A SIMPLIFIED PRESENTATION OF EINSTEIN'S UNIFIED FIELD EQUATIONS

TULLIO LEVI-CIVITA

The publication in December 1949 of Professor Einstein's most recent extension of his Unified Field Theory has aroused widespread interest. The theory was first proposed by Einstein in 1929. Later in the same year Professor Levi-Cevita produced a paper (of which the present pamphlet is a translation) suggesting ways in which, in his opinion, the presentation of the theory could be simplified and improved. Since then and until December 1949 there appears to have been no major development in this field of study.

Possessing, as Levi-Civita did, a mastery of the technical methods of this subject, his paper is likely to be of great value. It is therefore reprinted in its original form without change of any sort.

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EINSTEIN'S UNIFIED FIELD EQUATIONS

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In his recent paper, "Zur einheitlichen Feldtheorie",[1] Einstein made use of the fundamental idea that it is both possible and useful to give a geometrical interpretation of the complete system of the sixteen field equations (consisting of Einstein's celebrated gravitational equations and Maxwell's equations) in such a way as to include the definition (and the definition only) of an orthogonal quadruplet[2] embedded in the space-time world.

Conversely, the sixteen parameters determining a quadruplet are to give a complete definition not only of the Riemannian metric of space (as is well known, this takes place automatically), but of the phenomena of electromagnetism as well.

For this purpose the eminent author introduced covariant derivatives with respect to the quadruplet, and suggested relationships between them which to a first approximation lead to the required co-ordination of gravitational and electromagnetic phenomena.

It appears to me, however, that the root problem raised by Einstein can be solved in a simpler and more general way by making use of perfectly familiar methods of the absolute differential calculus on the one hand, while, on the other hand, retaining unaltered all results previously obtained.

1. Geometrical and formal preliminaries.[3]

Let x^ν $(\nu = 0, 1, \ldots, n-1)$ be general co-ordinates of a

[1] *Berliner Berichte*, I, 1929, pp. 1–8. [2] Ger. *Vierbein*.

[3] See in particular my *Absolute Differential Calculus* (English translation by Miss Long), Chap. III. Blackie & Son, Ltd., 1927.

Riemannian space R_n, and λ^ν_i $(i = 0, 1, \ldots, n-1)$ the parameters of n congruences, which define a lattice of lines in R_n and an n-uplet [1] at every point.

Following Einstein's example I shall use Greek letters for co-ordinate indices (such as ν), and Roman letters, on the other hand, for indices referring to the n-uplet (such as i). I shall leave out signs of summation with respect to Greek indices (provided they occur once above and once below), but other Σ's will be retained.

As usual, let the quantities $\lambda_{i|\nu}$ be the elements reciprocal to λ^ν_i (normalized cofactors). For every i they form a covariant system (moments of the n-uplet in question). By composition with the quantities λ^ν_i, $\lambda_{i|\nu}$ we obtain, from every mixed tensor of rank $p + q$ with the components

$$A^{\nu_1 \nu_2 \cdots \nu_q}_{\mu_1 \mu_2 \cdots \mu_p} \qquad (\mu_1, \ldots, \mu_p, \nu_1, \ldots, \nu_q = 0, 1, \ldots, n-1),$$

an "*n-uplet tensor*",[2] the components of which are defined by the formulæ

$$A_{i_1 \cdots i_p k_1 \cdots k_q} = A^{\nu_1 \cdots \nu_q}_{\mu_1 \cdots \mu_p} \lambda^{\mu_1}_{i_1} \cdots \lambda^{\mu_p}_{i_p} \lambda_{k_1|\nu_1} \cdots \lambda_{k_q|\nu_q}, \qquad (1)$$

and conversely, since these formulæ can be solved for the co-ordinate components in the form

$$A^{\nu_1 \cdots \nu_q}_{\mu_1 \cdots \mu_p} = \sum_0^{n-1}{}_{i_1 \cdots i_p k_1 \cdots k_q} A_{i_1 \cdots i_p k_1 \cdots k_q} \lambda_{i_1|\mu_1} \cdots \lambda_{i_p|\mu_p} \lambda^{\nu_1}_{k_1} \cdots \lambda^{\nu_q}_{k_q}.$$

The components of the n-uplet tensor are pure invariants with respect to transformations of co-ordinates; they essentially depend on the n-uplet considered, but, as is easily verified, they also behave like a tensor when the quantities λ^μ_i and $\lambda_{k|\nu}$ are simultaneously subjected to orthogonal transformations.

If we put

$$g_{\mu\nu} = \sum_0^{n-1}{}_i \lambda_{i|\mu} \lambda_{i|\nu}, \qquad (\mu, \nu = 0, 1, \ldots, n-1) \quad (2)$$

a definite metric

$$ds^2 = g_{\mu\nu} \, dx^\mu \, dx^\nu \quad . \quad . \quad . \quad . \quad (3)$$

(for real values of the quantities involved) is introduced into R_n in such a way that our n-uplet turns out orthogonal. Later (§3)

[1] Ger. *n-Bein.* [2] Ger. *Beintensor.*

I shall give the (unimportant) modifications required to transfer the n-uplet theory, avoiding any appearance of imaginaries, to an indefinite metric (with a given index of inertia).

Meanwhile I suppose that the covariant derivatives of the moments $\lambda_{i|\nu}$ have been introduced, and, following Ricci, I take the coefficients of rotation

$$\gamma_{ikl} = \lambda_{i|\nu\rho} \lambda^{\nu}_{k} \lambda^{\rho}_{l}. \quad . \quad . \quad . \quad . \quad (4)$$

In virtue of the identities

$$\gamma_{ikl} + \gamma_{kil} = 0 \quad . \quad . \quad . \quad . \quad . \quad (5)$$

(which result from the relationships between parameters and moments), Ricci's quantities γ form $n\dfrac{n(n-1)}{2}$ invariants with respect to transformations of co-ordinates, which of course essentially depend on the given n-uplet and necessarily include all its geometrical differential properties of the first order. With respect to orthogonal transformations *with constant coefficients* the quantities γ behave like a tensor of the third rank. In order to emphasize the limitation to transformations with constant coefficients I shall call such systems *local n-uplet tensors*. True n-uplet tensors behave as invariants with respect to all orthogonal transformations whose coefficients can vary in any way with the quantities x.

