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Exact solutions in nonlinear elastodynamics

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Two exact solutions of one-dimensional linear elastodynamics are given. The first problem concerns harmonic oscillations of a layer of finite thickness made of Mooney–Rivlin material, the other one deals with harmonic vibrations of an infinite cylinder made of the same material. Vibrations of the layer and the cylinder are produced by harmonic variation of loads applied to the boundaries. Both problems are reduced to the solution of known linear ordinary differential equations.

1. Introduction

There are few exact dynamical solutions available within the context of nonlinear constitutive theories in elasticity. In addition to describing the solution to simple boundary-initial value problems exactly, these solutions serve a very useful purpose, namely, that of providing a check for the numerical schemes which one encounters in more complex problems. Recently, several dynamical solutions have been established within the context of general theories of incompressible and compressible elastic solids (cf. Carroll [1–3]). However, there are few explicit exact elastodynamic solutions to initial boundary value problems within the context of specific nonlinear theories. Such avenues have been explored to great lengths in fluid mechanics within the context of the classical linearly viscous model, and also several specific non-Newtonian fluid models. While the same can be said of the area of elastodynamics of a linearized elastic material, not much effort has been expended within the context of nonlinear constitutive theories. Much of the emphasis has been put on establishing universal solutions and research has been striving more towards generality than the study of the specific constitutive theory.

Recently, there has been a resurgence of interest in determining exact elastostatic solutions within the context of specific nonlinear constitutive theories (cf. Currie and Hayes [5], Rajagopal and Wineman [6], Rajagopal, Wineman and Troy [7], Sensening [8], Ogden [9], McLeod, Rajagopal and Wineman [10], Chao, Rajagopal and Wineman [11], Fu, Rajagopal and Szeri [12], Rajagopal and Carroll [13], etc). These papers are primarily concerned with nonhomogeneous deformations of nonlinear elastic materials within the context of elastostatics.

In this brief note we discuss a couple of representative examples of elastodynamic problems within the context of the neo-Hookean and Mooney–Rivlin theories. The problems considered herein are by no means the only ones where exact solutions can be established. In fact, the analysis suggests a whole host of unidirectional motion problems for which one might reasonably expect to establish exact solutions.

The Cauchy stress **T** in a Mooney–Rivlin material is given by (cf. TRUESDELL and NOLL [4])

$$\mathbf{T} = -p\mathbf{I}+\mu\left(\beta+\frac{1}{2}\right)\mathbf{B}-\mu\left(\frac{1}{2}-\beta\right)\mathbf{B}^{-1}, \tag{1.1}$$

where

$$-\frac{1}{2} \leqslant \beta \leqslant \frac{1}{2},$$

$\mathbf{B} = \mathbf{F}\mathbf{F}^T$ is the right-relative Cauchy–Green strain tensor, and **F** denotes the deformation gradient. The spherical stress $-p\mathbf{I}$ is due to the assumption that the material is incompressible. When $\beta = 1/2$, the above constitutive equation simplifies to the neo-Hookean constitutive relation

$$\mathbf{T} = -p\mathbf{I}+\mu\mathbf{B}. \tag{1.2}$$

2. Equations of motion

Let us consider unidirectional time-dependent deformations of the form

$$x = X, \quad y = Y, \quad z = Z+w(Y, t), \tag{2.1}$$

where (x, y, z) denote the position at time t of a particle initially at (X, Y, Z). A simple computation yields

$$\mathbf{B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \dfrac{\partial w}{\partial y} \\ 0 & \dfrac{\partial w}{\partial y} & 1+\left(\dfrac{\partial w}{\partial y}\right)^2 \end{pmatrix}, \tag{2.2}$$

and

$$\mathbf{B}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1+\left(\dfrac{\partial w}{\partial y}\right)^2 & -\dfrac{\partial w}{\partial y} \\ 0 & -\dfrac{\partial w}{\partial y} & 1 \end{pmatrix}. \tag{2.3}$$

Notice that the det $\mathbf{F} = \det\mathbf{B} = 1$, and thus the motion under consideration is isochoric. It follows that the balance of linear momentum

$$\operatorname{div}\mathbf{T}+\varrho\mathbf{b} = \varrho\frac{d\mathbf{v}}{dt}, \tag{2.4}$$

reduces to

$$\frac{\partial p}{\partial x} = 0, \tag{2.5}$$

(2.6)
$$\frac{\partial p}{\partial y} = 2\mu\left(\beta-\frac{1}{2}\right)\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial y^2},$$

(2.7)
$$-\frac{\partial p}{\partial z} = -\mu\frac{\partial^2 w}{\partial y^2}+\varrho\frac{\partial^2 w}{\partial t^2}.$$

On defining a new function p^* through

(2.8)
$$p^* = p-\mu\left(\beta-\frac{1}{2}\right)\left(\frac{\partial w}{\partial y}\right)^2$$

we see that Eqs. (2.5)–(2.7) can be rewritten as

(2.9)
$$\frac{\partial p^*}{\partial x} = 0, \quad \frac{\partial p^*}{\partial y} = 0,$$

(2.10)
$$-\frac{\partial p^*}{\partial z} = -\mu\frac{\partial^2 w}{\partial y^2}+\varrho\frac{\partial^2 w}{\partial t^2}.$$

Let us suppose we are interested in the problem of a slab of Mooney–Rivlin material of thickness H subject to an oscillatory pressure gradient in the z direction of the form

(2.11)
$$\frac{\partial p}{\partial z} = \frac{\partial p^*}{\partial z} = -\varrho\{P_0+Q_0\cos\Omega t\}.$$

Then Eq. (2.10) reduces to

(2.12)
$$-\varrho\{P_0+Q_0\cos\Omega t\} = -\mu\frac{\partial^2 w}{\partial y^2}+\varrho\frac{\partial^2 w}{\partial t^2}.$$

