Arch. Mech., 32, 2, pp. 171–191, Warszawa 1980

Fundamental consequences of a new intrinsic time measure Plasticity as a limit of the endochronic theory

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A NEW measure of intrinsic time is introduced which broadens the endochronic theory and lends it a wider predictive scope. Idealized plastic models are shown to be constitutive subsets of the general theory and the phenomenon of yield is proved to be a consequence of a particular definition of the intrinsic time measure in terms of the plastic strain tensor. Various versions of the classical plasticity theory are shown to be asymptotic cases of the endochronic theory. In particular, the kinematic hardening model, the isotropic hardening model as well as their combinations, are derived directly from the general theory. In addition, the translation vector of the yield surface in stress space is found to be a constitutive property given by a linear functional of the history of the plastic strain. The Prager and Ziegler rules are obtained as special cases.

Wprowadzono nową metodę pomiaru czasu wewnętrznego, która rozszerza zakres stosowalności teorii endochronicznej i zakres ważności przewidywań tej teorii. Wykazano, że wyidealizowane modele plastyczne stanowią konstytutywne podzbiory teorii ogólnej. Wykazano też, że zjawisko płynięcia jest konsekwencją szczególnej definicji wewnętrznego miernika czasu w funkcji plastycznego tensora odkształcenia. Pokazano, że różne wersje klasycznej teorii plastyczności są przypadkami asymptotycznymi teorii endochronicznej. W szczególności kinematyczny model hartowania, model hartowania jednorodnego, jak też ich kombinacje dają się wyprowadzić bezpośrednio z teorii ogólnej. Ponadto stwierdzono, że wektor translacji powierzchni płynięcia w przestrzeni naprężeń jest własnością konstytutywną określoną przez liniowy funkcjonał historii odkształceń plastycznych. Reguły Pragera i Zieglera otrzymano jako przypadki szczególne.

Введен новый метод измерения внутреннего времени, который расширяет область приняемости эндохронической теории и область справедливости предсказаний этой теории. Показано, что идеальные пластические модели составляют определяющие подмножества общей теории. Показано тоже, что явление течения является следствием частного определения внутреннего мерила времени в функции пластического тензора деформаци. Показано, что разные варианты классической теории пластичности являются асимптотическими случаями эндохронической теории. В частности, кинематическая модель закалки, модель однородной закалки, как тоже их комбинации даются вывести непосредственно из общей теории. Кроме этого констатировано, что вектор трансляций поверхности течения в пространстве напряжений является определяющим свойством, описанным линейным функционалом истории пластических деформаци. Правила Прагера и Зиглера были получены как частные случаи.

1. Introduction

THE CONCEPT of the intrinsic time scale was introduced as a proper base of measurement of the memory of a material of its past deformation history, leading to constitutive theories which we have called "endochronic" [1, 2]. The case of strain rate independent yet history dependent materials was dealt with at length in previous references by the author and, subsequently, by other workers in the field whose contributions are duly referenced [3–8]. Other work, in other directions, involving the inelastic behavior of metals is currently being pursued by other authors. References [9, 10 and 11] are typical of this work. In the papers cited heretofore, we introduced two measures of intrinsic time. One pertains to the path traced by the deformation state in a nine-dimensional strain space. The other pertains to a stress path traced by the state of stress in a nine-dimensional stress space.

If we denote these times by ζ_ε and ζ_σ respectively, then limiting ourselves to small strains,

$$(1.0)\qquad \begin{aligned} d\zeta_\varepsilon &= P_{ijkl}\, d\varepsilon_{ij}\, d\varepsilon_{kl},\\ d\zeta_\sigma &= R_{ijkl}\, d\sigma_{ij}\, d\sigma_{kl},\end{aligned}$$

where σ_{ij} and ε_{ij} are the stress and infiniterminal strain tensors respectively, and P_{ijkl} and R_{ijkl} are the corresponding metrics. Repeated indices imply summation over their range of values in the usual fashion unless otherwise stated.

The constitutive equations that followed, in the light of these definitions, were obtained from thermodynamic arguments. In particular, the internal variable theory was used to arrive at the following set of constitutive equations which pertain to isothermal conditions:

$$(1.1)\qquad \boldsymbol{\sigma} = \frac{\partial\psi}{\partial\boldsymbol{\epsilon}},$$

$$(1.2)\qquad \frac{\partial\psi}{\partial\mathbf{q}_r} + \mathbf{b}_r\cdot\frac{d\mathbf{q}_r}{dz} = 0 \quad (r \text{ not summed}),$$

$$(1.3)\qquad z = z(\zeta), \quad \frac{dz}{d\zeta} > 0,$$

where ζ is given by either Eq. $(1.0)_1$ or Eq. $(1.0)_2$. In the above equations ψ is the free energy density and $\mathbf{q}_r$ the internal variables of the thermodynamic system. These are second-order tensors in the present treatment. In Eq. (1.2) $\mathbf{b}_r$ is a fourth-order dissipation tensor which is positive definite. A dot between two tensors represents an inner product.

The "rate" of irreversible entropy production, which is a measure of the internal dissipation, is obtained from the equation

$$(1.4)\qquad T\hat{\gamma} = -\frac{\partial\psi}{\partial\mathbf{q}_r}\cdot\hat{\mathbf{q}}_r,$$

where a roof over a symbol indicates differentiation with respect to z, and T is the absolute temperature. Evidently, as a result of Eqs. (1.2) and (1.4),

$$(1.5)\qquad T\hat{\gamma} = \hat{\mathbf{q}}_r\cdot\mathbf{b}_r\cdot\hat{\mathbf{q}}_r \quad (r \text{ summed}),$$

where the right-hand side of Eq. (1.5) is an inner product, clearly a positive scalar.

Observation of accumulated results of the application of the above theory to a variety of histories indicates certain broadly consistent trends. The theory is evidently simple, versatile and has powers of prediction additional to the theories of plasticity of the classical type *provided* there are no reversals in the rate of stress. In particular in one dimension, the slope of unloading, at a point of the uniaxial stress strain curve, was predicted by the "linear" version of the theory to be $2E_0-E_t$ where E_0 is the elastic modulus and E_t the tangent modulus. This is an overestimate of the normally observed unloading slope which is close to E_0.

The purpose of this paper is to eliminate this deficiency of the endochronic theory by introducing a measure of intrinsic time which is more closely representative of the dissipation properties of metals. The underlying cause of this discrepancy is of a thermodynamic nature and has to do with the fact that the theory, heretofore, predicts a rate of dissipation during loading which is identical to that which occurs at the onset of unloading. See Ref. [15].