Perhaps it is not superfluous to remark that the explicit expressions for the coefficients of rotation, γ, can also be obtained directly by ordinary differentiation without making use of the covariant derivatives of the quantities $\lambda_{i|\nu}$.

In order to do this, we have to introduce either the Pfaffian expressions

$$\psi_i = \lambda_{i|\nu} dx^{\nu},$$

or the operators

$$\frac{df}{ds_i} = X_i f = \sum_{0}^{n-1}{}_{\nu} \lambda^{\nu}_{i} \frac{\partial f}{\partial x^{\nu}}$$

(derivatives of a function $f[x^0, \ldots, x^{n-1}]$ in the direction of the lines of the congruences), and then to form the corresponding bilinear covariants or Poisson brackets. We can, however, attain the desired result even more rapidly by using (4) and noticing that, according to the definition of covariant differentiation, we have the identity

$$\lambda_{i|\nu\rho} - \lambda_{i|\rho\nu} = \frac{\partial \lambda_{i|\nu}}{\partial x^\rho} - \frac{\partial \lambda_{i|\rho}}{\partial x^\nu}.$$

We thus obtain

$$\gamma_{ikl} - \gamma_{ilk} = \sum_{0}^{n-1}{}_{\nu\rho} \lambda_k^\nu \lambda_l^\rho \left\{ \frac{\partial \lambda_{i|\nu}}{\partial x^\rho} - \frac{\partial \lambda_{i|\rho}}{\partial x^\nu} \right\},$$

and all the quantities γ are uniquely determined by these equations together with (5).

Equations (4) can be solved for the quantities $\lambda_{i|\nu\rho}$, giving

$$\lambda_{i|\nu\rho} = \sum_{0}^{n-1}{}_{jh} \gamma_{ijh} \lambda_{j|\nu} \lambda_{h|\rho}, \quad . \quad . \quad . \quad (4')$$

from which we obtain the conditions of integrability of (4′) by repeated covariant differentiation and formation of differences. For this we require the commutation-formula

$$\lambda_{i|\nu\rho\sigma} - \lambda_{i|\nu\sigma\rho} = R_{\mu\nu,\rho\sigma} \lambda_i^\mu, \quad . \quad . \quad . \quad (6)$$

where $R_{\mu\nu,\rho\sigma}$ denotes the Riemannian tensor.

In this way we obtain

$$\gamma_{ij,hk} = R_{\mu\nu,\rho\sigma} \lambda_i^\mu \lambda_j^\nu \lambda_h^\rho \lambda_k^\sigma, \quad . \quad . \quad . \quad (7)$$

where for brevity we write $\gamma_{ij,hk} =$

$$\frac{d\gamma_{ijh}}{ds_k} - \frac{d\gamma_{ijk}}{ds_h} + \sum_{0}^{n-1}{}_l [\gamma_{ijl}(\gamma_{lhk} - \gamma_{lkh}) + \gamma_{lik}\gamma_{ljh} - \gamma_{lih}\gamma_{ljk}]. \quad (8)$$

From (7) we conclude that the 4-index symbols, γ, form a (true) n-uplet tensor. In virtue of the well-known identities satisfied by the Riemannian symbols the formulæ (7) lead to similar identities for the 4-index symbols, γ, namely

$$\left.\begin{aligned} \gamma_{ij,hk} &= -\gamma_{ji,hk} = -\gamma_{ij,kh} = \gamma_{hk,ij} \\ \gamma_{ij,hk} &+ \gamma_{ih,kj} + \gamma_{ik,jh} = 0 \end{aligned}\right\} . \quad (9)$$

Now for the Einstein tensor

$$G_{\mu\sigma} = R_{\mu\nu,\rho\sigma} g^{\nu\rho}.$$

Its components G_{ik}, with respect to the two members i, k of the n-uplet are expressed, by (1), by

$$G_{ik} = G_{\mu\sigma} \lambda_i^\mu \lambda_k^\sigma,$$

whence, by (7),

$$G_{ik} = \sum_{0}^{n-1}{}_h \gamma_{ih,\,hk}. \quad . \quad . \quad . \quad . \quad . \qquad (10)$$

The linear (co-ordinate and n-uplet) invariant

$$G = G_{\mu\sigma} g^{\mu\sigma} = \sum_{0}^{n-1}{}_k G_{kk}$$

consequently takes the form

$$G = \sum_{0}^{n-1}{}_{hk} \gamma_{kh,\,hk}. \quad . \quad . \quad . \quad . \quad . \qquad (11)$$

In conclusion, I shall emphasize one other fact, namely that contraction of two indices in an n-uplet tensor leads to a reduced tensor—of the $(m - 2)$th rank if the original tensor is of the mth rank.

As we have already seen, the quantities γ_{ikl} form a local n-uplet tensor of the third rank, which in virtue of (5) is skew-symmetrical with respect to the two first indices i, k. The same is true for the differences $\gamma_{lik} - \gamma_{lki}$, which for $i, k \neq l$ are called *anormalities* (i.e. quantities which vanish when the lth congruence of the n-uplet is normal).

