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On some recent crack tip stress calculations in nonlocal elasticity

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A RECENT approximation scheme used by Eringen and co-workers [1] and [2] is shown to have a non-uniform character. Moreover, arguments are given to suggest that the problem posed (with finite displacements assumed) and solved numerically may not actually possess a solution.

Stwierdzono, że nowy schemat aproksymacyjny zastosowany przez Eringena i jego współpracowników [1, 2] ma charakter niejednostajny. Przytoczono ponadto argumenty pozwalające przypuszczać, że postawiony przez tych autorów problem (w którym przyjęto odkształcenia skończone) wraz z numerycznym rozwiązaniem może w istocie nie posiadać wcale rozwiązania.

Констатировано, что новая аппроксимационная схема, примененная Эрингеном и еге сотрудниками [1, 2], имеет немонотонный характер. Кроме этого приведены аргуменые позволяющие предполагать, что поставленная этими авторами задача (в которой приняты конечные деформации), совместно с численным решением, может в сущности совсем не иметь решения.

1. Introduction

RECENTLY ERINGEN and co-workers [1, 2] have discussed certain crack problems in the framework of a nonlocal elasticity theory. These authors used, as nonlocal elastic moduli, expressions for which the dispersion relations of plane waves coincided within the entire Brillouin zone with curves obtained in the Born–Von Kármán theory of lattice dynamics. The method of analysis used in [1] and [2] consisted in reducing the problem to an integral equation which was then solved numerically in an approximate inverse manner. The following conclusion was reached: "for cracks of length $2l > 50a$ we can safely utilize the classical displacement field to calculate the nonlocal stress field", $2l$ is the crack length and a the "atomic distance" between two neighbouring atoms of a perfect lattice. This approximation was used in [1] and [2] to calculate the stress at the crack tip for the plane strain problems of a crack under uniform shear and under uniform tension. Various deductions were then made about the stress concentration at the crack tip and comparisons were carried out with such classical results as those of Griffith.

The object of the present paper is to show by exact analysis that the approximation used in [1] and [2] is not a uniformly valid one and can thus give misleading results. To illustrate this we consider the inverse problem where the crack opening displacement is specified and calculate the resulting stress-field. Even this simplified calculation requires some care in order to determine accurate near crack-tip fields, so we first solve completely a model semi-infinite crack problem and then analyze the finite length crack problem in the limit $a/l \ll 1$ by using the method of matched asymptotic expansions.

For simplicity we restrict our attention to the model one-dimensional problem discussed in [1]. The cases of anti-plane shear, plane-strain shear and plane strain tension will be discussed in a future paper. Similar results are anticipated for these later cases.

Having established the non-uniformity of the approximation scheme, a natural question to ask is, what are the precise characteristics that the solution to the original problem must have? This question is addressed in Sect. 3 where we give arguments to suggest that there may be no solution (with finite displacement field) to the model problem treated numerically in [1]. We hope to develop these results to the plane-strain situation in a future paper.

2. A model one-dimensional problem

In [1] the following one-dimensional problem is proposed, defined by the equations

$$t_{yy} = (\lambda+2\mu)\int_{-\infty}^{\infty} \alpha(|x^1-x|)\frac{\partial v(x^1,y)}{\partial y}dx^1 \tag{2.1}$$

with

$$\gamma^2 v_{xx}+v_{yy} = 0, \quad \gamma^2 = \mu/(\lambda+2\mu) \tag{2.2}$$

(μ and λ are constants), together with the boundary condition:

$$\begin{aligned} t_{yy}(x,0) &= -t_0(x), \quad \text{given for} \quad |x|<l,\\ v(x,0) &= 0, \quad |x| \geqslant l \end{aligned} \tag{2.3}$$

and

$$v \to 0 \quad \text{as} \quad y \to \infty.$$

In Eq. (2.2) the subscripts denote partial differention. In [1] it is shown that the problem specified by Eq. (2.1) to (2.3) has similar characteristics to the more complicated plane strain crack situation.