In this paper we introduce a new definition of intrinsic time ζ which corrects both these deficiencies. In one (axial) dimension we stipulate that

$$d\zeta = \left| d\varepsilon - k\frac{d\sigma}{E_0} \right|, \tag{1.6}$$

where k is a positive scalar such that $0 \leqslant k \leqslant 1$ and E_0 is the elastic modulus. If, at the onset of unloading, we denote the unloading slope by E_- and the rate of dissipation by $\dot{\gamma}_-$, then it can be shown [15] that E_- tends to E_0 and $\dot{\gamma}_-$ tends to zero as k tends to unity.

In three dimensions we form a strain-like tensor θ_{ij} given by Eq. (1.7):

$$\theta_{ij} = \varepsilon_{ij} - \phi_{ijkl}\sigma_{kl}, \tag{1.7}$$

where $\boldsymbol{\phi}$ is a positive definite symmetric fourth-order material tensor. Dealing strictly with isotropic materials where $\boldsymbol{\phi}$ is isotropic, we proceed to define a deviatoric strain-like tensor η_{ij} by the equation

$$\eta_{ij} = e_{ij} - \frac{k_1}{2\mu} s_{ij}, \tag{1.8}$$

where e_{ij} and s_{ij} are the deviatoric strain and stress tensors, respectively. We also define a hydrostatic strain-like tensor θ_{kk} by the equation

$$\theta_{kk} = \varepsilon_{kk} - \frac{k_0}{3K}\sigma_{kk}. \tag{1.9}$$

The relation between k_0 and k_1 on one hand and the components of $\boldsymbol{\phi}$ on the other is shown in Sect. 2. Evidently, η_{ij} and θ_{kk} are the deviatoric and hydrostatic components, respectively, of θ_{ij}.

A hydrostatic intrinsic time measure $d\zeta_H$ and a deviatoric counterpart $d\zeta_D$ are now defined by the equations

$$\begin{aligned} d\zeta_H^2 &= k_{00}\, d\theta_{kk} d\theta_{ll} + k_{01}\, d\eta_{ij} d\eta_{ij}, \\ d\zeta_D^2 &= k_{10}\, d\theta_{kk} d\theta_{ll} + k_{11}\, d\eta_{ij} d\eta_{ij}, \end{aligned} \tag{1.10}$$

where k_{rs} are material parameters which provide for a coupling between hydrostatic and shear response.

The resulting constitutive equation for isotropic materials is given by Eqs. (2.8) and (2.9). These are identical in form to those of the simple endochronic theory (where both k_0 and k_1 are equal to zero) but predict behavior that is far closer to that of metals, when k_0 and k_1 are close to unity.

1.1. The very significant case of uncoupled deviatoric and hydrostatic response $k_{10} = k_{01} = 0$ and $k_1 = k_0 = 1$

1.1.1. Deviatoric response. In this case the deviatoric intrinsic time measure $d\zeta_D$ is equal to the norm of the deviatoric plastic strain tensor. The resulting deviatoric constitutive response is given by Eq. (2.35) or, equivalently, Eq. (2.37). With the aid of these equations the following propositions are proved in Sect. 2 of the text.

(i) A spherical yield surface in deviatoric stress space exists.

(ii) If $\varrho_1 = 0$ in Eq. (2.35) and $f(\zeta)$ increases monotonically with ζ, then the classical theory of plasticity with isotropic hardening follows. The increment of plastic strain is shown to be normal to the yield surface.

(iii) If $\varrho_1 \neq 0$ (in which case $\varrho_1 > 0$) and $f(\zeta) = 1$, then the spherical yield surface translates in deviatoric stress space and a theory of kinematic hardening results with a general rule which contains Prager's and Ziegler's rules as special cases.

(iv) If $\varrho_1 \neq 0$ and $f(\zeta)$ is a monotonically increasing function, then the yield surface translates and expands simultaneously according to rules inherent in the theory.

(v) The above are true in the case of softening, in which case the yield surface contracts, when $f(\zeta)$ is a monotonically decreasing function.

REMARK. No yield surface exists if $k_1 < 1$. Thus the classical theory of plasticity is the "boundary" of the endochronic theory, assuming that the hydrostatic response is elastic.

1.1.2. Hydrostatic response. The endochronic theory admits a more general hydrostatic response than the elastic hydrostatic response usually adopted in the classical plasticity theory. This is given by Eq. (2.53) which is discussed in the text.

2. Analysis in three dimensions and n internal variables

We define the strain-like tensor θ_{ij} by the following equation:

$$\theta_{ij} = \varepsilon_{ij} - \phi_{ijkl}\sigma_{kl}, \tag{2.1}$$

where $\boldsymbol{\phi}$ is a positive definite symmetric fourth-order tensor. In the case of isotropic materials, $\boldsymbol{\phi}$ is of the form

$$\phi_{ijkl} = \delta_{ij}\delta_{kl}\phi_0 + \phi_1(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{kj}). \tag{2.2}$$

In this event, $\boldsymbol{\theta}$ may be decomposed into its deviatoric and hydrostatic parts as follows:

$$\theta_{ij} = \frac{1}{3}\theta_{kk}\delta_{ij} + \eta_{ij}, \tag{2.3}$$

where

$$\theta_{kk} = \varepsilon_{kk} - \frac{k_0}{3K_0}\sigma_{kk}, \tag{2.4}$$

$$\eta_{ij} = e_{ij} - \frac{k_1}{2\mu_0}s_{ij}. \tag{2.5}$$

In Eq. (2.5) $\mathbf{e}$ and $\mathbf{s}$ are the deviatoric parts of $\boldsymbol{\epsilon}$ and $\boldsymbol{\sigma}$, respectively, μ_0 and K_0 are the elastic shear and bulk moduli of the material, respectively, and the constants k_0 and k_1 are related to ϕ_0 and ϕ_1 by the relations

$$2\phi_1 = \frac{k_1}{2\mu_0}, \quad (3\phi_0+2\phi_1) = \frac{k_0}{3K_0}.$$

Various degrees of generality are now possible. For instance, mindful of the consequence that in the case of isotropic materials, undergoing small deformation, the deviatoric and hydrostatic responses are separable [2], we may define a hydrostatic intrinsic time measure $d\zeta_H$, where

(2.6) $$d\zeta_H^2 = k_{00}\, d\theta_{kk} d\theta_{ll} + k_{01}\, d\eta_{ij} d\eta_{ij}$$

and a deviatoric intrinsic time measure $d\zeta_D$, where

(2.7) $$d\zeta_D^2 = k_{10}\, d\theta_{kk} d\theta_{ll} + k_{11}\, d\eta_{ij} d\eta_{ij}.$$

The hydrostatic measure $d\zeta_H$ would then be the appropriate one to use in the hydrostatic constitutive response, while $d\zeta_D$ would be used in the deviatoric constitutive response. In this case k_{ij} $(i, j = 0, 1)$ is a matrix of nondimensional scalars.

