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Reflection of a weak shock wave from an isothermal wall

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THE PROBLEM of reflection of a weak shock wave of its front parallel to the wall is analysed. Only the case of the isothermal wall is considered. The flow domain is divided into two parts: an outer domain containing both shock waves and an inner domain adjacent to the wall. To determine the outer flow the Lighthill technique and the multiple scales method are combined. The structure of the inner flow is exactly the same as in [1]. To determine some unknown functions the matching principle is used. The structure and the trajectory of the reflected shock wave are obtained as results.

W pracy analizuje się odbicie słabej fali uderzeniowej o froncie równoległym do ścianki. Rozważa się tylko przypadek izotermicznej ścianki. Obszar przepływu dzieli się na dwie części: obszar zewnętrzny zawierający obie fale uderzeniowe i obszar wewnętrzny przyległy do ścianki. W celu wyznaczenia przepływu zewnętrznego wykorzystano technikę Lighthilla łącznie z metodą wielu skal. Struktura przepływu wewnętrznego jest dokładnie taka sama jak w [1]. Zasada kojarzenia rozwiązań jest użyta do wyznaczenia pewnych niewiadomych funkcji. W wyniku otrzymuje się m.in. strukturę i trajektorię fali odbitej.

В работе анализируется отражение слабой ударной волны с фронтом параллельным стенке. Рассматривается только случай изотермической стенки. Область течения разделяется на две части: внешняя область, содержавшая обе ударные волны и внутренняя область, примыкающая к стенке. С целью определения внешнего течения комбинируется техника Лайтхилла с методом многих масштабов. Структура внутреннего течения точно такая же как в [1]. Принцип сращивания решений используется для определения некоторых неизвестных функций. В результате получаются, между прочим, структура и траектория отраженной волны.

1. Introduction

IN THE PRESENT paper we undertake the problem of reflection of a weak shock wave of its front parallel to the wall, a problem already theoretically considered in [1] and also in [2, 3]. In the paper by LESSER and SEEBASS [1] the problem under consideration was solved by means of perturbation methods. The same ideas were repeated in [2] and [3]. Let x denote the distance from the wall and let t be the time. The x-axis is directed from the gas to the wall, so the problem is considered in the domain $-\infty < x < 0$, $-\infty < t < \infty$. Lesser and Seebass divide this domain into several subdomains: "thermal boundary layer" (in the case of the isothermal wall) described by $x = 0(\sqrt{\varepsilon})$, $-\infty < t < \infty$, (ε is a "small" parameter) "acoustic region" $|x| \gg \sqrt{\varepsilon}$, $t = 0(1)$ and two domains involving either the incident shock wave ($-t \gg 1$) or the reflected one ($t \gg 1$). Then the solution, valid in the boundary layer, is matched to the "acoustic" approximation, and in turn the last one is matched separately to the incident shock and to the reflected one. As a result of this, the structure and the trajectory of the reflected shock wave, among others, are obtained. A disadvantage of this approach is the necessity of such a complicated division of the flow domain and, consequently, the necessity of many matchings. Moreover, in order to distinguish the "reflected shock" domain it is necessary to have some introductory information on the reflected shock wave: it is necessary to know that it exists and to have even very rough knowledge of its location.

In the present paper we consider the some problem as in [1], but another method of solution is used. We consider only the case of isothermal wall and we divide the domain into two parts only, inner domain (thermal boundary layer) and outer domain. We do not use any assumption concerning the reflected shock wave, we do not even assume its existence. Generally speaking, our method applied in the outer domain is a combination of the Lighthill technique [6] and the multiple scales method. Consequently, our calculations are longer and more tedious, but they are made once for all. Next, the outer expansion is matched to the inner one.

2. Basic assumptions

In this paper we consider a weak shock wave moving in a half-space bounded by an infinite plate. If the front of the shock wave is constantly parallel to the wall, then the problem may be treated as one-dimensional but unsteady. The coordinate system is

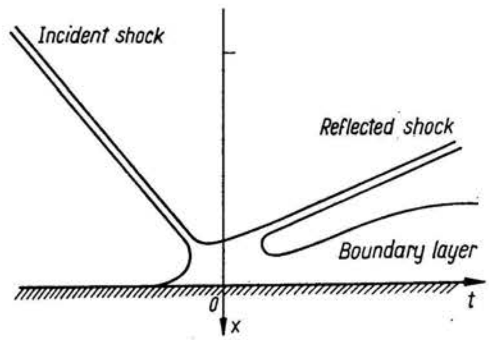

FIG. 1.

shown in Fig. 1. We assume that the shock wave is weak, so we may use the Navier-Stokes equations and neglect the molecular structure of the gas.

Also no inner degree of freedom is taken into account. Next it is assumed that the wall is impearmeable and isothermal. It is assumed that at $x^* = -\infty$ and $t^* = -\infty$, a weak shock wave was formed and it travels to the wall immersed at $x^* = 0$. The moment at which the shock wave occurs at the wall is denoted by $t^* = 0$.

We assume that before the arrival of the shock wave the gas was at rest and it was characterized by the constant density ϱ_1^*, constant temperature T_1^* and its velocity u^* was zero. Let $u_2^* = \text{const}$ be the gas velocity behind the incident shock. We form the parameter

$$\varepsilon = \frac{u_2^*}{a_1^*},$$

where a_1^* is the sound speed of the quiescent gas. The ratio ε is used as a basic parameter and it is assumed to be small. Following [1] we introduce dimensionless variables as follows:

$$x = \frac{\varrho_1^* a_1^* \varepsilon}{\mu^*} x^*, \quad t = \frac{\varrho_1^* a_1^{*2} \varepsilon}{\mu^*} t^*,$$

where μ^* is the coefficient of viscosity. In order to shorten calculations it is assumed to be constant. The dimensionless velocity, density and temperature are defined by

$$u^* = \varepsilon a_1^* u, \quad \varrho^* = \varrho_1^*(1+\varepsilon\varrho), \quad T^* = T_1^*(1+\varepsilon T).$$

The Navier-Stokes equations written in these coordinates take the form

$$\frac{\partial \varrho}{\partial t} + \frac{\partial u}{\partial x} + \varepsilon \frac{\partial}{\partial x}(\varrho u) = 0, \tag{2.1}$$

$$\frac{\partial u}{\partial t} + \frac{1}{\gamma}\frac{\partial}{\partial x}(\varrho + T) + \varepsilon\left[\varrho\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{1}{\gamma}\frac{\partial}{\partial x}(\varrho T) - \frac{4}{3}\frac{\partial^2 u}{\partial x^2}\right] + \varepsilon^2 \varrho u \frac{\partial u}{\partial x} = 0, \tag{2.2}$$

$$\frac{\partial T}{\partial t} - (\gamma-1)\frac{\partial \varrho}{\partial t} + \varepsilon\left[\varrho\frac{\partial T}{\partial t} - (\gamma-1)T\frac{\partial \varrho}{\partial t} + u\frac{\partial T}{\partial x} - (\gamma-1)u\frac{\partial \varrho}{\partial x} - \frac{\gamma}{\mathrm{Pr}}\frac{\partial^2 T}{\partial x^2}\right] + \varepsilon^2\left[\varrho u\frac{\partial T}{\partial x} - (\gamma-1)Tu\frac{\partial \varrho}{\partial x} - \frac{4}{3}\gamma(\gamma-1)\left(\frac{\partial u}{\partial x}\right)^2\right] = 0, \tag{2.3}$$

where γ is the specific heat ratio, Pr is the Prandtl number. To obtain the system of equations (2.1)–(2.3) the pressure p^* was eliminated by means of the perfect gas equation

$$p^* = R^* \varrho^* T^*,$$

where R^* is the gas constant.

