Arch. Mech., 40, 2–3, pp. 171–181, Warszawa 1988

Stability of a particular motion of a profile suspended by an elastic rod

P. CAPODANNO (BESANÇON)

THE AUTHOR considers a system formed by a rigid profile carrying an elastic rod, moving in an inviscid incompressible fluid in irrotational motion, the forces exerted by the fluid on the rod being negligible. The elastic rod is suspended on a rigid horizontal string, this constraint being frictionless. In the first part of the paper, there are written the equations of motion by means of the theorems of momentum and of the moment of momentum and Hamilton–Ostrogradski's principle; the first integrals are obtained. In the second part, the author studies the existence and the stability of motions of horizontal uniform translation of the profile with relative equilibrium of the rod in the undeformed state, the rod being directed vertically. The problem of stability is reduced to the problem of the minimum of a convenient functional; the author gives sufficient conditions of stability.

Rozważa się układ składający się ze sztywnego profilu zaopatrzonego w pręt sprężysty; układ ten porusza się w płynie nieściśliwym i nielepkim, zaniedbuje się siły wywierane przez płyn na pręt sprężysty. Pręt podwieszony jest na poziomej strunie i porusza się wzdłuż niej bez tarcia. W pierwszej części pracy wyprowadzono równania ruchu układu, opierając się na zasadach zachowania pędu i momentu pędu oraz na twierdzeniu Hamiltona–Ostrogradskiego. W drugiej części zbadano problem istnienia i stateczności ruchu profilu polegającego na jego równomiernym przesuwaniu poziomym z zachowaniem pionowego kierunku pręta. Problem stateczności sprowadzono do problemu minimalizacji pewnego funkcjonału; podano również warunki dostateczne stateczności ruchu układu.

Рассматривается система состоящая из жесткого профиля снабженного в упругий стержень. Эта система движется в несжимаемой и невязкой жидкости; пренебрегается силами действующими со стороны жидкости на упругий стержень. Стержень подвешен на горизонтальной струне и движется вдоль ней без трения. В первой части работы выведены уравнения движения системы, опираясь на законы сохранения импульса и момента импульса, а также на теорему Гамильтона-Остроградского. Во второй части исследована задача существования и устойчивости движения профиля, заключающегося в его равномерным горизонтальным сдвиге с сохранением вертикального направления стержня. Задача устойчивости сведена к задаче минимизации некоторого функционала; приведены тоже достаточные условия устойчивости движения системы.

1. Equations of motion

1.1. Statement of the problem

LET US CONSIDER the motion, with respect to a fixed coordinate system $O_1x_1y_1$, of an arbitrary rigid profile (c) without sharp edge, carrying a thin elastic rod PQ, in an inviscid incompressible fluid (density ϱ), in irrotational motion, at rest at infinity. The rod is inextensible, homogeneous (length L, density ϱ', mass $\mu' = \varrho' L$, centre of inertia G', moment of inertia of the cross-section I, Young's modulus E); its end P is fixed to the profile; its end Q is moving on a rigid horizontal string O_1x_1 and this constraint is frictionless. We assume that the forces exerted by the fluid on the rod are negligible. We shall use a particular coordinate system O, x, y fixed rigidly to the profile [1]. The motion of the profile is described by the components $l(t)$, $m(t)$ of the velocity of the point 0 and by the angular velocity $\omega(t)$ if $[O_1, x_1, Ox = \theta(t), \omega(t) = \dot{\theta}(t)]$. We denote by (x_C, y_C) and (x_G, y_G) the coordinates of the centre C and the centre of inertia G of the profile, by μ its mass; a and b are the coordinates of P.

In the undeformed state PQ_0, the rod is directed along an axis $P\xi$ $(Ox, P\xi = \alpha = $ $= \text{constant})$; the axis $P\eta$ is perpendicular to $P\xi$ (Fig. 1).

We denote by s $(0 \leqslant s \leqslant L)$ the abscissa of a particle M_0 of the rod in the undeformed state, by $u(s,t)$, $v(s,t)$ the components on $P\xi$, $P\eta$ of the elastic displacement vector $\mathbf{M_0M}$ of the particle M.

