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Similarity analysis of wave propagation problems in nonlinear rods

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SIMILARITY analysis of wave propagation problems in nonlinear rods are discussed from the view point of continuous groups of transformations. For the governing nonlinear partial differentia equations, the similarity variables and the characteristics of the equation are related at the wave front. A procedure leading to a similarity-characteristic relationship is developed which provides an additional condition at the wave front for the solution of the similarity representation. The similarity-characteristic relationship is derived for the problem of velocity impact of an inelastic rod and solutions for the wave propagation problem are obtained.

Omówiono analizę podobieństwa dla zagadnień propagacji fal w nieliniowych prętach z punktu widzenia ciągłych grup przekształceń. Dla podstawowego nieliniowego równania różniczkowego cząstkowego zmienne podobieństwa i charakterystyki równań związane są ze sobą na froncie fali. Opracowano procedurę prowadzącą do związków między podobieństwem i charakterystykami, stanowiących dodatkowy warunek, który na froncie fali spełniać powinno rozwiązanie problemu podobieństwa. Związek między zmiennymi podobieństwa a charakterystykami wyprowadza się z zagadnienia impulsu prędkości przyłożonego do pręta niesprężystego; otrzymuje się rozwiązanie problemu propagacji fal.

Обсужден анализ подобия для задач распространения волн в нелинейных стержнях с точки зрения непрерывных групп преобразований. Для основного нелинейного дифференциального уравнения в частных производных переменные подобия и характеристики уравнений связаны с собой на фронте волны. Разработана процедура приводящая к соотношениям между подобием и характеристиками составляющими дополнительное условие, которому на фронте волны должно удовлетворять решение задачи подобия. Соотношение между переменными подобия и характеристиками выводится из задачи импульса скорости, приложенного к неупругому стержню; получается решение задачи распространения волн.

Nomenclatures

- x coordinate along the axis of the rod (this is a Lagrangian coordinate, where x denotes the position of a particle in the initial unstrained state),
- t time,
- $u(x, t)$ particle displacement,
- $\sigma(x, t)$ nominal compressive stress, compressive stress is assumed positive,
- $v(x, t)$ particle velocity,
- $e(x, t)$ nominal compressive strain, compressive strain is assumed to be positive,
- μ, q material constants describing a nonlinear material,
- U, B column vectors,
- A $(n\times n)$ square matrix,
- λ^i eigenvalues or characteristics,
- η, ζ similarity variables,
- ε infinitesimal parameter,
- U^*, X, T infinitesimals of a continuous group of transformations,
- $D(t)$ distance to the wave front from the origin,
- ζ_w, η_w similarity coordinate at the wave front,

v_c, δ constants for the initial velocity condition,
ξ_1, ξ_2 the space variables obtained by integrating along the characteristics which locate the discontinuity.

1. Introduction

SIMILARITY analysis is essentially a procedure for finding transformations of independent variables, whereby a given system of partial differential equations and its associated auxiliary conditions are transformed into a system of ordinary differential equations and its auxiliary conditions. The method is applicable to linear as well as nonlinear partial differential equations. Recent methods of analysis based on group-theoretic motivations [1, 2] have placed similarity analysis on a former mathematical basis. However, the analysis of hyperbolic type of partial differential equations arising from wave propagation problems is complicated by the presence of the moving boundary condition. In this connection, SESHADRI and SINGH [3] suggested that invariance of a given system of partial differential equations of hyperbolic type under a continuous group of transformations would lead not only to a similarity variable but also to the characteristics along which a disturbance propagates. This fact has been used to establish a relationship between a characteristic and the similarity variable and to determine the numerical value of the similarity variable at the wave front. Application of this relationship to the problem of nonhomogeneous elastic rods was made by SINGH and BRAR [4].

In the present paper the relationship between similarity coordinate and characteristics is elaborated with reference to invariance of the equations under an infinitesimal group of transformations. Consequently, a method for determining the moving boundary condition in terms of the similarity coordinate is presented. This procedure leads to the similarity-characteristic relationship which is essential to the solution of the overall problem. The proposed method is applied to the similarity solution of the problem of velocity impact of a nonlinear rod.