Using the formulation of Ref. [1] as a point of departure, we may then write the rate independent (plastic) response of metals in the small deformation region as follows:

(2.8) $$\mathbf{s} = 2\int_0^{z_D} \mu(z_D - z'_D) \frac{\partial \mathbf{e}}{\partial z'_D} dz'_D$$

and

(2.9) $$\sigma_{kk} = 3\int_0^{z_H} K(z_H - z'_H) \frac{\partial \varepsilon_{kk}}{\partial z'_H} dz'_H,$$

where

(2.10) $$dz_D = \frac{d\zeta_D}{f_D(\zeta_D)}$$

and

(2.11) $$dz_H = \frac{d\zeta_H}{f_H(\zeta_H)}.$$

The functions f_D and f_H are both non-negative and satisfy the condition

(2.11') $$f_D(0) = f_H(0) = 1.$$

Of particular importance is the case where hydrostatic and deviatoric responses are uncoupled, i.e. $k_{01} = k_{10} = 0$. In this case k_{00} and k_{11} are both set equal to unity, without loss of generability, as one can verify by analysis if one wishes to do so. Thus Eqs. (2.6) and (2.7) now become

(2.12) $$d\zeta_D^2 = d\eta_{ij} d\eta_{ij},$$

(2.13) $$d\zeta_H^2 = d\theta_{kk} d\theta_{ll}.$$

We wish to discuss, at length, the case of the shear response; the arguments will apply equally well to its hydrostatic counterpart. With reference to Eq. (2.8), one obtains the following relation:

$$\mathbf{s} = 2\mu_0 \int_0^{z_D} \varrho(z_D - z'_D) \frac{d\boldsymbol{\eta}}{dz'_D} dz'_D, \tag{2.14}$$

where

$$\mu(z) \stackrel{\text{def}}{=} \mu_0 G(z), \quad G(0) = 1, \tag{2.15}$$

and $\varrho(z)$ is related to $G(z)$ by the integral equation

$$\varrho(z) - k_1 \int_0^z \varrho(z - z') \frac{dG}{dz'} dz' = G(z). \tag{2.16}$$

The solution of this equation in the case of two internal variables is given in Appendix A of Ref. [15]. More simply, the following algebraic relation exists between the Laplace transforms $\bar{\varrho}$ and $\bar{G}$ of ϱ and G, where p is the Laplace transform(1) variable:

$$\bar{\varrho} = \frac{\bar{G}}{1 - k_1 p \bar{G}}. \tag{2.17}$$

This specific form of the theory has the characteristic feature that it leads to a constitutive equation which gives the deviatoric stress response to the history of the deviatoric strain-like tensor $\boldsymbol{\eta}$ in terms of the path in $\boldsymbol{\eta}$ space. We point out, however, that there is no specific connection with an *a priori* existence of a yield surface. Furthermore, there is no dichotomy in the constitutive representation of the loading and unloading responses.

It is of interest to note that $d\boldsymbol{\eta}$ is akin to and is equal to the deviatoric "plastic strain" increment, in classical plasticity terms, if k_1 is unity and the "elastic strain" increment is defined as $\frac{d\mathbf{s}}{2\mu}$. Moreover, we will proceed to show that if $k_1 = 1$, then a yield surface exists. The proof will apply strictly to the case where $k_1 = 1$.

To this end we invoke the result of Ref. [1] according to which

$$G(z) = \sum_{r=1}^{n} G_r e^{-\alpha_r z}, \tag{2.18}$$

where G_r are positive; also, α_r are all positive with the exception of α_1 which may be zero. Furthermore, since $G(0) = 1$, it follows that

$$\sum_{r=1}^{n} G_r = 1. \tag{2.19}$$

Let $\bar{G}(p)$ be the Laplace transform of $G(z)$. Then, as a result of Eq. (2.18) $\bar{G}(p)$ is of the form

$$\bar{G}(p) = \frac{\bar{P}(p)}{\bar{Q}(p)}, \tag{2.20}$$

(1) For explanatory notes on this treatment see Appendix C of Ref. [15].

where

(2.21) $\bar{P} = G_1(p+\alpha_2) \dots (p+\alpha_n)+(p+\alpha_1)G_2 \dots (p+\alpha_n) \dots +(p+\alpha_1) \dots (p+\alpha_{n-1})G_n$

and

$$\bar{Q}(p) = (p+\alpha_1)(p+\alpha_2) \dots (p+\alpha_n). \tag{2.22}$$

Evidently, $\bar{P}$ and $\bar{Q}$ are polynomials of the order $(n-1)$ and n, respectively. Furthermore, the coefficient of the leading term of $\bar{P}$ is equal to unity. It follows from Eqs. (2.17) and (2.20) that

$$\bar{\varrho}(p) = \frac{\bar{P}(p)}{\bar{R}(p)}, \tag{2.23}$$

where

$$\bar{R}(p) = \bar{Q}(p)-p\bar{P}(p). \tag{2.24}$$

It may be shown by direct computation that $\bar{R}(p)$ is a polynomial of the order $n-1$, such that the coefficient of p^{n-1} is exactly the sum $\sum_{r=1}^{n} \alpha_r G_r$. Hence the ratio $\frac{\bar{P}(p)}{\bar{R}(p)}$ is not function-like, in the sense of Mikusinski, in that it contains a delta function of strength $\frac{1}{\sum_{r=1}^{n} \alpha_r G_r}$. In fact, if $-\beta_r$ are the zeros of $\bar{R}$, then $\bar{\varrho}(p)$ may be written as

$$\bar{\varrho}(p) = \frac{1}{\sum_{r=1}^{n} \alpha_r G_r} + \sum_{r=1}^{n-1} \frac{R_r}{\beta_r+p}, \tag{2.25}$$

where $\bar{\varrho}(p) - \frac{1}{\sum_{r=1}^{n} \alpha_r G_r}$ is the ratio of two polynomials, the numerator being of degree $n-2$, and the denominator $\bar{R}(p)$ of degree $n-1$. It is shown below that if the absolute values of the zeros of $\bar{Q}(p)$ are ordered in the sense that

$$\alpha_1 < \alpha_2 < \alpha_3 \dots < \alpha_n,$$

then β_r must always satisfy the inequalities

$$\alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \alpha_3 \dots < \beta_{n-1} < \alpha_n$$

and are therefore all positive. The proof is elementary. Evidently,

$$\bar{R}(-\alpha_1) = -\alpha_1(\alpha_2-\alpha_1)(\alpha_3-\alpha_1) \dots (\alpha_n-\alpha_1) < 0,$$
$$\bar{R}(-\alpha_2) = -\alpha_2(\alpha_1-\alpha_2)(\alpha_3-\alpha_2) \dots (\alpha_n-\alpha_2) > 0.$$

Similarly,

$$\bar{R}(-\alpha_3) < 0$$

and generally $\bar{R}(p)$ alternates in sign at the zeros of $\bar{Q}(p)$. It follows that $\bar{R}(p)$ must vanish at the points $p = -\beta_1$, $p = -\beta_2 \dots p = -\beta_{n-1}$ where β_r are all positive and are bounded from above and below by α_r according to the inequalities given above. Finally, since $\bar{R}(p)$ is a polynomial of degree $n-1$, it can have no other zeros. This completes the proof.

