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Circular misfitting inhomogeneity in a half-plane

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THE ELASTICITY problem of a misfitting circular inclusion in an isotropic half-plane is treated. Matrix and inclusion have different elastic constants, and the matrix outer boundary is either traction-free or undeformable. The stress field at the boundaries and the strain energy of the system are investigated, graphs showing that the elastic state is remarkably influenced by the elastic constant difference. Physical implications of the obtained results are pointed out.

Rozpatrzono problem niedopasowanej inkluzji kołowej w izotropowej półpłaszczyźnie sprężystej. Matryca i inkluzja charakteryzują się różnymi stałymi sprężystościami, a zewnętrzny brzeg matrycy jest albo wolny od obciążeń, albo nieodkształcalny. Bada się pole naprężenia w pobliżu brzegów oraz energię odkształcenia układu, a podane wykresy dowodzą, że stan sprężysty układu zależy w istotnym stopniu od różnicy stałych sprężystości. Wskazano na wnioski fizyczne płynące z przedstawionych wyników.

Рассмотрена задача несогласованного кругового включения в изотропной упругой полуплоскости. Матрица и включение характеризуются разными упругими постоянными, а внешняя граница матрицы или свободна от нагрузок, или недеформируема. Исследуется поле напряжений вблизи границ и энергия деформации системы, а приведенные графики показывают, что упругое состояние системы зависит в существенной степени от разницы упругих постоянных. Указаны физические следствия вытекающие из представленных результатов.

1. Introduction

DURING the investigation of phenomena characterized by the presence, within a solid, of particles or fibres of a different material, it is often observed that these are located near to outer surfaces. As a consequence, it appears desirable to integrate the well-known elasticity solutions for inclusions in infinite media [1, 2, 3] by a detailed study of the elastic fields around inclusions in bounded media.

In the present paper the simple, yet meaningful, case of a circular misfitting inclusion in an isotropic elastic half-plane is treated. The purpose is that of displaying, together with the effect of the outer boundary presence, the effect of the difference between the elastic constants of matrix and inclusion. Because of this difference, the problem cannot be reduced to a standard boundary-value one for a thoroughly homogeneous medium, as in Ref. [2].

The geometry of the problem is sketched in Fig. 1. The outer boundary $y = -h = -\beta a/2$ $(\beta \geqslant 2)$ is assumed either traction-free or clamped (undeformable), and the interface $r = a$ is supposed perfectly adhering. The radius misfit εa is of the order of the admissible displacements in linear elasticity.

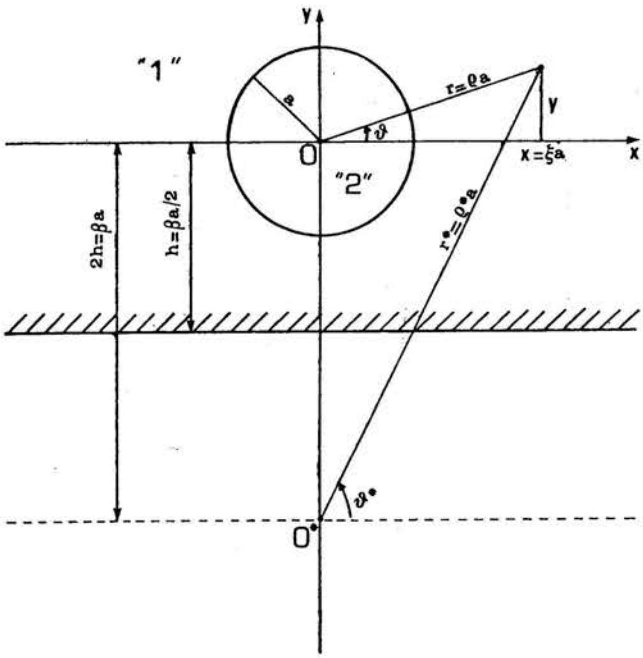

FIG. 1. Circular inclusion in a half-plane: geometry and notation.