Taking the Fourier transform of Eq. (2.2) and using Eq. $(2.3)_3$ gives the transform of the displacement in the half-space $y \geqslant 0$ as

$$\bar{v}(k,y) = A(k)\exp(-\gamma|k|y), \tag{2.4}$$

where

$$\bar{v}(k,y) = \int_{-\infty}^{\infty} v(x,y)e^{ikx}dx \tag{2.5}$$

and $|k|$ is defined so as to have positive real part in the complex k plane. Also, taking the Fourier transform of Eq. (2.1) and using Eq. (2.4) gives

$$\bar{t}_{yy} = -\gamma|k|(\lambda+2\mu)A(k)\int_{-\infty}^{\infty}\alpha(|x_1|)e^{-ikx_1}dx_1. \tag{2.6}$$

In [1] the problem is reduced to a Fredholm integral equation for the unknown function $A(k)$ and arguments are given, together with some numerical work, to support "strongly" the approximation of replacing $A(k)$ in Eq. (2.6) by the corresponding result for the classical elastic problem, i.e. the problem with $\alpha(|x_1|) \equiv \delta(|x_1|)$. This approximation amounts to using the displacement from the classical elastic problem, $v_c(x^1, y)$ say, in Eq. (2.1) to compute the stress field.

We first investigate the consequences of this approximation in a fairly straightforward example.

2.1. The semi-infinite problem

This problem which mimics a semi-infinite crack problem has such boundary conditions:

$$(2.7)\qquad \begin{aligned} t_{yy}(x,0) &= -e^{\lambda x} \quad \text{for} \quad x<0,\\ v(x,0) &= 0, \quad x>0,\\ v \to 0 \quad &\text{as} \quad v \to +\infty. \end{aligned}$$

In Eq. (2.7), $1/\lambda$ plays the role of a characteristic length. Writing

$$(2.8)\qquad \begin{aligned} t_{yy}(x,0) &= t_+(x) \quad \text{for} \quad x>0,\\ v(x,0) &= v_-(x) \quad \text{for} \quad x<0 \end{aligned}$$

with both $t_+(x)$ and $v_-(x)$ as yet unknown functions of x, the transforms of Eqs. (2.7) and (2.8) together with Eqs. (2.4) and (2.6) lead to the functional equation

$$(2.9)\qquad \bar{t}_{yy}(k,0) = \frac{-1}{(\lambda+ik)} + \bar{t}_+(k) = -\gamma_0|k|\,\hat{\alpha}(k)\bar{v}_-(k),$$

where

$$(2.10)\quad \text{and}\qquad \begin{aligned} \gamma_0 &\equiv \gamma(\lambda+2\mu) = \{\mu(\lambda+2\mu)\}^{1/2},\\ \hat{\alpha}(k) &\equiv \int_{-\infty}^{\infty} \alpha(|x_1|)e^{-ikx_1}dx_1. \end{aligned}$$

If $\hat{\alpha}(k) = 1$, (the classical elastic case), the functional equation (2.9), which holds on the line $\frac{\operatorname{Im} k}{L} = 0$, can be solved by writing $|k| = k_+^{1/2}k_-^{1/2}$ where $k_+^{1/2}$ has a branch cut from $-i0$ to $-i\infty$ and $k_-^{1/2}$ a cut from $+i0$ to $+i\infty$. These branches are chosen so that $|k|$ has a positive real part when viewed as a function of k in the complex k plane. Using this factorisation Eq. (2.9) can be rearranged as

$$(2.11)\qquad J = \frac{\bar{t}_+(k)}{k_+^{1/2}} + \frac{i}{(k-i\lambda)}\left(\frac{1}{k_+^{1/2}} - \frac{1}{(i\lambda)_+^{1/2}}\right) = -\gamma_0 k_-^{1/2}\bar{v}_{c-}(k) - \frac{i}{(i\lambda)_+^{1/2}(k-i\lambda)}.$$

Using analytic continuation, a generalized form of Liouville's theorem and edge conditions on the crack tip, it can be shown that J defined by Eq. (2.11) is in fact zero. Thus the solution of Eq. (2.9) with $\hat{\alpha}(k) = 1$ is

$$(2.12)\qquad \bar{v}_{c-}(k) = \frac{-i}{\gamma_0(i\lambda)_+^{1/2}k_-^{1/2}(k-i\lambda)}.$$