2. Relationship between similarity coordinates and characteristics, group-theoretic motivation

A common form of the equation representing wave propagation phenomena in a one-dimensional rod is given by

$$M(u) \equiv \psi(x, t, u, u_x, u_t) u_{xx} - u_{tt} = 0, \tag{2.1}$$

where ψ is a function of the arguments shown in the parenthesis, x is the Lagrangian coordinate, t is the time, u is the particle displacement and the variables in the subscript denote differentiation. Equation (2.1) is a quasi-linear partial differential equation of the hyperbolic type.

Many problems in nonlinear wave propagation are expressed in terms of a system of first-order quasi-linear equations [5], thereby limiting the solution to specific physical situations. The quasi-linear form can be written as

$$U_t + AU_x + B = 0, \tag{2.2}$$

where U and B are $n\times 1$ column vectors and A is a $(n\times n)$ square matrix. The eigenvalues $\lambda^l, l = 1, \ldots, n$, of the equation $|A-\lambda I| = 0$, are real and distinct for totally hyperbolic systems. The characteristics are given by

$$C^{(l)}: \left(\frac{dx}{dt}\right)_l = \lambda^{(l)}. \tag{2.3}$$

Equation (2.2) is invariant under a group of transformations:

$$G_1: x \to \alpha x, t \to \alpha t, U \to U; \quad (-\infty < x < \infty; t > 0; \alpha > 0), \tag{2.4}$$

provided that both A and B are invariant under G_1. A similarity solution can now be written, as reported in an earlier work [6] as

$$U = U(\eta), \quad \eta = \eta(x, t), \tag{2.5}$$

where η is a similarity variable, dependent on the indepedent variables x and t. The invariant boundary condition at the wave front is $u(\eta = \eta_w) = g(\eta_w)$. Consequently, Eq. (2.3) would also be invariant under G_1 since it defines the $x-t$ locus of the wave front [5]. The solution described by Eq. (2.5) is usually limited to problems resulting from disturbances that are suddenly initiated and uniformly sustained leading to a centered simple wave situation. It may be remarked here that whenever a given nonlinear hyperbolic partial differential equation of a wave propagation problem such as Eq. (2.1) is expressed in terms of the first-order quasi-linear form, Eq. (2.2), the boundary conditions for which the problem is solvable, becomes restricted. This is a definite disadvantage of the first-order quasi-linear formulation. On the other hand, similarity analysis of the original equation, (2.1), using invariance under a group of transformations, would lead to other physical situations that cannot be obtained by solving the first-order quasi-linear equation (2.2).

In the solution of the similarity representation (the transformed system of ordinary differential equation and the boundary conditions), the numerical value of the similarity coordinate at the wave front is not readily known. A method is hereby proposed which determines the similarity-characteristic relationship, and further, establishes the numerical value of the similarity variable at the wave front.

Define a continuous group of infinitesimal transformations G_2:

$$\begin{aligned} \bar{u} &= u+\varepsilon U^*(x, t, u)+0(\varepsilon^2), \\ \bar{x} &= x+\varepsilon X(x, t, u)+0(\varepsilon^2), \\ \bar{t} &= t+\varepsilon T(x, t, u)+0(\varepsilon^2), \end{aligned} \tag{2.6}$$

where ε is an infinitesimal parameter, U^*, X and T are known as infinitesimals. Let the solution of Eq. (2.1) be $u = \theta(x, t)$. The solution satisfies the fixed auxiliary conditions

$$B_\alpha(u_x, u_t, u, x, t) = 0, \tag{2.7}$$

on prescribed curves

$$W_\alpha(x, t) = 0, \quad \alpha = 1, 2 \ldots m, \tag{2.7'}$$

and the condition at the moving boundary or the wave front

$$u\big(x = D(t); t\big) = 0. \tag{2.8}$$

The displacement being zero ahead of the wave front, $x = D(t)$.