The condition that the rod is inextensible leads to the relation [5]

(1.1)
$$u' = -\frac{1}{2}v'^2 \qquad \left(u' = \frac{\partial u}{\partial s}\right).$$

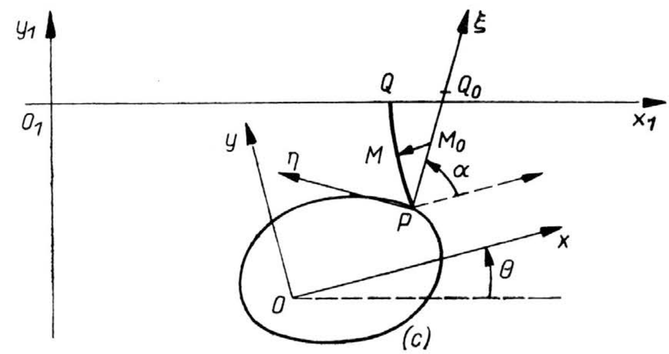

FIG. 1.

The boundary conditions at the end P are

(1.2)
$$v(0,t) = 0, \quad v'(0,t) = 0 \quad \text{for} \quad t \geqslant t_0.$$

1.2. Equations of motion

1. Let us denote by X, Y the components on Ox, Oy of the resultant, and by N the moment about 0 of the forces exerted by the rod on the profile. The equation of motion of the profile are [1]

(1.3)
$$\begin{aligned} A\dot{l} - B\omega m &= -\varrho\Gamma(m+\omega_c) - \mu g\sin\theta + X, \\ B\dot{m} + A\omega l &= \varrho\Gamma(l-\omega y_c) - \mu g\cos\theta + Y, \\ C\dot{\omega} + (B-A)lm &= \varrho\Gamma(lx_c + my_c) - \mu g(x_G\cos\theta + y_G\sin\theta) + N, \end{aligned}$$

where A, B, C are positive constants depending on the profile and on ϱ $(B \geqslant A)$, μg is the weight of the profile and Γ the circulation around the profile, constant by Helmholtz theorem.

2. It is easy to write the theorems of momentum and of the moment of momentum for the rod only.

The components of the momentum on Ox, Oy are

$$\mu'(\dot{x}_{G'}+l-\omega y_{G'}),\qquad \mu'(\dot{y}_{G'}+m+\omega x_{G'})$$

and the moment of momentum about 0 is

$$\tilde{g}+\dot{J}\omega+\mu'(x_{G'}m-y_{G'}m),$$

where $x_{G'}$, $y_{G'}$ are the coordinates of G', $\dot{J}$ and $\tilde{g}$ the moment of inertia and the relative moment of momentum of the rod about 0.

Let us denote by R the reaction of the string O_1x_1 on the rod, parallel to O_1y_1: We obtain

$$\begin{aligned}
&\mu'(\ddot{x}_{G'}-2\omega\dot{y}_{G'})+\mu'(\dot{l}-\omega m)-\mu'(y_{G'}\dot{\omega}+x_{G'}\omega^2) = -X+(R-\mu'g)\sin\theta,\\
(1.4)\quad &\mu'(\ddot{y}_{G'}+2\omega\dot{x}_{G'})+\mu'(\dot{m}+\omega l)+\mu'(x_{G'}\dot{\omega}-y_{G'}\omega^2) = -Y+(R-\mu'g)\cos\theta,\\
&\frac{d}{dt}(\tilde{g}+I\omega)+\mu'x_{G'}(m+\omega l)-\mu'y_{G'}(\dot{l}-\omega m) = N-\mu'g(x_{G'}\cos\theta-y_{G'}\sin\theta)\\
&\qquad\qquad +R(x_Q\cos\theta-y_Q\sin\theta),
\end{aligned}$$

where x_Q, y_Q are the coordinates of Q.