We solve Eqs. (2.1)–(2.3) subject to the conditions

$$\lim_{t\to-\infty} \varrho(x,t) = 0, \quad \lim_{t\to-\infty} u(x,t) = 0, \quad \lim_{t\to-\infty} T(x,t) = 0, \tag{2.4}$$

which express mathematically the fact that the gas is at rest ahead of the incident shock wave.

Next we look for solutions of Eqs. (2.1)–(2.3) satisfying the following conditions at the wall:

$$u(0,t) = 0, \tag{2.5}$$

which means that the wall is impermeable, and

$$T(0,t) = 0. \tag{2.6}$$

This condition is of double meaning. First it means that the wall is isothermal, i.e. its temperature is constant and it is the same before the arrival of the shock wave and after its reflection. Secondly it means that all the time the gas is in a thermal equilibrium with the wall.

The last group of assumptions concerns the oncoming flow. Let D^* be the incident shock wave speed. Then the ratio D^*/a_1^* is the Mach number of the incident shock wave. The Rankine-Hugoniot relations give

$$\frac{D^*}{a_1^*} = M = \sqrt{1+\left(\frac{\Gamma}{2}\varepsilon\right)^2} + \frac{\Gamma}{2}\varepsilon, \tag{2.7}$$

where

$$\Gamma = \frac{\gamma+1}{2}.$$

Let $\lim_{is}$ denote the following limiting process: $x \to -\infty$, $t \to -\infty$ with $\xi = x - M(\varepsilon)t$ fixed.

We assume that the following limits exist:

$$\lim_{is} u = u_s(x-Mt)+0(\varepsilon), \tag{2.8}$$

$$\lim_{is} \varrho = \varrho_s(x-Mt)+0(\varepsilon), \tag{2.9}$$

$$\lim_{is} T = T_s(x-Mt)+0(\varepsilon), \tag{2.10}$$

where

$$u_s(\xi) = \varrho_s(\xi) = \frac{1}{1+e^{\frac{\Gamma}{\beta}\xi}} = f(\xi), \tag{2.11}$$

$$T_s(\xi) = (\gamma-1)\varrho_s(\xi) \tag{2.12}$$

and β is a constant

$$\beta = \frac{4}{3} + \frac{\gamma-1}{\mathrm{Pr}}.$$

The formulae (2.11) and (2.12) mean that the incident shock wave is of the "classical" Taylor structure. Our task is to find an asymptotic solution to the problem under consideration. It needs dividing the domain $-\infty < x \leqslant 0$, $-\infty < t < \infty$ into two parts: an outer domain $x < 0$, $|x| \gg \sqrt{\varepsilon}$, $-\infty < t < \infty$ and an inner domain (thermal boundary layer) $x = 0(\sqrt{\varepsilon})$, $-\infty < t < \infty$.

In each of these domains another expansion is developed. They are called outer and inner expansion, respectively. Finally, they are matched one to another.

3. Outer expansion

Let $Q(t, x; \varepsilon)$ be any of the variables u, ϱ or T. The boundary conditions (2.8)–(2.10) show that Q is a function not only of x and t, but that it is a function of infinitely many arguments: x, t_0, t_1, t_2, t_3, etc., where $t_n = \varepsilon^n t$, $n = 0, 1, 2, \ldots$. Consequently, we look for solutions of Eqs. (2.1)–(2.3) in the form

$$Q = Q(x, t_0, t_1, \ldots, \varepsilon). \tag{3.1}$$

Then, for the derivative with respect to t we have

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t_0} + \varepsilon\frac{\partial}{\partial t_1} + \ldots . \tag{3.2}$$

Substituting Eqs. (3.1) and (3.2) into Eqs. (2.1)–(2.3), we obtain

$$\frac{\partial \varrho}{\partial t_0} + \frac{\partial u}{\partial x} + \varepsilon\left[\frac{\partial \varrho}{\partial t_1} + \frac{\partial}{\partial x}(\varrho u)\right] = 0(\varepsilon^2), \tag{3.3}$$

$$\frac{\partial u}{\partial t_0}+\frac{1}{\gamma}\frac{\partial}{\partial x}(\varrho+T)+\varepsilon\left[\frac{\partial u}{\partial t_1}+\varrho\frac{\partial u}{\partial t_0}+u\frac{\partial u}{\partial x}+\frac{1}{\gamma}\frac{\partial}{\partial x}(\varrho T)-\frac{4}{3}\frac{\partial^2 u}{\partial x^2}\right]=0(\varepsilon^2), \tag{3.4}$$

$$\frac{\partial}{\partial t_0}T-(\gamma-1)\frac{\partial \varrho}{\partial t_0}+\varepsilon\left[\frac{\partial}{\partial t_1}T-(\gamma-1)\frac{\partial \varrho}{\partial t_1}+\varrho\frac{\partial T}{\partial t_0}-(\gamma-1)T\frac{\partial \varrho}{\partial t_0}+u\frac{\partial T}{\partial x}-(\gamma-1)u\frac{\partial \varrho}{\partial x}-\frac{\gamma}{\mathrm{Pr}}\frac{\partial^2 T}{\partial x^2}\right]=0(\varepsilon^2). \tag{3.5}$$

In the above equations, t_0, t_1, etc. are treated as independent variables.

In the present paper we are not interested in large scales time variations described by t_2, t_3 etc. Thus our results will be valid only for times t of order of ε^{-1}.

Now, we apply the strained coordinate method to the system of equations (3.3)–(3.5), treating t_0 and t_1 as completely independent variables. We make the following transformation of independent variables:

$$\begin{aligned} x &= \Psi(\xi,\eta,\tau),\\ t_0 &= \Phi(\xi,\eta,\tau),\\ t_1 &= \tau, \end{aligned} \tag{3.6}$$

where the functions Ψ and Φ will be chosen according to the strained coordinate principles [6].

The system of equations (3.3)–(3.5) is equivalent to the following equations:

$$\left(\frac{\partial\Psi}{\partial\xi}\frac{\partial\varrho}{\partial\eta}-\frac{\partial\Psi}{\partial\eta}\frac{\partial\varrho}{\partial\xi}\right)+\left(\frac{\partial\Phi}{\partial\eta}\frac{\partial u}{\partial\xi}-\frac{\partial\Phi}{\partial\xi}\frac{\partial u}{\partial\eta}\right)+\varepsilon\left[\left(\frac{\partial\Psi}{\partial\eta}\frac{\partial\Phi}{\partial\tau}-\frac{\partial\Phi}{\partial\eta}\frac{\partial\Psi}{\partial\tau}\right)\frac{\partial\varrho}{\partial\xi}+\left(\frac{\partial\Phi}{\partial\xi}\frac{\partial\Psi}{\partial\tau}-\frac{\partial\Psi}{\partial\xi}\frac{\partial\Phi}{\partial\tau}\right)\frac{\partial\varrho}{\partial\eta}+\left(\frac{\partial\Psi}{\partial\xi}\frac{\partial\Phi}{\partial\eta}-\frac{\partial\Phi}{\partial\xi}\frac{\partial\Psi}{\partial\eta}\right)\frac{\partial\varrho}{\partial\tau}+\left(\frac{\partial\Phi}{\partial\eta}\frac{\partial(\varrho u)}{\partial\xi}-\frac{\partial\Phi}{\partial\xi}\frac{\partial(\varrho u)}{\partial\eta}\right)\right]=0(\varepsilon^2), \tag{3.7}$$