We use the notation $\bar{v}_c$ to indicate that Eq. (2.12) is derived from the approximation $\hat{\alpha}(k) = 1$. In the spirit of the approximations used in [1] we substitute from Eq. (2.12) into Eq. (2.9) and take this as the result for the stress $\bar{t}_{yy}$ in the model nonlocal problem defined in Eq. (2.1). The result is

$$\bar{t}_{yy} = \frac{i\hat{\alpha}(k)k_+^{1/2}}{(i\lambda)_+^{1/2}(k-i\lambda)}. \tag{2.13}$$

Clearly, if $\hat{\alpha}(k) \neq 1$, then Eq. (2.9) is not satisfied exactly, what is required is a measure of how the accuracy of the approximation (2.13) varies with the parameters defining $\hat{\alpha}(k)$. To investigate this we consider three forms for $\alpha(|x|)$.

$$\text{(i)} \qquad \alpha(|x|) = \frac{\beta}{2}e^{-\beta|x|} \quad \text{then} \quad \hat{\alpha}(k) = \frac{\beta^2}{(\beta^2+k^2)}, \tag{2.14}$$

and $\hat{\alpha}(k)$ is analytic in $-\beta < \operatorname{Im} k < \beta$. The dimensionless constant β/λ is assumed to be much greater than unity. For this example $\bar{t}_{yy}$ given by Eq. (2.13) can be split into the sum of plus and minus functions by inspection. The result is

$$\bar{t}_{yy}(k,0) = \bar{t}_-(k,0)+\bar{t}_+(k,0), \tag{2.15}$$

where

$$\bar{t}_+(k,0) = \frac{i\beta^2\{(i\lambda)_+^{1/2}-k_+^{1/2}\}}{(i\lambda)_+^{1/2}(k-i\lambda)(\lambda^2-\beta^2)} + \frac{i\beta\{(i\beta)_+^{1/2}-k_+^{1/2}\}}{2(i\lambda)_+^{1/2}(\beta-\lambda)(k-i\beta)} - \frac{i\beta k_+^{1/2}}{2(\lambda+\beta)(k+i\beta)(i\lambda)_+^{1/2}} \tag{2.16}$$

and

$$\bar{t}_-(k,0) = \frac{-i\beta^2}{(k-i\lambda)(\lambda^2-\beta^2)} - \frac{i\beta}{2}\left(\frac{\beta}{\lambda}\right)_+^{1/2} \frac{1}{(\beta-\lambda)(k-i\beta)}.$$

Inverting the expression for $\bar{t}_-(k,0)$ gives for the stress on $y = 0$, $x < 0$ the result

$$t_{yy}(x,0) = -\frac{e^{\lambda x}\beta^2}{(\beta^2-\lambda^2)} + \frac{\beta}{2}\left(\frac{\beta}{\lambda}\right)^{1/2}\frac{e^{\beta x}}{(\beta-\lambda)}.$$

Rearranging this as

$$P_c(x) = t_{yy}(x,0)+e^{\lambda x} = \frac{-\lambda^2 e^{\lambda x}}{(\beta^2-\lambda^2)} + \frac{\beta}{2}\left(\frac{\beta}{\lambda}\right)^{1/2}\frac{e^{\beta x}}{(\beta-\lambda)}, \tag{2.17}$$

it is easy to see that for $\beta/\lambda \gg 1$ the boundary condition (2.7) seems to be more nearly satisfied as β increases since $x < 0$. For fixed $x < 0$, the right hand side of Eq. (2.17) tends to zero as $\beta \to \infty$. However, if we write $\beta x = X$, then Eq. (2.17) becomes

$$P_c(x) = \frac{-\lambda^2 e^{\lambda X/\beta}}{(\beta^2-\lambda^2)} + \frac{\beta}{2}\left(\frac{\beta}{\lambda}\right)^{1/2}\frac{e^X}{(\beta-\lambda)} \tag{2.18}$$

and clearly $P_c(x)$ does not tend to zero, as $\beta \to \infty$, for $x < 0$ uniformly in x. In fact

$$P_c(x) \to \frac{1}{2}\left(\frac{\beta}{\lambda}\right)^{1/2} e^X \tag{2.19}$$

as $\beta \to \infty$, $X \leqslant 0$.