3. Adding the equations (1.3) and (1.4), we obtain

$$\begin{aligned}
&(A+\mu')\dot{l}-(B+\mu')\omega m+\mu'(\ddot{x}_{G'}-2\omega\dot{y}_{G'})-\mu'(y_{G'}\dot{\omega}+x_{G'}\omega^2) = -\varrho\Gamma(m+\omega x_c)\\
&\qquad\qquad +[R-(\mu+\mu')g]\sin\theta,\\
(1.5)\quad &(B+\mu')\dot{m}+(A+\mu')\omega l+\mu'(\ddot{y}_{G'}+2\omega x_{G'})+\mu'(x_{G'}\dot{\omega}-y_{G'}\omega^2) = \varrho\Gamma(l-\omega y_c)+[R\\
&\qquad\qquad -(\mu+\mu')g]\cos\theta,\\
&C\dot{\omega}+(B-A)lm+\frac{d}{dt}(\tilde{g}+J\omega)+\mu'x_{G'}(\dot{m}+\omega l)-\mu'y_{G'}(\dot{l}-\omega m)\\
&\qquad = \varrho\Gamma(lx_c+my_c)-(\mu+\mu')g(x_{\mathcal{G}}\cos\theta-y_{\mathcal{G}}\sin\theta)+R(x_Q\cos\theta-y_Q\sin\theta),
\end{aligned}$$

where $x_{\mathcal{G}}$, $y_{\mathcal{G}}$ are the coordinates of the centre of inertia $\mathcal{G}$ of the system profile-rod.

4. The equations of relative motion of the rod are the equation (1.1) and the equation obtained by applying Hamilton–Ostrogradski's principle to the motion of the rod with respect to the axes $P\xi\eta$.

Let us consider the set $\mathcal{V}$ of functions $v(x, t)$ four times continuously differentiable for $t \geqslant t_0, 0 \leqslant s \leqslant L$ and satisfying

$$v(0, t) = 0,\quad v'(0, t) = 0 \quad\text{for}\quad t \geqslant t_0.$$

The solution is the function $v(s, t) \in \mathcal{V}$ for which vanishes the integral

$$\int_{t_1}^{t_2}(\delta T+\delta W)dt \qquad (t_0 \leqslant t_1 \leqslant t_2),$$

where T is the relative kinetic energy of the rod and W the relative virtual work of all the forces acting on the rod. The varied functions are $v(s,t)+\delta v(s,t)$ where $\delta v(s,t)$ is a twice continuously differentiable function satisfying

$$\delta v(s,t_1)=\delta v(s,t_2)=0 \quad (0\leqslant s\leqslant L), \qquad \delta v(0,t)=\delta v'(0,t)=0 \quad (t\geqslant t_0).$$

The relative kinetic energy of the rod is

$$T=\frac{1}{2}\varrho'\int_0^L(\dot{u}^2+\dot{v}^2)\,ds$$

but, since u is a term of the second order, we take

$$T=\frac{1}{2}\varrho'\int_0^L\dot{v}^2\,ds.$$

The virtual work of the weight of the rod is

$$-\varrho' g\sin(\theta+\alpha)\int_0^L\delta u\,ds-\varrho' g\cos(\theta+\alpha)\int_0^L\delta v\,ds.$$

Integrating the first integral by parts and taking into account $\delta u(0,t)=0$ and $\delta u'=-v'\delta v'$, we obtain

$$\varrho' g\int_0^L\{\sin(\theta+\alpha)[(s-L)v']'-\cos(\theta+\alpha)\}\,\delta v\,ds.$$

The virtual work of the reaction is

$$R\sin(\theta+\alpha)\,\delta u(L,t)+R\cos(\theta+\alpha)\,\delta v(L,t).$$

Since

$$\delta u(L,t)=\int_0^L\delta u'\,ds=-\int_0^L v'\delta v'\,ds=-v'(L,t)\,\delta v(L,t)+\int_0^L v''\delta v\,ds,$$

we obtain

$$R[\cos(\theta+\alpha)-v'(L,t)\sin(\theta+\alpha)]\,\delta v(L,t)+\int_0^L R\sin(\theta+\alpha)v''\delta v\,ds.$$

The resultant of the fictitious forces applied to the mass element $dm=\varrho' ds$ is

$$(-\boldsymbol{\gamma}e-2\boldsymbol{\omega}\times\mathbf{V}_r)\,dm$$

where $\mathbf{V}_r$ is the relative velocity and $\boldsymbol{\gamma}_e$ the acceleration of transport.

