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Optimal design for global mechanical constraints(*)

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OPTIMAL design of beams or plates with prescribed in-plane shape is discussed assuming the behavioural constraint in the form of a functional of stress, strain, or displacement. The optimality conditions are derived and the adjoint problem for determination of the sensitivity operator is formulated. Simple examples of applicability of the derived conditions are presented.

Rozważa się problem optymalnego projektowania belek lub płyt o przyjętym konturze przy założonych więzach w postaci funkcjonału naprężenia, odkształcenia lub przemieszczenia. Wyprowadzono warunki optymalizacji i sformułowano stowarzyszony problem określenia operatora wrażliwości. Przedstawiono proste przykłady zastosowania wyprowadzonych warunków.

Рассматривается проблема оптимального проектирования балок или плит с заданным контуром, при заданных связях в виде функционала напряжений, деформаций или перемещений. Выведены условия оптимизации и сформулирована ассоциированная проблема определения оператора чуствительности. Представлены простые примеры применения выведенных условий.

1. Introduction

IN THE PRESENT paper we shall discuss the optimal design problem of structures for any form of global mechanical constraints imposed on the design. By the term "global constraint", we shall understand an integral equality or inequality expressed in terms of kinematic or static state variables and the design variables. We shall restrict our discussion to the case of structures of prescribed layout so that only dimension variables are to be determined. Thus, in the case of plates, shells or beams, the median plane and the in-plane shape are specified with loading and supports but cross-sectional properties such as thickness or reinforcement distribution may be varied in order to achieve an optimal solution. This problem was discussed, for instance, in [1—3] for the case of mean stiffness, local deflection or free frequency constraints. Here a more general formulation is presented for which the constraint functional does not coincide with the functional whose stationarity characterizes the solution of a boundary-value problem. In Sect. 2 the general optimality conditions will be derived and in Sect. 3 the adjoint problem associated with the sensitivity operator will be discussed. For the linear analysis problem, the optimality conditions can be explicitly expressed in terms of strains or stresses of the original and adjoint problems.

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It turns out that the identification and optimal design problems belong to the class of problems described in this paper. The analogy between these two types of problems is briefly discussed in Sect. 4.

2. General derivation of the necessary optimality conditions

Let the solution of the analysis problems be governed by a minimum of a certain functional with respect to the state function $\mathbf{q}$, that is

$$I(\mathbf{q}, s) \to \min_{\mathbf{q}}, \qquad \left.\frac{\delta I}{\delta \mathbf{q}}\right|_{\mathbf{q}=\mathbf{q}_0} = 0 \quad q \in Y_{\mathbf{q}}, \tag{2.1}$$

where $\mathbf{q} \in Y_{\mathbf{q}}$ may be, for instance, the displacement or the stress field and $s \in Y_s$ denotes the design function, for instance, the varying thickness or other cross-sectional parameter of the plate. In particular, the functional I may coincide with the potential or complementary energy of the nonlinear, elastic structure; $Y_{\mathbf{q}}$ and Y_s denote the appropriate Sobolev spaces.

Let the optimal design be governed by a minimum of another functional

$$G(\mathbf{q}, s) \to \min_s, \qquad s \in Y_s, \tag{2.2}$$

where G is understood to comprise both the usual cost function and the global constraint. For instance, when the optimal design problem is aimed at minimizing

$$J = \int F(s)dx \to \min_s \tag{2.3}$$

subject to the global constraint

$$H = \int K(\mathbf{q}, s)dx - H_0 \leqslant 0, \tag{2.4}$$

the functional (2.2) takes the form

$$G(\mathbf{q}, s) = \int F(s)dx + \lambda\Big[\int K(\mathbf{q}, s)dx - H_0\Big],$$

where dx may denote, respectively, the element of length or area and λ denotes the Lagrangian multiplier.

Let s_0 be a stationary value of Eq. (2.2) and $\mathbf{q}_0$ be the corresponding value of the state function. Then

$$I(\mathbf{q}_0, s_0) = \min_{\mathbf{q}} I(\mathbf{q}, s_0). \tag{2.5}$$

Consider a small variation of the design function, $s_1 = s_0 + \delta s$. Then

$$\mathbf{q}_1 = \mathbf{q}_0 + \frac{\delta \mathbf{q}}{\delta s}\delta s = \mathbf{q}_0 + \mathbf{S}\delta s = \mathbf{q}_0 + \delta \mathbf{q}_s, \tag{2.6}$$

where $\mathbf{S} = \dfrac{\delta \mathbf{q}}{\delta s}$ denotes *the sensitivity operator*. In fact, $\mathbf{S}$ maps any variation $\delta s \in Y_s$ into the space $Y_{\mathbf{q}}$, that is

$$\delta \mathbf{q}_s = \mathbf{S}\delta s. \tag{2.7}$$

For a physically or geometrically nonlinear problem the sensitivity operator depends on the actual values $\mathbf{q}$ and s, $\mathbf{S} = \mathbf{S}(\mathbf{q}, s)$.