2. The elasticity problem solution

The stresses σ_{ij} $(i, j = x, y)$ may be written in terms of two functions $\chi(z)$, $\phi(z)$ of the complex variable $z = x+iy$ as follows:

$$\sigma_{yy}+i\sigma_{xy} = \phi'(z)+\overline{\phi'(z)}+\bar{z}\phi''(z)+\chi''(z),$$

$$\sigma_{xx}+\sigma_{yy} = 2[\phi'(z)+\overline{\phi'(z)}],$$

where a prime denotes differentiation with respect to the argument and the bar denotes complex conjugates. The functions $\chi_1'(z)$, $\phi_1(z)$ and $\chi_2'(z)$, $\phi_2(z)$ [subscript 1 refers to matrix and subscript 2 to inclusion, except on expansion coefficients] may be written as:

$$\begin{cases} \chi_1'(z) = a\sum_{n=1}^{\infty}(A_n\zeta^{-n}+C_n\zeta^{*-n}), \\ \phi_1''(z) = a\sum_{n=1}^{\infty}(B_n\zeta^{-n}+D_n\zeta^{*-n}), \end{cases}$$

$$\begin{cases} \chi_2'(z) = a\sum_{n=1}^{\infty}E_n\zeta^{n}, \\ \phi_2''(z) = a\sum_{n=1}^{\infty}F_n\zeta^{n}, \end{cases}$$

where $\zeta = z/a$ and $\zeta^* = \zeta + i\beta$. Within this framework the solution of the problem, i.e. the complex coefficients $A_n, B_n, C_n, D_n, E_n, F_n$ (n from 1 to ∞), is given by the set of linear equations:

$$\begin{cases} C_n = \lambda[-n\bar{A}_n + (n^2 - \lambda^{-2})\bar{B}_n - i\beta n(n-1)\bar{B}_{n-1}], \\ D_n = \lambda[-\bar{A}_n + n\bar{B}_n - i\beta(n-1)\bar{B}_{n-1}], \\ A_1 = -p + \Psi(K_1 + \bar{K}_1), \\ A_{n+1} = (n-1)B_{n-1} + \Phi\bar{K}_{n+1}, \\ B_n = \Omega[\bar{H}_n + (n+2)\bar{K}_{n+2}], \\ H_n = (-1)^n \displaystyle\sum_{m=1}^{\infty} \binom{n+m-1}{n} C_m/(i\beta)^{m+n}, \\ K_n = (-1)^n \displaystyle\sum_{m=1}^{\infty} \binom{n+m-1}{n} D_m/(i\beta)^{m+n}, \end{cases}$$

$$\begin{cases} F_1 = [p + (1+\Psi)A_1]/2\Psi, \\ F_{n+1} = (1+\Phi^{-1})[\bar{A}_{n+1} - (n-1)\bar{B}_{n-1}], \\ E_n = (1+\Omega^{-1})\bar{B}_n - (n+2)F_{n+2}, \end{cases}$$

where

$$\lambda = \begin{cases} 1 & \text{for free boundary,} \\ -1/\varkappa_1 & \text{for clamped boundary,} \end{cases}$$

$$p = 4\Gamma\varepsilon G_1/(\varkappa_2 - 1 + 2\Gamma),$$

$$\Psi = [(\varkappa_1 - 1)\Gamma - (\varkappa_2 - 1)]/(\varkappa_2 - 1 + 2\Gamma),$$

$$\Phi = (\Gamma\varkappa_1 - \varkappa_2)/(\varkappa_2 + \Gamma),$$

$$\Omega = (\Gamma - 1)/(\Gamma\varkappa_1 + 1),$$

$$\Gamma = G_2/G_1,$$

$$\varkappa = \begin{cases} 3 - 4\nu & \text{for plane strain,} \\ (3-\nu)/(1+\nu) & \text{for plane stress} \end{cases}$$

and G and ν denoting shear moduli and Poisson's ratios, respectively.