We easily obtain the virtual work of the fictitious forces

$$-2\omega\varrho'\int_0^L u\delta\dot{v}\,ds+\varrho'\int_0^L\left\{\gamma\xi[(s-L)v']'-\omega^2\left[\frac{s^2-L^2}{2}v'\right]'-\gamma\eta+\omega^2 v-\dot{\omega}(s+2n)-2\omega\dot{u}\right\}\delta v\,ds,$$

where $\gamma\xi$, $\gamma\eta$ are the components on $P\xi\eta$ of the acceleration of the point 0

$$\gamma\xi=(\dot{l}-b\dot{\omega}-m\omega-a\omega^2)\cos\alpha+(\dot{m}+a\dot{\omega}+l\omega-b\omega^2)\sin\alpha,$$

$$\gamma\eta=-(\dot{l}-b\dot{\omega}-m\omega-a\omega^2)\sin\alpha+(\dot{m}+a\dot{\omega}+l\omega-b\omega^2)\cos\alpha.$$

Last, the virtual work of the internal forces is $\delta\left[-\frac{1}{2}\int\limits_0^L EIv''^2 ds\right]$. Consequently, Hamilton–Ostrogradski's principle gives

$$\int\limits_{t_1}^{t_2}\left\{\varrho'\int\limits_0^L \dot v\delta\dot v\,ds+\varrho' g\int\limits_0^L\{\sin(\theta+\alpha)[(s-L)v']'-\cos(\theta+\alpha)\}\delta v\,ds-2\omega\varrho'\int\limits_0^L u\delta\dot v\,ds\right.$$
$$+\varrho'\int\limits_0^L\left\{\{\gamma\xi[(s-L)v']'-\omega^2\left[\frac{s^2-L^2}{2}v'\right]'-\gamma\eta+\omega^2 v-\dot\omega(s+2u)-2\omega\dot u\right\}\delta v\,ds-EI\int\limits_0^L v''\delta v''ds$$
$$\left.+R[\cos(\theta+\alpha)-v'(L,t)\sin(\theta+\alpha)]\,\delta v(L,t)+\int\limits_0^L R\sin(\theta+\alpha)v''\delta v\,ds\right\}dt=0.$$

Taking into account $\delta v(s, t_1 = \delta v(s, t_2) = 0$ and $\delta v(0, t) = \delta v'(0, t) = 0$, we have

$$\int\limits_{t_1}^{t_2}\dot v\delta\dot v\,dt=-\int\limits_{t_1}^{t_2}\ddot v\delta v\,dt,\qquad \int\limits_{t_1}^{t_2}u\delta\dot v\,dt=-\int\limits_{t_1}^{t_2}\dot u\delta v\,dt,$$

$$\int\limits_0^L v''\delta v''ds=v''(L,t)\,\delta v'(L,t)-v'''(L,t)\,\delta v(L,t)+\int\limits_0^L v^{\mathrm{IV}}\delta v\,ds,$$

so that we obtain finally

$$\int\limits_{t_1}^{t_2}\left\{\int\limits_0^L\left\{\varrho'[-\ddot v+g[\sin(\theta+\alpha)\,((s-L)v')'-\cos(\theta+\alpha)]]+R\sin(\theta+\alpha)v''\right.\right.$$
$$\left.+\varrho'\left[\gamma\xi((s-L)v')'-\omega^2\left(\frac{s^2-L^2}{2}v'\right)'-\gamma\eta+\omega^2v-\dot\omega(s+2u)\right]-EIv^{\mathrm{IV}}\right\}\delta v\,ds$$
$$\left.+\left\{EIv'''(L,t)+R[\cos(\theta+\alpha)-v'(L,t)\sin(\theta+\alpha)]\right\}\delta v(L,t)-EIv''(L,t)\,\delta v'(L,t)\right\}dt=0.$$

Since $\delta v(s, t)$, $\delta v(L, t)$, $\delta v'(L, t)$ are arbitrary, we have the equations

$$(1.6)\qquad -\varrho'\ddot v+\varrho' g\{\sin(\theta+\alpha)[(s-L)v']'-\cos(\theta+\alpha)\}+R\sin(\theta+\alpha)v''$$
$$+\varrho'\left\{\gamma\xi[(s-L)v']'-\omega^2\left[\frac{s^2-L^2}{2}v'\right]'-\gamma\eta+\omega^2v-\dot\omega(s+2u)\right\}-EIv^{\mathrm{IV}}=0,$$

$$(1.7)\qquad EIv'''(L,t)+R[\cos(\theta+\alpha)-v'(L,t)\sin(\theta+\alpha)]=0,\qquad v''(L,t)=0.$$

The motion of the system profile-rod and the reation of the string on the rod are determined by the equations (1.1), (1.5), (1.6), the boundary conditions (1.2), (1.7) and also $u(0, t) = 0$ and the initial conditions.