Since $\mathbf{q} = \mathbf{q}_0$, $s = s_0$ is the optimal design, we have the stationarity condition(1)

$$\delta G = \frac{\delta G}{\delta s} : \delta s + \frac{\delta G}{\delta \mathbf{q}} : \mathbf{S}\delta s = 0, \tag{2.8}$$

where $\frac{\delta G}{\delta s}$ and $\frac{\delta G}{\delta \mathbf{q}}$ denote the variational Gateaux derivatives of the functional G. Since Eq. (2.8) is valid for any s and Eq. (2.8) is a linear functional of δs, the stationarity conditions can be written as follows:

$$\frac{\delta G}{\delta s} + \frac{\delta G}{\delta \mathbf{q}} \mathbf{S} = 0. \tag{2.9}$$

Let us note that when the constraint (2.4) coincides with the functional (2.1), the condition (2.9) can be reduced to a much simpler form, not involving the operator $\mathbf{S}$. In fact, then we have

$$\delta G = \frac{\delta G}{\delta s} : \delta s + \frac{\delta G}{\delta \mathbf{q}} : \delta \mathbf{q} = \frac{\delta J}{\delta s} : \delta s + \lambda \frac{\delta I}{\delta s} : \delta s + \lambda \frac{\delta I}{\delta \mathbf{q}} : \delta \mathbf{q} + \delta \lambda (I - I_0) = 0 \tag{2.10}$$

and since the solution of a boundary-value problem satisfies

$$\left. \frac{\delta I}{\delta \mathbf{q}} \right|_{\mathbf{q} = \mathbf{q}_0} = 0, \tag{2.11}$$

the following stationarity condition follows from Eq. (2.10)

$$\frac{\delta J}{\delta s} = -\lambda \frac{\delta I}{\delta s}, \quad I = I_0, \tag{2.12}$$

and this condition does not depend on the sensitivity operator $\mathbf{S}$. Let us note that the optimality conditions (2.12) are equivalent to those derived in [1, 2, 3] where also the conditions for global minimum were discussed (cf. also [8, 9]).

In a more general case, however, when I and G are arbitrary, the sensitivity operator $\mathbf{S}$ occurs in the stationarity condition (2.9). Obviously, this operator should be known for any given design s.

In order to derive the relationship between $\mathbf{S}$ and the derivatives of the functional I, consider a sequence of solutions corresponding to varying s. The equilibrium solutions are represented by points $P_1, P_2, \ldots P_n$ in the function space, Fig. 1, corresponding to a minimum of the functional I. The path P_1—P_2—P_3, ... is a locus of points representing solutions for consecutive values of the design function s. For any value of s the equilibrium solution satisfies the stationarity condition

$$\frac{\delta I}{\delta \mathbf{q}}(\mathbf{q}, s) = 0 \tag{2.13}$$

(1) The double dot between the two symbols denotes the global scalar product integrated over the structure domain whereas the single dot denotes the local scalar product at a particular structure point.

and the variation of Eq. (2.13) along the path $P_1 - P_2 - P_3, \ldots, P_n$ vanishes, that is

$$\frac{\delta^2 I}{\delta \mathbf{q}^2}\delta \mathbf{q}_s + \frac{\delta^2 I}{\delta \mathbf{q}\delta s}\delta s = \left(\frac{\delta^2 I}{\delta \mathbf{q}^2}\mathbf{S} + \frac{\delta^2 I}{\delta \mathbf{q}\delta s}\right)\delta s = 0. \tag{2.14}$$

Equation (2.14) together with the optimality condition (2.9) and the stationarity condition (2.1) constitute the set of equations governing both the analysis and the synthesis problems.

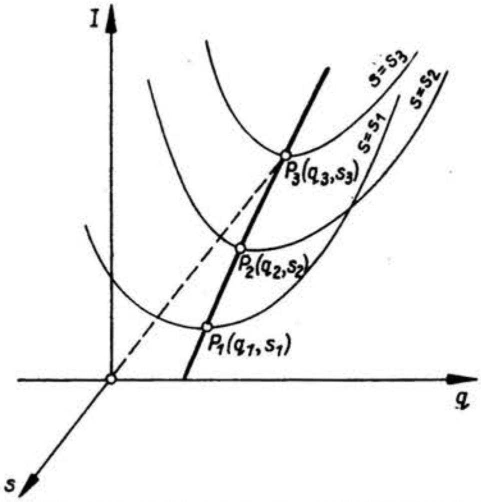

FIG. 1. Evolution of equilibrium solutions P_1, P_2, P_3 for varying design function s.