In [1] and [2] various plots are given of $P_c(x)$. Superficially, it looks as if the boundary condition (2.7), which is $P_c(x) = 0$, $x < 0$, were satisfied more accurately as a parameter, analogous to β, increases. However, we contend that the same non-uniform behaviour described above is present in their numerical results.

(2.20) (ii)
$$\alpha(|x|) = \frac{1}{a}\left(1-\frac{|x|}{a}\right), \quad |x| \leqslant a,$$
$$= 0, \qquad |x| \geqslant a.$$

This expression is used in [1], a is a lattice parameter and is given the value 2.48 Å in the case of steel. Thus with a of order 10^{-8} cms, the ratio of any macroscopic crack length to a will be much greater than unity. In this case

$$\hat{\alpha}(k) = \frac{\sin^2 ka}{k^2a^2} \tag{2.21}$$

which is analytic everywhere. The expression for $\bar{t}_{yy}(k,0)$ given in Eq. (2.13) is thus analytic in $0 < \operatorname{Im} k < \lambda$. The Fourier inversion theorem then gives

$$t_{yy} = \frac{1}{2\pi}\int\limits_{-\infty+id}^{\infty+id} \bar{t}_{yy}(k,0)e^{-ikx}dk \tag{2.22}$$

with $0 < d < \lambda$. From Eq. (2.13)

$$\bar{t}_{yy} = \frac{ik_+^{1/2}}{(i\lambda)_+^{1/2}(k-i\lambda)}\,\frac{\sin^2 ka}{k^2a^2}$$

and $\sin^2 ka = \frac{1}{2}-\frac{1}{4}e^{2ika}-\frac{1}{4}e^{-2ika}$, hence substituting into the above integral gives

$$t_{yy} = -\frac{\sinh^2(\lambda^2a^2)}{(\lambda^2a^2)}e^{\lambda x}, \quad \text{for} \quad x \leqslant -2a. \tag{2.23}$$

This result is obtained by closing the contour in the upper half-plane and picking up the pole at $k = i\lambda$. The condition $x \leqslant -2a$ is necessary in order that the exponential terms decay on a large semi-circular contour in this upper half-plane.

From Eq. (2.23) it is easily seen that as $a\lambda \to 0$, $t_{yy} \to -e^{\lambda x}$ uniformly in x provided $x \leqslant -2a$. The boundary condition is thus satisfied uniformly in the region $x \leqslant -2a$, it remains to investigate what happens in $-2a < x < 0$. To do this we take the complex integral (2.22) along the real axis ($d = 0$) and obtain

$$t_{yy} = \frac{1}{\pi}\int\limits_0^\infty \left(\frac{r^{1/2}}{\lambda^{1/2}}\,\frac{\sin^2 ra}{r^2a^2}\right)\frac{\left\{r\operatorname{Sin}\left(\frac{\pi}{4}+rx\right)-\lambda\operatorname{Cos}\left(\frac{\pi}{4}+rx\right)\right\}dr}{(r^2+\lambda^2)}. \tag{2.24}$$

This integral is valid for all x and is a continuous function of x with a constant value at $x = 0$. As $a \to 0$ the contribution to the integral from the integrand with the factor $\lambda\operatorname{Cos}\left(\frac{\pi}{4}+rx\right)$ will be finite. This is easily seen, since $\frac{\sin^2 ra}{r^2a^2} \leqslant 1$, the contribution to this part of the integral is of magnitude $< \frac{1}{\pi}\int\limits_0^\infty \frac{\lambda^{1/2}r^{1/2}}{(r^2+\lambda^2)}dr$ which is finite for $\lambda > 0$. The rest of the integrand is, however, much more sensitive to the limit $a \to 0$. To see this, write

$$r = \frac{R}{a}, \quad x = aX$$

to give

$$\text{(2.25)} \qquad t_{yy} \approx \frac{1}{\pi(a\lambda)^{1/2}}\int\limits_0^\infty \frac{R^{3/2}\sin^2 R}{R^2}\,\frac{\sin\left(\frac{\pi}{4}+RX\right)dR}{(R^2+a^2\lambda^2)}.$$

We use the $\approx$ sign to indicate that we are neglecting the contribution from the second term of the integrand of Eq. (2.24) since this is finite as $a \to 0$ as shown above.