If we apply the differential operator $\dfrac{d}{ds_j}$ to the elements $A_{(h)}$ (where (h) stands for $h_1 h_2 \ldots h_m$) of a local or true n-uplet tensor, we obtain a new *local* n-uplet tensor $\dfrac{dA_{(h)}}{ds_j}$, the rank of which exceeds that of the original tensor by unity. In particular, we obtain in this way the *local* n-uplet tensor of the fourth rank

$$\frac{d\gamma_{ikl}}{ds_j},$$

which is skew-symmetrical with respect to i and k. By contraction we obtain

$$\xi_{ik} = \sum_{0}^{n-1}{}_l \frac{d\gamma_{ikl}}{ds_l}, \quad . \quad . \quad . \quad . \quad . \qquad (12)$$

so that we have obviously formed a skew-symmetrical local n-uplet tensor ξ of the second rank. Its covariant and contravariant components are respectively

$$\xi_{\mu\nu} = \sum_{0}^{n-1}{}_{ik} \xi_{ik} \lambda_{i|\mu} \lambda_{k|\nu}, \quad \xi^{\mu\nu} = \sum_{0}^{n-1}{}_{ik} \xi_{ik} \lambda_i^{\mu} \lambda_k^{\nu}. \qquad (13)$$

We may mention in addition that the n quantities

$$c_l = \sum_{0}^{n-1}{}_j \gamma_{jlj} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (14)$$

may be interpreted as mean curvatures of the $n-1$-fold sections, drawn orthogonally to the lines of the n-uplet. By what we have said above, they are line-components of a local n-uplet vector. From the tensor of the third rank, $\gamma_{lik} - \gamma_{lki}$, and this vector we obtain by contraction a new local n-uplet tensor of the second rank, namely

$$\eta_{ik} = \sum_{0}^{n-1}{}_l c_l (\gamma_{lik} - \gamma_{lki}), \quad . \quad . \quad . \qquad (15)$$

which is also skew-symmetrical.

2. Formation of divergences. The special case $n = 4$.

If v^ν are the contravariant components of a vector $\mathbf{v}$, its divergence is defined by the invariant

$$\operatorname{div} \mathbf{v} = v^\nu{}_{|\nu} = \frac{1}{\sqrt{|g|}} \sum_{0}^{n-1}{}_\nu \frac{\partial(\sqrt{|g|}\, v^\nu)}{\partial x^\nu}, \quad . \qquad (16)$$

where, as usual, g denotes the determinant $\|g_{\mu\nu}\|$ and $|g|$ is written (instead of simply g) because the formula is then valid as it stands even for an indefinite ds^2.

For the divergence of a tensor $\boldsymbol{\xi}$ of the second rank with the contravariant components $\xi^{\mu\nu}$ we obtain a vector $\boldsymbol{\chi}$ with the contravariant components

$$\chi^\mu = \xi^{\mu\nu}{}_{|\nu}. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (17)$$

Following von Laue,[1] we shall write simply

$$\boldsymbol{\chi} = \operatorname{Div} \boldsymbol{\xi}. \quad . \quad . \quad . \quad . \quad . \qquad (17')$$

If we here replace the covariant derivatives $\xi^{\mu\nu}{}_{|\rho}$ by their explicit values, we obtain

$$\chi^\mu = \frac{1}{\sqrt{|g|}} \sum_{0}^{n-1}{}_\nu \frac{\partial}{\partial x^\nu} (\sqrt{|g|}\, \xi^{\mu\nu}). \quad . \quad . \qquad (17'')$$

[1] *Die Relativitätstheorie*, Bd. II (2nd edition, Vieweg, Brunswick, 1923), § 14.

in the case of a skew-symmetrical tensor ($\xi^{\mu\nu} + \xi^{\nu\mu} = 0$); hence, by (16),

$$\operatorname{div} \chi = \frac{1}{\sqrt{|g|}} \sum_{0}^{n-1}{}_{\mu\nu} \frac{\partial^2}{\partial x^\mu \partial x^\nu} (\sqrt{|g|}\, \xi^{\mu\nu}). \quad . \quad (18)$$

Owing to the skew-symmetry of the quantities $\xi^{\mu\nu}$, the right-hand side vanishes identically.

Thus if we again make use of covariant derivatives, we obtain the identity

$$\chi^{\mu}{}_{|\mu} = \xi^{\mu\nu}{}_{|\nu\mu} = 0,$$

or finally, in tensor notation,

$$\operatorname{div} (\operatorname{Div} \boldsymbol{\xi}) = 0. \quad . \quad . \quad . \quad . \quad (18')$$

That is, in an arbitrary Riemannian space the divergence of the divergence of a skew-symmetrical tensor of the second rank is identically zero.

In order to express the right-hand sides of (16) and (17) in n-uplet tensor components, it is sufficient to apply the operator

$$\frac{d}{ds_l} = \lambda^{\rho}_{l} \frac{\partial}{\partial x^\rho}$$

to the formulæ of definition

$$v_k = v^\nu \lambda_{k|\nu},$$
$$\xi_{ik} = \xi^{\mu\nu} \lambda_{i|\mu} \lambda_{k|\nu}.$$

By replacing ordinary differentiation by covariant differentiation on the right-hand side (which is permissible, as we are dealing with invariants), we obtain

$$\frac{dv_k}{ds_l} = v^{\nu}{}_{|\rho} \lambda_{k|\nu} \lambda^{\rho}_{l} + v^\nu \lambda_{k|\nu\rho} \lambda^{\rho}_{l},$$

$$\frac{d\xi_{ik}}{ds_l} = \xi^{\mu\nu}{}_{|\rho} \lambda_{i|\mu} \lambda_{k|\nu} \lambda^{\rho}_{l} + \xi^{\mu\nu} \lambda^{\rho}_{l} (\lambda_{i|\mu\rho} \lambda_{k|\nu} + \lambda_{i|\mu} \lambda_{k|\nu\rho}),$$

whence, by (4′), (16), and (17),

$$\sum_{0}^{n-1}{}_{k} \frac{dv_k}{ds_k} = \operatorname{div} \mathbf{v} + \sum_{0}^{n-1}{}_{hk} \gamma_{khk} v_h, \quad . \quad . \quad . \quad (19)$$

$$\sum_{0}^{n-1}{}_{k} \frac{d\xi_{ik}}{ds_k} = \chi_i + \sum_{0}^{n-1}{}_{hk} (\gamma_{ihk} \xi_{hk} + \gamma_{khk} \xi_{ih}), \quad . \quad (20)$$

which give the divergences div $\mathbf{v}$ and Div $\boldsymbol{\xi}$ of n-uplet tensors (of the first or second rank) directly by means of n-uplet components and n-uplet operations.