This equation can also be derived in a more formal way by considering the variation of both the design and the state functions, thus

$$s = s_0 + \delta s, \quad \mathbf{q} = \mathbf{q}_0 + \mathbf{S}\delta s + \delta \mathbf{q}' = \mathbf{q}_0 + \delta \mathbf{q}_s + \delta \mathbf{q}' \tag{2.15}$$

and the associated variation of the functional I

$$\Delta I = \frac{\delta I}{\delta s} : \delta s + \frac{\delta I}{\delta \mathbf{q}} : (\delta \mathbf{q}' + \mathbf{S}\delta s) + \frac{1}{2}\left[\frac{\delta^2 I}{\delta \mathbf{q}^2}(\delta \mathbf{q}' + \mathbf{S}\delta s)^2 + 2\frac{\delta^2 I}{\delta s \delta \mathbf{q}}\delta s : (\delta \mathbf{q}' + \mathbf{S}\delta s) + \frac{\delta^2 I}{\delta s^2}\delta s^2\right] + \ldots. \tag{2.16}$$

The stationarity of I at P_1 $(\mathbf{q}_0, s_0)$ requires the first derivative of I with respect to $\mathbf{q}$ to vanish and the stationarity of I at $P_2(\mathbf{q}_0 + \mathbf{S}\delta s, s + \delta s)$ with respect to $\mathbf{q}$ occurs when the variation terms linear in $\delta \mathbf{q}'$ vanish, that is

$$\frac{\delta^2 I}{\delta \mathbf{q}^2}\mathbf{S}\delta s : \delta \mathbf{q}' + \frac{\delta^2 I}{\delta s \delta \mathbf{q}}\delta s : \delta \mathbf{q}' = 0 \tag{2.17}$$

and since Eq. (2.17) represents a bilinear functional valid for arbitrary $\delta \mathbf{q}'$ satisfying the boundary conditions, the condition (2.14) is obtained from Eq. (2.17).

Consider now the particular case when

$$I(\mathbf{q}, s) = A(\mathbf{q}, s) - a(\mathbf{q}, s), \tag{2.18}$$

where A denotes the *quadratic* and a linear functionals of $\mathbf{q}$.

The condition (2.14) now takes the form

(2.19) $$\frac{\delta^2 A}{\delta \mathbf{q}^2}\delta \mathbf{q}_s + \frac{\delta^2 I}{\delta s \delta \mathbf{q}}\delta s = 0.$$

Let us note that $\frac{\delta^2 A}{\delta \mathbf{q}^2} = \mathbf{B}(s)$ is a self-adjoint operator not depending on $\mathbf{q}$. Introduce the *adjoint problem* defined by the operator equation

(2.20) $$\frac{\delta^2 A}{\delta \mathbf{q}^2}\mathbf{q}^* = \mathbf{B}(s)\mathbf{q}^* = -\frac{\delta G}{\delta \mathbf{q}}$$

and the boundary conditions assuring the integral equality

(2.21) $$\mathbf{B}\mathbf{q}^*:\mathbf{q} = \mathbf{B}\mathbf{q}:\mathbf{q}^*$$

valid for any $\mathbf{q}$ satisfying the boundary conditions of the initial analysis problem. Instead of Eq. (2.17), we can now write

(2.22) $$\mathbf{B}\delta\mathbf{q}_s:\mathbf{q}^* = \mathbf{B}\mathbf{q}^*:\delta\mathbf{q}_s = -\frac{\partial G}{\partial \mathbf{q}}:\delta\mathbf{q}_s = -\frac{\delta^2 I}{\delta s \delta \mathbf{q}}\delta s:\mathbf{q}^*$$

and the optimality condition (2.8) can now be presented in the form

(2.23) $$\delta G = \frac{\delta^2 I}{\delta s \delta \mathbf{q}}\delta s:\mathbf{q}^* + \frac{\delta G}{\delta s}:\delta s = \mathbf{B}\mathbf{q}\delta s:\mathbf{q}^* - \mathbf{b}\delta s:\mathbf{q}^* + \frac{\delta G}{\delta s}:\delta s = 0,$$

where $\mathbf{b} = \frac{\delta^2 a}{\delta s \partial \mathbf{q}}$ is a linear operator not depending on $\mathbf{q}$. Introducing the adjoint operators $\mathbf{B}^*$ and $\mathbf{b}^*$ so that

(2.24) $$\mathbf{B}\mathbf{q}\delta s:\mathbf{q}^* - \mathbf{b}\delta s:\mathbf{q}^* = \mathbf{B}^*\mathbf{q}^*:\mathbf{q}\delta s - \mathbf{b}^*\mathbf{q}^*:\delta s,$$

the stationarity condition (2.23) can be explicitly expressed in the form

(2.25) $$\mathbf{B}^*\mathbf{q}^*:\mathbf{q} - \mathbf{b}:\mathbf{q}^* + \frac{\delta G}{\delta s} = 0,$$

that is in terms of the solution of the adjoint problem (2.20).