$$\left(\frac{\partial\Psi}{\partial\xi}\frac{\partial u}{\partial\eta}-\frac{\partial\Psi}{\partial\eta}\frac{\partial u}{\partial\xi}\right)+\frac{1}{\gamma}\left(\frac{\partial\Phi}{\partial\eta}\frac{\partial(\varrho+T)}{\partial\xi}-\frac{\partial\Phi}{\partial\xi}\frac{\partial(\varrho+T)}{\partial\eta}\right)+\varepsilon\left[\left(\frac{\partial\Psi}{\partial\eta}\frac{\partial\Phi}{\partial\tau}-\frac{\partial\Phi}{\partial\eta}\frac{\partial\Psi}{\partial\tau}\right)\frac{\partial u}{\partial\xi}+\left(\frac{\partial\Phi}{\partial\xi}\frac{\partial\Psi}{\partial\tau}-\frac{\partial\Psi}{\partial\xi}\frac{\partial\Phi}{\partial\tau}\right)\frac{\partial u}{\partial\eta}+\left(\frac{\partial\Psi}{\partial\xi}\frac{\partial\Phi}{\partial\eta}-\frac{\partial\Phi}{\partial\xi}\frac{\partial\Psi}{\partial\eta}\right)\frac{\partial u}{\partial\tau}+\varrho\left(\frac{\partial\Psi}{\partial\xi}\frac{\partial u}{\partial\eta}-\frac{\partial\Psi}{\partial\eta}\frac{\partial u}{\partial\xi}\right)+u\left(\frac{\partial\Phi}{\partial\eta}\frac{\partial u}{\partial\xi}-\frac{\partial\Phi}{\partial\xi}\frac{\partial u}{\partial\eta}\right)+\frac{1}{\gamma}\left(\frac{\partial\Phi}{\partial\xi}\frac{\partial(\varrho T)}{\partial\xi}-\frac{\partial\Phi}{\partial\xi}\frac{\partial(\varrho T)}{\partial\eta}\right)-\frac{4}{3}\left(\frac{\partial\Phi}{\partial\eta}\frac{\partial w}{\partial\xi}-\frac{\partial\Phi}{\partial\xi}\frac{\partial w}{\partial\eta}\right)\right]=0(\varepsilon^2), \tag{3.8}$$

$$(3.9)\quad \left(\frac{\partial\Psi}{\partial\xi}\frac{\partial(T-(\gamma-1)\varrho)}{\partial\eta}-\frac{\partial\Psi}{\partial\eta}\frac{\partial(T-(\gamma-1)\varrho)}{\partial\xi}\right)+\varepsilon\left[\left(\frac{\partial\Psi}{\partial\eta}\frac{\partial\Phi}{\partial\tau}\right.\right.$$

$$\left.-\frac{\partial\Phi}{\partial\eta}\frac{\partial\Psi}{\partial\tau}\right)\frac{\partial(T-(\gamma-1)\varrho)}{\partial\xi}+\left(\frac{\partial\Phi}{\partial\xi}\frac{\partial\Psi}{\partial\tau}-\frac{\partial\Psi}{\partial\xi}\frac{\partial\Phi}{\partial\tau}\right)$$

$$\times\frac{\partial(T-(\gamma-1)\varrho)}{\partial\eta}+\left(\frac{\partial\Psi}{\partial\xi}\frac{\partial\Phi}{\partial\eta}-\frac{\partial\Phi}{\partial\xi}\frac{\partial\Psi}{\partial\eta}\right)\frac{\partial(T-(\gamma-1)\varrho)}{\partial\tau}$$

$$+\varrho\left(\frac{\partial\Psi}{\partial\xi}\frac{\partial T}{\partial\eta}-\frac{\partial\Psi}{\partial\eta}\frac{\partial T}{\partial\xi}\right)-(\gamma-1)T\left(\frac{\partial\Psi}{\partial\xi}\frac{\partial\varrho}{\partial\eta}-\frac{\partial\Psi}{\partial\eta}\frac{\partial\varrho}{\partial\xi}\right)$$

$$+u\left(\frac{\partial\Phi}{\partial\eta}\frac{\partial T}{\partial\xi}-\frac{\partial\Phi}{\partial\xi}\frac{\partial T}{\partial\eta}\right)-(\gamma-1)u\left(\frac{\partial\Phi}{\partial\eta}\frac{\partial\varrho}{\partial\xi}-\frac{\partial\Phi}{\partial\xi}\frac{\partial\varrho}{\partial\eta}\right)$$

$$\left.-\frac{\gamma}{\Pr}\left(\frac{\partial\Phi}{\partial\eta}\frac{\partial\theta}{\partial\xi}-\frac{\partial\Phi}{\partial\xi}\frac{\partial\theta}{\partial\eta}\right)\right]=0(\varepsilon^2),$$

$$(3.10)\quad \frac{\partial\Phi}{\partial\eta}\frac{\partial u}{\partial\xi}-\frac{\partial\Phi}{\partial\xi}\frac{\partial u}{\partial\eta}=\left(\frac{\partial\Psi}{\partial\xi}\frac{\partial\Phi}{\partial\eta}-\frac{\partial\Phi}{\partial\xi}\frac{\partial\Psi}{\partial\eta}\right)w,$$

$$(3.11)\quad \frac{\partial\Phi}{\partial\eta}\frac{\partial T}{\partial\xi}-\frac{\partial\Phi}{\partial\xi}\frac{\partial T}{\partial\eta}=\left(\frac{\partial\Psi}{\partial\xi}\frac{\partial\Phi}{\partial\eta}-\frac{\partial\Phi}{\partial\xi}\frac{\partial\Psi}{\partial\eta}\right)\theta.$$

We seek solutions of these equations in the form

$$(3.12)\quad Q(\xi,\eta,\tau,\varepsilon)=\sum_{n=0}^{\infty}\varepsilon^n Q_n(\xi,\eta,\tau),$$

and

$$(3.13)\quad \Phi(\xi,\eta,\tau,\varepsilon)=\frac{\eta-\xi}{2}+\varepsilon\Phi_1(\xi,\eta,\tau)+\ldots,$$

$$(3.14)\quad \Psi(\xi,\eta,\tau,\varepsilon)=\frac{\eta+\xi}{2}+\varepsilon\Psi_1(\xi,\eta,\tau)+\ldots,$$

where Q denotes one of the variables u, ϱ, T, w or θ.

Substituting Eqs. (3.12)–(3.14) into Eqs. (3.7)–(3.11) gives

$$\frac{\partial}{\partial\eta}(\varrho_0+u_0)-\frac{\partial}{\partial\xi}(\varrho_0-u_0)=0,$$

$$\frac{\partial}{\partial\eta}\left(u_0+\frac{\varrho_0+T_0}{\gamma}\right)-\frac{\partial}{\partial\xi}\left(u_0-\frac{\varrho_0+T_0}{\gamma}\right)=0,$$

$$(3.15)\quad \frac{\partial}{\partial\eta}(T_0-(\gamma-1)\varrho_0)-\frac{\partial}{\partial\xi}(T_0-(\gamma-1)\varrho_0)=0,$$

$$w_0=\frac{\partial u_0}{\partial\xi}+\frac{\partial u_0}{\partial\eta},$$

$$\theta_0=\frac{\partial T_0}{\partial\xi}+\frac{\partial T_0}{\partial\eta},$$

and

$$\frac{\partial}{\partial\eta}(\varrho_1+u_1)-\frac{\partial}{\partial\xi}(\varrho_1-u_1)+2\left(\frac{\partial\Psi_1}{\partial\xi}\frac{\partial\varrho_0}{\partial\eta}-\frac{\partial\Psi_1}{\partial\eta}\frac{\partial\varrho_0}{\partial\xi}\right)$$
$$+2\left(\frac{\partial\Phi_1}{\partial\eta}\frac{\partial u_0}{\partial\xi}-\frac{\partial\Phi_1}{\partial\xi}\frac{\partial u_0}{\partial\eta}\right)+\frac{\partial\varrho_0}{\partial\tau}+\frac{\partial}{\partial\xi}(\varrho_0u_0)+\frac{\partial}{\partial\eta}(\varrho_0u_0)=0;$$