When $X = 0$, the expression (2.25) shows that when $(a\lambda) \to 0$ the stress at the origin behaves like

$$\text{(2.26)} \qquad t_{yy} \approx \frac{1}{\pi(a\lambda)^{1/2}}\int\limits_0^\infty \left(\frac{\mathrm{Sin}^2 R}{R^2}\right)\frac{dR}{(2R)^{1/2}}.$$

Further, this result is irrespective of whether $X \to 0$ with $X > 0$ or with $X < 0$. Thus the same characteristics as shown in example (i) are present in this case, i.e. that as the region of the discrepancy in the boundary condition gets smaller the magnitude gets larger tending to infinity as $a \to 0$. In [1] it is asserted that the increase in magnitude of t_{yy} at $x = 0$ over the boundary value is the stress concentration. Our contention is that the stress is in fact continuous in this approximation and the value (2.26) is merely a property of the approximation, *not* of the original boundary value problem.

$$\text{(2.27)} \quad \text{(iii)} \qquad \alpha(|x|) = \alpha_0 \exp\left\{-\left(\frac{\beta}{a}\right)^2 x^2\right\} \quad \text{with} \quad \alpha_0 = \frac{\beta}{a\sqrt{\pi}}.$$

Typical values of constants in this expression are given in (2) as $a = 2.48$ Å and $\beta = 1.65$ for steel. Now $\hat{\alpha}(k) = \exp\{-k^2a^2/(4\beta^2)\}$ and is an entire function of k. Hence $\bar{t}_{yy}(k, 0)$ with this expression for $\alpha(k)$ is analytic in $0 < \mathrm{Im}\,k < \lambda$ so Eq. (2.22) applies. Evaluating this integral along the real axis leads to

$$\text{(2.28)} \quad t_{yy} = \frac{1}{\pi}\int\limits_0^\infty \frac{r^{1/2}}{\lambda^{1/2}}\,\frac{\exp\{-r^2a^2/(4\beta^2)\}\left\{r\,\mathrm{Cos}\left(\frac{\pi}{4}-rx\right)-\lambda\,\mathrm{Sin}\left(\frac{\pi}{4}-rx\right)\right\}dr}{(r^2+\lambda^2)}.$$

The second term in the integrand gives a finite contribution to t_{yy} as $a \to 0$. To investigate the contribution from the first term write

$$r = 2\beta R/a \quad \text{and} \quad x = aX/2\beta$$

to get

$$t_{yy} \approx \frac{1}{\pi} \int_0^\infty \left(\frac{2\beta}{a\lambda}\right)^{1/2} \frac{R^{1/2}\exp(-R^2)\operatorname{Cos}\left(\frac{\pi}{4}-RX\right)dR}{R^2+\left(\frac{a\lambda}{2\beta}\right)^2}. \tag{2.29}$$

Hence, this integral shows that t_{yy} goes to infinity like $(2\beta/a\lambda)^{1/2}$ as $(a\lambda/\beta) \to 0$ with $X \to +0$ or -0.

Thus in each of the examples we have considered, the approximation of using the displacement of the classical elastic problem in order to calculate the stress for the nonlocal problem has been shown to be non-uniform and hence unsatisfactory. In the next section we show that the same kind of behaviour is present in the finite crack problem.

2.2. The finite crack, model problem (Specified displacement)

To demonstrate the effect of the approximation suggested in [1] and [2], we consider the problem specified by Eqs. (2.1) and (2.2) together with the boundary conditions

$$\begin{aligned} v(x,0) &= 0, \quad |x| \geqslant l, \\ v(x,0) &= (l^2-x^2)^{1/2}, \quad |x| \leqslant l \end{aligned} \tag{2.30}$$

and

$$v \to 0 \quad \text{as} \quad y \to +\infty.$$

For reasonable crack lengths we expect a small parameter ε to appear in the problem because of the magnitude of the nonlocal moduli, for example a (the lattice parameter) is given in Angstroms (10^{-8} cms) in examples (ii) and (iii). Hence we define

$$\varepsilon_1 = \frac{1}{\beta l}, \quad \varepsilon_2 = \frac{a}{l}, \quad \varepsilon_3 = \frac{a}{\beta l}, \tag{2.31}$$

where $\varepsilon_i \ll 1$ and $i = 1, 2$ or 3 refers to examples (i), (ii) and (iii) of Sect. 2.1.