An alternative formulation of the optimization problem associated with Eq. (2.19) and (2.2) can be presented by starting from the operator equation of the problem(2), similarly as in [5]

(2.26) $$\mathbf{L}(s)\mathbf{q} = \mathbf{f}(\mathbf{q}, s)$$

and for small variations $\delta\mathbf{q}$ and δs, we have

(2.27) $$\mathbf{L}_s\delta s\mathbf{q} + \mathbf{L}_q\delta\mathbf{q} = \mathbf{f}_q\delta\mathbf{q} + \mathbf{f}_s\delta s$$

where the subscripts denote differentiation with respect to s or $\mathbf{q}$, and f denotes the differentiable function of $\mathbf{q}$ and s. The optimization problem (2.2) can be regarded as the problem constrained by Eq. (2.26), hence introducing the functional

(2.28) $$G' = G - \bar{\boldsymbol{\lambda}}:[\mathbf{L}(s)\mathbf{q} - \mathbf{f}(\mathbf{q}, s)],$$

(2) This operator equation can be identified, for instance, with the displacement equations of the linear elasticity theory.

we may consider its variation

$$(2.29)\qquad \delta G' = \frac{\delta G}{\delta \mathbf{q}} : \delta\mathbf{q} + \frac{\delta G}{\delta s} : \delta s - \bar{\boldsymbol{\lambda}} : [\mathbf{L}_s \delta s \mathbf{q} + \mathbf{L}\delta\mathbf{q} - \mathbf{f}_q \delta\mathbf{q} - \mathbf{f}_s \delta s] - \delta\bar{\boldsymbol{\lambda}} : [\mathbf{L}\mathbf{q} - \mathbf{f}]$$

$$= \left(\frac{\delta G}{\delta \mathbf{q}} - \mathbf{L}^*\bar{\boldsymbol{\lambda}} + \mathbf{f}_q^*\bar{\boldsymbol{\lambda}}\right) : \delta\mathbf{q} + \left(\frac{\delta G}{\delta s} - \mathbf{L}_s^*\bar{\boldsymbol{\lambda}} : \mathbf{q} - \bar{\mathbf{f}}_s^* : \boldsymbol{\lambda}\right)\delta s - \delta\bar{\boldsymbol{\lambda}} : (\mathbf{L}\mathbf{q} - \mathbf{f});$$

setting

$$(2.30)\qquad \mathbf{L}^*\bar{\boldsymbol{\lambda}} - \mathbf{f}_q^*\bar{\boldsymbol{\lambda}} - \frac{\delta G}{\delta \mathbf{q}} = 0$$

the optimality condition takes the form

$$(2.31)\qquad \frac{\delta G}{\delta s} - \mathbf{L}_s^*\bar{\boldsymbol{\lambda}} : \mathbf{q} - \mathbf{f}_s^* : \bar{\boldsymbol{\lambda}} = 0, \quad \mathbf{L}\mathbf{q} - \mathbf{f} = 0.$$

In Eqs. (2.29)–(2.31), the operators $\mathbf{L}^*$, $\mathbf{f}_q^*$, $\mathbf{f}_s^*$ are adjoint to $\mathbf{L}$, $\mathbf{f}_q$ and $\mathbf{f}_s$. Let us note that we neglected boundary terms assuming proper boundary conditions. Equation (2.30) now defines the adjoint problem and Eq. (2.31) is the optimality condition. The Lagrangian multiplier $\bar{\boldsymbol{\lambda}}$ is now the adjoint function to $\mathbf{q}$. The analogy between Eq. (2.31) and (2.25) can easily be traced.

3. Adjoint systems for stress and deflection constraints

In this section we shall discuss the optimality conditions and adjoint systems for the case when global constraints are imposed on stress state or displacements within the structure, and the material is linearly elastic.

3.1. Global stress constraint

Let $\mathbf{Q}$ and $\mathbf{q}$ be generalized stresses and strains interrelated by the constitutive law

$$(3.1)\qquad \mathbf{Q} = \mathbf{D}(s)\mathbf{q},$$

where $\mathbf{D}(s)$ denotes the stiffness matrix depending on the design function $s(x)$ which may be, for instance, the variable plate thickness. Varying s, we obtain from Eq. (3.1)

$$(3.2)\qquad \delta\mathbf{Q} = \mathbf{D}\delta\mathbf{q} + \mathbf{D}_s \delta s\mathbf{q} = \delta\mathbf{Q}' + \delta\mathbf{Q}''$$

and since $\delta\mathbf{Q}$ is the statically admissible stress field satisfying equilibrium equations and boundary conditions for vanishing surface tractions, we can write

$$(3.3)\qquad \int \delta\mathbf{Q} \cdot \delta\mathbf{q}\, dx = 0$$

since $\delta\mathbf{q}$ is the kinematically admissible strain field. The optimization problem can now be formulated as follows:

$$(3.4)\qquad \min \bar{G}|_s = \int \Phi(\mathbf{Q}, s)\, dx, \quad \text{subject to} \quad C = \int F(s)\, dx \leqslant C_0,$$

where $\Phi = \Phi(\mathbf{Q}, s)$ does not need to coincide with the stress energy function $W = W(\mathbf{Q}, s)$.