The values of the residues R_r are found from the formula

$$R_r = \frac{\bar{P}(-\beta_r)}{\bar{R}'(-\beta_r)}, \tag{2.26}$$

where $R'(p) \overset{\text{def}}{=} \frac{dR}{dp}$. For a detailed calculation see Appendix B of Ref. [15], where it is shown that the residues R_r are all positive.

Clearly, as a result of Eq. (2.25),

$$\varrho(z) = \frac{\delta(z)}{\sum_{r=1}^{n} \alpha_r G_r} + \sum_{r=1}^{n-1} R_r e^{-\beta_r z}, \tag{2.27}$$

which we write as

$$\varrho(z) = \varrho_0 \delta(z) + \varrho_1(z), \tag{2.28}$$

where

$$\varrho_1(z) = \sum_{r=1}^{n-1} R_r e^{-\beta_r z}. \tag{2.28'}$$

Evidently, it follows that

$$\mathbf{s} = 2\mu_0 \varrho_0 \frac{d\boldsymbol{\eta}}{dz} + 2\mu_0 \int_0^z \varrho_1(z-z') \frac{\partial \boldsymbol{\eta}}{dz} dz'. \tag{2.29}$$

Now, as a consequence of Eq. (2.29), at $z = 0$:

$$\mathbf{s} = 2\mu_0 \varrho_0 \left.\frac{d\boldsymbol{\eta}}{dz}\right|_{z=0}. \tag{2.30}$$

Also from Eq. (2.5) (and for $k_1 = 1$), the condition $\boldsymbol{\eta} = 0$ (and, therefore, $z = 0$) gives the relation

$$\mathbf{s} = 2\mu_0 \mathbf{e}. \tag{2.31}$$

Equation (2.31) merely attests to the fact that while $z = 0$ the deformation process is reversible and therefore the deviatoric stress response is elastic. See Sect. 3. It is also of interest that at $z = 0$, $\frac{d\boldsymbol{\eta}}{dz}$ is indeterminate and can take any value consistent with Eq. (2.31). Specifically, Eqs. (2.30) and (2.31) combine to give

$$\varrho_0 \left|\frac{d\boldsymbol{\eta}}{dz}\right|_0 = \mathbf{e}. \tag{2.32}$$

However, at $z > 0$ the derivative $\frac{d\boldsymbol{\eta}}{ds}$ exists and specifically at $z = 0^+$ one obtains the limit of $\frac{d\boldsymbol{\eta}}{dz}$ by approaching $z = 0$ from the right. In fact, the point 0^+ is the point of deviation from elastic response or the yield point. At this point

$$\left|\frac{d\boldsymbol{\eta}}{d\zeta}\right|^2 = 1 \tag{2.33}$$

in accordance with Eq. (2.12). Therefore, applying Eq. (2.30) and mindful of Eq. (2.11′),

$$|\mathbf{s}|^2 = 4\mu_0^2\varrho_0^2 \overset{\text{def}}{\equiv} s_Y^2. \tag{2.34}$$

In conclusion, in the process of monotonic loading, while $|\mathbf{s}| \leqslant s_Y$, $z = 0$ and Eq. (2.31) applies and the material response is elastic. However, when $|\mathbf{s}| > s_Y$, then Eq. (2.29) applies and the response is no longer elastic. Of course, s_Y is the *yield* stress and Eq. (2.34) is the von Mises yield criterion.

In particular, as a result of Eq. (2.34), the constitutive relation (2.29) may be written more succinctly as

$$\mathbf{s} = s_Y^0 \frac{d\boldsymbol{\eta}}{dz_D} + 2\mu_0 \int_0^{z_D} \varrho_1(z_D - z'_D) \frac{\partial\boldsymbol{\eta}}{\partial z'_D} dz'_D, \tag{2.35}$$

where $s_Y^0 = 2\mu_0\varrho_0$ and has the physical significance of an initial yield stress.

We shall show presently that Eq. (2.35) yields very rich results and reveals important characteristics of plastic behavior that first appeared as assumptions or conjectures in the classical theory of plasticity.

To this end, let the integral on the right-hand side of Eq. (2.35) be denoted by $\mathbf{r}$, i.e. set

$$\mathbf{r} = 2\mu_0 \int_0^{z_D} \varrho_1(z_D - z'_D) \frac{\partial\boldsymbol{\eta}}{\partial z'_D} dz'_D. \tag{2.36}$$

Equation (2.35) then reads

$$\mathbf{s} - \mathbf{r} = s_Y^0 \frac{d\boldsymbol{\eta}}{d\zeta_D} f(\zeta_D). \tag{2.37}$$

We recall that $k_1 = 1$ and therefore $d\boldsymbol{\eta}$ is exactly equal to the increment of plastic strain. We wish to examine Eq. (2.37) in the specific case where ζ_D and ζ_H are uncoupled,

$$d\zeta_D^2 = d\boldsymbol{\eta} \cdot d\boldsymbol{\eta} \tag{2.38}$$

and the hydrostatic response is elastic, i.e. $dz_H \equiv 0$, in which event both Eqs. (2.9) and (2.13) give rise to Eq. (2.39), i.e.

$$\sigma_{kk} = 3K\varepsilon_{kk} \tag{2.39}$$

in our previous notation.

Case (i): $f(\zeta) = 1$

Equations (2.37) and (2.38) give rise immediately to the following two results:

$$\|\mathbf{s}-\mathbf{r}\|^2 = s_Y^{0^2}, \tag{2.40}$$

$$d\boldsymbol{\eta} = \frac{1}{s_Y^0}(\mathbf{s}-\mathbf{r})d\zeta_D. \tag{2.41}$$

It is clearly obvious that if $\mathbf{r} = \mathbf{0}$ then Eq. (2.41) is that of an elastic perfectly plastic material with a von Mises yield criterion. In this case

$$d\boldsymbol{\eta} = \frac{1}{s_Y^0}\mathbf{s}\,d\zeta_D. \tag{2.42}$$

On the other hand, if $\mathbf{r} \neq \mathbf{0}$, then Eq. (2.40) shows readily that case (i) corresponds to *kinematic hardening*. This relation is in fact the equation of a hypersphere in deviatoric stress space. It also represents the equation of a circle in *principal* stress space. In either case s_Y^0 is the radius of the hypersphere (circle) and $\mathbf{r}$ is the radius vector which connects the origin of the stress space to the centre of the hypersphere (circle). The yield surface is, therefore, a translating spherical (circular) surface. Equation (2.41) shows that the increment in plastic strain is normal to the yield surface. These results are shown diagrammatically in Fig. 1. Note, however, that whereas in the classical theory of plasticity the concept of kinematic hardening was a conjecture, *here it is a derived result*.

