-\frac{9}{4} \frac{\mu}{\beta_3} + \sqrt{\left\{ \frac{\mu^2}{16\beta_3^2} (81 + \lambda \alpha_1 V(\Omega)) \right\}} \right]^{\frac{1}{2}}$$

satisfies the strict inequality (A.12). Thus we have established the fact that

$$H(0) > 0.$$

References

1. C. TRUESDELL, W. NOLL, *Nonlinear field theories of mechanics*, Flügge's Handbuch der Physik, III/3, Springer, Berlin-Heidelberg-New York 1965.
2. B. D. COLEMAN, W. NOLL, *An approximation theorem for functionals, with applications in continuum mechanics*, Arch. Rational Mech. Anal., 6, 355–370, 1960.
3. R. L. FOSDICK, K. R. RAJAGOPAL, *Thermodynamics and stability of fluids of third-grade*, Proc. Roy. Soc., London A. **339,** 351-377, 1980.

4. J. E. DUNN, R. L. FOSDICK, *Thermodynamics, stability and boundedness of fluids of complexity 2 and fluids of second-grade*, Arch. Ratl. Mech. Anal., **56**, 3, 191–252, 1974.
5. R. L. FOSDICK, K. R. RAJAGOPAL, *Anomalous features in the model of second order fluids*, Arch. Ratl. Mech. Anal., **70**, 145-152, 1979.
6. R. H. DYER, D. E. EDMUNDS, *Lower bounds for solutions of the Navier-Stokes equation*, Proc. London, Math. Soc., 3, 169–178, 1968.
7. HAJIMU OGAWA, *On lower bounds and uniqueness for the solutions of the Navier-Stokes equations*, J. Math. Mech., **18**, 445–452, 1968.
8. M. E. GURTIN, *Modern thermodynamics*, Notas De Matemetica Fisica, vol. II, Instituto de Matematica, Universidade Federal do Rio de Janeiro, 1972.
9. B. D. COLEMAN, *On the stability of equilibrium states of general fluids*, Arch. Ratl. Mech. Anal., **36**, 1–32, 1970.
10. W. FLEMING, *Functions of several variables*, 2nd edition, Springer, New York-Heidelberg-Berlin 1979.

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