To investigate the behaviour near the crack tip $x = l$ we write

$$\begin{aligned} x &= l+\varepsilon lX, \quad y = \varepsilon lY, \\ x' &= l+\varepsilon lX', \quad y' = \varepsilon lY', \\ v &= (\varepsilon l)^{1/2}V, \quad t_{yy} = (\varepsilon l)^{-1/2}T, \end{aligned} \tag{2.32}$$

where ε without a subscript refers to either of Eqs. (2.31) whichever is appropriate. In these new coordinates Eqs. (2.1) and (2.2) become

$$T(X,Y) = (\lambda+2\mu) \int_{-\infty}^{\infty} \alpha_i(|X^1-X|) \frac{\partial V(X^1,Y)}{\partial Y} dX^1 \tag{2.33}$$

and

$$\gamma^2 V_{,XX}+V_{,YY} = 0, \tag{2.34}$$

where

$$\text{(2.35)}\quad \begin{aligned} &\text{(i)} && \alpha_1(|X|) = \frac{1}{2}\exp(-|X|),\\ &\text{(ii)} && \alpha_2(|X|) = (1-|X|), \quad |X| \leqslant 1,\\ &&& \qquad\quad = 0, \qquad\qquad |X| \geqslant 1,\\ &\text{(iii)} && \alpha_3(|X|) = \frac{1}{\sqrt{\pi}}\exp(-X^2). \end{aligned}$$

The boundary conditions (2.30) become

$$\text{(2.36)}\quad \begin{aligned} &V(X,0) = 0, \quad X > 0, \quad X < -\frac{2}{\varepsilon},\\ &V(X,0) = (-X)^{1/2}(2l+\varepsilon l X)^{1/2}, \quad -\frac{2}{\varepsilon} < X < 0. \end{aligned}$$

As $\varepsilon \to 0$ these boundary conditions become

$$\text{(2.37)}\quad \begin{aligned} &V(X,0) = 0, \quad X > 0,\\ &V(X,0) = +(2l)^{1/2}(-X)^{1/2}, \quad -\infty < X < 0. \end{aligned}$$

This problem now has similar characteristics to the semi-infinite one discussed in Sect. 2.1 and it is straightforward to derive the corresponding results

$$\text{(2.38)}\qquad \overline{T}(k,0) = -\gamma_0|k|\,\hat{\alpha}_i(k)V_-(k)$$

and from Eqs. (2.37)

$$\text{(2.39)}\qquad \overline{V}_-(k) = -l^{1/2}\left(\frac{\pi}{2}\right)^{1/2} e^{\pi i/4}k_-^{-3/2}$$

with

$$\hat{\alpha}_1(k) = \frac{1}{(1+k^2)}, \quad \hat{\alpha}_2(k) = \frac{\sin^2 k}{k^2}$$

and

$$\hat{\alpha}_3(k) = \exp(-k^2/4).$$

The calculation of $T(X,0)$ from Eq. (2.38) is similar to the evaluation of t_{yy} in Sect. 2.1. For example (i), $\overline{T}$ can be split by inspection into plus and minus functions as

$$\text{(2.40)}\qquad \overline{T}_+ = \frac{\gamma_0 l^{1/2}}{2}\left(\frac{\pi}{2}\right)^{1/2} e^{-i\pi/4}\left\{\frac{-k_+^{-1/2}}{(k+i)} + \frac{1}{(k-i)}\left(k_+^{-1/2}-(i)^{-1/2}\right)\right\}$$

and

$$\overline{T}_- = \frac{\gamma_0}{2} l^{1/2}\left(\frac{\pi}{2}\right)^{1/2}\frac{e^{-\pi i/2}}{(k-i)}$$

since $(i)^{-1/2} = e^{-i\pi/4}$. Inverting $\overline{T}_-$ gives

$$\text{(2.41)}\qquad T = \frac{\gamma_0 l^{1/2}}{2}\left(\frac{\pi}{2}\right)^{1/2} e^X \quad \text{for} \quad X < 0.$$