Actually, the boundary conditions at $y = -\beta a/2$ (either zero tractions or zero displacements) are satisfied by the first two equations of the set, while the remaining equations satisfy the conditions at the interface (continuity and equilibrium [1]). Thus, the required coefficients may conveniently be evaluated by the iteration method.

It is worth noting that the parameter p coincides with the equilibrium pressure at the interface when the matrix is unbounded, and that if the inclusion is *homogeneous* (i.e. $G_2 = G_1$ and $\nu_2 = \nu_1$), then a closed-form solution is obtainable.

3. Numerical examples and discussion

Some results of numerical calculation (all for plane strain and $\beta = 3$) of stresses at the boundaries are presented in Figs. 2 to 5. The graphs are drawn for $\Gamma = 0.5$ and $\Gamma = \infty$ (rigid inclusion) and for the following pairs of Poisson's ratio values: $\nu_1 = 0.3$, $\nu_2 = 0.2$ (solid lines); $\nu_1 = 0$, $\nu_2 = 0.5$ (dashed lines); $\nu_1 = 0.5$, $\nu_2 = 0$ (dot-dashed lines). Computer calculations were performed by retaining the first 30 terms of the infinite series, which in all the examples presented here gives satisfactory convergency. In

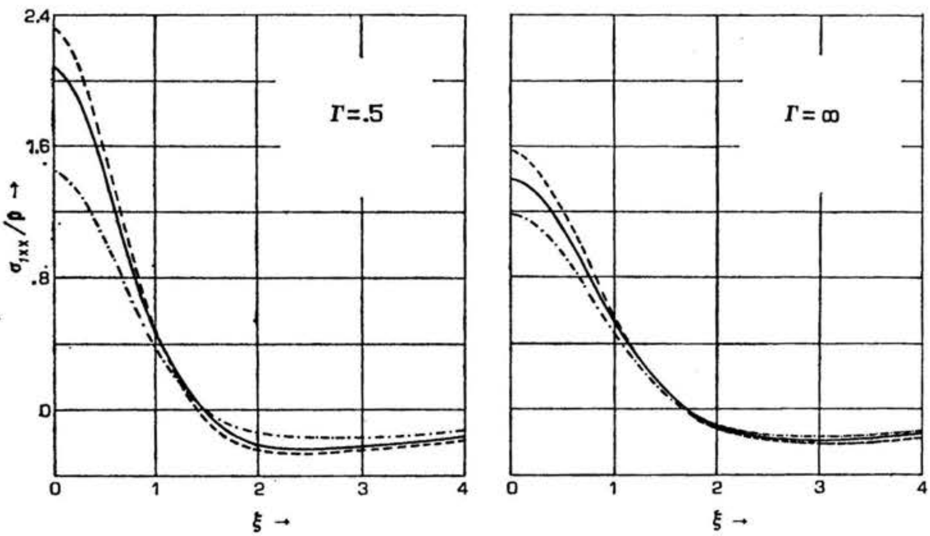

FIG. 2. σ_{1xx}/p at free boundary.