Let us now define the adjoint system. Consider the structure of the same shape and support conditions with no external loading but subjected to the initial strain

$$\mathbf{q}^i = \frac{\partial \Phi}{\partial \mathbf{Q}}. \tag{3.5}$$

The corresponding residual stress and strain states are $\mathbf{Q}^r$ and $\mathbf{q}^r$, and the total strain $\mathbf{q}^0$ satisfies the equality, Fig. 2.

$$\mathbf{q}^0 = \mathbf{q}^i + \mathbf{q}^r, \quad \mathbf{Q}^r = \mathbf{D}\mathbf{q}^r. \tag{3.6}$$

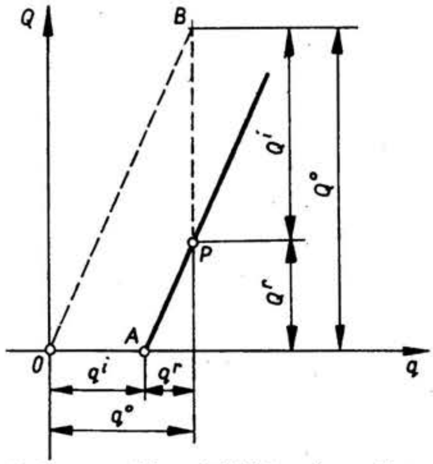

FIG. 2. Decomposition of initial strains and stresses, $q^0 = q^i + q^r$, $Q^0 = Q^i + Q^r$.

Using Eqs. (3.2), (3.5) and (3.6), we can write

$$\begin{aligned}\delta\overline{G} &= \int \frac{\partial \Phi}{\partial \mathbf{Q}} \cdot \delta\mathbf{Q}dx + \int \frac{\partial \Phi}{\partial s}\delta s dx = \int (\mathbf{q}^0 - \mathbf{q}^r)\cdot \delta Q dx + \int \frac{\partial \Phi}{\partial s}\delta s dx \\ &= \int -\mathbf{q}^r \cdot (\delta\mathbf{Q}' + \delta\mathbf{Q}'')dx + \int \frac{\partial \Phi}{\partial s}\delta s dx = -\int \mathbf{D}\mathbf{q}^r \cdot \delta\mathbf{q} dx \\ &- \int \mathbf{q}^r \cdot \mathbf{D}_s \delta s \mathbf{q} dx + \int \frac{\partial \Phi}{\partial s}\delta s dx = -\int \mathbf{q}^r \cdot \mathbf{D}_s \mathbf{q}\delta s + \int \frac{\partial \Phi}{\partial s}\delta s dx.\end{aligned} \tag{3.7}$$

In Eq. (3.7) we used the virtual work equalities

$$\int \delta\mathbf{Q}\cdot \mathbf{q}^0 dx = \int \mathbf{Q}^r \cdot \delta\mathbf{q} dx = 0. \tag{3.8}$$

Considering now the functional

$$G = \overline{G} + \lambda(C - C_0) \tag{3.9}$$

its first variation in view of Eq. (3.7) can be expressed as follows:

$$\delta G = \int \left(-\mathbf{q}^r \cdot D_s \mathbf{q} + \frac{\partial \Phi}{\partial s} + \lambda F_s\right)\delta s dx = 0 \tag{3.10}$$

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and the stationarity condition takes the form

(3.11) $$\mathbf{q}^r \cdot \mathbf{D}_{,s}\mathbf{q} - \frac{\partial \Phi}{\partial s} = \lambda \frac{dF}{ds} = \lambda F_{,s},$$

which depends on the solution of initial and adjoint problems. In the particular case, when $\Phi = W$, then $\mathbf{q}^r = 0$ and the optimality condition derived in [2] is obtained.

An alternative derivation can be provided by starting from the constraint expressed in terms of strains, that is

(3.12) $$\min G'|_s = \int \Psi(\mathbf{q}, s)dx, \quad \text{subject to} \quad C = \int F(s)dx \leqslant C_0.$$

Let us write the inverse relations to Eqs. (3.1) and (3.2)

(3.13) $$\mathbf{q} = \mathbf{E}(s)\mathbf{Q},$$

(3.14) $$\delta\mathbf{q} = \mathbf{E}\delta\mathbf{Q} + \mathbf{E}_{,s}\delta s\mathbf{Q} = \delta q' + \delta q''$$

and $\mathbf{E} = \mathbf{D}^{-1}$ is the compliance matrix.