1.3. The first integrals

1. Let us denote by T_p and T_r the absolute kinetic energies of the profile and of the rod.

It is well-known [2, 3] that, if $\mathscr{P}$ is the power of the forces exerted by the fluid on the profile, we have

$$\frac{dT_p}{dt} - \mathscr{P} = \frac{d}{dt}\left[\frac{1}{2}(Al^2 + Bm^2 + C\omega^2)\right].$$

Then, there is the energy integral for the system profile-rod

$$(1.8) \quad \frac{1}{2}(Al^2 + Bm^2 + C\omega^2) + T_r = -(\mu + \mu')g[y_0 + x_{\mathscr{G}}\sin\theta + y_{\mathscr{G}}\cos\theta] - \frac{1}{2}\int_0^L EIv''^2 ds + Ct,$$

where y_0 is the absolute ordinate of 0.

2. Multiplying the first two equations (1.5) by $e^{i\theta}$ and $ie^{i\theta}$ and adding them, we easily obtain

$$\frac{d}{dt}\left\{(A+\mu')l + i(B+\mu')m + \mu'(\dot{x}_{G'} + i\dot{y}_{G'}) + i\mu'\omega(x_{G'} + iy_{G'})]e^{i\theta}\right\}$$
$$= \frac{d}{dt}[i\varrho\Gamma(x_{1C} + iy_{1C})] + i[R - (\mu + \mu')g],$$

where x_{1C}, y_{1C} are the absolute coordinates of the centre C of the profile.

This equation can be obtained directly, remarking that Al and Bm are the components on Oxy of the quasi-momentum of the system profile-fluid [2, 3]. Taking the real part, we obtain the second first integral

$$(1.9) \quad (Al + \mu')l\cos\theta - (B + \mu')m\sin\theta + \mu'[(\dot{x}_{G'} - \omega y_{G'})\cos\theta - (\dot{y}_{G'} + \omega x_{G'})\sin\theta]$$
$$= -\varrho\Gamma(y_0 + x_c\sin\theta + y_c\cos\theta) + Ct.$$

2. Stability of a particular motion of the system

2.1. Existence of motions of uniform horizontal translation of the profile with relative equilibrium of the rod in the undeformed state

1. If V is the velocity of this translation, the motion is defined by

$$\theta = \theta_0 = \text{const}, \quad \omega = 0, \quad l = V\cos\theta_0, \quad m = -V\sin\theta_0, \quad \dot{u} = \dot{v} = 0.$$

The Eq. (1.5) give

$$(2.1) \quad R_0 = (\mu + \mu')g - \varrho\Gamma V,$$

$$(2.2) \quad (B - A)V^2\sin\theta_0\cos\theta_0 + [\varrho\Gamma V x_c - (\mu + \mu')g x_{\mathscr{G}_0} + R_0 x_{Q_0}]\cos\theta_0$$
$$- [\varrho\Gamma V y_c - (\mu + \mu')g y_{\mathscr{G}_0} + R_0 y_{Q_0}]\sin\theta_0 = 0,$$

where R_0, $(x_{\mathscr{G}_0}, y_{\mathscr{G}_0})$, (x_{Q_0}, y_{Q_0}) are the values of R and of the coordinates of $\mathscr{G}$ and Q in the motion of translation.

The configuration of the rod is given by Eq. (1.6), we obtain the differential equation

$$(2.3) \quad \varrho' g\{\sin(\theta_0 + \alpha)[(s - L)v']' - \cos(\theta_0 + \alpha)\} + R_0\sin(\theta_0 + \alpha)v'' - EIv^{\text{IV}} = 0$$

with the boundary conditions

$$(1.2') \quad v(0) = v'(0) = 0,$$

$$(1.7') \quad \begin{aligned} R_0[\cos(\theta_0 + \alpha) - v'(L)\sin(\theta_0 + \alpha)] + EIv'''(L) &= 0, \\ v''(L) &= 0. \end{aligned}$$

2. It is possible to integrate the equation (2.3) with the boundary conditions (1.2′), (1.7′).

But we shall restrict ourselves to considering the case of the rod in the undeformed state. In this case, $v(s) = 0$ and $\cos(\theta_0+\alpha) = 0$. We shall take $\theta_0 = \pi/2-\alpha$ and the rod is directed vertically upwards.

