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On necking phenomena and bifurcation solutions(*)

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VARIOUS theoretical approaches to necking phenomena in inelastic solids are discussed with particular emphasis on Hill's theory of bifurcation. Overall, largely intuitive, stability criteria are considered, the concept of eigenstates under all-round dead loading being shown as providing simple stability conditions for quasi-static deformation processes. In the bifurcation approach, the possible existence of non-trivial solutions (eigenmodes) to homogeneous boundary-value problems is investigated. A number of such solutions which appear successfully to model necking in various simple situations (for elastic/plastic materials) are presented. On extending the theory to other situations such as homogeneous biaxial tension, however, at least for the classical elastic/plastic material satisfying a Mises yield criterion and normality rule, the difficulty appears that the predicted critical stresses are unrealistically high from a physical point of view. Some ways of overcoming this difficulty are discussed, including modification of constitutive equation, modification of prescribed boundary conditions, and the investigation of the possible existence of shear-band modes, as often observed in thin metal strips when necking occurs. Finally, a number of criteria for localization of deformation based on the study of the growth of initial geometrical or material imperfections are presented.

Przedyskutowano różne teoretyczne podejścia do zagadnień powstawania szyjki w ciałach niesprężystych, ze szczególnym podkreśleniem teorii bifurkacji Hilla. Rozważono ogólne, w znacznym stopniu intuicyjne, kryteria stateczności wykazując, że koncepcja stanów własnych przy wszechstronnym obciążeniu zachowawczym przewiduje proste warunki stateczności dla quasi-statycznych procesów deformacji. Przy podejściu bifurkacyjnym zbadano możliwość istnienia nietrywialnych rozwiązań (modów własnych) jednorodnych zagadnień na wartości brzegowe. Przedstawiono pewną liczbę rozwiązań dobrze modelujących zjawisko powstawania szyjki w różnych prostych sytuacjach (dla materiałów sprężysto-plastycznych). Jednakże przy próbie rozszerzenia teorii na inne sytuacje, takie jak jednorodne dwuosiowe rozciąganie klasycznych materiałów sprężysto-plastycznych spełniających warunek plastyczności Misesa i zasadę normalności, powstaje taka trudność, że przewidywane naprężenia krytyczne przybierają wartości nierealistycznie wysokie z fizycznego punktu widzenia. Przedyskutowano pewne sposoby pokonania tej trudności, a w szczególności modyfikację równań konstytutywnych, modyfikacji danych warunków brzegowych oraz zbadanie możliwości istnienia modów w zakresie ścinania, które często obserwuje się przy powstawaniu szyjki w cienkich metalowych paskach. Przedstawiono wreszcie pewną liczbę kryteriów dla lokalizacji deformacji, opartych na badaniu wzrostu początkowych niedoskonałości geometrycznych lub materiałowych.

Обсуждены разные теоретические подходы к задачам возникновения шейки в неупругих телах, с особенным подчеркиванием теории бифуркации Хилла. Рассмотрены общие, в значительной степени интуитивные, критерия устойчивости, показывая, что концепция собственных состояний при всехсторонной консервативной нагрузке предсказывает простые условия устойчивости для квазистатических процессов деформаций. При бифуркационном подходе исследована возможность существования нетривиальных решений (собственных модов) однородных задач для граничных значений. Представлено некоторое количество решений хорошо моделирующих явление возникновения шейки в разных простых ситуациях (для упруго-пластческих материалов). Однако при попытке расширения теории на другие ситуации, такие как однородные двухосевые растяжения классических упруго-пластических материалов, удовлетворяющих условию пластичности Мизеса и принципу нормальности, возникает такая трудность, что предсказываемые критические напряжения принимают нереалистически высокие значения с физической точки зрения. Обсуждены некоторые способы обхода

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этой трудности, а в частности модификация определяющих уравнений, модификация данных граничных условий, а также исследование возможности существования модов в области сдвига, которые часто наблюдаются при возникновении шейки в тонких металлических полосах. Представлено наконец некоторое количество критериев для локализации деформаций, опирающихся на исследовании роста начальных геометрических или материальных неидеальностей.

1. Introduction

IN PRESENTING a survey of the present state of theories of instability and necking phenomena in inelastic materials, I am well aware of the excellent review given by STORÅKERS [1] to a previous Euromech Colloquium. I apologise in advance for the inevitable coverage of much of the same ground. The issues raised by Storåkers are still very relevant, as a glance at the recent literature will confirm. Here I shall attempt to give a reasonably self-contained account of the field, concentrating on applications involving predominantly tensile states of stress rather than those concerned with buckling under compression.

It is hardly necessary to stress the importance of an adequate theory of necking in plastic flow, since its occurrence in diverse practical problems of metal-forming is so familiar. The phenomenon is not necessarily one of instability. In practical terms the stability of deformation after necking depends on the stiffness of the loading device as well as the material and geometric properties which influence the rate of development of the neck and the rate of growth of cracks in the neck. For certain materials, e.g. polymers or hot glass, stable plastic flow may continue long after a neck has conspicuously formed. Bifurcation is probably a more appropriate concept in this context than stability That is, the phenomenon may be regarded as an abrupt transition from one, perhaps fairly homogeneous, mode of deformation to a different mode characteristic of necking, whether "diffuse" or "localized". The problem is then to formulate a criterion for bifurcation. However, intuitive notions of stability still seem to be of service, particularly in the materials science literature.

Here I have attempted to classify the various approaches under three main headings, i.e. (i) instability (ii) bifurcation (iii) analysis of imperfections. Each area will be surveyed in a somewhat selective manner, in view of the quantity of relevant work in the literature. In particular, I shall concentrate on necking under relatively simple states of stress, for the most part uniaxial or biaxial homogeneous states (apart from the discussion of imperfections, where the stresses are necessarily inhomogeneous from the outset). While sophisticated computer programs for evaluating critical stresses in rather complex situations are currently available (see, for example, [2]), it is felt that simple stress-states present a sufficient number of outstanding problems of interest for the purposes of this discussion.

2. Instability

2.1. Intuitive stability criteria

Our familiar starting-point is the uniaxial tensile test on a bar of unspecified material. Deformation is assumed to be initially homogeneous, at least within some gauge-length of the specimen. When will necking occur? The intuitive approach to be found in books such as [3] is that plastic flow involves both work-hardening, which is stabilizing, and softening (destabilizing). In the early stages of the flow the hardening outweighs the softening, and the flow becomes unstable when the two effects are in balance. Although this approach is apparently not concerned in a direct way with bifurcation, it is worth noting in passing that exclusion conditions (which preclude the possibility of bifurcation) are still commonly expressed in terms of critical values of hardening parameters.

Hardening is assumed here to be a material property, whereas softening may be either "material" or "geometrical". In a tension test, softening is attributed to the continuing reduction in cross-sectional area A, and so is geometrical. For an increment of strain $\delta\varepsilon$, with corresponding true-stress increment $\delta\sigma$, it is convenient to define the hardening corresponding to that incremental deformation as $A\,\delta\sigma$, and the softening as $-\sigma\delta A$, both relating to the change in load-carrying capacity of the bar. The critical state occurs when these quantities are equal, i.e. when

$$\frac{d\sigma}{d\varepsilon} = E_t = \sigma, \tag{2.1}$$

(assuming isochoric flow) where E_t is the tangent modulus. This is of course the condition for the load σA on the specimen to be an analytic maximum (with respect to, say, ε). In fact it was presumably the maximum load condition which motivated the above definitions of hardening and softening. This condition still effectively plays a major role in discussions on necking. Thus, high values of E_t, as for example in austenitic stainless steels transforming martensitically during deformation, are associated with high resistance to necking. Also, the stability of the necks formed in tensile tests on polymer specimens is regarded as characteristic of a load-extension curve which exhibits a load maximum followed by gradually increasing load with further extension, the increasing values of E_t inhibiting neck development.