Let the system of equations and auxiliary conditions, (2.1), (2.7) and (2.8), be invariant under the one-parameter infinitesimal group of transformation, (2.6). This implies that

a) $$M(\bar{u}) = M(u) = 0,$$

where $M(\bar{u})$ is obtained by replacing (x, t, u) by $(\bar{x}, \bar{t}, \bar{u})$.

b) $$B_\alpha(\bar{u}_{\bar{x}}, \bar{u}_{\bar{t}}, \bar{u}, \bar{x}, \bar{t}) = 0,$$

for the invariance of the auxiliary conditions, and

c) $$\bar{u}\big(\bar{x} = D(\bar{t}); \bar{t}\big) = 0,$$

at the wave front.

Assuming that a unique solution of $M(u) = 0$ exists,

$$u(x, t) = \bar{u}(\bar{x}, \bar{t}) = u\big(x, t, \theta(x, t), \varepsilon\big), \tag{2.9}$$

the invariant surface can be obtained by solving the following equation, obtainable from Eq. (2.9):

$$X(x, t, \theta)\frac{\partial\theta}{\partial x} + T(x, t, \theta)\frac{\partial\theta}{\partial t} = U^*(x, t, \theta). \tag{2.10}$$

The characteristic equation corresponding to Eq. (2.10) obtained from invariance of Eq. (2.1) under the transformations (2.6) is

$$\frac{dx}{X(x, t, \theta)} = \frac{dt}{T(x, t, \theta)} = \frac{d\theta}{U^*(x, t, \theta)}. \tag{2.11}$$

Solution of Eq. (2.11) gives the similarity transformations [2].

In the case of wave propagation in one-dimensional rods, the continuity of the rod at the wave front must be preserved. This implies that at $x = D(t)$ there is zero displacement, for there is no displacement ahead of the wave front. Therefore the condition

$$u\big(x = D(t); t\big) = 0, \tag{2.8}$$

must hold at the wave front. Physically speaking, the path of the wave front would be described by a characteristic

$$\phi\big(x = D(t); t\big) = C, \quad \text{where } C \text{ is a constant.} \tag{2.12}$$

For the wave propagation problem to be expressible in terms of a similarity representation,

$$\phi(x, t) = \phi(\bar{x}, \bar{t})$$

at the moving boundary, $x = D(t)$, giving the expression

$$X(x, t, u)\frac{\partial\phi}{\partial x} + T(x, t, u)\frac{\partial\phi}{\partial t} = 0. \tag{2.13}$$

The infinitesimals $X(x, t, u)$ and $T(x, t, u)$ are related by the following equation:

$$\frac{dx}{X(x, t, u)} = \frac{dt}{T(x, t, u)} = \frac{d\phi}{0}. \tag{2.14}$$

The solution of the subsystem (2.11) leads to the similarity transformation. Also, solution of the subsystem (2.14) defines the moving boundary, $x = D(t)$. In other words, for a hyperbolic system of partial differential equations, the invariance of the governing equations under a group of transformations identifies the characteristics in simple wave regions for the class of wave propagation problems considered in this paper.

3. The similarity-characteristic relationship

Consider now the proposed procedure for determining the numerical value of the similarity variable at the wave front.

With reference to Eqs. (2.1) and (2.2), the characteristics can be determined from the equation

$$|A-\lambda I| = 0,$$

giving

(3.1) $$C^l: \left(\frac{dx}{dt}\right)_l = K^{(l)}(x, t, u, u_x, u_t).$$

Being consistent with the physical problem under consideration ($x \geq 0$, $t \geq 0$) and dropping the other characteristics, two distinct cases for the determination of the similarity-characteristic relationship are discussed.

Case (a):

(3.1') $$\frac{dx}{dt} = K(x, t).$$

Integrating along the characteristics,

(2.12') $$\phi_1(x, t) = C.$$

Also, from the equivalence of Eqs. (2.11) and (2.14)

(3.2) $$\eta_1(x, t) \equiv \eta_w \xi_1(x, t),$$

where $\eta(x, t)$ is the similarity variable obtained by solving Eq. (2.11), η_w is its value at the wave front and $\xi_1(x, t)$ is obtained by integrating Eq. (3.1').