$$\frac{\partial}{\partial\eta}\left(u_1+\frac{\varrho_1+T_1}{\gamma}\right)-\frac{\partial}{\partial\xi}\left(u_1-\frac{\varrho_1+T_1}{\gamma}\right)+2\left(\frac{\partial\Psi_1}{\partial\xi}\frac{\partial u_0}{\partial\eta}-\frac{\partial\Psi_1}{\partial\eta}\frac{\partial u_0}{\partial\xi}\right)$$
$$+\frac{2}{\gamma}\left(\frac{\partial\Phi_1}{\partial\eta}\frac{\partial(\varrho_0+T_0)}{\partial\xi}-\frac{\partial\Phi_1}{\partial\xi}\frac{\partial(\varrho_0+T_0)}{\partial\eta}\right)+\frac{\partial u_0}{\partial\tau}+\varrho_0\left(\frac{\partial u_0}{\partial\eta}\right.$$
$$\left.-\frac{\partial u_0}{\partial\xi}\right)+u_0\left(\frac{\partial u_0}{\partial\xi}+\frac{\partial u_0}{\partial\eta}\right)+\frac{1}{\gamma}\left(\frac{\partial(\varrho_0T_0)}{\partial\xi}+\frac{\partial(\varrho_0T_0)}{\partial\eta}\right)-\frac{4}{3}\left(\frac{\partial w_0}{\partial\xi}+\frac{\partial w_0}{\partial\eta}\right)=0;$$

(3.16)
$$\frac{\partial}{\partial\eta}(T_1-(\gamma-1)\varrho_1)-\frac{\partial}{\partial\xi}(T_1-(\gamma-1)\varrho_1)+2\left(\frac{\partial\Psi_1}{\partial\xi}\frac{\partial(T_0-(\gamma-1)\varrho_0)}{\partial\eta}\right.$$
$$\left.-\frac{\partial\Psi_1}{\partial\eta}\frac{\partial(T_0-(\gamma-1)\varrho_0)}{\partial\xi}\right)+\frac{\partial}{\partial\tau}(T_0-(\gamma-1)\varrho_0)+\varrho_0\left(\frac{\partial T_0}{\partial\eta}-\frac{\partial T_0}{\partial\xi}\right)$$
$$-(\gamma-1)T_0\left(\frac{\partial\varrho_0}{\partial\eta}-\frac{\partial\varrho_0}{\partial\xi}\right)+u_0\left(\frac{\partial T_0}{\partial\eta}+\frac{\partial T_0}{\partial\xi}\right)-(\gamma-1)u_0\left(\frac{\partial\varrho_0}{\partial\xi}\right.$$
$$\left.+\frac{\partial\varrho_0}{\partial\eta}\right)-\frac{\gamma}{\mathrm{Pr}}\left(\frac{\partial\theta_0}{\partial\xi}+\frac{\partial\theta_0}{\partial\eta}\right)=0,$$

etc.

The functions w i θ play only an auxiliary role and because of that we have not written equations for w_1 and θ_1.

We solve the systems of equations (3.15) and (3.16) subject to conditions following from Eqs. (2.4).

(3.17)
$$\lim_{\substack{\xi\to+\infty\\ \eta\to-\infty\\ \xi+\eta-\text{fixed}}} u_0(\xi,\eta,\tau)=0,\qquad \lim_{\substack{\xi\to+\infty\\ \eta\to-\infty\\ \xi+\eta-\text{fixed}}} \varrho_0(\xi,\eta,\tau)=0,\qquad \lim_{\substack{\xi\to+\infty\\ \eta\to-\infty\\ \xi+\eta-\text{fixed}}} T_0(\xi,\eta,\tau)=0,$$

and

(3.17)
$$\lim_{\substack{\xi\to+\infty\\ \eta\to-\infty\\ \xi+\eta-\text{fixed}}} u_1(\xi,\eta,\tau)=0,\qquad \lim_{\substack{\xi\to+\infty\\ \eta\to-\infty\\ \xi+\eta-\text{fixed}}} \varrho_1(\xi,\eta,\tau)=0,\qquad \lim_{\substack{\xi\to+\infty\\ \eta\to-\infty\\ \xi+\eta-\text{fixed}}} T_1(\xi,\eta,\tau)=0,$$

etc.

From Eqs. $(3.15)_3$ and $(3.17)_1$ we obtain

(3.18)
$$T_0=(\gamma-1)\varrho_0.$$

Using this relation in Eq. $(3.15)_2$ gives

$$\frac{\partial}{\partial\eta}(u_0+\varrho_0)+\frac{\partial}{\partial\xi}(\varrho_0-u_0)=0.$$

From the above equation and from Eq. $(3.15)_1$ we have

(3.19)
$$\begin{aligned}\varrho_0+u_0&=2g_0(\xi,\tau),\\ \varrho_0-u_0&=2h_0(\eta,\tau),\end{aligned}$$

where g_0 and h_0 are arbitrary bounded functions. In virtue of Eqs. (3.18) and (3.19) we can write

$$
\begin{aligned}
u_0(\xi,\eta,\tau) &= g_0(\xi,\tau)-h_0(\eta,\tau),\\
\varrho_0(\xi,\eta,\tau) &= g_0(\xi,\tau)+h_0(\eta,\tau),\\
T_0(\xi,\eta,\tau) &= (\gamma-1)[g_0(\xi,\tau)+h_0(\eta,\tau)].
\end{aligned}
\tag{3.20}
$$

Substitution of Eq. $(3.20)_1$ into Eq. $(3.15)_3$, and Eq. $(3.20)_1$ into Eq. (3.15) gives

$$
\begin{aligned}
w_0 &= \frac{\partial}{\partial\xi}g_0(\xi,\tau)-\frac{\partial}{\partial\eta}h_0(\eta,\tau),\\
\theta_0 &= (\gamma-1)\left[\frac{\partial}{\partial\xi}g_0(\xi,\tau)+\frac{\partial}{\partial\eta}h_0(\eta,\tau)\right].
\end{aligned}
\tag{3.20'}
$$

Finally, let us note that from Eqs. (3.17) and (3.20) it follows that

$$
\lim_{\xi\to+\infty} g_0(\xi,\tau)=0,\qquad \lim_{\eta\to-\infty} h_0(\eta,\tau)=0.
\tag{3.21}
$$

Now we turn to the analysis of Eqs. (3.16).