For other deformation processes in which there may be no geometrical softening it may be possible to define "material" softening. In [3], for example, it is assumed that a constitutive equation may be expressed as an equation of state:

$$\sigma = \sigma(\varepsilon, \dot{\varepsilon}, T, \gamma, \ldots), \tag{2.2}$$

where σ is a representative stress (in the tensile test the longitudinal stress averaged over a cross-section), and the other parameters included are representative strain, strain-rate, temperature, and surface energy, respectively. The differential of stress is then given by

$$d\sigma = \frac{\partial\sigma}{\partial\varepsilon}\,d\varepsilon + \frac{\partial\sigma}{\partial\dot{\varepsilon}}\,d\dot{\varepsilon} + \frac{\partial\sigma}{\partial T}\,dT + \frac{\partial\sigma}{\partial\gamma}\,d\gamma + \ldots,$$

and material softening is associated with any term on the right-hand side which is negative. This approach may yield apparently useful physical insights, and detailed physical mechanisms may be investigated. In the case of uniaxial compression at high temperatures, JONAS, HOLT and COLEMAN [4] have considered the possibility that a point of instability may arise when the rate of material softening, i.e. negative hardening, exceeds the geometrical *hardening* due to increasing cross-sectional area. Here the approach seems to be in competition with well-established theories of elastic and plastic buckling. Buckling theories, of course, involve a three- (or perhaps two-) dimensional model.

The essentially one-dimensional model which incorporates a mean longitudinal stress σ still has its place, particularly in the treatment of strain-rate sensitive materials. For example, HART's theory [5] of instability under uniaxial tension has been quite influential. An account of this approach may be found in [6], which contains an equation of state and a rate-constitutive equation incorporating possible dependence on hydrostatic stress, and which also draws a distinction, following Hart, between the maximum load criterion and a criterion for localization of deformation under uniaxial tension. We shall return to this in our discussion of imperfections. For strain-rate insensitive deformation we remark that no such distinction is necessary, i.e. it is accepted by followers of this approach that the localization commences at the maximum load point.

The problem of how to generalize the maximum load condition to necking phenomena under more complex states of stress than uniaxial ones has been frequently discussed. For thin or thick cylindrical or spherical shells under internal pressure it is natural to take a maximum pressure criterion (see JOHNSON and MELLOR [7]). But for cylindrical tubes under combined internal pressure and longitudinal tension quite arbitrary criteria have been proposed. Another problematic case is the tensile specimen under lateral fluid pressure, where simple but misleading arguments indicated that the effect of the fluid pressure would be to delay the onset of necking, whereas bifurcation theory predicted no such effect (see [8, 9, 10]). Here we shall discuss the generalization of the maximum load condition implied by HILL's theory of eigenstates [11], and the following section will contain mathematical preliminaries relevant to the subsequent discussion of bifurcation theory.

2.2. Eigenstates

We assume here that the deformation processes to be considered are time-independent and isothermal. A state of deformation is an eigenstate if it admits the possibility of incremental deformation under all-round dead loading, i.e. with the loads on surface elements momentarily constant in magnitude and direction. This definition offers a convenient generalization of the basic notion of a tensile specimen at maximum load, and intuitively seems to retain the association with instability, at least in situations where "soft" loading devices are involved.

At an eigenstate, then, a possible velocity mode $\mathbf{v}$ and a corresponding nominal stress-rate field $\dot{n}_{ij}$ (taking the current configuration as reference) exist, satisfying the condition of stationary loads

$$\dot{n}_{ij}\delta\Sigma_i = \dot{T}_j\delta\Sigma = 0 \tag{2.3}$$

over the whole surface of the body, where $\dot{T}_j$ are the components on a background Cartesian frame of the nominal traction-rate $\dot{\mathbf{T}}$ (rate of increase of load per unit reference area), and $\delta\mathbf{\Sigma}$ is a vector surface area element of magnitude $\delta\Sigma$.

In addition the equations of continuing equilibrium (neglecting body forces)

$$\dot{n}_{ij,i} = 0 \tag{2.4}$$

must be satisfied, where the comma denotes partial differentiation, here with respect to the x_i-coordinate of the Cartesian frame.

The material is assumed to have a rate-constitutive equation of the form

$$\frac{\mathscr{D}\tau_{ij}}{\mathscr{D}t} = \frac{\partial V}{\partial \varepsilon_{ij}}, \tag{2.5}$$

where τ_{ij} is the Kirchhoff stress, $\mathscr{D}/\mathscr{D}t$ denotes the Jaumann (rigid-body) derivative, ε_{ij} is the strain-rate $(v_{i,j}+v_{j,i})/2$, and V is a potential function, homogeneous of second degree in ε_{ij}. It follows that

$$\dot{n}_{ij} = \frac{\mathscr{D}\tau_{ij}}{\mathscr{D}t} + \sigma_{ik}\omega_{jk} - \sigma_{jk}\varepsilon_{ik} = \frac{\partial U}{\partial v_{j,i}} \tag{2.6}$$

in terms of a potential function U of velocity gradient, satisfying

$$U = V + \frac{1}{2}\sigma_{ij}v_{k,i}v_{k,j} - \sigma_{ij}\varepsilon_{ik}\varepsilon_{jk}. \tag{2.7}$$

Here ω_{ij} is the local spin $(v_{i,j}-v_{j,i})/2$, and σ_{ij} is the Cauchy stress tensor.

The existence of such potential functions is convenient from a mathematical point of view so that rate-boundary value problems which arise naturally in quasi-static deformation processes are self-adjoint and the existence of corresponding variational principles is assured. The class of materials admitting such potentials apparently includes hyperelastic and elastic/plastic solids. (For more details of the basic framework, see [12] and, for a recent and more general view of the field, [13]).

The classical elastic/plastic material element, for example, is conventionally taken to deform incrementally at yield in accordance with

$$\frac{\mathscr{D}\tau_{ij}}{\mathscr{D}t} = \left(\mathscr{L}_{ijkl} - \frac{\alpha}{g}\lambda_{ij}\lambda_{kl}\right)\varepsilon_{kl}, \tag{2.8}$$

where

$$\alpha = \begin{cases} 0 & \text{if} \quad \lambda_{kl}\varepsilon_{kl} \leqslant 0 \text{ (elastic unloading)}, \\ 1 & \text{if} \quad \lambda_{kl}\varepsilon_{kl} > 0 \text{ (further plastic loading)}, \end{cases}$$

and $\mathscr{L}_{ijkl}$ are the elastic moduli.

Thus the corresponding function V is given by

$$V = \frac{1}{2}\frac{\mathscr{D}\tau_{ij}}{\mathscr{D}t}\varepsilon_{ij} = \frac{1}{2}\left\{\mathscr{L}_{ijkl}\varepsilon_{ij}\varepsilon_{kl} - \frac{\alpha}{g}(\lambda_{kl}\varepsilon_{kl})^2\right\}. \tag{2.9}$$

It is also convenient to define μ_{ij} and h by the equations

$$\lambda_{ij} = \mathscr{L}_{ijkl}\mu_{kl}, \quad g = h + \lambda_{ij}\mu_{ij};$$

then h can be regarded as a measure of hardening and μ_{ij} as the normal to the yield surface in stress-space if a normality rule is assumed.

For metal single crystals deforming by multi-slip, or for polycrystals at a yield vertex, it would be necessary to take V to be a different quadratic function of ε_{ij} in each of a number of different pyramidal domains of strain-rate space. The most suitable rate-constitutive equation for metals, especially when deep in the plastic range, is in fact still a subject of debate, and we shall return to this question later.

In an unstressed state it may be expected that U and V are positive definite functions. Then a path of deformation may exist such that for all states on the path the volume integral of U over the body satisfies

(2.10) $$\int U(v_{j,i})dV > 0$$

for arbitrary non-vanishing velocity gradient fields. States for which Eq. (2.10) holds are "quasi-stable" ([14]) in that the work of internal deformation exceeds to second order the work done by the dead loads in any virtual motion. (Such energy criteria for stability of continuous systems have been criticised by, for example, KNOPS and WILKES [15] on the basis that they may not correspond to any acceptable dynamical stability criterion.) The relevance of Eq. (2.10) to bifurcation will be discussed in Sect. 2.