For $n = 4$ we have an elementary tensor of the fourth rank at our disposal, namely the well-known Riccian ϵ-system, the covariant and contravariant components of which, $\epsilon_{\mu\nu\rho\sigma}$, $\epsilon^{\mu\nu\rho\sigma}$ respectively, are equal to zero if the four indices are not all different. The other components have the respective values $\pm\sqrt{|g|}$, $\pm\dfrac{1}{\sqrt{|g|}}$, the upper or lower sign being taken according as the permutation $(\mu\nu\rho\sigma)$ is even or odd with respect to (0123).

Let $\boldsymbol{\xi}$ again be a skew-symmetrical tensor of the second rank with the contravariant components $\xi^{\nu\rho}$. If we put

$$p_\mu = \epsilon_{\mu\nu\rho\sigma}\,\xi^{\nu\rho|\sigma}, \quad . \quad . \quad . \quad . \quad (21)$$

which means the same as

$$p^\mu = \epsilon^{\mu\nu\rho\sigma}\,\xi_{\nu\rho|\sigma}, \quad . \quad . \quad . \quad . \quad (21')$$

or

$$\mathbf{p} = \mathrm{Div}^*\,\boldsymbol{\xi} \quad . \quad . \quad . \quad . \quad . \quad (21'')$$

in von Laue's notation, we are justified in calling the vector $\mathbf{p}$ with the above covariant and contravariant components the *Pfaffian divergence of* $\boldsymbol{\xi}$, because the p^μ's vanish identically if, and only if, the $\xi_{\nu\rho}$'s coincide with the coefficients of the bilinear covariants of a Pfaffian expression $\phi_\nu dx^\nu$. This is most easily proved by replacing the covariant derivatives $\xi_{\nu\rho|\sigma}$ in (21′) by their explicit values and noting that, owing to the skew-symmetry of the quantities $\xi_{\nu\rho}$, all that we have left is

$$p^\mu = \sum_{0}^{3}{}_{\nu\rho\sigma}\,\epsilon^{\mu\nu\rho\sigma}\frac{\partial \xi_{\nu\rho}}{\partial x^\sigma}. \quad . \quad . \quad . \quad (21''')$$

The right-hand sides obviously vanish if the quantities $\dfrac{\partial \xi_{\nu\rho}}{\partial x^\sigma}$ are replaced by the differences $\dfrac{\partial^2 \phi_\nu}{\partial x^\rho \partial x^\sigma} - \dfrac{\partial^2 \phi_\rho}{\partial x^\nu \partial x^\sigma}$.

By substituting the expression (21‴) for the p^μ's in the second form (16) of the divergence of a vector we immediately obtain div $\mathbf{p} = 0$, which, bearing (21″) in mind, may be written

$$\mathrm{div}\,(\mathrm{Div}^*\,\boldsymbol{\xi}) = 0; \quad . \quad . \quad . \quad . \quad (22)$$

that is, the divergence of the Pfaffian divergence of a skew-symmetrical tensor of the second rank in R_4 *vanishes identically.*

Further, we shall proceed to represent the vector $\mathfrak{p}$ (the Pfaffian divergence) directly in terms of the *n*-uplet components

$$\xi_{ik} = \xi_{\mu\nu} \lambda^{\mu}_{i} \lambda^{\nu}_{k}$$

of the given tensor. Here it suggests itself to start from the solved form of the equations which we have just written down, namely

$$\xi_{\nu\rho} = \sum_{0}^{3}{}_{hk} \xi_{hk} \lambda_{h|\nu} \lambda_{k|\rho},$$

and to calculate the quantities $\xi_{\nu\rho|\sigma}$ by covariant differentiation of the right-hand side.

From

$$\xi_{hk|\sigma} = \sum_{0}^{3}{}_{l} \frac{d\xi_{hk}}{ds_l} \lambda_{l|\sigma}$$

and (4′) we obtain

$$\xi_{\nu\rho|\sigma} = \sum_{0}^{3}{}_{hkl} \frac{d\xi_{hk}}{ds_l} \lambda_{h|\nu} \lambda_{k|\rho} \lambda_{l|\sigma} + \sum_{0}^{3}{}_{hkjl} \xi_{hk} \lambda_{l|\sigma} \{ \gamma_{hjl} \lambda_{j|\nu} \lambda_{k|\rho} + \gamma_{kjl} \lambda_{j|\rho} \lambda_{h|\nu} \};$$

hence, by (21′),

$$\begin{aligned} p_i = p^{\mu} \lambda_{i|\mu} &= \epsilon^{\mu\nu\rho\sigma} \xi_{\nu\rho|\sigma} \lambda_{i|\mu} \\ &= \sum_{0}^{3}{}_{hkl} \epsilon_{ihkl} \frac{d\xi_{hk}}{ds_l} + \sum_{0}^{3}{}_{jhkl} \xi_{hk} (\epsilon_{ijkl} \gamma_{hjl} + \epsilon_{ihjl} \gamma_{kjl}) \\ &= \sum_{0}^{3}{}_{hkl} \epsilon_{ihkl} \left\{ \frac{d\xi_{hk}}{ds_l} + \sum_{0}^{3}{}_{j} (\gamma_{jhl} \xi_{jk} + \gamma_{jkl} \xi_{hj}) \right\}, \end{aligned}$$

where for brevity we have put

$$\begin{aligned} \epsilon_{ihkl} &= \epsilon^{\mu\nu\rho\sigma} \lambda_{i|\mu} \lambda_{h|\nu} \lambda_{k|\rho} \lambda_{l|\sigma} \\ &= \frac{1}{\sqrt{|g|}} \begin{vmatrix} \lambda_{0|0} & \lambda_{0|1} & \lambda_{0|2} & \lambda_{0|3} \\ \lambda_{1|0} & \lambda_{1|1} & \lambda_{1|2} & \lambda_{1|3} \\ \lambda_{2|0} & \lambda_{2|1} & \lambda_{2|2} & \lambda_{2|3} \\ \lambda_{3|0} & \lambda_{3|1} & \lambda_{3|2} & \lambda_{3|3} \end{vmatrix}. \end{aligned} \qquad (23)$$

Thus these quantities ϵ_{ihkl} are equal to zero if two of the four indices are equal. If, on the other hand, *ihkl* is a permutation of the numbers 0123, ϵ_{ihkl} has the value ± 1, according as the class of the substitution $\begin{pmatrix} ihkl \\ 0123 \end{pmatrix}$ is even or odd. We accordingly see that in the expression which we have just obtained for the p_i's the two last terms are equal to each other, so that we finally obtain

$$p_i = \sum_0^3{}_{hkl} \epsilon_{ihkl} \left\{ \frac{d\xi_{hk}}{ds_l} + 2 \sum_0^3{}_j \gamma_{jhl} \xi_{jk} \right\}. \quad . \quad (24)$$

3. Transformations for an indefinite metric.

According to Eisenhart [1] all the formulæ of the n-uplet theory can be transferred in a readily intelligible way to indefinite metrics, without leaving the real region even temporarily.