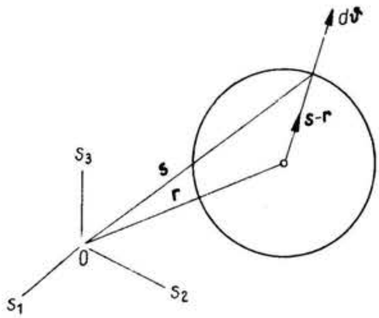

FIG. 1. Geometric illustration of Eqs. (2.40) and (2.41).

Furthermore, we feel it is important to emphasize that in the classical theory it is not known how the surface translates, i.e., it is not known *a priori* how $\mathbf{r}$ depends on the history of loading or plastic strain. For instance, Prager assumed that

$$d\mathbf{r} = c\,d\boldsymbol{\eta}, \tag{2.43}$$

where c is a material constant. Ziegler suggested that

$$d\mathbf{r} = d\mu(\mathbf{s}-\mathbf{r}), \tag{2.44}$$

where $d\mu$ is a positive quantity not specified. We point out that Eq. (2.43) is equivalent to Eq. (2.44) if $d\mu$ is proportional to $d\zeta_D$ as Eq. (2.41) readily indicates. However, Prager's (or Ziegler's) rule is a particular case of the present theory as is pointed out in the following Remark.

REMARK 1. Prager's rule of kinematic hardening is a particular case of the present theory obtained from Eq. (2.36) by setting

(2.45) $$\varrho_1(\zeta_D) = \text{constant}$$

in which event

(2.46) $$\mathbf{r} = 2\mu_0 \varrho_1 \boldsymbol{\eta}.$$

This, as is well known, is called linear hardening in the sense that the stress strain curve in shear is linear beyond the onset of yield.

In this same vein, if $\varrho_1(\zeta_D)$ is not a constant but consists of a single exponential, that is

(2.47) $$\varrho_1(\zeta_D) = \varrho_1 e^{-\alpha\zeta_D},$$

then it is easily shown as a result of Eqs. (2.36) and (2.37) that

(2.48) $$d\mathbf{r} = \frac{2\mu_0\varrho_1}{s_Y^0} dz(\mathbf{s}-\beta\mathbf{r}),$$

where

(2.49) $$\beta = 1+\frac{\alpha s_Y^0}{2\mu_0\varrho_1}.$$

This is a new type of kinematic hardening. Note that the yield surface no longer translates along the outward normal at the extremity of the stress vector but in a direction which is skew. The skewness depends on the value of β.

Of course, the above are particular cases. In general the translation vector is determined by Eq. (2.36), in terms of a convolution product which involves the material function $\varrho_1(\zeta_D)$. Within the assumptions of the present case, it is important to observe that $\varrho_1(\zeta_D)$ can be determined by a simple shear experiment. In fact, one can show readily from Eq. (2.36) that

(2.50) $$2\mu_0\varrho_1(\zeta_D) = \left.\frac{d\tau}{d\gamma_p}\right|_{\gamma_p=\zeta_D}, \quad \tau > \tau_Y,$$

where τ is the shear stress and γ_p the tensorial plastic shear strain component.

REMARK 2. The function $\varrho_1(\zeta_D)$ is determinate from a single monotonic shear test.

REMARK 3. The mode of translation of the yield surface([2]) is determined from a single monotonic shear experiment.

Case (ii): $f(\zeta)$ monotonically increasing

The counterparts of Eqs. (2.40) and (2.41) are now the following:

(2.51) $$||\mathbf{s}-\mathbf{r}|| = f(\zeta_D)s_Y^0,$$

(2.52) $$d\boldsymbol{\eta} = \frac{1}{s_Y^0 f(\zeta_D)}\cdot(\mathbf{s}-\mathbf{r})d\zeta_D.$$

([2]) for all histories

Clearly, if $r = 0$, then Eq. (2.52) is that of a plastic material with *isotropic hardening* and a von Mises yield criterion. In this case

(2.52a)
$$d\boldsymbol{\eta} = \frac{1}{s_Y^0 f(\zeta_D)} \mathbf{s} d\zeta_D.$$

On the other hand, if $\mathbf{r} \neq \mathbf{0}$, then Eq. (2.51) shows that the yield surface now expands as well as translates. The increment of plastic strain is still normal for the yield surface. The translation vector $\mathbf{r}$ is still given by Eq. (2.36). We shall not go into the details of determining $f(\zeta)$ and $\varrho_1(\zeta_D)$ in this case, at this juncture. The reader is referred to Appendix D, of Ref. [15].

The constitutive equation for $\sigma > \sigma_Y$ for the hydrostatic response is identical in form to the above and can be written down by inspection using analogous terminology. To wit, when $k_0 = 1$, then

(2.53)
$$\sigma = \sigma_Y^0 \frac{d\theta}{dz_H} + K_0 \int_0^{z_H} \phi_1(z_H - z'_H) \frac{\partial\theta}{dz'_H} dz'_H,$$

where $\sigma = \sigma_{kk}/3$. If in the process of monotonic loading $\sigma \leqslant \sigma_Y$, then the response is elastic and

(2.54)
$$\sigma = K_0 \varepsilon_{kk}.$$

Of course, the above discussion applies only to the case when k_0 is equal to unity.

3. Elastic response at points of interdeterminancy of $\dfrac{d\eta}{dz_D}$

In this section we shall show that when $k_1 = 1$, there exist physical processes other than the ones discussed above for which $d\zeta_D$ is equal to zero and that in fact these processes are associated with *elastic* deformation. Analogous conclusions can be drawn with regard to $d\zeta_H$ when $k_0 = 1$. The above remarks are made in the context of assumed strict independence between deviatoric and hydrostatic response, in the sense that a history of deviatoric strain has no effect on the hydrostatic response and, correspondingly, a history of hydrostatic strain has no effect on the deviatoric response(³).

It is important for our purposes to introduce certain definitions:

Deviatoric plastic strain space

The space of deviatoric strain components with metric δ_{ij}. The coordinates of this space will be denoted by χ_i, $(i = 1, 2, \ldots, 9)$ corresponding to the components of the tensor η_{ij}.