In order that Eq. (2.10) be not violated by rigid-body rotations, the relevant stress--states under consideration must satisfy the three inequalities

(2.11) $$\bar{\sigma}_1+\bar{\sigma}_2 > 0, \quad \bar{\sigma}_2+\bar{\sigma}_3 > 0, \quad \bar{\sigma}_3+\bar{\sigma}_1 > 0,$$

where $\bar{\sigma}_1, \bar{\sigma}_2, \bar{\sigma}_3$ are the principal values of the mean stress tensor $\bar{\sigma}_{ij} = \int \sigma_{ij}dV/$(volume of body). Thus we are concerned here with predominantly *tensile* states of stress. Note that a state of homogeneous uniaxial tension does not strictly satisfy Eq. (2.11), but may remain in the set of stress-states to be considered if we agree in this particular case to exclude rigid-body rotations about the tensile axis from class of admissible fields in Eq. (2.10), on the grounds that such rotations have a neutral effect.

Suppose that we first consider "linear" materials for which U is simply a quadratic function of velocity gradients. A path of deformation on which Eq. (2.10) holds *may* terminate in a state at which the inequality

(2.12) $$\int U dV \geqslant 0$$

holds, the equality holding for some non-trivial velocity gradient field. Since such a field then minimizes $\int UdV$, the calculus of variations can be used to show that it is an *eigenmode*, the associated stress-rate satisfying Eqs. (2.3) and (2.4). The state is then called a *primary eigenstate*. As will be discussed further in Sect. 2, there can be no bifurcation of the deformation path under a variety of loading conditions *before* the primary eigenstate is reached.

Primary eigenstates certainly exist for homogeneous bodies under uniform stress--states, for which Eq. (2.12) is completely equivalent to

(2.13) $$U \geqslant 0$$

for non-trivial velocity gradient fields. The primary eigenstate is reached when the quadratic form U becomes positive semi-definite. Since the coefficients of this quadratic form involve only material moduli and stresses, it is evident that in this case the primary eigenstate is independent of the specimen geometry.

For piecewise-linear materials as in (2.8), HILL's procedure ([12]) was to decompose U into $Q+R$, where Q is a single quadratic function and R is a convex function of ε_{ij}. In particular, we choose for the elastic/plastic material Eq. (2.8).

$$Q = \frac{1}{2}\left\{\mathscr{L}_{ijkl}\varepsilon_{ij}\varepsilon_{kl} - \frac{1}{g}(\lambda_{kl}\varepsilon_{kl})^2\right\} + \frac{1}{2}\sigma_{ij}(v_{k,i}v_{k,j} - 2\varepsilon_{ik}\varepsilon_{jk}),$$

and

$$R = \begin{cases} 0 & \text{if} \quad \lambda_{kl}\varepsilon_{kl} > 0, \\ \dfrac{1}{2g}(\lambda_{kl}\varepsilon_{kl})^2 & \text{if} \quad \lambda_{kl}\varepsilon_{kl} \leqslant 0. \end{cases}$$

Primary eigenstates of U and Q then coincide if the directions of the eigenmodes for Q are chosen to make R vanish. It is thus sufficient to look for the eigenstates of the "comparison linear solid" given by the velocity gradient potential Q (whose incremental moduli are the relevant plastic moduli for both the "loading" and "unloading" conditions).

Hill found that for homogeneous rigid/plastic bodies under a uniform triaxial state of stress satisfying Eq. (2.11) the condition excluding eigenstates (assuming a Mises yield condition and normality rule) could be expressed as

$$h > \sigma + \frac{(\sigma_1-\sigma)^3+(\sigma_2-\sigma)^3+(\sigma_3-\sigma)^3}{(\sigma_1-\sigma)^2+(\sigma_2-\sigma)^2+(\sigma_1-\sigma)^2}, \tag{2.15}$$

where $\sigma = (\sigma_1+\sigma_2+\sigma_3)/3$. (It should be stated that while rigid/plastic materials do not strictly admit the potentials U or V, eigenstates can still be defined as above). Primary eigenstates exist when the equality holds in Eq. (2.15), which for uniaxial tension ($\sigma_2 = \sigma_3 = 0$) implies

$$h = \frac{2}{3}E_t = \frac{2}{3}\sigma_1, \tag{2.16}$$

i.e. the maximum load condition, and for biaxial tension ($\sigma_3 = 0$) implies

$$h = \frac{1}{6}\frac{(\sigma_1+\sigma_2)(4\sigma_1^2-7\sigma_1\sigma_2+4\sigma_2^2)}{(\sigma_1^2-\sigma_1\sigma_2+\sigma_2^2)}. \tag{2.17}$$

The latter result had in fact been previously obtained by SWIFT [15] on the basis that for instability the loads in the directions of σ_1 and σ_2 should be simultaneously stationary. However, as we shall see, it does not follow that at an eigenstate both loads are necessarily stationary with respect to some monotonically increasing deformation parameter in an actual experiment.

Essentially the same results were obtained by MILES [16] for elastic/plastic solids on the assumption that the elastic moduli were much greater than the plastic moduli. The eigenmodes existing in states (2.16) or (2.17) have homogeneous strain-rate fields coaxial with the stress. For a non-classical constitutive equation such as that corresponding to a Hencky "deformation theory", the value of the plastic modulus for incremental shearing might be considerably decreased from its classical value (the elastic shear modulus), and there would be the possibility of a shearing eigenmode at an earlier stage in the deformation process.

The practical significance of the above eigenstates is not always straightforward to assess. With "soft" loading devices, applying dead loads incrementally, eigenstates are presumably critical if a load maximum is involved. The eigenstate (2.17), however, is not necessarily associated with a load maximum. If the tensile loads L_1, L_2 acting on two opposite pairs of faces of a rectangular body in biaxial tension are increased proportionally, then naturally they both attain a maximum simultaneously at the eigenstate (2.17). For a different loading history, e.g. one in which the stresses are increased proportionally, the increments in L_1 and L_2 may have opposite signs at the eigenstate, implying that one load is still increasing while the other is decreasing beyond a maximum. In fact, a simple rigid/plastic Mises analysis shows that at the eigenstate (2.17), unless L_1 and L_2 reach a maximum simultaneously, we have

$$\frac{d(\ln L_1)}{d(\ln L_2)} = \frac{\alpha(1-2\alpha)}{(2-\alpha)}, \tag{2.18}$$

where $\alpha = \sigma_2/\sigma_1$. This is negative if $\frac{1}{2} < \alpha < 2$.

The experimental results of NEGRONI and THOMSEN [17] on the biaxial stretching of aluminium sheet specimens indicated that tensile instability did occur roughly at the eigenstate (2.17), but that a more satisfactory criterion (which they called Dorn's criterion) was that based on one of the load parameters L_1, L_2 first attaining a maximum. It should be noted, however, that a number of factors, including the particular choice of yield surface, will effect the calculation of the primary eigenstate. TADROS and MELLOR [18, 19] have recently applied a combination of the Swift stability criterion (2.17) and the analysis of Marciniak and Kuczynski (to be discussed later) to the problem of limit strains in the in-plane stretching of sheet metal, and have compared the theory with experimental results on steel, aluminium, and brass sheets. Limit strains were taken here to be the strains at fracture rather than at the onset of necking.

The situation is further complicated by detailed consideration of the loading conditions. For loading by *fluid pressure*, Eq. (2.15) is not appropriate. The existence of primary eigenstates under stationary fluid pressure has been considered by MILES [10] with reference to the above-mentioned problem of the tensile specimen under lateral fluid pressure. HILL and MILSTEIN, in a series of papers on the behaviour of single crystals at large strains with particular emphasis on stability (see, for example, [20, 21]), have investigated the existence of primary eigenstates under all-round fluid pressure loading (tensile or compressive). The corresponding eigenmodes involve either incremental uniform dilation or shearing. Again the body may be regarded as quasi-stable up to the attainment of the primary eigenstate.