Using Eq. (2.1), we can replace Eq. (2.2) by

$$(2.2') \quad (B-A)V^2\sin\alpha\cos\alpha+[\varrho\Gamma V(x_c-a)-\mu g(x_G-a)]\sin\alpha-[\varrho\Gamma V(y_c-b)-\mu g(y_G-b)] \times\cos\alpha = 0.$$

This equation gives values of the angle α so that the motion is possible. It is easy to observe that Eq. (2.2′) has at least two solutions; setting $X = \sin\alpha$, $Y = \cos\alpha$, it is obvious that the rectangular hyperbola

$$(2.2'') \quad (B-A)V^2XY+[\varrho\Gamma V(x_c-a)-\mu g(x_G-a)]X-[\varrho\Gamma V(y_c-b)-\mu g(y_G-b)]y = 0$$

which contains the point $X = Y = 0$ intersects the circle $X^2+Y^2 = 1$ at least in two points. We remark that the roots of Eq. (2.2′) do not depend on the rod. In the result, we shall assume that α is a root of Eq. (2.2′) and we shall denote by "motion $\mathcal{M}_0$" the motion of the uniform horizontal translation of the profile with the rod in the relative equilibrium in the undeformed state and directed vertically upwards.

We are going to study the stability of the "motion $\mathcal{M}_0$".

2.2. The functional permitting to solve the stability problem

Let us set

$$l = \bar{l}+V\cos\theta, \qquad m = \bar{m}-V\sin\theta$$

and, if U_1 and V_1 are the components on Oxy of the absolute velocity of a point of the rod,

$$U_1 = \bar{U}_1+V\cos\theta, \qquad V_1 = \bar{V}_1-V\sin\theta.$$

In the "motion $\mathcal{M}_0$", $\theta = \theta_0 = (\pi/2)-\alpha$ and $\bar{l}$, $\bar{m}$, $\bar{U}_1$, $\bar{V}_1$ vanish. It is easy to observe that if we set

$$\mathscr{C}_1 = \frac{1}{2}(A\bar{l}^2+B\bar{m}^2+C\omega^2)+\frac{\varrho'}{2}\int_0^L(\bar{U}+\bar{V})ds,$$

the first integral (1.8) can be written in the form

$$\mathscr{C}_1-\frac{V^2}{2}(A\cos^2\theta+B\sin^2\theta)+V\{(A+\mu')l\cos\theta-(B+\mu')m\sin\theta+\mu'[(\dot{x}_{G'}-\omega y_{G'})\cos\theta$$

$$-(\dot{y}_G+\omega x_G)\sin\theta]\} = -(\mu+\mu')g[y_0+x_{\mathscr{G}}\sin\theta+y_{\mathscr{G}}\cos\theta]-\frac{EI}{2}\int_0^L v''^2ds = Ct,$$

or, using the first integral (1.9), in the form

$$\mathscr{C}_1+W = Ct$$

with

$$W = -\frac{V^2}{2}(A\cos^2\theta+B\sin^2\theta)+(\mu+\mu')g(x_{\mathscr{G}}\sin\theta+y_{\mathscr{G}}\cos\theta)-\varrho\Gamma V(x_c\sin\theta+y_c\cos\theta)$$

$$+\frac{EI}{2}\int_0^L v''^2 ds+[(\mu+\mu')g-\varrho\Gamma V]y_0.$$

We have

$$y_0 = -a\sin\theta-b\cos\theta-[L+u(L,t)]\sin(\theta+\alpha)-v(L,t)\cos(\theta+\alpha)$$

and we can replace $(\mu+\mu')x_{\mathscr{G}}$ and $(\mu+\mu')y_{\mathscr{G}}$ by $\mu x_G+\mu' x_G$ and $\mu y_G+\mu' y_G$ and use the formula

$$\mu'(x_{G'}+iy_{G'}) = \mu'(a+ib)+\varrho'\int_0^L [(u+s)+iv]e^{i\alpha}ds.$$

We finally obtain the first integral

(2.4) $$\mathscr{C}_1+W = Ct$$

with

(2.5) $$W = -\frac{V^2}{2}(A\cos^2\theta+B\sin^2\theta)+[\mu g(x_G-a)-\varrho\Gamma V(x_c-a)]\sin\theta+[\mu g(y_G-b)$$
$$-\varrho\Gamma V(y_c-b)]\cos\theta+\varrho' g\int_0^L [(u+s)\sin(\theta+\alpha)+v\cos(\theta+\alpha)]ds-[(\mu+\mu')g-\varrho\Gamma V]$$
$$\times[\big(L+u(L,t)\big)\sin(\theta+\alpha)+v(L,t)\cos(\theta+\alpha)]+\frac{EI}{2}\int_0^L v''^2 ds.$$