Recalling that $X = (x-l)/\varepsilon l$ and that $t_{yy} = (\varepsilon l)^{-1/2}T$ we see that t_{yy} tends to zero, the stress free crack boundary condition, except when $x-l = 0(\varepsilon)$, i.e. except within distances of order ε from the crack tip. Within such distances the crack boundary stress goes to infinity like $\varepsilon^{-1/2}$, as $\varepsilon \to 0$.

Thus for the finite crack problem the correct boundary condition cannot be satisfied uniformly by the approximation of using the elastic crack displacement. Similar results can be obtained for examples (ii) and (iii) following the analysis of Sect. 2.1.

3. Miscellaneous results

The results of Sect. 2 demonstrate (in our opinion) the inadequacy of the approximation suggested in [1] and [2], however, the question remains as to what is the precise nature of the solution to the problem originally formulated in Sect. 2 with the boundary conditions (2.3). Note that the case with constant $t_0(x)$ was treated numerically in [1] and in the results displayed which apparently justified the approximation scheme subsequently used which we have critized in Sect. 2.

To investigate this further, we consider solutions of Eq. (2.2) in terms of a continuous distribution of virtual screw dislocations within the crack $y = 0$, $-l < x < l$. Thus we can write

$$\frac{\partial v(x, y)}{\partial y} = \gamma, \quad \int_{-}^{l} \frac{(x-\xi)f(\xi)d\xi}{(x-\xi)^2+\gamma^2 y^2}, \tag{3.1}$$

where

$$v(x, 0) = \int_{-l}^{x} f(\xi)d\xi. \tag{3.2}$$

We have presupposed here that the crack displacements will be finite by assuming $f(\xi)$ is integrable. It is possible that there may be solutions to the nonlocal problem in which $v(x, 0)$ is *not* finite particularly as $x \to \pm l$. However, assuming $f(\xi)$ is integrable we can substitute for Eq. (3.1) into Eq. (2.1) to get

$$t_{yy} = (\lambda+2\mu)\gamma \int_{-\infty}^{\infty} \alpha(|x^1-x|)\,dx^1 \int_{-l}^{l} \frac{(x^1-\xi)\,t(\xi)}{(x^1-\xi)^2+\gamma^2 y^2}\,d\xi, \tag{3.3}$$

$$= \gamma_0 \int_{-l}^{l} f(\xi)d\xi \int_{-\infty}^{\infty} \frac{\alpha(|x^1-x|)(x^1-\xi)}{(x^1-\xi)^2+\gamma^2 y^2}\,dx^1 \tag{3.4}$$

the last equation following by interchanging the order of integration.

Taking the limit y tending to zero Eq. (3.4) can be written as

$$t_{yy} = \gamma_0 \int_{-l}^{l} f(\xi)d\xi \int_{-\infty}^{\infty} \frac{\alpha(|x_0|)dx_0}{x_0+(x-\xi)}, \tag{3.5}$$

the inner integral being a Cauchy principal value (Hilbert transform). For convenience rewrite Eq. (3.5) as

$$t_{yy} = \gamma_0 \int_{-l}^{l} f(\xi)\, d\xi K(x-\xi), \tag{3.6}$$

where

$$K(x-\xi) = \int_{-\infty}^{\infty} \frac{\alpha(|x_0|)dx_0}{x_0+x-\xi}. \tag{3.7}$$

For $\alpha(|x_0|)$ given in our previous three examples we have

(i) $\alpha(|x|) = \frac{\beta}{2}\exp(-\beta|x|)$

then

$$K(x-\xi) = \frac{\beta}{2}\operatorname{sgn}(-x+\xi)\{\exp(\beta|\xi-x|)E_i(-\beta|\xi-x|) - \exp(-\beta|\xi-x|)\overline{E}_i(\beta|\xi-x|)\} \tag{3.8}$$

(see tables of the Hilbert transform, ERDELYI, MAGNUS, OBERHETTINGER [3].