Consider an example when $\psi(x, t, u, u_x, u_t) = A^* x^m t^n$ where A^*, m and n are constants; the characteristic can be obtained by integrating

(3.3) $$\frac{dx}{dt} = \sqrt{A^*}\, x^{m/2} t^{n/2},$$

(3.4) $$x = \left(\sqrt{A^*}\,\frac{2-m}{2+n}\right)^{\frac{2}{2-m}} t^{\frac{2+n}{2-m}}.$$

For the characteristic passing through the origin, the constant of integration vanishes. From the invariance of Eq. (2.1) with $\psi = A^* x^m t^n$, the similarity variable can be written as

(3.5) $$\eta(x, t) = \frac{x}{t^{\frac{2+n}{2-m}}}.$$

Using Eq. (3.2)

$$\eta_w = \frac{\eta(x,t)}{\xi_1(x,t)} = \left(\sqrt{A^*}\,\frac{2-m}{2+n}\right)^{\frac{2}{2-m}}, \tag{3.6}$$

where η_w is the value of the similarity variable at the wave front.

Case (b):

$$\frac{dx}{dt} = K(x, t, u, u_x, u_t). \tag{3.1''}$$

Since the characteristic is dependent on u, u_x and u_t, an explicit relationship between x and t along the characteristics is not possible. The integration along the characteristics can be carried out if the similarity transformation, which is the solution of Eq. (2.11), is introduced in Eq. (3.1″). Let the similarity solution of Eq. (2.1) be expressed in the form

$$u(x,t) = \beta(x,t)F(\eta), \tag{3.7}$$

where η is the similarity variable. Equation (3.1″) can then be expressed as

$$\frac{dx}{dt} = \bar{K}\big(x, t, F(\eta), F'(\eta)\big). \tag{3.8}$$

At the wave front the similarity and characteristic relation assumes the form

$$\eta(x,t) = \eta_w \xi_2(x,t). \tag{3.9}$$

Equations (3.6) and (3.9) are the similarity characteristic relationships required for the two cases of $\psi(x, t, u, u_t, u_x)$. The similarity characteristic relationship must be satisfied along with the rest of the similarity representation for the solution of the wave-propagation problem. It should be understood that at the wave front $\eta = \eta_w$, and that Eq. (3.9) must be satisfied.

4. Impact on a rod with a nonlinear stress-strain relationship

The governing equations for small deformations, within the framework of the uniaxial theory of thin rods, are [7]

$$\begin{aligned} \frac{\partial \sigma}{\partial x} &= -\varrho \frac{\partial v}{\partial t}, \\ \frac{\partial e}{\partial t} &= -\frac{\partial v}{\partial x}, \\ e &= \left[\frac{\sigma}{\mu}\right]^q, \end{aligned} \tag{4.1}$$

where μ, ϱ, q are material constants, x is the Lagrangian space coordinate, t is time, σ and e are nominal compressive stress and nominal compressive strain, respectively. u is the displacement of a point, and v its velocity. Also

$$e = -\frac{\partial u}{\partial x}, \quad v = \frac{\partial u}{\partial t}. \tag{4.2}$$

The auxiliary conditions are given by the following equations:

The condition at the origin, $x = 0$, is

$$v(0, t) = \frac{\partial u}{\partial t}(0, t) = v_c t^\delta, \qquad (v_c > 0;\ \delta \text{ is a parameter}). \tag{4.3}$$

The condition at the wave front is

$$u\big(x = D(t); t\big) = 0, \tag{4.3'}$$

where $x = D(t)$ gives the location of the wave front.

The initial conditions are

$$u(x, t = 0) = \frac{\partial u}{\partial x}(x, t = 0) = 0, \qquad x > 0. \tag{4.3''}$$

Combining Eqs. (4.1) and (4.2), the equation of motion assumes the form

$$\frac{\mu}{\varrho q}\left[\left(-\frac{\partial u}{\partial x}\right)^{\frac{1-q}{q}} \frac{\partial^2 u}{\partial x^2}\right] = \frac{\partial^2 u}{\partial t^2}. \tag{4.4}$$