To define the adjoint system, let us consider the same structure under no external loading, but loaded by the initial stress $\mathbf{Q}^i$, such that

(3.15) $$\mathbf{Q}^i = \mathbf{D}\mathbf{q}^i = \frac{\partial \psi}{\partial \mathbf{q}}$$

and $\mathbf{Q}^r = \mathbf{Q}^0 - \mathbf{Q}^i$, where $\mathbf{Q}^r = \mathbf{D}\mathbf{q}^r$ and $\mathbf{Q}^0 = \mathbf{D}\mathbf{q}^0$, Fig. 2. Now the variation of G' equals

(3.16) $$\delta G' = \int \frac{\partial \psi}{\partial \mathbf{q}} \cdot \delta\mathbf{q}dx + \int \frac{\partial \psi}{\partial s}\delta s dx = \int (\mathbf{Q}^0 - \mathbf{Q}^r) \cdot \delta\mathbf{q}dx + \int \frac{\partial \psi}{\partial s}\delta s dx$$
$$= \int \mathbf{Q}^0 \cdot \mathbf{E}\delta\mathbf{Q}dx + \int \mathbf{Q}^0 \cdot \mathbf{E}_{,s}\delta s\mathbf{Q}dx + \int \frac{\partial \psi}{\partial s}\delta s dx = \int \delta\mathbf{Q} \cdot \mathbf{q}^0 dx$$
$$+ \int \mathbf{Q}^0 \cdot \mathbf{E}_{,s}\mathbf{Q}\delta s dx + \int \frac{\partial \psi}{\partial s}\delta s dx = \int \mathbf{Q}^0 \cdot \mathbf{E}_{,s}\mathbf{Q}\delta s dx + \int \frac{\partial \psi}{\partial s}\delta s dx.$$

In deriving Eq. (3.16) we used the virtual work principle

(3.17) $$\int \mathbf{Q}^r \cdot \delta\mathbf{q}dx = \int \delta\mathbf{Q} \cdot \mathbf{q}^0 dx = 0.$$

Considering the functional $G = G' + \lambda(C - C_0)$, its variation is expressed as follows:

(3.18) $$\delta G = \int \left[\mathbf{Q}^0 \cdot E_{,s}\mathbf{Q} + \frac{\partial \psi}{\partial s} + \lambda \frac{dF}{ds}\right] dx + \delta\lambda(C - C_0) = 0$$

and the stationarity condition requires that

(3.19) $$\mathbf{Q}^0 \cdot \mathbf{E}_{,s}\mathbf{Q} + \frac{\partial \psi}{\partial s} = -\lambda \frac{\partial F}{\partial s}, \quad C = C_0.$$

It is easy to show the equivalence of Eqs. (3.11) and (3.19). Since $\mathbf{DE} = \mathbf{I}$, where $\mathbf{I}$ is the unit matrix, we have

(3.20) $$D_{,s}\mathbf{E} + \mathbf{D}\mathbf{E}_{,s} = 0, \quad \text{hence} \quad \mathbf{E}_{,s} = -\mathbf{E}\mathbf{D}_{,s}\mathbf{E}, \quad \mathbf{D}_{,s} = -\mathbf{D}\mathbf{E}_{,s}\mathbf{D}$$

and in view of Eq. (3.14) there is

$$\left(\frac{\partial \psi}{\partial s}\right)_{\mathbf{q}} = \frac{\partial \Phi}{\partial s} + \frac{\partial \Phi}{\partial \mathbf{Q}}\left(\frac{\partial \mathbf{Q}}{\partial s}\right)_{\mathbf{q}} = \frac{\partial \phi}{\partial s} - \mathbf{q}^i \mathbf{D} \mathbf{E}_s \mathbf{Q}, \tag{3.21}$$

where $(\ )_{\mathbf{q}}$ denotes the derivative taken at constant $\mathbf{q}$. Using Eq. (3.21), the optimality condition (3.19) can easily be retransformed into Eq. (3.11).

The adjoint systems for global stress constraints were applied in [4], though their direct mechanical interpretation was not presented.

Example 1. Consider a beam for which the bending moment $M = M(x)$ is a linear function of curvature $k = k(x)$ and a nonlinear function of thickness $h = h(x)$, that is $M = \alpha h^m k$, where α is a cross-sectional stiffness parameter. Let the optimal design problem be formulated as follows:

$$\min \bar{G} = \int M^n dx, \quad \text{subject to} \quad C = \int h dx \leqslant C_0. \tag{3.22}$$

The functional G now has the form

$$G = \int M^n dx + \lambda\left[\int h dx - C_0\right] = \alpha^n \int h^{mn} k^n dx - \lambda\left[\int h dx - C_0\right]. \tag{3.23}$$

The initial curvature field of the adjoint beam now is

$$k^i(x) = nM^{n-1} = \alpha^n n h^{m(n-1)} k^{n-1}. \tag{3.24}$$

The adjoint problem is reduced to determining the curvature field $k^0 = k^0(x)$ and the residual moment field $M^r = M^r(x)$ induced by the initial curvature field $k^i(x)$ defined by Eq. (3.24). Since there is

$$s = h, \quad D_s = m\alpha h^{m-1}, \quad \frac{\partial \Phi}{\partial s} = 0, \quad \frac{dF}{ds} = 1, \quad k^r \Rightarrow k^0 - k^i, \tag{3.25}$$

the optimality condition (3.11) now reads

$$k^r m\alpha h^{m-1} k = \lambda \tag{3.26}$$

or, using Eqs. (3.24) and (3.25), we have

$$\alpha k^0 k m h^{m-1} - \alpha^{n+1} m n k^n h^{mn-1} = \lambda \tag{3.27}$$

that is the condition expressed in terms of curvatures of the initial and adjoint systems. Let us note that for varying thickness, the adjoint system will correspond to non-vanishing $M^r(x)$ and $k^r(x)$ for all values of n. Eqation (3.27) relates k^0, k and h; thus an optimal value of h can be found from Eq. (3.27) through an iterative procedure.