In virtue of Eqs. (3.20) we can write Eq. $(3.16)_3$ as follows:

$$
\frac{\partial}{\partial\eta}\left[T_1-(\gamma-1)\varrho_1-\frac{(\gamma-1)(\gamma-2)}{2}\varrho_0^2+\frac{\gamma(\gamma-1)}{\mathrm{Pr}}w_0\right]
-\frac{\partial}{\partial\xi}\left[T_1-(\gamma-1)\varrho_1-\frac{(\gamma-1)(\gamma-2)}{2}\varrho_0^2+\frac{\gamma(\gamma-1)}{\mathrm{Pr}}w_0\right]=0.
$$

We solve this equation subject to the conditions $(3.17)_2$; this yields

$$
T_1=(\gamma-1)\varrho_1+\frac{(\gamma-1)(\gamma-2)}{2}\varrho_0^2-\frac{\gamma(\gamma-1)}{\mathrm{Pr}}w_0.
$$

If the last relation is used in Eq. $(3.16)_2$, then it takes the form

$$
\begin{aligned}
&\frac{\partial}{\partial\eta}(\varrho_1+u_1)+\frac{\partial}{\partial\xi}(\varrho_1-u_1)+2\left(\frac{\partial\Psi_1}{\partial\xi}\frac{\partial u_0}{\partial\eta}-\frac{\partial\Psi_1}{\partial\eta}\frac{\partial u_0}{\partial\xi}\right)\\
&\quad+2\left(\frac{\partial\Phi_1}{\partial\eta}\frac{\partial\varrho_0}{\partial\xi}-\frac{\partial\Phi_1}{\partial\xi}\frac{\partial\varrho_0}{\partial\eta}\right)+\frac{\partial u_0}{\partial\tau}+\frac{\gamma-1}{2}\left(\frac{\partial}{\partial\eta}\varrho_0^2+\frac{\partial}{\partial\xi}\varrho_0^2\right)\\
&\quad+\varrho_0\left(\frac{\partial u_0}{\partial\eta}-\frac{\partial u_0}{\partial\xi}\right)+u_0\left(\frac{\partial u_0}{\partial\eta}+\frac{\partial u_0}{\partial\xi}\right)-\beta\left(\frac{\partial w_0}{\partial\eta}+\frac{\partial w_0}{\partial\xi}\right)=0.
\end{aligned}
$$

Now we combine the last equation and Eq. $(3.16)_1$ and obtain the following two equations:

$$
\begin{aligned}
&\frac{\partial}{\partial\eta}(\varrho_1+u_1)-\frac{\partial}{\partial\eta}(\Psi_1-\Phi_1)\frac{\partial}{\partial\xi}(\varrho_0+u_0)+\frac{1}{2}\frac{\partial}{\partial\tau}(\varrho_0+u_0)\\
&\quad+\frac{\gamma-1}{4}\left(\frac{\partial}{\partial\eta}\varrho_0^2+\frac{\partial}{\partial\xi}\varrho_0^2\right)+\frac{1}{2}\varrho_0\left(\frac{\partial u_0}{\partial\eta}-\frac{\partial u_0}{\partial\xi}\right)+\frac{1}{2}u_0\left(\frac{\partial u_0}{\partial\eta}\right.\\
&\quad\left.+\frac{\partial u_0}{\partial\xi}\right)+\frac{1}{2}\left(\frac{\partial}{\partial\xi}(\varrho_0u_0)+\frac{\partial}{\partial\eta}(\varrho_0u_0)\right)-\frac{\beta}{2}\left(\frac{\partial w_0}{\partial\eta}+\frac{\partial w_0}{\partial\xi}\right)=0
\end{aligned}
$$

and

$$\frac{\partial}{\partial\xi}(\varrho_1-u_1)-\frac{\partial}{\partial\xi}(\Psi_1+\Phi_1)\frac{\partial}{\partial\eta}(\varrho_0-u_0)-\frac{1}{2}\frac{\partial}{\partial\tau}(\varrho_0-u_0)+\frac{\gamma-1}{4}\left(\frac{\partial}{\partial\eta}\varrho_0^2+\frac{\partial}{\partial\xi}\varrho_0^2\right)+\frac{1}{2}\varrho_0\left(\frac{\partial u_0}{\partial\eta}-\frac{\partial u_0}{\partial\xi}\right)+\frac{1}{2}u_0\left(\frac{\partial u_0}{\partial\eta}+\frac{\partial u_0}{\partial\xi}\right)$$
$$-\frac{1}{2}\left(\frac{\partial}{\partial\xi}(\varrho_0u_0)+\frac{\partial}{\partial\eta}(\varrho_0u_0)\right)-\frac{\beta}{2}\left(\frac{\partial w_0}{\partial\eta}-\frac{\partial w_0}{\partial\xi}\right)=0.$$

General solutions to these equations are of the form

$$(3.22)_1\quad \varrho_1+u_1=2g_1(\xi,\tau_1)-\frac{\gamma-3}{4}h_0^2(\eta,\tau)-\frac{\beta}{2}\frac{\partial}{\partial\eta}h_0(\eta,\tau)-\frac{\gamma-3}{2}g_0(\xi,\tau)h_0(\eta,\tau)$$
$$+2\left[\Psi_1-\Phi_1-\frac{\gamma-3}{4}H_0(\eta,\tau)\right]\frac{\partial}{\partial\xi}g_0(\xi,\tau)-\eta\left(\frac{\partial g_0}{\partial\tau}+\Gamma g_0\frac{\partial g_0}{\partial\xi}-\frac{\beta}{2}\frac{\partial^2}{\partial\xi^2}g_0\right)$$

and

$$(3.22)_2\quad \varrho_1-u_1=2h_1(\xi,\tau_1)-\frac{\gamma-3}{2}g_0^2(\xi,\tau)-\frac{\gamma-3}{2}g_0(\xi,\tau)h_0(\eta,\tau)$$
$$+\frac{\beta}{2}\frac{\partial}{\partial\xi}g_0(\xi,\tau)+2\left[\Psi_1+\Phi_1-\frac{\gamma-3}{4}G_0(\xi,\tau)\right]\frac{\partial}{\partial\eta}h_0(\eta,\tau)$$
$$-\xi\left(\frac{\partial h_0}{\partial\tau}-\Gamma h_0\frac{\partial h_0}{\partial\eta}-\frac{\beta}{2}\frac{\partial^2 h_0}{\partial\eta^2}\right),$$

where g_1 and h_1 are arbitrary bounded functions, and G_0 and H_0 are defined by the relations

$$(3.23)\qquad \frac{\partial G_0}{\partial\xi}=g_0,\qquad \frac{\partial H_0}{\partial\eta}=h_0.$$

Substituting in Eq. (3.22)

$$(3.24)\qquad \begin{aligned}\Psi_1-\Phi_1&=\frac{\gamma-3}{4}H_0,\\ \Psi_1+\Phi_1&=\frac{\gamma-3}{4}G_0,\end{aligned}$$

and also

$$(3.25)\qquad \frac{\beta}{2}\frac{\partial^2 g_0}{\partial\xi^2}=\Gamma g_0\frac{\partial g_0}{\partial\xi}+\frac{\partial g_0}{\partial\tau},$$

$$(3.26)\qquad \frac{\beta}{2}\frac{\partial^2 h_0}{\partial\eta^2}=-\Gamma h_0\frac{\partial h_0}{\partial\eta}+\frac{\partial h_0}{\partial\tau},$$

we obtain bounded expressions for u_1 and ϱ_1.

Now the necessity of combining the two methods is evident. Applying solely the multiple scales method is equivalent to substituting in Eqs. (3.22) $\Psi_1=\Phi_1=0$. Then, however, we should not be able to eliminate terms involving G_0 and H_0 which in general are unbounded functions. Thus, in the case under consideration the multiple scales method fails. Similar arguments show that on the other hand the Lighthill method also breaks down, because it does not make possible the elimination of the last terms in Eq. (3.22), which also are not bounded.