Using the similarity procedure [6], the similarity representation can be determined as

$$u(x, t) = v_c t^{\delta+1} F(\zeta), \tag{4.5}$$

where $F(\zeta)$ is the similarity-function and

$$\zeta = \frac{kx}{t^m}, \tag{4.5'}$$

is the similarity-variable, wherein,

$$k = \left(\frac{\varrho q}{\mu}\right)^{\frac{q}{q+1}} \left(\frac{1}{v_c}\right)^{\frac{1-q}{1+q}}, \tag{4.5''}$$

and

$$m = \frac{\delta+1+q(1-\delta)}{q+1}. \tag{4.5'''}$$

Using the transformation equations (4.5), the equation of motion (4.4) is transformed into the ordinary differential equation

$$(-F')^{\frac{1-q}{q}} F'' = m^2\zeta^2 F'' + m(m-2\delta-1)\zeta F' + \delta(\delta+1)F. \tag{4.6}$$

The auxiliary conditions, under similarity transformations, assume the form

$$F(0) = \frac{1}{1+\delta}, \tag{4.6'}$$

$$F(\zeta_w) = 0, \tag{4.6''}$$

where ζ_w, the value of the similarity variable at the wave front, is as yet unknown. From Eq. (4.5') the location of the wave front $x = D(t)$ can be written as

$$D(t) = \zeta_w \frac{t^m}{k}. \tag{4.7}$$

Also, the characteristics of Eq. (4.4) can be expressed as

$$\frac{dx}{dt} = \pm \sqrt{\frac{\mu}{\varrho q}} \left(-\frac{\partial u}{\partial x}\right)^{\frac{1-q}{2q}}. \tag{4.8}$$

As discussed in the case (b) of the previous section, by substituting Eq. (4.5) into Eq. (4.8) and integrating along the positive characteristic, the following relationship is obtained:

$$D(t) = \frac{t^m}{k} \left\{ \frac{[-F'(\zeta_w)]^{\frac{1-q}{2q}}}{1+\frac{\delta(1-q)}{(1+q)}} \right\}. \tag{4.9}$$

From the equivalence of Eqs. (4.7) and (4.9), the similarity characteristic relationship can now be written as

$$\zeta_w = \frac{[-F'(\zeta_w)]^{\frac{1-q}{2q}}}{1+\frac{\delta(1-q)}{(1+q)}}. \tag{4.10}$$

For a linear elastic material, $q = 1$, $\mu = E$ (Young's Modulus), we get

$$\zeta_w = 1 \quad \text{and} \quad D(t) = \sqrt{\frac{E}{\varrho}}\, t.$$

(i) For the case when $q = 1$, and $\delta > -1$

$$F(\zeta) = \frac{1}{1+\delta} (1-\eta)^{\delta+1}, \tag{4.11}$$

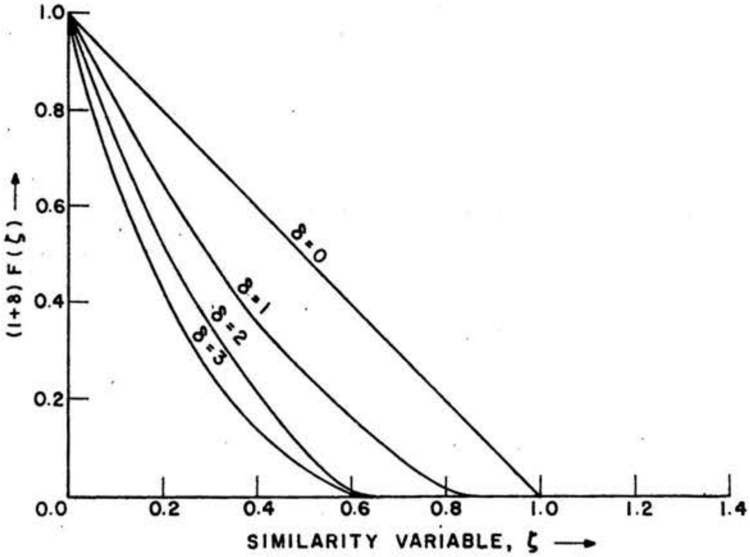


FIG. 1.

satisfies Eqs. (4.6) and (4.10), so that

$$u(x,t) = \frac{v_c}{1+\delta}\left[t - \frac{x}{c_e}\right)^{\delta+1}, \tag{4.12}$$

where c_e = velocity of propagation of this elastic wave. Plot between ζ and $F(\zeta)$ for different values of q is shown in Fig. 1.