(³) Henceforth the suffix D will be omitted. Also, this chapter will apply exclusively to deviatoric response and occasional omission of the word "deviatoric" must not be construed to imply otherwise.

Deviatoric plastic strain path

This is a continuous oriented curve in χ_i space.

REMARK. The path determines the variation of χ_i in this space. We limit the discussion to paths that pass through the origin. In this event $\chi_i = 0$ at the origin.

Let s be a parameter which is zero at the origin and increases (decreases) monotonically as the curve is traversed in the positive (negative) direction. Let the curve be traversed in the positive sense and let P_1 and P_2 be two *adjacent* points on the curve such that P_1 precedes P_2. We define ds by the relation

$$ds = |P_1 P_2|, \tag{3.1}$$

where $|P_1 P_2|$ is the distance between the two points. If the curve is traversed in the negative direction, then

$$ds = -|P_1 P_2|. \tag{3.2}$$

We define ζ as the cumulative distance travelled by the extremity of $\vec{\chi}$, i.e.

$$d\zeta = |ds| \tag{3.3}$$

or

$$\zeta = \int |ds|. \tag{3.4}$$

3.1. Equation (2.35) at reversal points

DEFINITION. *A reversal point R is a point on the strain path s at which there is a reversal in the sign of the strain increment. In effect at R:*

$$d\chi_i^- = -d\chi_i^+. \tag{3.5}$$

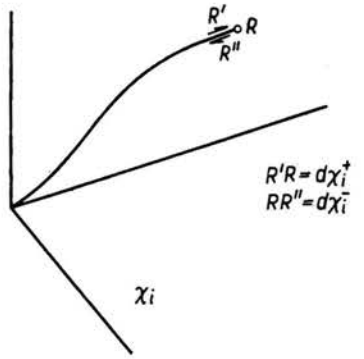

FIG. 2. R as a reversal point.

With reference to Fig. 2, the following relations are of interest:

$$\frac{d\chi_i}{ds}(R') = \frac{d\chi_i}{ds}(R) + \theta_i, \qquad \lim_{R' \to R} \theta_i = 0. \tag{3.6}$$

Similarly,

$$\lim_{R''\to R} \frac{d\chi_i}{ds}(R'') = \frac{d\chi_i}{ds}(R). \tag{3.6'}$$

However, as a result of our definitions of ds and dz,

$$\begin{aligned} &\lim_{R'\to R} \frac{ds}{d\zeta} = 1, \\ &\lim_{R''\to R} \frac{ds}{d\zeta}\Big|_{R''} = -1. \end{aligned} \tag{3.7}$$

Hence, invoking Eq. (2.10),

$$\begin{aligned} &\lim_{R'\to R} \frac{ds}{dz}\Big|_{R'} = f(\zeta)|_R, \\ &\lim_{R''\to R} \frac{ds}{dz}\Big|_{R''} = -f(\zeta)|_R. \end{aligned} \tag{3.8}$$

Also,

$$\begin{aligned} &\lim_{R'\to R} \mathbf{r}(R') = \mathbf{r}(R), \\ &\lim_{R''\to R} \mathbf{r}(R'') = \mathbf{r}(R). \end{aligned} \tag{3.9}$$

Let in the vicinity of R

$$d\boldsymbol{\eta} = \mathbf{l}ds, \tag{3.10}$$

where l_{ij} are the direction cosines of the tangent to the path at R. Then, as a result of Eqs. (3.6) through (3.10) and Eq. (2.35),

$$\begin{aligned} \mathbf{s}(R') &= s_Y^0 \mathbf{l} f_R(\zeta) + \mathbf{r}(R), \\ \mathbf{s}(R'') &= -s_Y^0 \mathbf{l} f_R(\zeta) + \mathbf{r}(R), \end{aligned} \tag{3.11}$$

in the limit $R' \to R$, $R'' \to R$.

Since in fact l_{ij} is normal to the stress hypersphere at R and $s_Y^0 f(\zeta)$ is the current radius of the hypersphere, Eqs. $(3.11)_{1,2}$ admit the geometric construction shown in Fig. 3.

Also, perusal of Fig. 2 shows that

$$\begin{aligned} &\lim_{R'\to R} z(R') = z(R), \\ &\lim_{R''\to R} z(R'') = z(R). \end{aligned} \tag{3.11'}$$

Therefore in the stress range $R'R''$, z is constant and, hence, $dz = 0$. The response, therefore, in this range is elastic and the constitutive equation is simply

$$d\mathbf{s} = 2\mu_0 d\mathbf{e}. \tag{3.12}$$

A plot of ζ versus s for this process is shown in Fig. 4.

Clearly, at R the slope $\dfrac{ds}{d\zeta}$ is indeterminate.

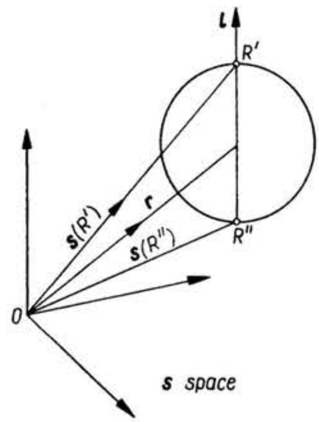

FIG. 3. Stress responses in the limits $R' \to R$, $R'' \to R$.

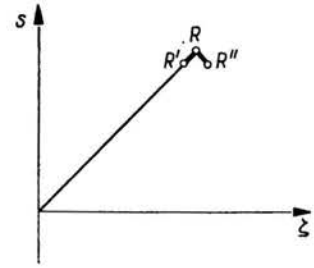

FIG. 4. A history with a point of strain reversal.

3.2. More general deformation processes with constant ζ

We begin with Eqs. (2.35) and (2.36) which we write in the form

$$\mathbf{s} = s_Y^0 \frac{d\boldsymbol{\eta}}{dz} + \mathbf{r} \tag{3.13}$$

or, equivalently,

$$\mathbf{s} = s_Y^0 \int_0^z \delta(z-z') \frac{d\boldsymbol{\eta}}{dz'} dz' + \mathbf{r}. \tag{3.14}$$

Since, however, the delta function in Eq. (3.14) was obtained by a limiting process in which $k_1 \to 1$, we replace $\delta(z)$ by a function which in the above limiting process becomes a delta function, i.e., we write

$$\mathbf{s} = \frac{2\mu_0}{1-k_1} \int_0^z e^{-\alpha(z-z')} \frac{d\boldsymbol{\eta}}{dz'} dz' + \mathbf{r}(z), \tag{3.15}$$

where

$$\alpha = \frac{2\mu_0}{(1-k_1)s_Y^0} \tag{3.16}$$

and $\mathbf{s}$ is obtained from Eq. (3.15) by a limiting process in which k_1 tends to unity. Equation (3.15) may now be written as a differential equation in the form

$$\frac{1}{s_Y^0}(\mathbf{s}-\mathbf{r})dz + \frac{1-k_1}{2\mu_0} d(\mathbf{s}-\mathbf{r}) = d\boldsymbol{\eta}. \tag{3.17}$$