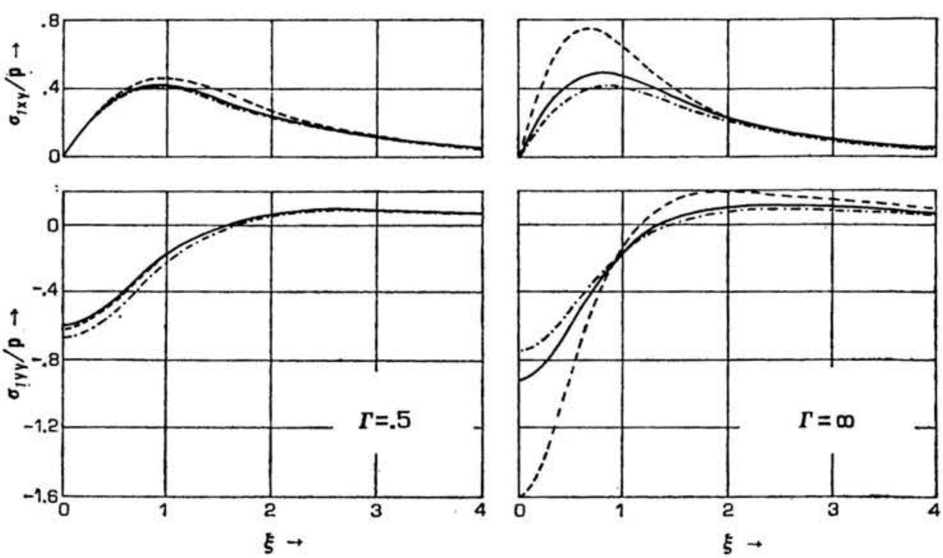

FIG. 3. Dimensionless shear stress and normal stress at clamped boundary.

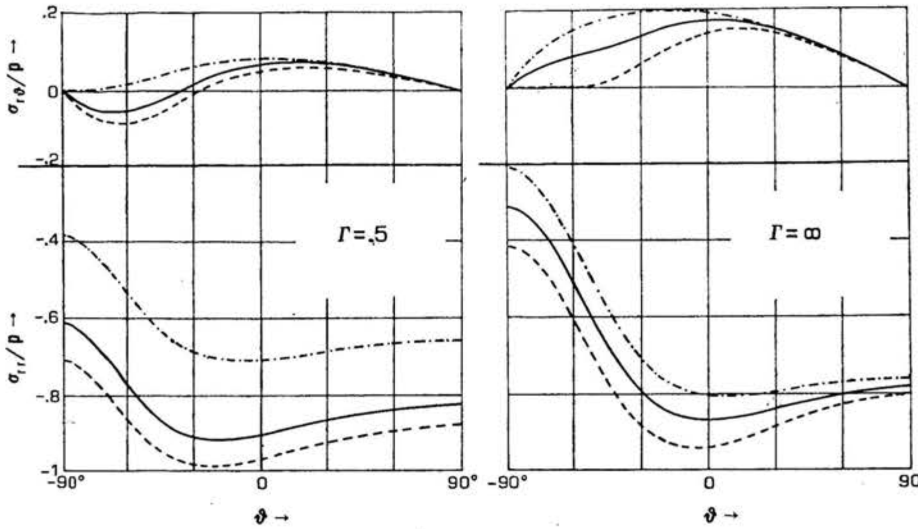

FIG. 4. Dimensionless shear stress and normal stress at common interface. Free outer boundary.

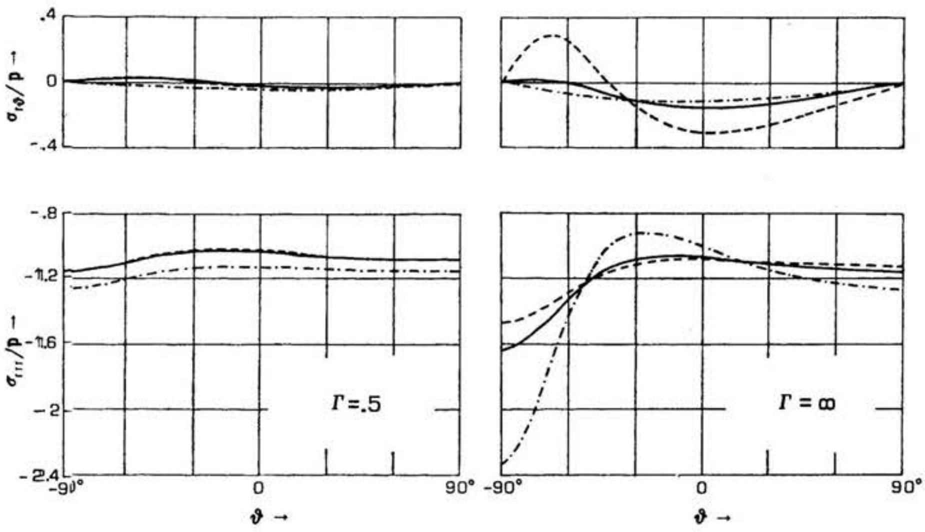

FIG. 5. Dimensionless shear stress and normal stress at common interface. Clamped outer boundary.