Equations (3.23) do not uniquely determine the functions G_0 and H_0. We choose them such that

$$G_0(-\eta,\eta) = 0, \quad H_0(-\eta,\eta) = 0. \tag{3.27}$$

Then the straight line $\xi+\eta = 0$ in the (ξ, η)-plane corresponds to the boundary $x = 0$ in the (x, t)-plane, namely we have

$$x(-\eta, \eta, \tau) = 0, \quad t(-\eta, \eta, \tau) = \xi.$$

Equation (3.23) subject to the conditions (3.27) determine uniquely the functions G_0 and H_0 and they are given by

$$G_0(\xi, \eta, \tau) = \frac{\gamma-3}{4}\int_{-\eta}^{\xi} g_0(\sigma, \tau)d\sigma, \quad H_0(\xi, \eta, \tau) = \frac{\gamma-3}{4}\int_{-\xi}^{\eta} h_0(\sigma, \tau)d\sigma. \tag{3.28}$$

Thus the transformation (3.6) is of the form

$$\begin{aligned} x &= \frac{\eta+\xi}{2} + \varepsilon\frac{\gamma-3}{8}\int_{-\eta}^{\xi}[g_0(\sigma, \tau)+h_0(-\sigma, \tau)]d\sigma + \ldots, \\ t &= \frac{\eta-\xi}{2} + \varepsilon\frac{\gamma-3}{8}\int_{-\eta}^{\xi}[g_0(\sigma, \tau)-h_0(-\sigma, \tau)d\sigma + \ldots, \\ \tau &= t_1 = \varepsilon t = \varepsilon\frac{\eta-\xi}{2} + \ldots. \end{aligned} \tag{3.29}$$

For the time being, only the boundary conditions (2.4) have been used. Now we apply the conditions (2.8)–(2.10) to our solutions (3.20), (3.29). Since the upper bound of absolute values of t we are interested in is of order of ε^{-1}, we can simplify these conditions assuming that if x and t tend to $-\infty$ with $x-t-\varepsilon\frac{\Gamma}{2}t$ fixed, then

$$u \to f\left(x-t-\varepsilon\frac{\Gamma}{2}t\right).$$

This condition, when applied to Eq. (3.12), takes the form

$$u_0 \to f\left(x-t_0-\frac{\Gamma}{2}t_1\right) \tag{3.30}$$

provided that $x \to -\infty$, $t_0 \to -\infty$, $t_1 \to -\infty$ with $x-t_0-\frac{\Gamma}{2}t_1$ fixed.

However, in order to keep $x-t_0-\frac{1}{2}\Gamma t_1$ fixed it must be $x-t_0 \to -\infty$, moreover, we have of course $x+t_0 \to -\infty$. It follows then from Eq. (3.29) that $\xi \to -\infty$ and $\eta \to -\infty$.

We re-formulate this condition assuming that

$$\lim u_0(\xi, \eta, \tau) = f\left(\xi - \frac{\Gamma}{2}\tau\right) \tag{3.31}$$

if $\xi \to -\infty$, $\eta \to -\infty$ and $\tau \to -\infty$ with $\xi - \frac{\Gamma}{2}\tau$ fixed.

Assuming that

$$\lim_{\substack{\eta \to -\infty \\ t \to -\infty}} h_0(\eta, \tau) = 0 \tag{3.32}$$

we obtain from Eqs. (3.20) and (3.31), Eq. (3.32)

$$\lim g_0(\xi, \tau) = f\left(\xi - \frac{\Gamma}{2}\tau\right) \tag{3.33}$$

$\left(\xi \to -\infty,\ \tau \to -\infty,\ \xi - \frac{\Gamma}{2}\tau \text{ fixed}\right)$. If we take

$$g_0(\xi, \tau) = f\left(\xi - \frac{\Gamma}{2}\tau\right) = \left\{1 + \exp\left[\frac{\Gamma}{\beta}\left(\xi - \frac{\Gamma}{2}\tau\right)\right]\right\}^{-1}, \tag{3.34}$$

then both Eq. (3.25) and the condition (3.33) are satisfied.

However, the boundary conditions (2.5), (2.6) at the wall cannot be satisfied. Indeed, in the (ξ, η)-plane the equation of the wall is $\xi + \eta = 0$. Thus the boundary conditions (2.5), (2.6) may be formulated as follows:

$$u_0(-\eta, \eta) = 0, \quad T_0(-\eta, \eta) = 0. \tag{3.35}$$

And now we can see that it is impossible to satisfy the two conditions (3.35). This means that the boundary layer exists close to the wall. It is considered in the next paragraph.

4. Inner domain. Matching

The basic features of the flow in the inner domain (thermal boundary layer) were established by LESSER and SEEBASS [1]. Consequently, we give only an outline of the analysis. We define the inner coordinates by [1].

$$r = \frac{\eta + \xi}{2\sqrt{\varepsilon}} \leqslant 0, \quad s = \frac{\eta - \xi}{2} \tag{4.1}$$

and

$$\sqrt{\varepsilon}\,\hat{u} = u, \quad \hat{\varrho} = \varrho, \quad \hat{T} = T. \tag{4.2}$$

Then from Eqs. (3.7)–(3.11) we obtain

$$\begin{gathered} \frac{\partial \hat{\varrho}}{\partial s} + \frac{\partial \hat{u}}{\partial r} = 0(\varepsilon), \\ \frac{\partial}{\partial r}(\hat{\varrho} + \hat{T}) = 0(\varepsilon), \\ \frac{\partial}{\partial s}(\hat{T} - (\gamma - 1)\hat{\varrho}) - \frac{\gamma}{\Pr}\frac{\partial^2 \hat{T}}{\partial r^2} = 0(\varepsilon). \end{gathered} \tag{4.3}$$

We solve these equation subject to the boundary conditions

(4.4) $$\hat{u}(0, s) = 0, \quad \hat{T}(0, s) = 0.$$

It can be shown that the solution to the problem (4.3)–(4.4) is (see [1])

$$\hat{u} = -2\varphi'(s)r - \frac{2(\gamma-1)}{\sqrt{\pi}} \int_0^r dz \int_0^\infty \varphi'\left(s - \frac{z^2\mathrm{Pr}}{4w}\right) \frac{e^{-w}}{\sqrt{w}} dw + 0(\varepsilon),$$

(4.5) $$\hat{\varrho} = 2\varphi(s) + \frac{2(\gamma-1)}{\sqrt{\pi}} \int_0^\infty \varphi\left(s - \frac{r^2\mathrm{Pr}}{4w}\right) \frac{e^{-w}}{\sqrt{w}} dw + 0(\varepsilon),$$

$$\hat{T} = 2(\gamma-1)\left[\varphi(s) - \frac{1}{\sqrt{\pi}} \int_0^\infty \varphi\left(s - \frac{r^2\mathrm{Pr}}{4w}\right) \frac{e^{-w}}{\sqrt{w}} dw\right] + 0(\varepsilon),$$

where $\varphi(s)$ is a function to be determined, $\varphi'(s)$ is its derivative.

Hence we have two undetermined functions, namely h_0 in the outer expansion and φ in the inner one. (To be precise h_0 is not completely arbitrary because it has to satisfy Eq. (3.26) and only an initial condition for this equation is needed).

We find both the function φ and the initial condition for Eq. (3.26) from the matching principle. We apply this principle in the form due to COLE [7].