(ii) For the case $\delta = 0, q \neq 0$, Eq. (4.6) becomes

$$\left[(-F')^{\frac{1-q}{q}} - \zeta^2\right]F'' = 0. \tag{4.13}$$

Two cases arising out of Eq. (4.13) are considered. The first is for $0 < q < 1$ and the second for $q > 1$.

Consider the first case, $0 < q < 1$. On the basis of Eq. (4.13) we can write

$$(-F')^{\frac{1-q}{q}} = \zeta^2. \tag{4.13'}$$

Integrating Eq. (4.13') with respect to ζ

$$F(\zeta) = -\frac{(\zeta)^{\frac{1+q}{1-q}}}{\dfrac{1+q}{1-q}} + C_2, \tag{4.14}$$

where C_2 is a constant of integration.

The boundary conditions on the basis of Eqs. (4.6') and (4.6'') become

$$F(0) = 1 \quad \text{and} \quad F(\zeta_w^{(1)}) = 0. \tag{4.15}$$

The solution that satisfies the similarity-characteristic relationship along with the boundary conditions (4.15) is

$$F(\zeta) = 1 - \frac{\zeta^{\frac{1+q}{1-q}}}{\dfrac{1+q}{1-q}}, \tag{4.14'}$$

and

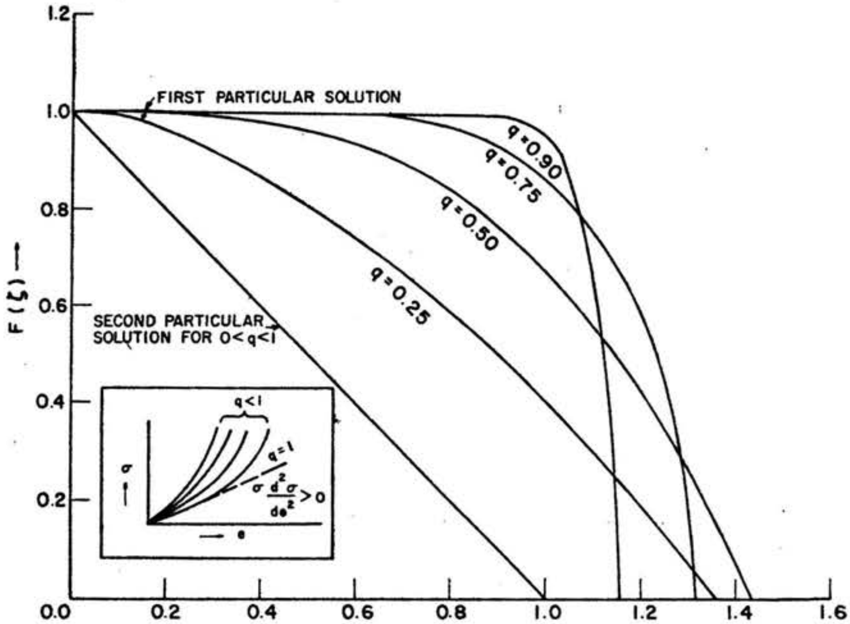
$$\zeta_w^{(1)} = [-F'(\zeta_w^{(1)})]^{\frac{1-q}{2q}} = \left(\frac{1+q}{1-q}\right)^{\frac{1-q}{1+q}}. \tag{4.16}$$

Equation (4.16) is the first particular solution of the first case. The second particular solution of this case is obtained from Eq. (4.13''), given subsequently, as