We shall suppose that the layers at $y = 0$ and $y = H$ are at rest, i.e., the appropriate boundary conditions are

(2.13)
$$w(0, t) = 0, \quad w(H, t) = 0.$$

It is straightforward to verify that the solution to Eq. (2.12) subject to Eq. (2.13) is

(2.14)
$$w(y, t) = \frac{\varrho P_0}{2\mu}y(H-y)+\frac{Q_0}{\Omega^2}\left[\cos\sqrt{\frac{\varrho\Omega^2}{\mu}}\,y + \frac{\left(1-\cos\sqrt{\frac{\varrho\Omega^2}{\mu}}\,H\right)}{\sin\sqrt{\frac{\varrho\Omega^2}{\mu}}\,H}\sin\sqrt{\frac{\varrho\Omega^2}{\mu}}\,y-1\right]\cos\Omega t.$$

Let us next consider the problem wherein the layer at $y = 0$ oscillates with the velocity $U\cos\Omega t$ and the layer at $y = H$ is at rest, i.e.,

(2.15)
$$w(0, t) = U\cos\Omega t, \quad w(H, t) = 0.$$

We shall seek a solution to Eq. (2.10) wherein there is no pressure gradient in the z direction. It is once again easy to verify that the solution to Eq. (2.10), subject to Eq. (2.15) is

$$w(y,t) = \frac{U}{\sin \sqrt{\frac{\varrho\Omega^2}{\mu}}\, H}\left[\sin \sqrt{\frac{\varrho\Omega^2}{\mu}}\,(H-y)\right]\cos\Omega t. \tag{2.16}$$

It is easy to see there are several other elastodynamic boundary value problems involving layers for which one can establish such exact solutions.

Next, we turn our attention to elastodynamic problems involving a cylinder of neo-Hookean material of radius R_0. Consider a deformation of the form

$$r = R, \quad \theta = \Theta, \quad z = Z + w(R,t). \tag{2.17}$$

A simple computation yields

$$\mathbf{B} = \begin{pmatrix} 1 & 0 & \frac{\partial w}{\partial R} \\ 0 & 1 & 0 \\ \frac{\partial w}{\partial R} & 0 & 1+\left(\frac{\partial w}{\partial R}\right)^2 \end{pmatrix}, \quad \mathbf{B}^{-1} = \begin{pmatrix} 1+\left(\frac{\partial w}{\partial R}\right)^2 & 0 & -\frac{\partial w}{\partial R} \\ 0 & 1 & 0 \\ -\frac{\partial w}{\partial R} & 0 & 1 \end{pmatrix}, \tag{2.18}$$

and once again we note that $\det\mathbf{F} = \det\mathbf{B} = 1$. Thus the motion is isochoric.

The balance of linear momentum reduces to

$$-\frac{\partial p}{\partial R} = -\frac{\partial p}{\partial \theta} = 0, \tag{2.19}$$

$$-\frac{\partial p}{\partial Z} = -\mu\left[\frac{\partial^2 w}{\partial R^2} + \frac{1}{R}\frac{\partial w}{\partial R}\right] + \varrho\frac{\partial^2 w}{\partial t^2}. \tag{2.20}$$

As before, we shall suppose that

$$\frac{\partial p}{\partial Z} = -\varrho[P_0 + Q_0\cos\alpha t]. \tag{2.21}$$

Assuming a solution of the form

$$w(R,t) = W(R)\cos\alpha t + G(R) \tag{2.22}$$

and carrying out an analysis similar to that for the layer problem yields exact solutions for $G(R)$ and $W(R)$. The appropriate equations of motion are

$$\frac{d^2G}{dR^2} + \frac{1}{R}\frac{dG}{dR} = -\frac{\varrho}{\mu}P_0, \tag{2.23}$$

$$\frac{d^2W}{dR^2} + \frac{1}{R}\frac{dW}{dR} + \left(\frac{\varrho\alpha^2}{\mu}\right)W = -\frac{\varrho Q_0}{\mu}. \tag{2.24}$$

The appropriate boundary condition is

$$W(R_0, t) = 0, \tag{2.25}$$

and we shall require that the velocity field be bounded. It immediately follows that

$$G(R) = -\frac{\varrho P_0}{4\mu}(R^2 - R_0^2), \tag{2.26}$$

and

$$W(R) = -\frac{\pi\varrho Q_0}{2\mu m} RJ_1(mR)Y_0(mR) - \left[\frac{\varrho Q_0}{\mu m^2 J_0(mR_0)} + \frac{\pi\varrho Q_0}{2\mu m} RY_1(mR)\right] J_0(mR), \tag{2.27}$$

where J_0, J_1, Y_0 and Y_1 are Bessel functions of order 0 and 1 of the first and second kind, and

$$m^2 = \frac{\varrho\alpha^2}{\mu} > 0. \tag{2.28}$$

Thus

$$W(R, t) = \frac{-\pi\varrho Q_0}{2\mu m} RJ_1(mR)Y_0(mR)\cos\alpha t - \left[\frac{\varrho Q_0}{\mu m^2 J_0(mR_0)} + \frac{\pi\varrho Q_0}{2\mu m} RY_1(mR)\right] J_0(mR)\cos\alpha t - \frac{\varrho P_0}{4\mu}(R^2 - R_0^2). \tag{2.29}$$

The examples discussed above are just a few of the many cases where it is possible to determine explicit exact solutions to elastodynamic problems involving neo-Hookean and Mooney–Rivlin materials. Such simple exact solutions would serve a very useful purpose in providing means for checking the complicated algorithm one develops in the numerical study of the elastodynamics of nonlinear materials.

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