(ii) $\alpha(|x|) = \frac{1}{a}\left(1-\frac{|x|}{a}\right), \quad |x| \leqslant a,$

$\qquad = 0, \qquad |x| \geqslant a$

then

$$K(x-\xi) = \frac{1}{a^2}\{(a+x-\xi)\log|x+a-\xi| - (a-x+\xi)\log|x-a-\xi| - 2(x-\xi)\log|x-\xi|\}. \tag{3.9}$$

(iii) $\alpha(|x|) = \frac{\beta}{a\sqrt{\pi}}\exp\left\{-\left(\frac{\beta}{a}\right)^2 x^2\right\}$

then

$$K(x-\xi) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} \frac{e^{-x_0^2}dx_0}{\frac{ax_0}{\beta}+x-\xi}. \tag{3.10}$$

For interest we consider a fourth example

(iv) $\alpha(|x|) = \frac{1}{\pi}\frac{a}{(x^2+a^2)}$

then

$$K(x-\xi) = \frac{x-\xi}{(x-\xi)^2+a^2}. \tag{3.11}$$

The functions $\alpha(|x|)$ given above can be ordered in terms of how quickly they tend to zero as $|x| \to \infty$. Such an ordering is (ii), (iii), (i), (iv). The boundary condition $t_{yy} = 1$ for $|x| \leqslant l$, $y = 0$ leads to the integral equation

$$(3.12)\qquad 1 = \gamma_0 \int_{-l}^{l} K(x-\xi)f(\xi)d\xi, \quad |x| \leqslant l.$$

We first consider the kernel (3.11) since it is perhaps the easiest to deal with. When $a = 0$, $K(x-\xi) = \dfrac{1}{x-\xi}$ and Eq. (3.12) becomes the usual Cauchy integral equation associated with the classical elastic problem. However, when $a \neq 0$, $K(x-\xi)$ is finite for all real x and it is easily seen that the right hand side of Eq. (3.12) is an analytic function of x for all real x provided $f(\xi)$ is integrable in $-l \leqslant \xi \leqslant l$. In fact, replacing x by z in the right hand side of Eq. (3.12) we can analytically continue it into the complex z plane as the function

$$\gamma_0 \int_{-l}^{l} \frac{f(\xi)(z-\xi)d\xi}{(z-\xi)^2+a^2}$$

which is analytic everywhere except for poles on the lines $z = \pm ia+\xi$ for $|\xi| \leqslant l$. However, the analytic continuation of the left hand side is unity which contradicts the equality sign in Eq. (3.12). This contradiction shows that Eq. (3.12) has *no solution* for integrable $f(\xi)$.

As another example consider (iii), again if $a = 0$ (3.10) reduces to the Cauchy kernel and the classical elastic case results. However, if $a/\beta \neq 0$, Eq. (3.10) can be continued into the complex z plane and is analytic everywhere. Moreover, it can be represented in the half plane. $\mathrm{Im}(z-\xi) > 0$ as

$$K(z-\xi) = \frac{2\beta i}{a}\left[\int_{-\infty}^{i\beta(\xi-z)/a} \exp(-\beta_1^2)\right] d\beta_1 \exp(-\beta^2(\xi-z)^2/a^2)$$

and then clearly, if $f(\xi)$ is integrable, the right hand side of Eq. (3.12) with x replaced by z has the growth properties of $e^{-\beta^2 z^2/a^2}$ in the upper half plane and clearly cannot be equal to unity (the analytic continuation of the left hand side of Eq. (3.12)). This contradiction shows that Eq. (3.12) cannot have a solution for integrable $f(\xi)$, for example (iii) either.

The difficulties with moduli such as those of example (iii) have been discussed in a different context by Rogula [4].

4. Concluding remarks

We have attempted to show in Sect. 2 that the approximation used in [1] and [2] is a non-uniform one and hence unlikely to give correct results for the problem as formulated in Eqs. (2.1) to (2.3). The results of Sect. 3, although not perhaps sufficiently rigorous, do suggest strongly that the direct numerical calculations made in [1] are in fact attempt-