$\mathscr{C}_1$ depends only on the velocities; it is a positive functional, vanishing only in the undisturbed motion. W is a functional depending on the angle θ and on the configuration of the rod.

Consequently, we can apply the results concerning the stability of motion of a rigid body containing elastic parts, obtained by V. V. RUMIANTSEV, V. N. RUBANOVSKII, V. M. MOROZOV [6, 7, 8].

2.3. Study of the stability of the „motion $\mathscr{M}_0$"

The sufficient condition that the "motion $\mathscr{M}_0$" is stable, is that W has a minimum for this motion. Let us set

$$\bar{\theta} = \theta-\theta_0 = \theta-\frac{\Pi}{2}+\alpha.$$

It is easy to observe that the terms of the first order in W vanish by virtue of the condition (2.2') (it is obvious *a priori*).

The second variation $\delta^2 W$ of W can be written in the form

$$\delta^2 W = \{K+[(\mu+\mu')g-\varrho\Gamma V]L\}\bar{\theta}^2+2\Big\{[(\mu+\mu')g-\varrho\Gamma V]v(L,t)-\varrho' g\int_0^L v\,ds\Big\}\bar{\theta}$$
$$+2\varrho' g\int_0^L u\,ds-2[(\mu+\mu')g-\varrho\Gamma V]u(L,t)+EI\int_0^L v''^2 ds,$$

where

(2.6) $$K = (B-A)V^2\cos 2\alpha + [\varrho\Gamma V(x_c-a)-\mu g(x_G-a)]\cos\alpha + [\varrho\Gamma V(y_c-b)-\mu g(y_G-b)] \times \sin\alpha - \frac{\mu' g L}{2}.$$

We can transform $\delta^2 W$ by using the relations

$$u(L,t) = -\frac{1}{2}\int_0^L v'^2 ds \quad \text{and} \quad \int_0^L u(s,t)ds = -\frac{1}{2}\int_0^L (L-s)v'^2 ds,$$

so that

$$\delta^2 W = K\left(\bar{\theta} - \frac{\varrho' g}{K}\int_0^L v\,ds\right)^2 - \frac{\varrho'^2 g^2}{K}\left(\int_0^L v\,ds\right)^2 + [(\mu+\mu')g - \varrho\Gamma V]L\left[\bar{\theta} + \frac{v(L,t)}{L}\right]^2$$
$$+ \frac{(\mu+\mu')g - \varrho\Gamma V}{L}\left[L\int_0^L v'^2 ds - v^2(L,t)\right] - \varrho' g\int_0^L (L-s)v'^2 ds + EI\int_0^L v''^2 ds.$$

Using the Schwarz inequality, it is easy to observe that as $(\mu+\mu')g - \varrho\Gamma V > 0$, so we have

$$\delta^2 W \geqslant K\left(\bar{\theta} - \frac{\varrho' g}{K}\int_0^L v\,ds\right)^2 + [(\mu+\mu')g - \varrho\Gamma V]L\left[\bar{\theta} + \frac{v(L,t)}{K}\right]^2$$
$$+\left(EI - \frac{\varrho' g L^3}{2}\right)\int_0^L v''^2 ds - \frac{\varrho'^2 g^2 L}{K}\int_0^L v^2 ds.$$

Let us then consider the problem of the minimum of the functional

$$\frac{\int_0^L v''^2 ds}{\int_0^L v^2 ds}$$

in the set of functions which are four times continuously differentiable and satisfy $v(0) = v'(0) = 0$.