The theoretical existence of primary eigenstates thus continues to be of interest from a stability point of view. The stability criterion always involves specifying the behaviour of the surface tractions, and so the loading conditions relevant to a particular problem must be carefully considered. The other principal difficulty is the specification of the appropriate constitutive equation, i.e. of the potentials U and V.

3. Bifurcation

3.1. The rate-problem

First we review the general analysis due to HILL [12, 13] of the quasi-static isothermal deformation of time-independent materials, with particular reference to uniqueness. This work, now well-known, involves the formulation of mathematical boundary-value problems, and leads to variational principles which have been fruitful in yielding finite--element procedures (e.g. [22]) for applying the theory to problems of practical interest.

The configuration of a body is supposed to be completely known at a certain instant during a deformation process, together with the distributions of stress and strain within the body, surface tractions, and the distribution of all relevant material parameters. To determine the incremental deformation of the body, we suppose firstly that nominal traction--rates $\dot{\mathbf{T}}$ are prescribed on a part of the surface Σ_T with velocities $\mathbf{v}$ on the remainder Σ_V (or perhaps complementary components of $\dot{\mathbf{T}}$ and $\mathbf{v}$ on a part of the surface). A "rate"--boundary value problem is thereby established in which the field equations are, combining Eqs. (2.4) and (2.6),

$$\left(\frac{\partial U}{\partial v_{j,i}}\right)_{,i} = 0, \tag{3.1}$$

while the boundary conditions are

$$\begin{aligned} \dot{n}_{ij} n_i = \frac{\partial U}{\partial v_{j,i}} n_i \text{ prescribed on } \Sigma_T, \\ v_i \text{ prescribed on } \Sigma_V, \end{aligned} \tag{3.2}$$

where $\mathbf{n}$ is a unit outward normal on Σ_T. The possibility of two velocity solutions $\mathbf{v}^{(1)}$, $\mathbf{v}^{(2)}$, with corresponding stress-rate fields $\dot{n}_{ij}^{(1)}$, $\dot{n}_{ij}^{(2)}$, is considered. These fields must satisfy the equation

$$0 = \int (\dot{n}_{ij}^{(1)} - \dot{n}_{ij}^{(2)})(v_j^{(1)} - v_j^{(2)}) n_i d\Sigma = \int \Delta\left(\frac{\partial U}{\partial v_{j,i}}\right) \Delta(v_j) n_i d\Sigma,$$

where Δ denotes the difference $(\ \ )^{(2)} - (\ \ )^{(1)}$.

Transforming by the Divergence Theorem gives

$$\int \Delta\left(\frac{\partial U}{\partial v_{j,i}}\right) \Delta v_{j,i} dV = 0.$$

If the material is incrementally linear, with

$$\dot{n}_{ij} = c_{ijkl} v_{l,k} \tag{3.3}$$

and

$$U = \frac{1}{2} c_{ijkl} v_{j,i} v_{l,k}, \tag{3.4}$$

where the "pseudomoduli" c_{ijkl} are instantaneously constant (independent of $v_{j,i}$), the above integral becomes

$$\int U(\Delta v_{j,i}) dV = 0.$$

Uniqueness is then guaranteed whenever

$$\int U(v_{j,i}) dV > 0 \tag{3.5}$$

for arbitrary (piecewise differentiable) velocity fields vanishing on Σ_V.

A connection between uniqueness and the "quasi-stability" criterion (2.10) is now apparent. Satisfaction of Eq. (2.10) implies that of Eq. (3.5). Thus for states of stress satisfying Eq. (2.11) a primary eigenstate given by the inequality (2.12) will provide lower bounds for the stresses (or an upper bound for the hardening parameter) at which bifurcation (failure of uniqueness) is possible under the given boundary conditions. This simple bounding property of the primary eigenstates is associated only with predominantly tensile stress-states.

Elastic/plastic solids are only piecewise linear, but on applying the above theory with Q as in Eq. (2.14) instead of U it follows that bifurcation in a homogeneous Mises rigid/plastic or elastic/plastic solid in a uniform triaxial state of stress satisfying Eq. (2.11) cannot occur under the given boundary conditions before the state given by equality in Eq. (2.15). Indeed, such materials admit trivial bifurcations under boundary conditions of prescribed nominal traction-rate on the whole surface as soon as the inequality in Eq. (2.15) is reversed. For example, a bar under uniaxial tension can continue to deform plastically or else unload elastically under negative load increments at any point after the attainment of maximum load, and similar modes are available in biaxial and triaxial stress-states. Necking modes are not specifically involved here, and in order to generate them further attention must be paid to the detailed boundary conditions. Restrictions on velocities are required.

A path of deformation for which Eq. (3.5) holds at each point, assuming now that Σ_V does not vanish, may terminate in a state for which

$$\int U(v_{j,i})dV \geqslant 0 \tag{3.6}$$

in the same class of velocity fields, the equality holding for some velocity field vanishing on Σ_V. Such a mode then minimizes $\int U dV$, and by the calculus of variations it again follows that this mode is an eigenmode satisfying Eq. (3.1) and the *homogeneous* boundary conditions. The velocity solution to the rate-problem is then undetermined to within an arbitrary multiple of the eigenmode. For plastic materials, if we assume that there is a "fundamental solution" for which continued plastic loading takes place everywhere, it is, more precisely, a question of adding a sufficiently small multiple of the eigenmode so that the loading condition is nowhere violated (consistent with the approach of SHANLEY [21]). The arbitrariness of the bifurcation mode would presumably be resolved by the specification of higher-order rate problems at the given instant, involving more detailed specification of the quasi-static loading conditions (nominal traction-accelerations, etc.).

In applications of the theory it is normally assumed that primary eigenstates given by Eq. (3.6) exist. This is not necessarily the case, however. In an analysis of plane-strain bifurcation for a rather general class of materials (which includes incompressible elastic//plastic solids), HILL and HUTCHINSON [24] demonstrated the existence of certain states, which, while not actually eigenstates, are points of accumulation for a spectrum of eigenstates. Thus it is possible to envisage a deformation process for which Eq. (3.5) holds everywhere up to a certain point which, though not itself an eigenstate, is critical in the sense that on proceeding with the process a finite distance along the deformation path an infinite number of eigenstates would be encountered. Similar critical states have also been noted by NEEDLEMAN [25]. It is possible that other pathological situations may exist.

3.2. Applications

We confine our attention here to applications involving inelastic materials, although finite elasticity has been a major area of interest. The first detailed application of Hill's theory of uniqueness in rigid/plastic solids was to tension specimens in plane strain [26]. Although rigid/plastic solids do not strictly admit potentials U and V, exclusion conditions sufficient for uniqueness in rigid/plastic bodies had been obtained [28] before the general theory outlined above had been established. The difficulty with rigid/plastic materials is that an extremely restricted class of modes is available in uniform stress-states when a normality rule is assumed. This class does not include axially symmetric necking modes for cylindrical bars under uniaxial tension.

Incorporation of elasticity into the material model does allow such modes to exist, and axisymmetric necking modes for circular cylinders of incompressible elastic/plastic material were found by HUTCHINSON and MILES [27], following earlier work by CHENG, ARIARATNAM, and DUBEY [28]. Boundary conditions were taken as follows:

(a) the lateral surface of the bar traction-free ($\dot{\mathbf{T}} = 0$);

(3.7) (b) on the plane ends of the bar, zero components of $\dot{\mathbf{T}}$ tangential to the surface (shear-free conditions) and prescribed (constant) axial component of velocity.

Similar boundary conditions had been adopted in [26]. A mode consisting of uniform stretching is always available.