3.2. Global displacement constraint

Consider now the constraint imposed on the lateral displacement of the beam or plate

$$\bar{G} = \int g(\mathbf{u}) dS_c \leqslant g_0, \tag{3.28}$$

where S_c denotes the portion of the middle surface area. For the adjoint system, let us introduce the traction over S_c

$$\mathbf{P}_a = \frac{\partial g(\mathbf{u})}{\partial \mathbf{u}} \tag{3.29}$$

and denote the corresponding solution by $\mathbf{Q}^a$, $\mathbf{q}^a$, $\mathbf{u}^a$. The adjoint plate or beam is identically supported as the initial system but loaded only by the traction $\mathbf{P}_a$.

Consider the following optimization problem:

$$\min \bar{G}, \text{ subject to } \int F(ds)dx \leqslant C_0 \tag{3.30}$$

and introduce the functional $G = \bar{G} + \lambda[\ F(s)dx - C_0]$. Now we have

$$\delta\bar{G} = \int \frac{\partial g}{\partial \mathbf{u}} \cdot \delta\mathbf{u} dS_c = \int \mathbf{P}_a \cdot \delta\mathbf{u} dS_c = \int \mathbf{Q}^a \cdot \delta\mathbf{q} dx. \tag{3.31}$$

Since by virtue of Eq. (3.14), there is $\mathbf{q} = \mathbf{EQ} + \mathbf{E}_s \delta s \mathbf{Q}$, the integral (3.21) can be reduced to the form

$$\delta\bar{G} = \int \mathbf{Q}^a \cdot \mathbf{E}\delta\mathbf{Q} dx + \int \mathbf{Q}^a \cdot \mathbf{E}_s \delta s \mathbf{Q} dx = \int \delta\mathbf{Q} \cdot \mathbf{q}^a dx$$
$$+ \int \mathbf{Q}^a \cdot \mathbf{E}_s \mathbf{Q} \delta s dx = \int \mathbf{Q}^a \cdot \mathbf{E}_s \mathbf{Q} \delta s dx = -\int \mathbf{q}^a \cdot \mathbf{D}_s \mathbf{q} \delta ds x \tag{3.32}$$

and

$$\delta G = \int \mathbf{Q}^a \cdot \mathbf{E}_s \mathbf{Q} \delta s dx + \lambda \frac{dF}{ds} \delta s dx + \delta\lambda \left[\int F(s) dx - C_0 \right] = 0. \tag{3.33}$$

Thus the stationarity conditions take the form

$$\mathbf{Q}^a \cdot E_s \mathbf{Q} = -\lambda \frac{dF}{ds}, \quad \delta\lambda \left[\int F(s)dx - C_0 \right] = 0. \tag{3.34}$$

Example. Consider, for instance, a cantilever beam, built-in at $x = 0$ and loaded by a concentrated force P at its tip $x = l$. Let the optimization problem be formulated as follows:

$$\min \bar{G} = \int_0^l \beta |u| dx \quad \text{subject to} \quad \int_0^l h dx - C_0 \leqslant 0, \tag{3.35}$$

where β is a constant factor and $\bar{G}$ can be regarded as a mean compliance. In this case the adjoint beam is loaded by the uniform pressure $p_a = \beta$ over its length. Thus the bending moments are

$$M = -P(l-x), \quad M_a = -\frac{\beta(l-x)^2}{2} \tag{3.36}$$

and if the compliance $E = \alpha h^{-m}$, the optimality condition (3.34) provides

$$\lambda h^{m+1} = mP\alpha\beta \frac{(l-x)^3}{2}. \tag{3.37}$$

It is seen that now the adjoint structure is analogous to that discussed by SHIELD and PRAGER [6] who considered stationarity of the mutual energy of two systems with respect to a design variable.

4. Identification problems

The discussion carried out in the preceding section can directly be applied to a class of identification problems for which the cross-sectional stiffness parameters are to be determined from a set of experimental measurements of stress, strain or displacement.