We note in passing that in the limit $k_1 \to 1$

$$\|\mathbf{s}-\mathbf{r}\| = s_Y^0 f(\zeta) = s_Y \tag{3.18a,b}$$

a result given previously in Eq. (2.40). We also recall the definition of $d\boldsymbol{\eta}$:

(3.19)
$$d\boldsymbol{\eta} = d\mathbf{e} - \frac{k_1}{2\mu_0} d\mathbf{s}.$$

Equations (3.17) and (3.19) combine to give the result

(3.20)
$$k_1 \frac{\hat{\mathbf{s}}}{s_Y} d\zeta + (1-k_1) d\hat{\mathbf{e}} = d\boldsymbol{\eta},$$

where

(3.21)
$$\hat{\mathbf{s}} = \mathbf{s} - \mathbf{r},$$

(3.22)
$$d\hat{\mathbf{e}} = d\mathbf{e} - \frac{k_1}{2\mu_0} d\mathbf{r}.$$

However, from Eq. (2.36)

(3.23)
$$d\mathbf{r} = 2\mu_0(\varrho_1(0) d\boldsymbol{\eta} + \mathbf{h} dz),$$

where

(3.24)
$$\mathbf{h} = \int_0^z \varrho_1'(z-z') \frac{d\boldsymbol{\eta}}{dz'} dz'$$

and

(3.25)
$$\varrho_1' = \frac{d\varrho_1}{dz} \quad (0 < z < \infty).$$

Thus, ϱ' is a well-behaved (continuous, monotonically decreasing) function of z. Note that $\mathbf{h}(0) = \mathbf{0}$.

Equations (3.20) and (3.23) combine to give the following equation for dz:

(3.26)
$$\mathbf{a} dz + (1-k_1) d\mathbf{e} = c\, d\boldsymbol{\eta},$$

where

(3.27)
$$\mathbf{a} = \left\{ k_1 \frac{\hat{\mathbf{s}}}{s_Y^0} - k_1(1-k_1)\mathbf{h} \right\} / f(\zeta),$$

(3.28)
$$c = 1 + k_1(1-k_1)\varrho_1(0).$$

By taking the norm of both sides of Eq. (3.26) one obtains the following quadratic equation in $d\zeta$

(3.29)
$$\frac{(c^2 - ||a||^2)}{1-k_1} d\zeta^2 - 2\mathbf{a} \cdot d\mathbf{e} d\zeta - (1-k_1)||d\mathbf{e}||^2 = 0.$$

The two roots of Eq. (3.29) are given by Eq. (3.30):

(3.30)
$$d\zeta = \frac{\mathbf{a} \cdot d\mathbf{e} \pm \sqrt{\mathbf{a} \cdot d\mathbf{e} + (1-k_1)\, C ||d\mathbf{e}||^2}}{C},$$

where

(3.31)
$$C = \frac{c^2 - ||a||^2}{1-k_1}.$$

It may be shown that

$$\lim_{k_1 \to 1} C = C_1, \tag{3.32}$$

where

$$C_1 = 2\left((1+\varrho_1(0)+\frac{\hat{\mathbf{s}}\cdot\mathbf{h}}{s_Y^0 f(\zeta)^2}\right). \tag{3.33}$$

In the limit of $k_1 \to 1$, the two roots of $d\zeta$ are therefore

$$d\zeta = \begin{cases} \hat{\mathbf{s}}\cdot d\mathbf{e}/C_1 \\ 0 \end{cases} \tag{3.34}$$

with the constraint $d\zeta \geqslant 0$.

It follows directly from Eq. (1.15) that the increment $d\gamma$ in irreversible entropy has the same sign as dz. Since $d\gamma$ must either be positive or zero, processes for which dz is negative are *not* admissible.

We shall first investigate the situation where C_1 is positive and, therefore, bounded by the inequalities

$$0 < C_1 < 2\{1+\varrho_1(0)\}.$$

Case (i). $\hat{\mathbf{s}}\cdot d\mathbf{e} > 0$

In this case $d\zeta$ is positive and is given by Eq. (3.35)

$$d\zeta = \frac{\hat{\mathbf{s}}\cdot d\mathbf{e}}{C_1}. \tag{3.35}$$

The process is admissible and constitutes "loading" in the sense that plastic response applies.

Case (ii). $\hat{\mathbf{s}}\cdot d\mathbf{e} < 0$

In this case the dissipation is negative, since $d\zeta$ is negative, and the process is inadmissible. Therefore the second root of $d\zeta$ must be chosen and, as a result,

$$d\zeta = 0. \tag{3.36}$$

In this event the dissipation is zero and the deformation is elastic since Eq. (3.36) implies that

$$d\boldsymbol{\eta} = 0 \tag{3.37}$$

in which case

$$d\mathbf{s} = 2\mu_0\, d\mathbf{e}. \tag{3.38}$$

REMARK. The inequality $\hat{\mathbf{s}}\cdot d\mathbf{e} < 0$ of case (ii) is the "unloading condition" given unequivocally by the endochronic theory. More specifically, given a state of stress on the yield surface, the process is dissipative and $d\zeta > 0$, if the strain increment $d\mathbf{e}$ makes an acute angle with the radius vector $\hat{\mathbf{s}}$ through the point which represents the state of stress; see Eq. (3.35). The process is elastic if the angle is equal to or greater than ninety degrees, in which case $\hat{\mathbf{s}} \cdot d\mathbf{e} \leqslant 0$.

Case $C_1 = 0$

The situation where $C_1 \to 0$ results in $d\zeta$ becoming unboundedly large in which event the material flows without limit. This event will occur when

$$\hat{\mathbf{s}} \cdot \mathbf{h} = -s_Y^0(1+\varrho_1(0))f(\zeta)^2. \tag{3.39}$$

3.3. Stress paths within the yield surface

If Eq. (2.35) is to apply, $d\zeta$ must be different from zero (in fact positive) since otherwise the derivative $\frac{d\boldsymbol{\eta}}{d\zeta}$ is not determinate. Therefore, if Eq. (2.35) applies, then the incremental process emanating from state $\mathbf{s}$ is dissipative and vice-versa. In the same vein, Eq. (2.51) is a direct consequence of Eq. (2.35). Therefore, if an incremental process originating at state s is to be dissipative, Eq. (2.51) must necessarily apply. It follows as a corollary stating that stress states which do not satisfy Eq. (2.51) cannot lead immediately to dissipative processes.