Incompressibility permits the introduction of a function $\Phi(r, z)$ such that

$$v_r = -\frac{\partial \Phi}{\partial z}, \quad v_z = \frac{1}{r}\frac{\partial}{\partial r}(r\Phi), \tag{3.8}$$

and if $\mathbf{v}$ is to be an eigenmode satisfying homogeneous boundary conditions, Φ may be taken in the separable form

$$\Phi = \phi(r)\sin\left(\frac{k\pi z}{L}\right), \quad k = 1, 2, 3, \ldots, \tag{3.9}$$

where L is the cylinder length and the origin for the axial coordinate z is taken at one end. The field equations reduce to a fourth-order ordinary differential equation for $\phi(r)$, with the solution

$$\phi = \mathrm{Re}\left\{CJ_1\left(\frac{k\pi\varrho r}{L}\right)\right\}, \tag{3.10}$$

when J_1 is the Bessel function of the first kind of order one, and ϱ is a complex constant depending on the elastic and plastic moduli. An eigenvalue equation is obtained by applying the boundary conditions at $r = R$, the cylinder radius. Asymptotic analysis of the equation yields the following expression for the bifurcation stress σ_c:

$$\sigma_c = E_t^c + \frac{1}{8}\gamma^2\sigma_c + \frac{\gamma^4\mu}{192} + 0(\gamma^4\sigma_c, \gamma^6\mu), \tag{3.11}$$

where E_t^c is the tangent modulus at bifurcation, μ is the elastic shear modulus, and $\gamma = k\pi R/L$. The lowest critical stress is given by $k = 1$, corresponding, perhaps surprisingly, to necking at one *end* of the bar, whereas necking at the centre (the case considered in [26] for a plane-strain specimen) corresponds to $k = 2$.

It can be seen that the necking mode is initiated *after* the maximum load point (at which Eq. (2.1) holds), the delay depending on the geometry of the bar (being shorter the smaller the ratio R/L). Uniqueness holds at least as far as the maximum load point. The delay is further shortened by considering the effect of decreasing tangent modulus after the maximum load point. Thus Eq. (3.11) seems to be a satisfactory result which conforms broadly to practical experience. Whether detailed agreement with experiment could ever be demonstrated, however, is an open question.

Approximate (upper-bound) solutions for a flat rectangular bar under uniaxial tension were obtained by MILES [16], showing much the same features as the cylindrical bar solution, provided that the specimen is sufficiently thin. Explicitly, the upper bound takes the simple form

$$\sigma_c = 2E_t^c\left(\frac{\sqrt{2}\alpha+\sin\sqrt{2}\alpha}{\sqrt{2}\alpha+3\sin\sqrt{2}\alpha}\right) \tag{3.12}$$

for sufficiently thin specimens, where $\alpha = k\pi b/L$, the length of the specimen being L and the width $2b$; k is an integer. For certain values of b/L the lowest value of σ_c is not obtained with $k = 1$.

The solution of COWPER and ONAT [26] for rigid/plastic rectangular blocks under plane strain was extended to elastic/plastic materials by ARIARATNAM and DUBEY [29], and subsequently in greater depth by HILL and HUTCHINSON [24], who carried out the analysis for a class of orthotropic, incrementally linear, incompressible materials with rate-constitutive equations independent of hydrostatic pressure. The equations are

$$\begin{gathered}\frac{\mathcal{D}}{\mathcal{D}t}(\sigma_{11}-\sigma_{22}) = 2\mu^*(\varepsilon_{11}-\varepsilon_{22}), \qquad \frac{\mathcal{D}\sigma_{12}}{\mathcal{D}t} = 2\mu\varepsilon_{12},\\ \varepsilon_{11}+\varepsilon_{22} = 0,\end{gathered} \tag{3.13}$$

where the two independent moduli μ and μ^* (functions of the deformation history, as usual) represent instantaneous moduli for shearing parallel to the axes of the rectangular block and at 45° to them, respectively.

A stream function $\psi(x_1, x_2)$ exists such that

$$v_1 = \frac{\partial\psi}{\partial x_2}, \quad v_2 = -\frac{\partial\psi}{\partial x_1}, \tag{3.14}$$

and in terms of ψ the field equations become

$$\left(\mu+\frac{1}{2}\sigma\right)\frac{\partial^4\psi}{\partial x_1^4}+2(2\mu^*-\mu)\frac{\partial^4\psi}{\partial x_1^2\partial x_2^2}+\left(\mu-\frac{1}{2}\sigma\right)\frac{\partial^4\psi}{\partial x_2^4} = 0, \tag{3.15}$$

where σ is the (uniaxial) tensile stress in the x_1-direction.

Equation (3.15) can be classified as (a) elliptic (b) hyperbolic (c) parabolic, according to the number of real roots ν_1/ν_2 of the equation

$$\left(\mu+\frac{1}{2}\sigma\right)\nu_1^4+2(2\mu^*-\mu)\nu_1^2\nu_2^2+\left(\mu-\frac{1}{2}\sigma\right)\nu_2^4 = 0 \tag{3.16}$$

is (a) zero (b) four (c) two. The elliptic regime is the most relevant for the incompressible elastic/plastic material, regarding Eq. (3.13) as defining the "comparison linear solid".

We might then expect μ to represent an elastic shear modulus and $4\mu^*$ to correspond to the tangent modulus, so that the maximum load point occurs when $\sigma = 4\mu^*$.

In the elliptic regime Eq. (3.16) has an exact solution in terms of functions of two complex variables, but it has not yet been possible apparently to exploit this using methods familiar in the theory of anisotropic elasticity. Hill and Hutchinson obtained separable solutions of the form

$$\psi = f(x_2)\sin\left(\frac{n\pi x_1}{L}\right), \quad n = 1, 2, 3, \ldots, \tag{3.17}$$

where L is the length of the block, using the boundary condition (3.7) of prescribed uniform longitudinal velocities and zero shearing tractions on the ends. Eigenvalue equations for the bifurcation stresses were found for each of the elliptic, hyperbolic, and parabolic regimes, for both symmetric and anti-symmetric modes (corresponding to odd and even functions $f(x_2)$, respectively). If $\mu > 2\mu^*$, the lowest bifurcation stress for a given geometry factor b/L (when b is the width) lies in the elliptic regime, and for small b/L is slightly greater than the stress at maximum load. Asymptotically,

$$\frac{\sigma_c}{4\mu^*} = 1 + \frac{1}{3}\gamma^2 + \frac{7}{45}\gamma^4 + 0(\gamma^6, \gamma^6\mu^*/\mu), \tag{3.18}$$

where $\gamma = n\pi b/2L$.

For the case $\mu < 2\mu^*$, however, there is the interesting result that the elliptic/parabolic boundary in parameter space (where $\sigma = 2\mu$) is the locus of points of accumulation of bifurcation points for both symmetric (n even) and anti-symmetric (n odd) modes. As soon as the boundary is crossed from the elliptic regime (where the exclusion condition holds in this case), modes of sufficiently short wavelength become available. Such solutions had previously been found by BIOT [30] in his work on "internal stability" in a rigidly confined medium.

In the hyperbolic and parabolic regimes the possibility of additional solutions of the field equations in the form of localized shear-bands exists. We shall discuss this presently in more detail.

Bifurcation under compressive stress (buckling) in the same situation was investigated by YOUNG [31]. The lowest bifurcation stress still occurs in the elliptic regime, and for sufficiently slender blocks the corresponding mode is anti-symmetric (Euler buckling). Solutions are again of the separable form (3.17).

The plane-strain problem has been extended by NEEDLEMAN [25] to deal with rather more general constitutive equations than Eq. (3.13), while retaining the mathematical convenience of incompressibility. His equations can be interpreted in terms of the total loading of an elastic/plastic solid for which a normality rule does not hold. Given the current interest in shear-band solutions, the intention here may be to make the hyperbolic or parabolic regimes more accessible, raising the possibility of the initiation of a localized shear-band mode *before* a "diffuse" necking mode.