the discussion, cases of low and high values of Γ will be referred to as *soft* and *hard* inclusion cases, respectively.

At the free boundary it is interesting to note the change of the sign of the stress σ_{1xx} (see Fig. 2) at some distance from $x = 0$, and the high value that its maximum could reach for relatively small misfits ($|\varepsilon| \geqslant 10^{-4}$). This behaviour of the stress field may cause fracture of the material, starting from the surface.

Figure 3 shows plots of the tractions exerted on a clamped boundary. It is observed that if the inclusion is hard, then the maximum absolute values of σ_{1xy} and σ_{1yy} clearly exceed $|\varepsilon G_1|$, which for $|\varepsilon| \geqslant 10^{-4}$ is generally comparable to, or larger than the critical shear stress of the matrix; therefore, especially for low ν_1, plastic deformation on or near the surface is likely to occur. Moreover, since the normal stress changes its sign at some surface point, a bending torque results, working to detach the material from the clamping substrate.

Figures 4 and 5 show the normal and tangential stresses at the interface (which are common to matrix and inclusion) for free and clamped outer boundary, respectively. As expected, the presence of a free outer boundary lowers the *pressure* $|\sigma_{rr}|$ at the interface, while a clamped boundary enhances it. On the other hand, the graphs point out the remarkable effect of the elastic constant difference. In particular, the shear $\sigma_{r\theta}$, which is zero for an unlimited matrix, now for a hard inclusion may be comparable to p, depending on the Poisson's ratio values. A practical consequence is that the bond to the matrix of a hard inclusion close to a plane surface may no longer be considered as a perfect one, and a smooth, or partially adhering, interface should be assumed.

Finally, let us briefly consider the effect of the outer boundary on the strain energy of the system. The strain energy W (per unit length normal to the plane) may be calculated as the total work necessary to deform the matrix inner boundary and the inclusion's outer boundary. This easily leads to the simple expression:

$$W = -2W_\infty F_1/p,$$

where $W_\infty = \pi a^2 \varepsilon p$ is the strain energy when the matrix is unbounded. Results of numerical calculation of this equation have shown that the presence of a free boundary lowers the strain energy, while a clamped boundary enhances it. In both cases the function $W(\beta)$ is monotonic. For a hard inclusion the change $|W-W_\infty|/W_\infty$ is significant up to about $\beta = 7$, and its maximum value (reached for $\beta = 2$) is about 0.5. Sensibly lower figures are obtained for a soft inclusion.

Thus, for instance, nucleation and growth of a new phase should find relatively unfavourable conditions when the specimen's surface is clamped, and in this case the growth centres, if any, should mostly be found in the specimen's bulk. On the contrary, the centres should form near or on the surface if the latter is free of tractions, especially if they are hard. These features are actually observed, for example, in the $\beta \to \alpha$ tin transformation [4], and in all likelihood are responsible for the copious acoustic emission detected during heating of tin specimens [5].

It is also of some interest to consider the trend of the thermodynamic driving force acting on the inclusion due to the presence of the outer boundary. This force, perpendicular to the boundary, is given by

$$f = -\partial W/\partial h = -2a^{-1}(\partial W/\partial \beta).$$

Thus, under the action of f the inclusion is stimulated to migrate toward a free surface, or toward the bulk if the surface is clamped. Moreover, it can be shown that at the same distance from the surface a harder inclusion is subjected to a stronger attraction (free surface) or repulsion (clamped surface).

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