REMARK. States $\mathbf{s}$ which lie *within* the yield surface cannot lead immediately to dissipative incremental processes.

It follows from the above remark that all incremental processes which emanate from stress states within the yield surface are elastic ($d\zeta = 0$), in which event for all $d\mathbf{s}$ emanating from such state $\mathbf{s}$:

$$d\mathbf{s} = 2\mu d\mathbf{e}. \tag{3.40}$$

3.4. Incremental determination of the constitutive response

Given that the constitutive properties of a material are known, let, at some point in the course of the deformation (strain) history, $\mathbf{s}$ and $\mathbf{h}$ also be known. Then C_1 can be determined from Eq. (3.33). Given an increment of strain $d\mathbf{e}$, Eq. (3.34) determines $d\zeta$, depending on the sign of $\hat{\mathbf{s}} \cdot d\mathbf{e}$. If the sign is negative or zero, then $d\zeta = 0$ and $d\mathbf{s}$ is given by Eq. (3.38). If the sign is positive, then

$$d\zeta = \frac{\hat{\mathbf{s}} \cdot d\mathbf{e}}{C_1} \tag{3.41}$$

and

$$dz = \frac{d\zeta}{f(\zeta)}. \tag{3.42}$$

In the limit of $k_1 \to 1$ then, as a result of Eq. (3.17) or Eq. (3.13),

$$d\boldsymbol{\eta} = \frac{\hat{\mathbf{s}}}{s_Y^0} dz. \tag{3.43}$$

Hence $d\boldsymbol{\eta}$ may be calculated. Knowledge of $d\boldsymbol{\eta}$ leads to the direct determination of $d\mathbf{s}$ from Eq. (3.19) in the limit of $k_1 \to 1$, i.e.

$$d\mathbf{s} = 2\mu_0(d\mathbf{e} - d\boldsymbol{\eta}). \tag{3.44}$$

Clearly $d\mathbf{r}$ may now be found from Eq. (3.23) and the procedure may thus be repeated at will.

We note that at the initiation of plastic deformation $\mathbf{r} = \mathbf{0}$, $\mathbf{h} = \mathbf{0}$ and $\mathbf{s} = 2\mu_0 \mathbf{e}$.

4. The closure of hysteresis loops

The question of closure is treated by means of an application to aluminum undergoing stress reversals in a uniaxial stress field. One has difficulties in obtaining closed hysteresis loops in the first quadrant of the stress-strain diagram if one uses the definition of intrinsic time of Ref. [1], i.e. Eq. $(1.0)_1$. The purpose of this section is to demonstrate that no such difficulties arise when

$$d\zeta = |d\theta_1| \tag{4.1}$$

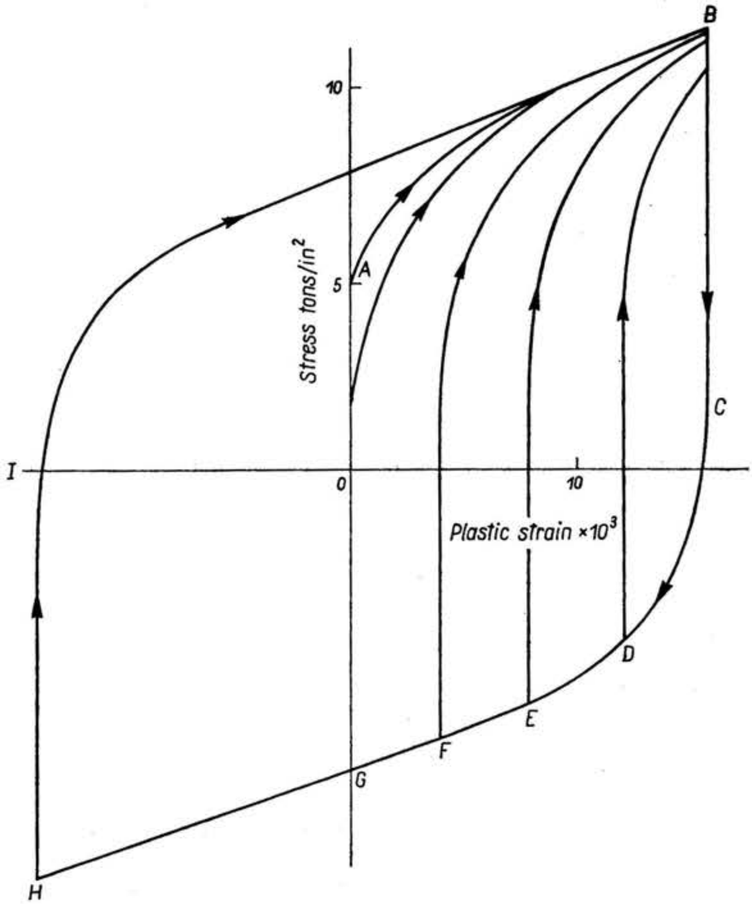

FIG. 5. Theoretically predicted unloading-reloading behaviour of aluminum.

and

$$\theta_1 = \varepsilon_1 - \frac{k\sigma_1}{E} \tag{4.2}$$

and $k = 1$.

For the purpose of demonstration we use the constitutive equation

$$\sigma_1 = \sigma_Y \frac{d\theta_1}{d\zeta} + \int_0^{\zeta} E(\zeta - \zeta') \frac{d\theta_1}{d\zeta'} d\zeta', \tag{4.3}$$

where

$$E(\zeta) = E_1 e^{-\alpha\zeta} + E_2 \tag{4.4}$$

with the following values of the constants: $\sigma_Y = 5$ tons/in^2, $E_1 = 1500$ tons/in^2, $E_2 = 200$ tons/in^2, $\alpha = 500$. The elastic modulus E_0 is equal to 4.46×10^3 tons/in^2; however, this value is not pertinent to the present application since σ_1 is plotted directly against θ_1. The computation of the stress response to loading-unloading-reloading histories is straight-forward and will not be treated in detail here. For a more elaborate treatment of this problem the reader is referred to Ref. [2]. The results are shown in Fig. 5.

The point A is the initial yield event. Point B denotes the point of unloading and C, D, E, F, G, and H are points of reloading. Note that points such as C, where $\sigma_C \geqslant \sigma_B - 2\sigma_Y$, give rise to reloading paths which coincide with the unloading paths. In such a case the area of the hysteresis loops is zero. However, reloading from points where $\sigma < \sigma_B - 2\sigma_Y$ gives rise to loops that are closed, as shown in the figure. In this constitutive theory, as it is constituted by Eqs. (4.1), (4.2) and (4.3), no open loops exist for $k = 1$.

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Received May 16, 1979.