Other situations in which the classical elastic/plastic model with smooth yield surface and normality rule apparently gives physically reasonable results are the spherical shell and long cylindrical shell under internal pressure (NEEDLEMAN [37] and CHU [33]). Here, if the rate of increase of internal pressure is prescribed, an eigenstate is reached when the pressure reaches a maximum with respect to, say, the internal radius. For thin-walled shells the critical states are as given in [7]. Under prescribed internal volume, however, bifurcation does not necessarily occur at the maximum pressure point. No simple exclusion condition precluding bifurcation before this point has apparently been found, except for rigid/plastic shells under certain conditions (MILES [34], STORÅKERS [35], STRIFORS and STORÅKERS [36]).

For elastic/plastic spherical shells under prescribed internal volume, the results in [32] show that a bifurcation mode involving thinning around a point and thickening at the opposite pole becomes available at an instant significantly beyond the maximum pressure point. This is true even for thin shells, for which the state of stress is essentially biaxial.

It is worth mentioning at this stage that when fluid pressure p acts on a surface with an instantaneous rate of change $\dot{p}$, the loading conditions take the form

$$\dot{n}_{ij}n_i = -\dot{p}n_j+p(v_{k,j}n_k-v_{k,k}n_j) \tag{3.19}$$

and that the relevant uniqueness functional for the boundary value problem in which $\dot{p}$ is prescribed on Σ_p and velocity $\mathbf{v}$ on the remainder Σ_v is, instead of $\int U dV$,

$$\int\left\{U-\frac{1}{2}\,p(v_{k,j}v_{j,k}-v_{k,k}v_{j,j})\right\}dV, \tag{3.20}$$

while that for the problem in which fluid pressures $p_1, p_2, \ldots, p_k$ and their rates are prescribed on the complete internal or external surfaces $\Sigma(^1)$, $\Sigma(^2)$, ..., $\Sigma(k)$, respectively, as in the case of the shells above, can be expressed as

$$\int\left\{U+\frac{1}{2}\,\sigma_{ij}(v_{j,k}v_k-v_{k,k}v_j)_{,i}\right\}dV. \tag{3.21}$$

The latter problem is still variational and the loading conservative, but the analysis is more difficult because of the presence of second derivatives of velocity components in the uniqueness functional.

Another application of the theory for which detailed calculations have been carried out [37] is necking in rotating elastic/plastic discs under increasing angular velocity, where it appears that for a sufficiently ductile material a nonaxially-symmetric bifurcation mode may occur before the instant at which the angular velocity attains a maximum with respect to, say, the radius of the disc.

In all the applications so far mentioned the bifurcation stresses seem to be roughly in accord with physical intuition. The theory has been less successful so far in dealing with problems of biaxial tension. The bifurcation stress calculated by DUBEY and ARIARATNAM [38] for a rectangular solid under equal biaxial tension is of the same order of magnitude as the elastic shear modulus, and thus physically unreasonable. The boundary conditions adopted were of prescribed normal velocity components and shear-free nominal traction-rates on the two opposite pairs of faces subjected to traction, enabling a separable solution (in the Cartesian coordinates x_1, x_2, x_3) to be obtained. A similar result holds for circular plates under prescribed radial velocity components around the edge, according to BRUHNS and THERMANN [39] and NEEDLEMAN and TVERGAARD [40]. Here the velocity solutions are separable in the polar coordinates r, θ, z. One might speculate about the possibility of non-separable solutions, and regard the above calculated bifurcation stresses as merely upper bounds to the true values. The gap between these upper bounds and the lower bounds given by (2.17) is, however, uncomfortably wide.

It is at this point that we make contact with the well-known paradox of plastic buckling, that, as has been commonly accepted since the 1940's, the predicted buckling stresses of the classical elastic/plastic model are for many problems highly unconservative compared with the more acceptable predictions of the so-called deformation theory due to Hencky. Much has been written on this subject (e.g. see HUTCHINSON [41]). The present, rather unsatisfactory, situation appears to be that computer programs for the calculation of buckling stresses, e.g. [2], provide for the optional use of either "J_2-flow theory" (basically the classical Prandtl-Reuss equations) or "J_2-deformation theory". The use of Hencky's equations, which lack a certain respectability, is often justified by appealing to the fact that the associated incremental behaviour corresponds to the fully-active loading of a material in which the current stress lies at a vertex on the yield surface(e.g. SEWELL [42]). From an experimental point of view the existence of such vertices is still an open question, although particular physical models of polycrystal deformation do predict the formation of a vertex at the loading point on the yield surface (HUTCHINSON [43]). Since Hencky's is a small strain theory, it is also necessary to specify the precise form that is to be used at finite strain.

The root of the problem seems to be that bifurcation modes from a state of uniaxial compression usually involve a substantial shearing component. The corresponding stress thus depends on the instantaneous shear modulus, which according to the classical theory retains its elastic value, but which is considerably reduced in value using deformation theory. A further point is that structures are often highly imperfection-sensitive as regards buckling. This factor may also help to resolve the "paradox", while leaving the theoretical bifurcation stress of dubious practical value. Similar considerations may apply in necking problems.

The possibility of varying the boundary conditions for biaxial tension was considered in [16] and an approximate (upperbound) solution was obtained for the bifurcation stresses in equal biaxial tension when "hard" loading devices act in one direction (prescribed uniform normal velocity components, with shear-free tractions) and "soft" loading devices in the perpendicular direction (prescribed nominal traction-rates). A separable solution is not available for this problem. The bifurcation stresses, while occurring beyond the eigenstate (2.17), are of a physically reasonable order of magnitude for sufficiently thin specimens. The upper-bound expression for σ_c is, explicitly,

$$\sigma_c = E_t^c\left(1+\frac{\sinh(2\alpha)}{2\alpha}\right), \tag{3.22}$$

where $\alpha = k\pi b/L$, $k = 1, 2, 3, \ldots$, if the specimen is thin enough. The sensitivity of the critical stress to the particular boundary conditions appears, therefore, to be large.

The simplest mathematical loading conditions corresponding to an elastic loading device would seem to be, as suggested by HILL [45], taking $(\dot{T}_j + K_{ij}v_i)$ to be prescribed on the loaded surface, when K_{ij} could represent the moduli of a distributed set of elastic springs performing the loading. The rate-problem is still variational, even for a set of n loading devices with different moduli $K_{ij}^{(s)}$, $s = 1, 2, \ldots, n$, acting on n different parts of the surface Σ_s. The appropriate functional is

$$\int U dV + \frac{1}{2} \sum_{s=1}^{n} \int K_{ij}^{(s)} v_i v_j d\Sigma_s, \tag{3.23}$$

whose variation is to be zero for bifurcation in the class of continuous, piecewise continuously differentiable vector fields $\mathbf{v}$. It is obvious that if all $K_{ij}^{(s)}$ are positive definite and the principal mean stresses satisfy Eq. (2.11), then bifurcation still cannot occur before the primary dead-loading eigenstate. Separable solutions for this class of problems are again unavailable, but upper bounds for critical stresses can be formulated in terms of a Rayleigh quotient. In the case of uniaxial tension, for example, taking $K_{ij} = K\delta_{i1}\delta_{j1}$ for simplicity, where x_1 is in the axial direction, it is easy to show that the critical stress lies between that at maximum load (the primary eigenstate for dead loading, corresponding to $K = 0$) and that for the hard loading device ($K \to \infty$) in which axial velocities are prescribed on the ends. Thus, as might be expected intuitively, a loading device with finite K_{ij} will serve to reduce the bifurcation stress from the above-mentioned unacceptably high values associated with hard loading devices.

It may be that specification of normal velocity components on a surface is, from a theoretical and possibly practical point of view, an over-severe restriction on the possible velocity modes in the classical elastic/plastic solid. On the other hand, the in-plane stretching experiments of AZRIN and BACKOFEN [46] do seem to involve rather stiff loading conditions, corresponding to high values of K_{ij}. It appears, therefore, that there are still serious difficulties in applying bifurcation theory directly to the problem of limit strains in sheet-metal forming.