It is well-known that this problem is reduced to the determination of the smallest eigenvalue of the problem [4]

$$v^{IV} - \lambda v = 0,$$
$$v(0) = v'(0) = 0, \quad v''(L) = v'''(L) = 0.$$

The eigenvalues $\lambda = \nu^4$ are the roots of the equation

$$1 + \cos\nu L \operatorname{ch}\nu L = 0$$

and the smallest root is given by

$$\nu_0 L = 1.875 \ldots .$$

Then, we have

$$\int_0^L v''^2 ds \geqslant \nu_0^4 \int_0^L v^2 ds$$

so that, if $EI - \frac{\varrho' g l^3}{2} > 0$, we can write

$$\delta^2 W \geqslant K\left(\bar{\theta} - \frac{\varrho' g}{K}\int_0^L v\,ds\right)^2 + [(\mu+\mu')g - \varrho\Gamma V]L\left(\bar{\theta} + \frac{v(L,t)}{K}\right)^2$$

$$+\left[\left(EI - \frac{\varrho' g L^3}{2}\right)\nu_0^4 - \frac{\varrho'^2 g^2 L}{K}\right]\int_0^L v^2\,ds.$$

Consequently, under the conditions

$$(2.7) \qquad K > 0, \quad (\mu+\mu')g - \varrho\Gamma V > 0, \quad \left(EI - \frac{\varrho' g L^3}{2}\right)\nu_0^4 - \frac{\varrho'^2 g^2 L}{K} > 0,$$

$\delta^2 W$ is the positive definite with respect to

$$\bar{\theta} - \frac{\varrho' g}{K}\int_0^L v\,ds, \quad \bar{\theta} + \frac{v(L,t)}{K} \quad \text{and} \quad ||v|| = \left(\int_0^L v^2 ds\right)^{1/2}.$$

Remarking that $\int_0^L v ds < \sqrt{L}||v||$, it is easy to see (using of Rumiantsev's stability theorem [8]) that under the conditions (2.7), the "motion$\mathcal{M}_0$" is stable with respect to θ, $||v||$, $v(L, t)$, l, m, ω and $\varrho' \int_0^L (U_1^2 + V_1^2) ds$.

Let us study the condition $K > 0$, where K is given by the formula (2.6).

Let us set again $X = \sin\alpha$, $Y = \cos\alpha$ and consider the rectangular hyperbola

$$(B-A)V^2(Y^2 - X^2) + [\varrho\Gamma V(x_c - a) - \mu g(x_G - a)]Y + [\varrho\Gamma V(y_c - b) - \mu g(y_G - b)]X - \frac{\mu' g L}{2} = 0.$$

It is easy to construct this curve for $L = 0$, and to study its evolution when L increases. It can be shown that, for L sufficiently small, at least one point where the hyperbola (2.2″) intersects the unit circle, belongs to the domain $K > 0$.

Consequently, if L is sufficiently small and if Joukowski's force is less than the weight of the system profile-rod, there is at least one "motion $\mathcal{M}_0$" stable.

References

1. G. COUCHET, *Sur un cas d'intégration par quadratures du mouvement d'un profil ausein d'un fluide parfait incompressible en mouvement irrotationnel*, Comtes Rendus Acad. Sci. Paris., **258**, 1722–1724, 1964.
2. G. COUCHET, *Mouvements plans d'un fluide en présence d'un profil mobile*, Mémorial des Sciences Mathématiques, Fasc. 135, Gauthier-Villars, Paris 1956.

3. G. COUCHET, *Les profils en aérodynamique instationnaire et la condition de Joukowski*, A. Blanchard, Paris 1976.
4. R. COURANT and D. HILBERT, *Methods of mathematical physics*, vol. I, Interscience Publishers Inc., New York 1965.
5. A. I. LOURIÉ, *Mécanique analytique*, Tome II, Masson, 1968.
6. V. M. MOROZOV and V. N. RUBANOVSKII, *Stability of relative equilibrium on a circular orbit of a rigid body with elastic rods*, Mech. Solids, 1, 143–149, 1974.
7. V. N. RUBANOVSKII, *On the stability of certain motions of a rigid body with elastic rods and liquid*, J. Appl. Math. Mech., **36**, 38–53, 1972.
8. V. V. RUMIANTSEV, *On the motion and stability of an elastic body with a cavity containing fluid*, J. Appl. Math. Mech., **33**, 6, 927–937, 1969.

LABORATOIRE DE MECANIQUE THEORIQUE
FACULTE DES SCIENCES, BESANÇON CEDEX, FRANCE.

Received September 12, 1985.