Let $\delta(\varepsilon)$ be a positive function such that

(4.6) $$\lim_{\varepsilon\to 0} \delta(\varepsilon) = 0, \quad \lim_{\varepsilon\to 0} \frac{\delta(\varepsilon)}{\sqrt{\varepsilon}} = +\infty.$$

"Intermediate" variables $\xi \leqslant 0$, and σ are introduced by

(4.7) $$\eta + \xi = 2\xi\delta(\varepsilon), \quad \eta - \xi = 2\sigma.$$

Then we have also

(4.8) $$r = \xi\frac{\delta(\varepsilon)}{\sqrt{\varepsilon}}, \quad s = \sigma.$$

From Eqs. (4.2), (4.5), (4.6) and (4.8) we obtain

$$u = -2\varphi'(\sigma)\xi\delta(\varepsilon) + \frac{2(\gamma-1)}{\sqrt{\pi\mathrm{Pr}}}\sqrt{\varepsilon} \int_0^\infty \frac{\varphi'(\sigma-\alpha)}{\sqrt{\alpha}} d\alpha + 0(\varepsilon),$$

(4.9) $$\varrho = 2\varphi(\sigma) + 0(\varepsilon),$$

$$T = 2(\gamma-1)\varphi(\sigma) + 0(\varepsilon).$$

On the other hand, from Eqs. (3.20), $(3.29)_3$, (3.34) and (4.7) we have

$$u = f(\xi\delta - \sigma - \varepsilon\sigma) - h_0(\xi\delta + \sigma, \varepsilon\sigma) + 0(\varepsilon)$$

$$= f(-\sigma) - h_0(\sigma, 0) + \xi\delta(\varepsilon)\left[f'(-\sigma) - \frac{\partial}{\partial\sigma} h_0(\sigma, 0)\right] + 0(\delta^2),$$

(4.10) $$\varrho = f(-\sigma) + h_0(\sigma, 0) + \xi\delta(\varepsilon)\left[f'(-\sigma) + \frac{\partial}{\partial\sigma} h_0(\sigma, 0)\right] + 0(\delta^2),$$

$$T = (\gamma-1)\left[f(-\sigma) + h_0(\sigma, 0)\right] + (\gamma-1)\xi\delta(\varepsilon)\left[f'(-\sigma) + \frac{\partial}{\partial\sigma} h_0(\sigma, 0)\right] + 0(\delta^2).$$

Equating Eqs. (4.9) and (4.10) gives

$$h_0(\sigma, 0) = f(-\sigma) - \frac{2(\gamma-1)}{\sqrt{\pi \Pr}} \sqrt{\varepsilon} \int_0^\infty \frac{\varphi'(\sigma-\alpha)}{\sqrt{\alpha}} d\alpha, \tag{4.11}$$

$$\varphi(\sigma) = f(-\sigma) - \frac{\gamma-1}{\sqrt{\pi \Pr}} \sqrt{\varepsilon} \int_0^\infty \frac{\varphi'(\sigma-\alpha)}{\sqrt{\alpha}} d\alpha. \tag{4.12}$$

From Eqs. (4.11) and (4.12) we see that both $h_0(\sigma, 0)$ and $\varphi(\sigma)$ depend on $\sqrt{\varepsilon}$, contrary to our earlier asumptions! To avoid this contradiction it would be necessary to return to the very beginning of our considerations and to supply the expansion (3.12) with terms $0(\sqrt{\varepsilon})$.

However, following LESSER and SEEBASS [1] we note that Eqs. (3.7)–(3.11) do not contain terms of order of $\sqrt{\varepsilon}$. Consequently, equations for co-efficients by $\sqrt{\varepsilon}$ will be a linearization of the equations for Q_0. Thus, instead of solving Eq. (3.26) and its linearized version, we solve only Eq. (3.26) subject to an initial condition involving the term $0(\sqrt{\varepsilon})$.

The solution of Eq. (4.12) is

$$\varphi(\sigma, \varepsilon) = f(-\sigma) - \frac{\sqrt{\varepsilon}}{4} U(\sigma) + 0(\varepsilon), \tag{4.13}$$

where

$$U(\sigma) = \frac{\gamma-1}{2} \left(\frac{2\Gamma}{\pi\beta \Pr}\right)^{1/2} \int_0^\infty \alpha^{-1/2} \operatorname{sech}^2 \left(\alpha - \frac{\Gamma}{2\beta}\sigma\right) d\alpha. \tag{4.14}$$

A diagram of this function is given in [1], here we note only that

$$U(\sigma) = \begin{cases} 2(\gamma-1) \sqrt{\dfrac{\Gamma}{\beta \Pr}}\, e^{\frac{\Gamma}{\beta}\sigma} + 0(e^{\frac{2\Gamma}{\beta}}), & \sigma \to -\infty, \\ 2(\gamma-1) \dfrac{1}{\sqrt{\pi \Pr \sigma}} + 0(\sigma^{-3/2}), & \sigma \to +\infty. \end{cases}$$

From Eqs. (4.11) and (4.13) we have

$$h_0(\sigma, 0) = f(-\sigma) - \sqrt{\varepsilon} U(\sigma) + 0(\varepsilon).$$

Neglecting terms of order of $0(\varepsilon)$ which contribute to higher order approximation, we solve Eq. (3.26) subject to the initial condition

$$h_0(\eta, 0) = f(-\eta) - \sqrt{\varepsilon} U(\eta). \tag{4.15}$$

Equation (3.26) is the so-called Burgers equation, it is thoroughly discussed in [8]. The Hopf-Cole transformation (see [8])

$$h_0(\eta, \tau) = \frac{\beta}{\Gamma} \frac{1}{F} \frac{\partial}{\partial \eta} F \tag{4.16}$$

reduces the Burgers equation (3.26) to the heat equation

$$\frac{\beta}{2} \frac{\partial^2 F}{\partial \eta^2} = \frac{\partial F}{\partial \tau}. \tag{4.17}$$

We solve this equation subject to the initial condition

$$F(\eta, 0) = \frac{1}{f(\eta)} e^{-\sqrt{\varepsilon}\frac{\beta}{\Gamma}(X\eta)}, \tag{4.18}$$

where

$$X(\eta) = \int_{-\infty}^{\eta} U(\sigma)d\sigma = 2(\gamma-1)\left(\frac{2\beta}{\pi\Gamma \mathrm{Pr}}\right)^{1/2} \int_{0}^{\infty} \sqrt{\alpha}\,\mathrm{sech}^2\left(\alpha - \frac{\Gamma}{2\beta}\eta\right)d\alpha. \tag{4.19}$$

A diagram showing the behaviour of this function is also presented in [1]. It can be shown that the following asymptotic formulae hold:

$$X(\sigma) = \begin{cases} 2(\gamma-1)\sqrt{\dfrac{\beta}{\Gamma \mathrm{Pr}}} e^{\frac{\Gamma}{\beta}\sigma} + 0(e^{\frac{2\Gamma}{\beta}\sigma}), & \sigma \to -\infty, \\ 4(\gamma-1)\sqrt{\dfrac{\sigma}{\pi \mathrm{Pr}}} + 0(\sigma^{-1/2}), & \sigma \to +\infty. \end{cases}$$

5. Reflected shock wave. Final remarks

We are interested mostly in the trajectory and the structure of the reflected shock wave. Consequently, we confine ourselves to the case of positive values of times t, and τ.

The solution to the problem (4.17), (4.18) is (see, for example [9])

$$F(\eta, \tau) = \frac{1}{\sqrt{2\pi\beta\tau}} \int_{-\infty}^{\infty} e^{\frac{-(\alpha-\eta)^2}{2\beta\tau}} F(\alpha, 0)d\alpha, \quad (\tau > 0)$$

and after some manipulations it can be written as follows ([3]):

$$F = 1 + e^{\frac{\Gamma}{\beta}V} - \sqrt{\varepsilon}\frac{\Gamma}{\beta}[e^{\frac{\Gamma}{\beta}V} I(V_5) + I(V_2)], \tag{5.1}$$

where

$$V = \eta + \frac{\Gamma}{2}\tau,$$
$$V_2 = \eta,$$
$$V_5 = \eta + \Gamma\tau$$

and

$$I(V) = \frac{1}{2}\int_{-\infty}^{\infty} \mathrm{erfc}\left(\frac{\alpha - V}{\sqrt{2\beta\tau}}\right) U(\alpha) e^{-\sqrt{\varepsilon}\frac{\Gamma}{\beta}X(\alpha)} d\alpha, \quad (\tau > 0).$$