3.3. Localized solutions

There is currently much interest in localized shear-band solutions, regarded as bifurcation on some fundamental path of deformation. It is suggested that such modes can serve as models for failure in over-consolidated clay soils [47], in rocks under compressive stresses [48], and in sheet metal under biaxial tension [49]. At this colloquium a similar approach to the phenomenon of localized shearing in single crystals has been presented by Dr. Asaro. The instability involved is regarded as essentially "material", or "constitutive", rather than "geometrical", and the modes contain discontinuities in velocity gradients.

The possibility of a discontinuity surface that does not move relative to the material (a "stationary discontinuity") exists for a material with the constitutive equation

$$\dot{n}_{ij} = c_{ijkl} v_{l,k}, \tag{3.24}$$

whenever there exist vectors $\mathbf{v}$ satisfying

$$\det C_{jl} = 0 \tag{3.25}$$

with $C_{jl} = c_{ijkl}v_i v_k$ (see [45]). If the matrix **C** is positive definite for all **v** at a point, the system of equations of continuing equilibrium.

$$(c_{ijkl}v_{l,k})_{,i} = 0 \tag{3.26}$$

is said to be strongly elliptic (SE) there, and no discontinuity surfaces are admissible when the SE condition holds everywhere. In the plane-strain bifurcation problem of [24], Eq. (3.25) reduces to Eq. (3.16), making allowances for incompressibility. Thus discontinuity surfaces are precluded in the elliptic regime, but not in the hyperbolic and parabolic regimes, where the corresponding straight lines $v_1x_1+v_2x_2$ = constant are characteristics, across which there can be discontinuities in velocity gradients (but continuity of traction-rates). Although the lowest bifurcation stress (when $2\mu^* < \mu$) corresponds to a diffuse mode in the elliptic regime, it is argued that shear-band solutions (with deformation concentrated between two characteristics) may become available soon afterwards on the path of uniform deformation (the precise moment depending on the behaviour of μ nad μ^* with continuing deformation), and that the shear-band might become the ultimate failure mode. Thus diffuse necking would be followed by localized necking and fracture, in accordance with observations on flat metal bars [3].

A possible objection to these discontinuous solutions is that, although the field equations (3.26) hold everywhere, the boundary conditions, a basic part of the deformation of the rate-problem, cannot be satisfied, in particular where the band meets the free surface. Writers have so far responded to this difficulty only in an intuitive way by regarding the band as "vanishingly thin". Alternatively one might regard the modes as solutions in an infinite medium for which detailed boundary conditions are not relevant. The theory is, however, not easy to reconcile with Hill's general analysis of bifurcation, and shear-band solutions receive no mention in [13]. The band analysis is, of course, distinct from the imperfection method of MARCINIAK and KUCZYNSKI [51], although it appears that there is some correspondence in the limiting case as the amplitude of the imperfection tends to zero (TVERGAARD [61]).

Discontinuous solutions for plane-stress problems have been given by STÖREN and RICE [49] in their model of localized necking in thin sheets. Here the c_{ijkl} represent the appropriate moduli for continued plastic loading. The classical elastic/plastic solid still presents difficulties, however. HILL [52] had previously shown that for rigid/plastic solids localized necking is possible in the hyperbolic regime, shear bands being in the characteristic directions, which are the lines of zero extension in the plane of the sheet. Thus localized necking when no such lines of zero extension exist, as in the case of equal biaxial tension, should not be possible. To permit shear-band formation, and to use this as a basis for the calculation of "forming-limit diagrams", Stören and Rice proposed a constitutive equation based on Hencky's deformation theory, regarded as equivalent to the incremental equations for a solid with a vertex on the yield surface at the current stress. The calculations seem to agree reasonably well with experimentally derived forming-limit diagrams. An alternative modification of the classical model involves non-normal strain increments to the current yield surface, a feature of the constitutive equations introduced, with the help of physical arguments, by RUDNICKI and RICE [48] to predict the failure of brittle rocks under compressive stresses.

Further discussion of this approach and applications for various constitutive equations may be found in [53] and [54]. An advantage over the formulation in terms of a rate-boundary value problem is that the critical states may be found by straightforward algebra once the constitutive equation has been formulated. For example, in plane stress the critical conditions may be obtained by assuming a shear-band solution, writing down the condition for continuity of nominal traction-rate across the band, and then putting equal to zero the 2×2 determinant of coefficients in the homogeneous linear equation obtained.

Another line of enquiry which it seems relevant to mention here bears on a remark by HILL [11] to the effect that the divergence between two quasi-static equilibrium paths "might grow from just a single point of the continuum." The coefficients c_{ijkl} are called semi-strongly elliptic (SSE) if they satisfy the condition

$$c_{ijkl}v_i v_k \eta_j \eta_l \geqslant 0 \tag{3.27}$$

for all $\mathbf{v}$, $\boldsymbol{\eta}$. According to Hadamard, it is necessary (at least for unconstrained materials) that this condition holds everywhere if the exclusion condition

$$\int U dV = \int \frac{1}{2} c_{ijkl} v_{j,i} v_{l,k} dV > 0 \tag{3.28}$$

is to hold for the "Dirichlet problem" in which $\mathbf{v}$ is specified over the entire surface. YOUNG [56] has extended a proof of this result by NOLL [55] to cover incompressible materials, for which Eq. (3.24) becomes

$$\dot{n}_{ij} = c_{ijkl} v_{l,k} + p\delta_{ij},$$

where p is an undetermined scalar field. The coefficient c_{ijkl} are assumed to be piecewise continuous. The proof involves dividing up the body into a finite number of cells, choosing a point 0 in the interior of a particular cell, and then considering appropriate continuous velocity fields which vanish outside the cell (thereby satisfying homogeneous boundary conditions). It is then shown that the exclusion condition (3.28) implies that the c_{ijkl} are SSE at 0. Thus, once the c_{ijkl} fail to be SSE at a point in the body, a bifurcation localized in the neighbourhood of that point might be possible.

4. Imperfection analysis

We conclude with a brief survey of current methods of assessing the influence of imperfections on localization and necking. Such approaches often appear completely unrelated to bifurcation theories, but generally involve some intuitive choice of stability criterion. In the field of elastic structural stability the general theory, initiated by Koiter, of the effect of unavoidable geometrical or material imperfections is well-advanced, offering a comprehensive account of bifurcation, imperfection sensitivity, and post-bifurcation behaviour. For plastic stability, the only comparable work seems to be that of HUTCHINSON [42]. A number of simpler approaches, however, deserve consideration.

In the materials science literature much discussion of instability under uniaxial tension is based on the work of HART [5], who proposed a stability criterion that takes account of the effect of strain-rate sensitivity. A specimen is assumed to have uniform cross-sectional area $A_0(t)$ at time t except for a non-uniform region where the minimum cross-sectional area is $A(t)$. The mean longitudinal stresses at these sections satisfy $\sigma A = \sigma_0 A_0$ at all times for equilibrium. Hart's linearized analysis yields

$$\frac{d}{dt}(\Delta A) = \frac{\dot{A}_0}{A_0}\frac{(m+\gamma-1)}{m}\Delta A, \tag{4.1}$$

where $\Delta A = A - A_0$, and γ and m are material parameters, expressible as $\gamma = \sigma^{-1}\partial\sigma/\partial\varepsilon$, $m = \dot{\varepsilon}\sigma^{-1}\partial\sigma/\partial\dot{\varepsilon}$ in terms of an equation of state $\sigma = \sigma(\varepsilon, \dot{\varepsilon})$. Instability is taken to occur when ΔA is instantaneously increasing in magnitude, giving the instability criterion (for m positive)

$$m+\gamma < 1. \tag{4.2}$$

If $m = 0$ (no strain-rate sensitivity), the instability point coincides with the maximum load point ($\gamma = 1$).

JONAS *et al.* [4] observed that Eq. (4.1) is valid only when the non-uniformities are mechanically imposed on an initially uniform specimen, and also proposed an alternative instability criterion that the strain difference $|\Delta\varepsilon| = |\varepsilon - \varepsilon_0|$ should be growing in magnitude. In fact this is equivalent to the criterion that the relative size $|\Delta A/A_0|$ of the area nonuniformity should be increasing. The instability condition is then

$$\gamma < 1, \tag{4.3}$$

independent of m. According to SAGAT and TAPLIN [57], however, neither of the conditions (4.2) and (4.3) agree with experimental results for rate-sensitive materials in that they considerably underestimate the critical strains at localization.