$$F(\zeta) = 1 - \zeta. \tag{4.14''}$$

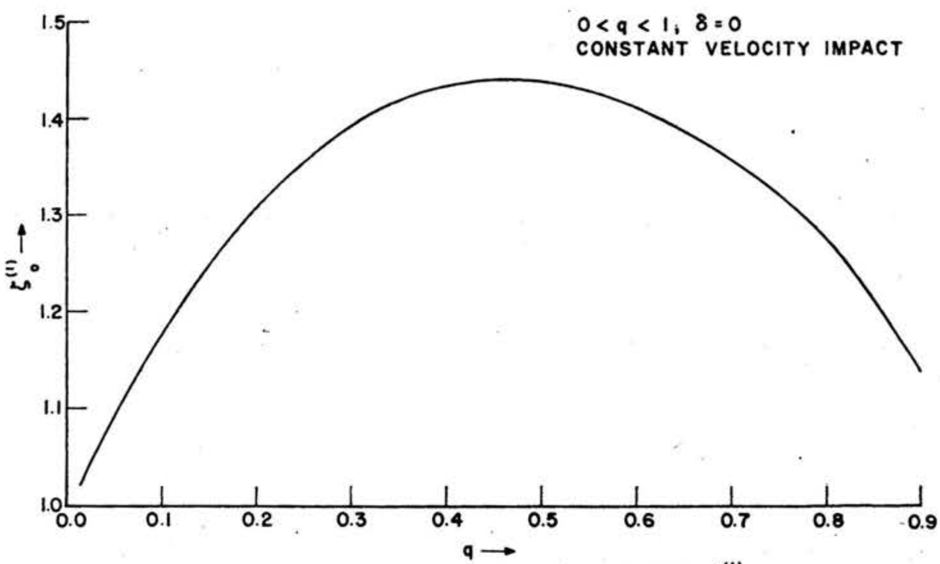
Plot of $F(\zeta)$ and ζ for the case as given by Eq. (4.14') is shown in Fig. 2 for different values of q. Furthermore, the relation between q and $\zeta_w^{(1)}$ is shown in Fig. 3. The second particular solution of the first case as given by Eq. (4.14'') is also plotted in Fig. 2.

It can be seen that in this case for $q < 1$, Eq. (4.16) gives physically meaningful results. Materials such as rubbers and even certain metals exhibit the constitutive law ($q < 1$).



F(ζ) vs. ζ, CONSTANT VELOCITY IMPACT (0 < q < 1, δ = 0)

FIG. 2.



SIMILARITY COORDINATE AT WAVEFRONT, $\zeta_w^{(1)}$, vs q

FIG. 3.

For such materials $\sigma\dfrac{d^2\sigma}{de^2} > 0$ for any e. In this case the distance between wave front decreases during propagation and there is a tendency to form shock waves.

The solution of the second case, $q > 1$, arising out of the Eq. (4.13) is the one due to VON KÁRMÁN and DUWEZ [8].

Most metals can be described by the stress-strain law with $q > 1$. For these constitutive relationships $\sigma\dfrac{d^2\sigma}{de^2} < 0$ and as the stress at the end of the rod increases continuously, the waves generated successively at the end of the rod will propagate continually with decreasing velocities. Also, for such materials the distance between the wave fronts will increase during their propagation.

If we consider Eq. (4.13) again, this equation is satisfied for

$$F''(\zeta) = 0. \tag{4.13''}$$

The solution of the above equation under the boundary conditions (4.15) gives

$$F(\zeta) = 1-\frac{\zeta}{\zeta_w^{(2)}}, \tag{4.17}$$

where, according to Eq. (4.10), $\lambda_w^{(2)} = 1$, and on the basis of Eq. (4.5)

$$u(x,t) = v_c\left(t-\frac{x}{\lambda_w^{(1)}}\right), \quad \frac{1}{\lambda_w^{(2)}} = \left[\frac{1}{v_c}\right]^{\frac{1-q}{1+q}}\left[\frac{\varrho q}{\mu}\right]^{\frac{q}{1+q}}, \tag{4.17'}$$