If we consider an indefinite

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

we (as is well known) call a (real) direction dx^ν *time-like* or *space-like*, according as the corresponding ds^2 turns out greater or less than zero; null directions are those directions, ∞^{n-2} in number, for which $ds^2 = 0$.

In any case we call the ratios

$$\lambda^\nu = \frac{dx^\nu}{|ds|} \qquad (\nu = 0, 1, \ldots, n-1)$$

parameters of a proper (i.e. non-null) direction.

Hence we have

$$g_{\mu\nu} \lambda^\mu \lambda^\nu = \frac{ds^2}{|ds^2|} = \pm 1 = e, \quad . \quad . \quad (25)$$

if we henceforth denote positive or negative unity by e.

As in the definite case we introduce as *moments* of a given direction the covariant quantities

$$\lambda_\nu = g_{\mu\nu} \lambda^\mu, \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

so that the quadratic identity (25) takes the form

$$\lambda_\nu \lambda^\nu = e. \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

If the quantities λ_i^ν $(i = 0, 1, \ldots, n-1)$ are the parameters

[1] *Riemannian Geometry*, Princeton University Press, 1926, Chap. III.

of an orthogonal n-uplet consisting of proper directions only, we have

$$\lambda_{i|\nu}\lambda_k^\nu = 0 \qquad (i \neq k)$$

on account of the orthogonality of the n-uplet, and also

$$\lambda_{i|\nu}\lambda_i^\nu = \pm 1 = e_i,$$

by (27).

The total number of negative (and consequently also of the remaining positive) quantities e_i for a given ds^2 is always equal to its index of inertia, and hence is always the same no matter what (proper) n-uplet is considered.

The two groups of relationships between parameters and moments of an n-uplet which we have just written down may be summarized in the single formula

$$\lambda_i^\nu \lambda_{k|\nu} = e_k\,\delta_{ik} = e_i\,\delta_{ik}, \quad . \quad . \quad . \quad (28)$$

where the symbols δ_{ik} have their usual meaning; or, since $e_k{}^2 = 1$,

$$\lambda_i^\nu e_k \lambda_{k|\nu} = \delta_{ik}.$$

From this we conclude that the elements reciprocal to the parameters λ_i^ν are not exactly equal to the moments $\lambda_{i|\nu}$, but to $e_i\lambda_{i|\nu}$. Thus the quantities $e_i\lambda_i^\nu$ are the elements reciprocal to the moments $\lambda_{i|\nu}$. If we imagine the equations (26) written down for every n-uplet, we have

$$\lambda_{i|\nu} = g_{\rho\nu}\lambda_i^\rho$$

(denoting the index of summation by ρ). By multiplying by $e_i\lambda_{i|\mu}$ and summing with respect to i we obtain

$$g_{\mu\nu} = \sum_0^{n-1}{}_i\, e_i\,\lambda_{i|\mu}\lambda_{i|\nu},$$

which replaces formula (2) for the definite case, and so on.

From this point it will suffice if I confine myself to quite brief hints, and I shall of course write down only those formulæ which do not remain unaltered throughout. These will be marked with an asterisk and given the same number as the corresponding formula referring to a definite metric.

In the first place, n-uplet components of any given tensor and coefficients of rotation γ_{ikl} must in any case be introduced by the equations of definition (1) and (4); the solved expressions for the quantities $\lambda_{i|\nu\rho}$, on the other hand, are in general

$$\lambda_{i|\nu\rho} = \sum_{0}^{n-1}{}_{jh} \gamma_{ijh} e_j e_h \lambda_{j|\nu} \lambda_{h|\rho}. \quad . \quad . \quad (4')^*$$

The covariant equations (6), and also the equations of definition of the 4-index symbols γ (7) are true without restriction; but the n-uplet tensor expressions for the quantities $\gamma_{ij,\,hk}$ suffer a small modification. In fact we must in general put

$$\gamma_{ij,\,hk} = \frac{d\gamma_{ijh}}{ds_k} - \frac{d\gamma_{ijk}}{ds_h}$$
$$+ \sum_{0}^{n-1}{}_l e_l [\gamma_{ijl}(\gamma_{lhk} - \gamma_{lkh}) + \gamma_{lik}\gamma_{ljh} - \gamma_{lih}\gamma_{ljk}]. \quad (8)^*$$

Of course these quantities are still connected by the relationships (9), in virtue of equations (7).