A linearized analysis for an initially non-uniform specimen (for which Hart's assumption that $\Delta\varepsilon = -\Delta A/A_0$ is not valid to first order) shows that

$$\frac{d}{dt}(\Delta A) + h(t)\Delta A = \frac{\gamma}{m}\dot{A}_0\eta, \tag{4.4}$$

where η is the initial relative non-uniformity $[1 - A(0)/A_0(0)]$, and $h(t) = \dot{\varepsilon}_0(m+\gamma-1)/m$; see HUTCHINSON and OBRECHT [58]. Thus it is not surprising that Eq. (4.2) is not a satisfactory condition for localization, since the characteristic time for growth of $|\Delta A|$ is $0(h^{-1}) = 0(\dot{\varepsilon}_0{}^{-1})$ when m is not too small. Therefore necking may develop very slowly. If m is small (and positive), a neck may develop more quickly, but the criterion simply reduces to the maximum load condition.

Hutchinson and Obrecht also carried out an exact three-dimensional linearized analysis on the particular example of a circular cylinder of power-law creeping material (defined by $\dot{\varepsilon}_{ij} = \frac{3}{2}\alpha\sigma_e^{n-1}s_{ij}$, where s_{ij} is the deviatoric stress and $\sigma_e = \left(\frac{3}{2}s_{ij}s_{ij}\right)^{1/2}$ is the effective stress), in order to provide a check on the validity of the above, essentially one-dimensional, approximation in which the three-dimensional constraint effect of the material on either side of a developing neck is neglected. They showed that Eq. (4.4) holds in this particular case provided that the wavelength of the initial (sinusoidal) nonuniformity in the specimen geometry is not too small.

HUTCHINSON and NEALE [59] have applied the "long-wavelength" approximation to the development of a neck in a bar of material satisfying $\sigma = K\varepsilon^N \dot{\varepsilon}^m$. An initial non-uniformity apparently develops slowly at first, but then rapid growth occurs, with the strain ε_0 in the uniform section eventually reaching a maximum with respect to the strain ε in the necking region. This condition ($d\varepsilon_0/d\varepsilon = 0$) can serve as a new criterion for instability or localization, corresponding essentially to a one-dimensional version of the criterion proposed by MARCINIAK and KUCZYNSKI [51] for failure in thin sheets under quasi-static biaxial deformation.

In the so-called M–K analysis a thickness inhomogeneity is assumed in the form of a narrow groove lying at right-angles to the direction of the greater (logarithmic) strain, say ε_1. A state of plane stress is assumed inside and outside the narrow band, so that, as in the "long-wavelength" approximation for uniaxial tension, the three-dimensional constraints acting on an incipient neck are neglected. If it is assumed that proportional straining takes place outside the band, with $\varepsilon_1/\varepsilon_2 = \dot{\varepsilon}_1/\dot{\varepsilon}_2 = \varrho$, a constant, and if quantities inside the band are denoted by $(\quad)^b$, we have, for compatibility and continuing equilibrium,

$$(\varepsilon_2)^b = \varepsilon_2 \quad \text{and} \quad (\sigma_1)^b t^b = \sigma_1 t,$$

where t denotes current thickness. Assuming incompressibility

$$\frac{\dot{t}^b}{t^b} = -\dot{\varepsilon}_1^b - \dot{\varepsilon}_2^b \ \text{ and } \ \frac{\dot{t}}{t} = -\dot{\varepsilon}_1 - \dot{\varepsilon}_2.$$

Then, with incremental behaviour given by

$$\dot{\sigma}_1 = L_{11}\dot{\varepsilon}_1 + L_{12}\dot{\varepsilon}_2,$$
$$\dot{\sigma}_2 = L_{21}\dot{\varepsilon}_1 + L_{22}\dot{\varepsilon}_2,$$

solving for $\dot{\varepsilon}_1^b$ shows that $d\varepsilon_1^b/d\varepsilon_1 \to \infty$ as $\sigma_1^b \to L_{11}^b$, so that the critical stress for localization is $t^b L_{11}^b/t$.

This approach has been regarded as equivalent to a bifurcation analysis in the limit $t^b/t \to 1$, when $\dot{\varepsilon}_1^b$ becomes indeterminate at the critical stress $\sigma_1 = L_{11}$. NEEDLEMAN [60], for example, has considered the problem of localization in pressurized spherical membranes, taking into account the curvature of the sheet. See also TVERGAARD [61], who showed that forming limit diagrams calculated on this basis for classical elastic/plastic solids are closer to those found experimentally if kinematic hardening is assumed rather than the usual isotropic hardening law.

The physical assumptions underlying the M–K method were examined by AZRIN and BACKOFEN [46], who showed that unrealistically large imperfections were required if the model was to predict limit strains corresponding to those observed experimentally. It is now commonly assumed that the simple geometrical imperfection of the model represents the overall (geometrical and material) imperfection of the sheet.

Another analytical method for following the growth of an imperfection is a regular perturbation analysis as employed in the investigation of the extreme imperfection-sensitivity of the cruciform column in buckling by BUDIANSKY and HUTCHINSON [62]. The method is valid only for small deviations from the basic state of stress, and for elastic/plastic solids it must be assumed that no elastic unloading occurs, so that effectively a "hypoelastic" material is considered. The perturbation expansion is not valid in the neighbourhood of a bifurcation point, when a singular perturbation analysis is required. However, the perturbation method can give an indication of the rate of growth of imperfections of different wavelengths (see, for example, NEEDLEMAN and TVERGAARD [40]).

Finally we must mention the powerful computational procedures, in particular finite element methods, so impressively employed in recent years in calculating the growth of imperfections in different situations. These methods do not of themselves provide criteria for instability and localization, but have been used to assess the validity of the various approximate theories and models discussed above.

5. Conclusion

Hill's bifurcation theory is now evidently well-established and, of the theories discussed, is the only one leading to critical stresses dependent on the overall dimensions of the specimen, a commonly observed feature of necking (see NADAI [63]). However, the theory seems to predict excessively high critical stresses for the classical elastic/plastic solid in a number of situations, both tensile and compressive. If the method is to serve as a basis for design purposes, the safest course at present seems to be to use a deformation-theory constitutive equation. Such constitutive models may be justified by appealing to the possible existence of vertices on the yield surface, and there are other constitutive equations, for example exhibiting non-normality, which are equally effective. A complete imperfection analysis may yet justify the use of the classical model, but the need for a relatively simple theory, which can be applied by engineers in practical situations, is apparent.

The band analysis and the M–K method produce values for critical stresses in a straightforward manner. HUTCHINSON and NEALE, in an important series of papers [64] on sheet necking under uniaxial and biaxial tension, have studied the validity of these approximations, as compared with the fully three-dimensional model, and have compared the predictions of these two approaches with experimental data for both the "J_2-flow theory" and a finite strain version of deformation theory. They point out, for equal biaxial tension, the large imperfection-sensitivity of the flow theory (using the M–K analysis), which does not even admit a localized bifurcation solution in this case, and conclude that the classical theory is unlikely to be able to predict localized necking in the range $\varepsilon_1/\varepsilon_2 = \varrho > 0$. The implications for computer programs being developed for studying particular sheet-metal forming operations are not encouraging. On the other hand, a deformation theory is unlikely to give satisfactiory predictions if the loading history leading to the current value of ϱ is strongly non-proportional.

The computer still has an important role to play, of course, in elucidating the effects of particular choices of constitutive equations and also of imperfections. Other important features such as plastic anisotropy, temperature, and, at a microscopic level, void growth, may also be incorporated. New insights on the nature of localization and instability may also come from experimental work. Current theories of necking may prove to be adequate if appropriate constitutive equations can be formulated.

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