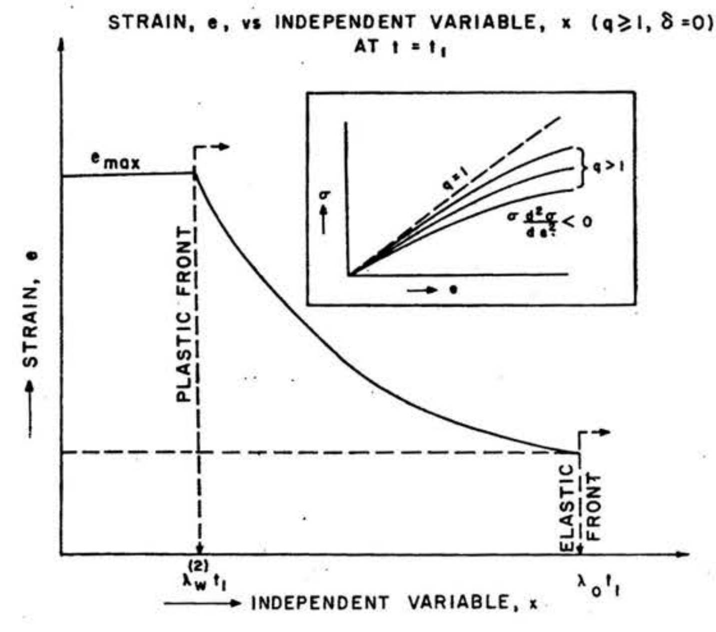


FIG. 4.

$\lambda_w^{(2)}$ is the velocity of propagation of the plastic wave. The strain, $e_{max} = -\dfrac{\partial u}{\partial x} = -\dfrac{Vc}{\lambda_w^{(2)}}$, is propagated along the wave front. The complete solution for the problem of constant velocity is made up of the following components:

(4.18)
(a) $e(x, t) = 0$, for $x > \lambda_0 t$, λ_0 = elastic wave velocity for $q = 1$,

(b) $e(x, t) = \dfrac{v_c}{\lambda_0}$ at $x = \lambda_0 t$,

(c) $e(x, t) = e_{max} = \dfrac{v_c}{\lambda_w^{(2)}}$ for $0 < x < \lambda_w^{(2)} t$.

The strain profile is shown in Fig. 4. The characteristic curves for $\delta = 0$ and $q > 1$ are shown in Fig. 5.

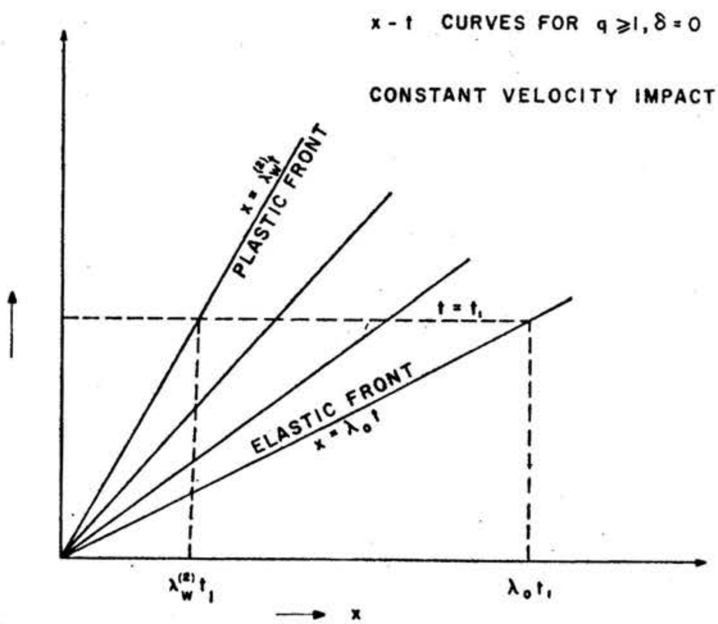


FIG. 5.

(iii) For the most general impact problem $q \neq 0$, and any δ, $(\delta \neq -1)$, Eqs. (4.6) and the similarity-characteristic relationship, equation (4.9), can be solved by a numerical procedure.

5. Discussion

For the analysis of nonlinear wave propagation problems, the traditional methods using first-order formulation lead to solutions for which the boundary and the initial conditions are somewhat restrictive. Based on group-theoretic motivation, a relationship between similarity variables and characteristics is derived using the concept of invariance of the governing equations under a transformation group. The procedure is applied to the problem of velocity impact of a semi-infinite inelastic rod and some useful closed form solutions are derived. There are some extensions of the proposed similarity-characteristic relationship and the associated theory; possible applications can be made to hyperbolic equations which are invariant under a general group of transformations in contrast to the dimensional group applied in the paper. Also, extensions could be made to two- or three-dimensional problems.

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