It is essential to note, however, that the local transference from one n-uplet to another does not correspond to any orthogonal transformation, but to a pseudo-orthogonal transformation, i.e. to a transformation which leaves the quadratic form

$$Q(z) = \sum_{0}^{n-1}{}_i e_i z_i^2$$

invariant. Thus the coefficients a_{ik} of a pseudo-orthogonal transformation of this kind must satisfy the conditions

$$\sum_{0}^{n-1}{}_l e_l a_{il} a_{kl} = \sum_{0}^{n-1}{}_l e_l a_{li} a_{lk} = \delta_{ik}.$$

The most general expression which can be attributed to the coefficients a_{ik} in the case of infinitesimal pseudo-orthogonal transformations follows immediately from the condition that the form $Q(z)$ is to be invariant. We have merely to put

$$a_{ik} = \delta_{ik} + e_i \beta_{ik}$$

and to regard the quantities β_{ik} as indefinitely small. If in Q we carry out the substitution

$$z_i = \sum_{0}^{n-1}{}_k a_{ik} z'_k = z'_i + e_i \sum_{0}^{n-1}{}_k \beta_{ik} z'_k \quad . \quad . \quad (29)$$

and require that $Q(z')$ should retain the form

$$\sum_{0}^{n-1}{}_i\, e_i z_i'^2,$$

what we obtain (as in the case of pure orthogonal substitutions) is the condition of skew-symmetry, namely

$$\beta_{ik} + \beta_{ki} = 0. \quad . \quad . \quad . \quad . \quad (30)$$

The components of an n-uplet tensor are systems of numbers which behave like tensors with respect to pseudo-orthogonal transformations; for *local* n-uplet tensors this behaviour is maintained only with respect to pseudo-orthogonal transformations with *constant* coefficients. The operators

$$\frac{df}{ds_i} = X_i f = \sum_{0}^{n-1}{}_\nu\, \lambda_i^\nu \frac{\partial f}{\partial x^\nu}$$

behave like n-uplet vectors.

If (i) and (k) denote any group of n-uplet indices and $A_{(i)j}$, $B_{(k)l}$ two local n-uplet tensors, then contraction with respect to j, l is defined by the formula

$$\sum_{0}^{n-1}{}_l\, e_l A_{(i)l} B_{(k)l}.$$

We accordingly obtain

$$G_{ik} = \sum_{0}^{n-1}{}_h\, e_h \gamma_{ih,\,hk}, \quad . \quad . \quad . \quad . \quad (10)^*$$

$$G = \sum_{0}^{n-1}{}_{hk}\, e_h e_k \gamma_{kh,\,hk}, \quad . \quad . \quad . \quad (11)^*$$

instead of (10) and (11).

Further, the formulæ (12), (14), and (15) must be replaced by

$$\xi_{ik} = \sum_{0}^{n-1}{}_l\, e_l \frac{d\gamma_{ikl}}{ds_l}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)^*$$

$$c_l = \sum_{0}^{n-1}{}_j\, e_j \gamma_{jlj}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)^*$$

and

$$\eta_{ik} = \sum_{0}^{n-1}{}_l\, e_l c_l (\gamma_{lik} - \gamma_{lki}), \quad . \quad . \quad (15)^*$$

while the expressions (13) for covariant and contravariant components in terms of the n-uplet components ξ_{ik} are to be deduced from (1), the universally valid definition of the n-uplet components of a tensor. Hence they become

$$\left.\begin{aligned} \xi_{\mu\nu} &= \sum_{0}^{n-1}{}_{ik}\, \xi_{ik}\, e_i\, e_k\, \lambda_{i|\mu}\, \lambda_{k|\nu}, \\ \xi^{\mu\nu} &= \sum_{0}^{n-1}{}_{ik}\, \xi_{ik}\, e_i\, e_k\, \lambda_i^{\mu}\, \lambda_k^{\nu}. \end{aligned}\right\} \quad . \quad . \quad (13)^*$$

As contraction of pseudo-orthogonal n-uplet tensors is brought about by inserting the factor e with the appropriate index, it is at once clear that (19), (20), and (24) take the forms

$$\operatorname{div} \mathbf{v} = \sum_{0}^{n-1}{}_k\, e_k \frac{dv_k}{ds_k} - \sum_{0}^{n-1}{}_{hk}\, e_h\, e_k\, \gamma_{khk}\, v_h, \quad . \quad . \quad . \quad . \quad (19)^*$$

$$\chi_i = \sum_{0}^{n-1}{}_k\, e_k \frac{d\xi_{ik}}{ds_k} - \sum_{0}^{n-1}{}_{hk}\, e_h\, e_k\, (\gamma_{ihk}\, \xi_{hk} + \gamma_{khk}\, \xi_{ih}), \quad (20)^*$$

$$p_i = \sum_{0}^{3}{}_{hkl}\, e_h\, e_k\, e_l\, \epsilon_{ihkl} \left\{ \frac{d\xi_{hk}}{ds_l} + 2 \sum_{0}^{3}{}_j\, e_j\, \gamma_{jhl}\, \xi_{jk} \right\}. \quad . \quad (24)^*$$

Of course the equations (18′) and (22), i.e.

$$\operatorname{div} (\operatorname{Div} \boldsymbol{\xi}) = 0, \quad \operatorname{div} (\operatorname{Div}^* \boldsymbol{\xi}) = 0, \quad . \quad (31)$$

which express invariant properties, always remain valid.

4. Gravitational equations.

As usual, let the covariant components of the energy tensor be denoted by $T_{\mu\nu}$. If influences of any origin are admitted, these quantities $T_{\mu\nu}$ are to be imagined broken up into two parts, one of which, $\tau_{\mu\nu}$, is purely electromagnetic, and the other, $\mathbf{T}_{\mu\nu}$, represents the remainder, if any. We therefore put

$$T_{\mu\nu} = \tau_{\mu\nu} + \mathbf{T}_{\mu\nu}, \quad . \quad . \quad . \quad . \quad (32)$$

where $\boldsymbol{\tau}$ is the well-known Maxwell tensor; further, for empty space $\mathbf{T}_{\mu\nu}$ is of course equal to zero.

As is well known, the Einstein equations (without the cosmological term) are

$$G_{\mu\nu} - \tfrac{1}{2} G g_{\mu\nu} = -\kappa T_{\mu\nu},$$

where the constant of proportionality κ may be expressed in terms of f, the gravitational constant, and c, the velocity of light $\left(\kappa = \frac{8\pi f}{c^4}\right)$.