From Eqs. (4.16) and (5.1) we have

$$h_0 = 1 - \frac{1 + \sqrt{\varepsilon}\left[e^{\frac{\Gamma}{\beta}V}\dfrac{\partial I(V_5)}{\partial V_5} + \dfrac{\partial I(V_2)}{\partial V_2} - \dfrac{\Gamma}{\beta}I(V_2)\right]}{1 + e^{\frac{\Gamma}{\beta}V} - \sqrt{\varepsilon}\dfrac{\Gamma}{\beta}[e^{\frac{\Gamma}{\beta}V} I(V_5) + I(V_2)]}. \tag{5.2}$$

Using Eqs. (3.34) and (4.16) in Eqs. (3.29) we obtain

$$
(5.3)\qquad
\begin{aligned}
x+t &= \eta+\varepsilon\frac{\gamma-3}{4}\frac{\beta}{\Gamma}\ln\frac{f\left(\frac{\Gamma}{2}\tau-\xi\right)}{f\left(\frac{\Gamma}{2}\tau+\eta\right)}+\ldots,\\
x-t &= \xi+\varepsilon\frac{\gamma-3}{4}\frac{\beta}{\Gamma}\ln\frac{F(\eta,\tau)}{F(-\xi,\tau)}+\ldots.
\end{aligned}
$$

The above three formulae together with Eqs. (3.20) and (3.34) complete the description of the outer flow after the reflection.

For large positive values of τ, i.e. a long time after the reflection, the above formulae simplify.

First we have for $\tau \gg 1$

$$
(5.4)\qquad g_0(\xi,\tau)\approx 1.
$$

Hence

$$
(5.5)\qquad x+t = \eta+\varepsilon\frac{\gamma-3}{4}(\xi+\eta)+\ldots.
$$

The asymptotic form of the function $h_0(\eta, \tau)$ is more complicated. We do not give details of its derivation (they are given in [3]) and we present only the result. We have

$$
(5.6)\qquad
\begin{aligned}
&u\approx\sqrt{\varepsilon}\,U(V_s)\quad \text{for}\quad \eta>0,\\
&u\approx\left\{1+\sqrt{\varepsilon}\frac{\Gamma}{\beta}U(V_s)\exp\left[\frac{\Gamma}{\beta}\left(V-\sqrt{\varepsilon}X(V_s)\right)\right]\right\}f(V-\varepsilon X(V_s))\\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{for}\quad 0>\eta>-\Gamma\tau,
\end{aligned}
$$

and

$$
(5.6)'\qquad u\approx 1\quad\text{for}\quad -\Gamma\tau>\eta.
$$

The formulae (5.6) remind corresponding formulae obtained in [1], but now the variable η is different, because we have

$$
(5.7)\qquad
\begin{aligned}
x-t &= \xi+\varepsilon\frac{\gamma-3}{4}\{\eta+\xi+\sqrt{\varepsilon}[X(-\xi+\Gamma\tau)-X(\eta+\Gamma\tau)]\}\quad\text{for}\quad \eta>0,\\
x-t &= \xi+\varepsilon\frac{\gamma-3}{4}\frac{\beta}{\Gamma}\ln\frac{f\left[\frac{\Gamma}{2}\tau-\xi-\sqrt{\varepsilon}X(\Gamma\tau-\xi)\right]}{f\left[\frac{\Gamma}{2}\tau+\eta-\sqrt{\varepsilon}X(\Gamma\tau+\eta)\right]}\quad\text{for}\quad 0>\eta>-\Gamma\tau,
\end{aligned}
$$

and

$$
(5.7)'\qquad x-t=\xi\quad\text{for}\quad -\Gamma\tau>\eta.
$$

However, from Eqs. (5.5) and (5.7) we have approximately

$$
(5.8)\qquad x+t+\frac{\gamma-3}{2}\tau=\left(1+\varepsilon\frac{\gamma-3}{2}\right)\eta.
$$

This definition of the variable η is different from that used in [1] but if we substitute Eq. (5.8) into Eq. (5.6), then we obtain approximately the same expressions for u which are given in the quoted paper.

Moreover, we can obtain the same equation of the reflected shock wave trajectory. Indeed, we define the trajectory of the reflected shock wave by the equation (see [1])

$$\eta+\frac{\Gamma}{2}\tau = \sqrt{\varepsilon}\,X(\eta+\Gamma\tau).$$

From this equation we have

$$\eta = -\frac{\Gamma}{2}\tau+4(\gamma-1)\sqrt{\varepsilon}\sqrt{\frac{\Gamma\tau}{2\pi\mathrm{Pr}}}+\ldots .$$

Using this relation in Eq. (5.8) we have

$$x_{\mathrm{shock}} = -t+\varepsilon\frac{5-3\gamma}{4}t+2(\gamma-1)\varepsilon\left(\frac{2\Gamma}{\pi\mathrm{Pr}}\right)^{1/2}\sqrt{t}. \tag{5.9}$$

Exactly the same results were obtained by LESSER and SEEBASS [1] and experimentally by STURTEVANT and SLACHMUYLDERS [4]. We can see some shift of the trajectory of the reflected shock wave compared to that predicted by the ideal gas model. Also the velocity of the reflected shock is less than that in the theory referred to. Such a result is in a good agreement with the experimental data [4, 5]. This displacement can be explained by the influence of the thermal boundary layer.

Finally, we give some remarks concerning the applied method. It is more formal than that applied by Lesser and Seebass. However, it needs more tedious calculations, but it does not mean that the whole work is more cumbersome, because we do not need to solve various approximations to the Navier-Stokes equations in the corresponding domains and then to match them. On the contrary, we obtain solutions valid at once in the whole outer domain. Moreover, our method is fully constructive, we have not used any introductory knowledge about the reflected shock wave, we have not even assumed its existence. The existence of the reflected shock wave, its structure and trajectory are the results of our considerations, however, they cannot be treated as a rigorous proof.

It is important that both the multiple scales method and the strained coordinate method fail if they are applied separately, only when used together they can cope with the problem.

We have determined only the two first terms of the expansions. In order to find terms of order $\varepsilon^2, \varepsilon^3, \ldots, \varepsilon^n$, we take into account slower time variations characterized by $t_2, t_3, \ldots$ up to t_n and we treat these variables as being independent. Next we generalize Eqs. (3.6) and (3.12)–(3.14) assuming that

$$x = \Psi(\xi, \eta, \tau_0, \ldots, \tau_{n-1}; \varepsilon) = \frac{\eta+\xi}{2}+\sum_{k=1}^{n}\varepsilon^k\Psi_k(\xi, \eta, \tau_0, \ldots, \tau_{n-1})+0(\varepsilon^{n+1}),$$

$$t_0 = \Phi(\xi, \eta, \tau_0, \ldots, \tau_{n-1}; \varepsilon) = \frac{\eta-\xi}{2}+\sum_{k=1}^{n}\varepsilon^k\Phi_k(\xi, \eta, \tau_0, \ldots, \tau_{n-1})+0(\varepsilon^{n+1}),$$

$$Q = \sum_{k=0}^{n} \varepsilon^k Q_k(\xi, \eta, \tau_0, \dots, \tau_{n-1}) + 0(\varepsilon^{n+1}),$$

where

$$\tau_0 = t_1, \dots, \quad \tau_{n-1} = t_n.$$

Thus only x and $t_0 \equiv t$ are strained.

Then the general principles of the Lighthill technique and the multiple scales method should be followed.

Although the coefficients $Q_1, Q_2, \dots, Q_n$ satisfy linear partial differential equations, calculations become more and more tedious.

The method used by LESSER and SEEBASS [1] also becomes more and more involved when calculating higher order approximations, because it is necessary to distinguish new subdomains characterized by $t_2, t_3, \dots, t_n$. Of course, these divisions demand newer and newer matchings.

Consequently, the problem of determining higher order terms is very cumbersome and it is independent of the applied method.

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