Assume that the displacement is measured at n points of the plate, $\mathbf{u} = \mathbf{u}_i$, $i = 1, 2, \ldots, \ldots, n$. Let the calculated displacement field for the assumed stiffness matrix $\mathbf{D}$ be $\mathbf{u}(x)$. Assume the plate to be uniform but the stiffness matrix to depend on several stiffness parameters $s_1, s_2, s_3, \ldots$, thus $\mathbf{D} = \mathbf{D}(s_1, s_2, s_3, \ldots)$. Our aim is to determine these parameters in order to minimize the distance between the measured and the predicted displacements. Let the measure of this distance be

$$G_1 = \frac{1}{2} \sum_{i=1}^{n} \alpha_i (\mathbf{u} - \mathbf{u}_i)^2. \tag{4.1}$$

Introduce the adjoint plate or beam loaded by concentrated loads

$$\mathbf{P}_i = \alpha_i (\mathbf{u} - \mathbf{u}_i) = \left. \frac{\partial G_1}{\partial \mathbf{u}} \right|_{x = x_i} \tag{4.2}$$

at test points. Denote the actual state in the original structure by $\mathbf{Q}$, $\mathbf{q}$, $\mathbf{u}$ and the adjoint state by $\mathbf{Q}^a$, $\mathbf{q}^a$, $\mathbf{u}^a$. Following Eqs. (3.32) and (3.20), we can write

$$\delta G_1 = \sum_{i=1}^{n} \delta s_i \int \mathbf{Q}^a \cdot \mathbf{E}_{si} \mathbf{Q} dx = -\sum_{i=1}^{n} \delta s_i \int \mathbf{q}^a \cdot \mathbf{D}_{si} \mathbf{q} dx. \tag{4.3}$$

The stationarity condition requires each integral occurring in Eq. (4.3) to vanish. However, Eq. (4.3) can be used directly in evaluating the functional gradient $\partial G / \partial s_i$ at each iteration step, similarly as it was done in [7]. Let us note that s_i can be assumed in some cases as orthotropy coefficients of the plate. Assume now that strain measurements are obtained at n points, $\mathbf{q} = \mathbf{q}_i$, $i = 1, 2, \ldots, n$. Assuming the distance function in the form

$$G_2 = \frac{1}{2} \sum_{i=1}^{u} \beta_i (\mathbf{q} - \mathbf{q}_i)^2 \tag{4.4}$$

let us introduce the initial stresses $\mathbf{Q}^i$ for the adjoint system applied at the test points

$$\mathbf{Q}^i = \beta_i (\mathbf{q} - \mathbf{q}_i) = \left. \frac{\partial G_2}{\partial \mathbf{q}} \right|_{x = x_i} \tag{4.5}$$

and, according to Eq. (3.16), the variation of G_2 equals

$$\delta G_2 = \sum_{i=1}^{n} \delta s_i \int \mathbf{Q}^0 \cdot \mathbf{E}_{si} \mathbf{Q} dx = -\sum_{i=1}^{n} \delta s_i \int \mathbf{q}^0 \cdot \mathbf{D}_{si} \mathbf{q} dx. \tag{4.6}$$

Obviously, the initial stresses $\mathbf{Q}^i$ correspond to localized distortions $\mathbf{q}^r$ at the measurement points, cf. Fig. 2.

5. Concluding remarks

The present work provides a general discussion of the optimality conditions for global behaviour constraints expressed as any functional of stress, strain or displacement. For linear elastic structures the optimality conditions can be expressed directly in terms of the states of original and adjoint problems. In particular, for global stress or strain constraints the adjoint problem is reduced to the initial stress or strain problems. For non-linear structures, the sensitivity operator can be obtained by solving a linear boundary value problem, from which the variations in state functions can be expressed in terms of variations of design variables. The identification problems can be treated within the same general formulation.

References

1. W. Prager and J. E. Taylor, *Problems of optimal structural design*, J. Appl. Mech., **35**, 102–106, 1968.
2. Z. Mróz, *Multiparameter optimal design of plates and shells*, J. Struct. Mech., **1**, 371–393, 1972.
3. E. F. Masur, *Optimum stiffness and strength of elastic structures*, J. Eng. Mech. Div., Proc. ASCE, **96**, 621–640, 1970.
4. Y. W. Chun and E. J. Haug, *Two-dimensional shape optimal design*, Int. J. Num. Meth. Eng., **13**, 311–337, 1978.
5. S. M. Rohde and G. T. Mc Allister, *Some representations of variations with application to optimization and sensitivity analysis*, Int. J. Eng. Sci., **16**, 443–449, 1978.
6. R. T. Shield and W. Prager, *Optimal structural design for given deflection*, Zeitschr. Angew. Math. Phys., **21**, 513–528, 1970.
7. J. P. Kernevez, C. Knopf-Lenoir, G. Touzot and G. Verchery, *An identification method applied to an orthotropic plate bending experiment*, Intern. J. Num. Math. Eng., **12**, 129–139, 1978.
8. J. E. Taylor, *The strongest column: an energy approach*, J. Appl. Mech., **34**, 486–487, 1967.
9. M. Życzkowski, *Optimale Formen des dünnwandigen geschlossenen Querschnittes eines Balkens bei Berücksichtingug von Stabilitätsbedingungen*, Z. angew. Math. Mechanik, **48**, 7, 455–462, 1968.

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