If we introduce the corresponding n-uplet tensors in accordance with the formulæ

$$G_{ik} = G_{\mu\nu} \lambda_i^{\mu} \lambda_k^{\nu},$$
$$T_{ik} = T_{\mu\nu} \lambda_i^{\mu} \lambda_k^{\nu}, \text{ \&c.},$$

we have, on the one hand,

$$T_{ik} = \tau_{ik} + \mathbf{T}_{ik}, \quad . \quad . \quad . \quad . \quad (32')$$

from (32), and (what is most important) the gravitational equations in the n-uplet tensor form[1]

$$G_{ik} - \tfrac{1}{2}\delta_{ik} G = -\kappa T_{ik}, \qquad (i, k = 0, 1, 2, 3) \qquad (\mathrm{I})$$

where, in accordance with (10)* and (11)*,

$$G_{ik} = \sum_{0}^{3}{}_h e_h \gamma_{ih,hk}, \quad G = \sum_{0}^{3}{}_k e_k G_{kk} = \sum_{0}^{3}{}_{hk} e_h e_k \gamma_{kh,hk}.$$

As the space-time manifold on which the general theory of relativity is to be based possesses an indefinite metric with an index of inertia 3, we have to put

$$e_0 = 1, \quad e_1 = e_2 = e_3 = -1. \quad . \quad (33)$$

The quantities $\gamma_{ij,hk}$ are introduced by the equations (8)* as lattice differential elements of the second order. Their combinations G_{ik} behave like tensors with respect to all pseudo-orthogonal (i.e. in the present case Lorentz) transformations (even if the coefficients are permitted to vary in any way with position).

Accordingly, as indeed is clear from the outset, the ten equations (I) do not, as far as their original form is concerned, favour any special quadruplet. They are valid in one and the same form for all orthogonal quadruplets of the relativistic R_4, and, as is well-known, serve to define their metric.

As in every case they give ten relationships between the

[1] Given in 1918 by Cisotti (*Rend. Acc. Lincei*, Vol. XXVII, pp. 366–371), but confined to the (imaginary) notation of (8), (10), (11).

sixteen parameters λ^{ν}_{i}, we need only find six other apparently reasonable conditions connecting the latter, in order to mark out a special lattice (the *world lattice*) from among all the possible quadruplets and lattices corresponding to the space-time-manifold R_4.

We shall shortly (§6) carry out this final step, which is in fact the only essential one. Meanwhile we may appropriately lead up to it by putting Maxwell's equations into a suitable form.

5. Electromagnetic equations.

Let $F_{\mu\nu}$, $F^{\mu\nu}$, F_{ik} be the (covariant, contravariant, and n-uplet) components of the skew-symmetrical tensor **F** which defines the electromagnetic field in the space-time world; let **S** (a vector) be the current-vector [1] and S_{μ}, &c., its four components, where all the quantities are understood to be measured in so-called rational units.

Maxwell's equations (as adopted in the general theory of relativity after Einstein) then take the forms

$$\mathrm{Div}\,\mathbf{F} = \mathbf{S}, \quad \mathrm{Div}\,{}^{*}\mathbf{F} = 0. \quad . \quad . \quad (34)$$

Each group contains four equations, so that at first glance one would take the total number of equations to be eight. But we necessarily have div $\mathbf{S} = 0$, so that by (31) there must exist two identical relationships, namely those which express the fact that the divergences in question vanish. Thus two equations of the system (34) may (with appropriate subsidiary conditions) be regarded as resulting from the other six; and in fact we know that if **S** is regarded as given or as associated in some other way with the tensor **F**, then the equations (34) merely serve to determine the six components of **F** for $x^0 + dx^0$ uniquely from their values for a given x^0 (and any x^1, x^2, x^3).

We have still to write down the symmetrical stress-energy tensor explicitly. As is well known, its covariant components are defined as follows:

$$\tau_{\mu\nu} = -g^{\rho\sigma} F_{\mu\rho} F_{\nu\sigma} + \tfrac{1}{4} g_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma}.$$

By composition with $\lambda^{\mu}_{i}\,\lambda^{\nu}_{k}$ (by replacing $g^{\rho\sigma}$ on the right-

[1] Ger. *Viererstrom.*

hand side by $\sum_{0}^{3} {}_l e_l \lambda_l^\rho \lambda_l^\sigma$ and $F^{\rho\sigma}$ by $\sum_{0}^{3} {}_{jh} e_j e_h F_{jh} \lambda_j^\rho \lambda_h^\sigma$) we obtain the required n-uplet tensor formula:

$$\tau_{ik} = -\sum_{0}^{3} {}_l e_l F_{il} F_{kl} + \tfrac{1}{4} \delta_{ik} \sum_{0}^{3} {}_{jh} e_j e_h F_{jh}{}^2. \quad . \quad (35)$$

6. **Interpretation of the electromagnetic tensor in the world lattice. Purely geometrical formulation of the field equations.**

A priori we may quite arbitrarily connect the six n-uplet components F_{ik} of the electromagnetic field with any geometrical properties of a quadruplet (thereby defined) of the R_4. A very simple way of doing this is to make the quantities F_{ik} proportional to the corresponding elements of a (differential) skew-symmetrical *local* n-uplet tensor, e.g. to the differential expressions, of the second or first order respectively, which are defined by the equations

$$\xi_{ik} = \sum_{0}^{3} {}_l e_l \frac{d\gamma_{ikl}}{ds_l} \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)^*$$

or

$$\eta_{ik} = \sum_{0}^{3} {}_l e_l c_l (\gamma_{lik} - \gamma_{lki}) \quad . \quad . \quad . \quad (15)^*$$

of §3.

As we shall see, the best way is to select the first expression, and we accordingly put

$$F_{ik} = v\xi_{ik} \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{P})$$

where v denotes a constant.

As the Ricci coefficients of rotation γ_{ikl} are merely ratios of an angle and a length, the quantities ξ_{ik} are of dimensions l^{-2}. The quantities F_{ik}, on the other hand, behave like the square root of an energy-density. Consequently we have

$$[F_{ik}] = l^{-\frac{1}{2}} t^{-1} m^{\frac{1}{2}}.$$

Hence the factor of proportionality v has dimensions

$$l^{\frac{3}{2}} t^{-1} m^{\frac{1}{2}},$$

which are those of an electric charge e, e.g. the electronic charge, so that we may write

$$v = \mathfrak{z} e, \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

where the factor of proportionality $\mathfrak{z}$ is now a pure number. Moreover, we may also replace e in (36) by any other quantity of the same dimensions; e.g. we may put

$$v = \mathfrak{z}_1 \sqrt{hc}, \quad . \quad . \quad . \quad . \quad . \qquad (36')$$

where h is Planck's constant, c the velocity of light in empty space, and $\mathfrak{z}_1$ a pure number.

Hence the final forms of the geometrical equations which arise from the Maxwellian system (34) and our proposed addition (P), are

$$\operatorname{Div} \boldsymbol{\xi} = \frac{1}{v} \mathbf{S}, \quad \operatorname{Div}^* \boldsymbol{\xi} = 0, \quad . \quad . \qquad (II)$$

where $\boldsymbol{\xi}$ means the local n-uplet tensor (12)*. *In conclusion, then, the geometrical definition of the quadruplet* (*world lattice*) *associated with the field is to be taken from the two systems* (I) *and* (II), *which together give sixteen* (apparently eighteen, but in reality only sixteen) *differential equations* (of the second and third order respectively) *involving the sixteen n-uplet parameters* λ^r_i.

7. The case of empty space: absence of an electromagnetic field.

In empty space ($T_{ik} = 0$, $\mathbf{S} = 0$), (I) reduces in virtue of (32) to the form

$$G_{ik} - \tfrac{1}{2} \delta_{ik} G = -\kappa \tau_{ik}, \quad . \quad . \quad . \qquad (I')$$

where the term τ_{ik} on the right-hand side is given by

$$\tau_{ik} = -v^2 \sum_0^3{}_l\, e_l\, \xi_{il}\, \xi_{kl} + \tfrac{1}{4} v^2 \delta_{ik} \sum_0^3{}_{jh}\, e_j\, e_h\, \xi_{jh}{}^2 \qquad (35')$$

by (35) and (P); while the system (II) becomes

$$\operatorname{Div} \boldsymbol{\xi} = 0, \quad \operatorname{Div}^* \boldsymbol{\xi} = 0. \quad . \quad . \qquad (II')$$

If the electromagnetic field vanishes in addition to the external energy tensor $\mathbf{T}_{ik}$, the quantities ξ_{ik}, and hence, by (35′), the quantities τ_{ik} also, are equal to zero. If this happens everywhere in the space-time world, we know [1] that the equations (I′), which simply become $G_{ik} = 0$, necessarily imply that the metric of the space is Euclidean or, more correctly, pseudo-Euclidean.

[1] Cf. Serini, *Rend. Acc. Lincei*, Vol. XXVII, 1918, pp. 235–238.

What, then, is the geometrical meaning of the absence of electromagnetic phenomena in this limiting case, i.e. what is the geometrical meaning of the equations

$$\xi_{ik} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (37)$$

They simply state the fact that the world lattice is Cartesian or, more correctly, pseudo-Cartesian.

In order to give as concise a proof of this as possible, I shall only consider quadruplets in which the deviations from a pseudo-Cartesian lattice are infinitely small.

If, in particular, we take the co-ordinates x^ν to be Cartesian co-ordinates with respect to that lattice, we have

$$\lambda'^{\nu}_{i} = \delta_{i\nu}$$

for the parameters of the corresponding quadruplet.

Let λ^ν_i be the parameters of any neighbouring quadruplet. Since the passage from the quantities λ'^{ν}_{i} to the quantities λ^ν_i corresponds to an infinitesimal pseudo-orthogonal transformation, the quantities λ^ν_i must, by (29), be expressible as follows:

$$\lambda^\nu_i = \delta_{i\nu} + e_i \sum_{0}^{3}{}_k \beta_{ik} \delta_{k\nu} = \delta_{i\nu} + e_i \beta_{i\nu}, \quad . \quad (38)$$

where the quantities β_{ik} *form a skew-symmetrical n-uplet tensor.* From this we can immediately calculate the reciprocal elements. To a first approximation we obtain

$$e_i \lambda_{i|\nu} = \delta_{i\nu} + e_\nu \beta_{i\nu},$$

whence, multiplying by e_i,

$$\lambda_{i|\nu} = \delta_{i\nu} e_i + e_i e_\nu \beta_{i\nu}. \quad . \quad . \quad . \quad (38')$$

On the other hand, if we altogether neglect infinitely small quantities, the operators

$$\frac{d}{ds_l} = \sum_{0}^{3}{}_\nu \lambda^\nu_l \frac{\partial}{\partial x^\nu}$$

reduce to the simple form

$$\frac{\partial}{\partial x^l},$$

and the covariant derivatives reduce to their usual forms.

Thus (4), the definition of the rotational invariants γ, gives (except for infinitely small quantities of the second order)

$$\gamma_{ikl} = e_i\, e_k \frac{\partial \beta_{ik}}{\partial x^l},$$

and from (12)* we further obtain

$$\xi_{ik} = e_i\, e_k \sum_0^3{}_l\, e_l \frac{\partial^2 \beta_{ik}}{(\partial x^l)^2}.$$

The differential operator $\sum_0^3{}_l\, e_l \frac{\partial^2 \beta_{ik}}{(\partial x^l)^2}$ is none other than the Dalembertian or Lorentz operator $\Box$. Thus the equations (37) take the form

$$\Box\, \beta_{ik} = 0, \quad . \quad . \quad . \quad . \quad . \quad (37')$$

and together with suitable initial and boundary conditions they give

$$\beta_{ik} = 0,$$

i.e. *the Cartesian* (or, more correctly, pseudo-Cartesian) *character of the world lattice.* I think that this conclusion justifies our assumption (P). If we had put, say,

$$F_{ik} = v'\, \eta_{ik}, \quad (v' = \text{constant})$$

where the quantities η_{ik} are given by the expressions (15)*, we should not have obtained any satisfactory result.

A more general assumption, such as

$$F_{ik} = v\, \xi_{ik} + v'\, \eta_{ik},$$

would, on the other hand, be more complicated, though just as admissible as (A) from the logical point of view. To a first approximation, in fact, we should obtain the same result, as the η's are of higher order than the ξ's.