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On work-hardening adaptation of discrete structures under dynamic loadings*

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THE PRESENT paper deals with problems of work-hardening adaptation of discrete structures subjected to dynamic loadings in the hypothesis of infinitesimal displacements. The behaviour of structural elements is assumed to be rigid-plastic with piecewise linear yield surface, piecewise linear work-hardening and strain rate sensitivity. An adaptation criterion is given and bounds on strain parameters are formulated. These bounds can be made the most stringent by solving a problem of finite plasticity which may be considered as the holonomic formulation of the adaptation problem suitably perturbed. The holonomic solution is directly utilized to form the bounding quantities. A minimum principle similar to Haar-Kármán's is shown to play a role in characterizing the adaptability of the structure and the optimality of the bounds. A simple application is presented.

Rozpatruje się problemy przystosowania konstrukcji dyskretnych obciążonych dynamicznie w procesie wzmocnienia plastycznego, przyjmuje się przy tym hipotezę przemieszczeń infinitezymalnych. Przyjmuje się, że zachowanie się elementów konstrukcyjnych jest sztywno-plastyczne z odcinkowo liniową powierzchnią przepływu, z odcinkowo liniową czułością na wzmocnienie i prędkość deformacji. Podano kryterium adaptacyjne oraz sformułowano ograniczenia na parametry odkształcenia. Ograniczenia te mogą zostać maksymalnie uściślone drogą rozwiązania problemu skończonej plastyczności, który to problem może być rozważany jako holonomiczne sformułowanie odpowiednio zaburzonego problemu adaptacji. Rozwiązanie holonomiczne jest bezpośrednio wykorzystane do utworzenia odpowiednich wielkości ograniczających. Wykazano, że pewna zasada minimum, podobna do zasady Haara-Kármána, gra istotną rolę przy charakteryzacji przystosowania konstrukcji i optymalności ograniczeń. Przedstawiono proste zastosowanie teorii.

Рассматриваются проблемы адаптации дискретных структур, нагруженных динамически, в процессе пластического упрочнения, при использовании гипотезы инфинитизимальных перемещений. Предполагается, что поведение конструкционных элементов жестко-пластическое с кусочно-линейной поверхностью течения, и с кусочно-линейной чувствительностью на упрочнение и скорость деформаций. Приводится критерий адаптации и формулировка ограничения на параметры деформации. Эти ограничения могут быть максимально уточнены путем решения задачи конечной пластичности, рассмотренной в качестве голономной формулировки соответственно измущенной задачи. Голономическое решение использовано для нахождения соответствующих ограничивавеличин. Показано, что принцип минимума, подобный принципу Хаара-Кармана, играет существенную роль при определении конструкции и оптимальности ограничений. Представлено простое применение теории.

1. Introduction

THE "WORKHARDENING ADAPTATION" concept was introduced by PRAGER [1, 2] not long ago. According to this concept, a structure adapts to loads which vary arbitrarily within

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a given domain when, after an initial phase during which a limited amount of plastic deformation may be produced, it finally behaves like a purely rigid body in the sense that no further strains occur. Work-hardening adaptation, therefore, may be considered as a limit case of classical shakedown, obtained by making the elastic constitutive component tend to vanish. This leads to two consequences: i) since self-stresses, as a response to plastic strains, cannot exist, adaptation may occur only as a result of the work-hardening behaviour; ii) since a straightforward passage from the classical shakedown theory to the work-hardening adaptation theory could present difficulties, a specific treatment of the latter subject is useful and desirable.

After Prager's work, further contributions were made. POLIZZOTTO [3, 4] formulated "static" and "kinematic" adaptation criteria considering a rather broad class of work-hardening laws and quasi-static as well as dynamic loadings. Second-order geometrice ffects were considered by KÖNIG and MAIER [5] and by POLIZZOTTO *et al.* [6]. The work-hardening adaptation concept was assumed as a design criterion first by PRAGER [7] for truss-like structures, then by POLIZZOTTO *et al.* [8] for more general discrete structures. Bound techniques were also given in the paper quoted [5].

In spite of this research work, the practical utility of the work-hardening adaptation theory in structural engineering is not yet fully understood, in particular as regards dynamics.

On the other hand, we know how widespread the use of the rigid-plastic model is in structural engineering, not only within the classical limit analysis and design, but also in the field of dynamic plasticity where a great amount of research work has been done and very useful results have been given (see, e.g. [9–13]). Elementary rigid-plastic theories provide, often simply and quickly, an estimate of major deformation caused by large dynamic loadings, and this estimate then constitutes a useful basis for more refined analyses.

Along this line of reasoning there is a sufficient motivation for a work-hardening adaptation theory, provided questions about its inherent limitations are better understood in the future. This theory is an approximation on the unsafe side of the elastic-plastic shakedown theory [14], so a study on the order of magnitude of the safety factors for adaptation provided by the two theories is to be hoped for. A crucial point of the work-hardening adaptation theory is the evaluation of the greatest deformation we must expect in the adapted structure. For this purpose, as is known, bounding techniques are the only practicable way, but only in one paper [5] has this subject been treated so far.

The present paper is devoted to work-hardening adaptation criteria and bounds on deformation for discrete structures subject to dynamic loads in the hypothesis of infinitesimal displacements. For the sake of a greater generality, viscous forces are also considered. The plastic behaviour of a typical element is described by a piecewise linear yield surface and is supposed to show a piecewise linear work-hardening and a linear strain rate sensitivity.

A method recently given by POLIZZOTTO [15, 16], based on a perturbation procedure, is here reformulated for rigid-plastic structures. In this way, after some preliminaries and definitions, a general inequality is given. From this inequality, by specializing the applied perturbances, an adaptation criterion or bounds on plastic strains can be deduced.

Then these bounds are made the most stringent by solving a minimization problem which can be viewed as a minimum principle in finite plasticity. A simple application and some final remarks conclude the paper.

The matrix notation will be adopted. Capital bold letters will be used to indicate matrices, small bold letters to indicate vectors. The symbol T as an exponent indicates transposition. The other symbols will be defined where they first appear.

2. Basic relations

Let a discrete (or discretized) structure, made up of rigid-plastic finite elements, be in a known geometrical configuration, characterized by zero displacements, at time $t = 0$. Then, loadings $\mathbf{f}$ are applied at the nodes of the structure. $\mathbf{f}$ is variable with time t, but the real time sequence is unknown. We only know that as t varies the load takes values within a given bounded region, which is equivalent to saying it can be thought of as a one-to-one and homogeneous function of a vector variable, $\boldsymbol{\tau}$, ranging within a given r-dimensional *loading domain* Π. Thus

$$\mathbf{f} = \mathbf{f}(\boldsymbol{\tau}), \quad \forall \boldsymbol{\tau} \in \Pi. \tag{2.1}$$

Any admissible loading history is obtained as soon as a time function $\boldsymbol{\tau} = \boldsymbol{\tau}(t)$ is chosen, provided $\boldsymbol{\tau}(t) \in \Pi$ for any $t \geqslant 0$.

Considering the displacements as infinitesimal, the compatibility equations are

$$\begin{aligned} \boldsymbol{\epsilon} &= \mathbf{C}\mathbf{u}, \quad \forall t \geqslant 0, \\ \dot{\mathbf{u}} &= \dot{\mathbf{u}}_0, \quad \text{for} \quad t = 0, \end{aligned} \tag{2.2}$$

where $\mathbf{C}$ is the compatibility matrix and $\dot{\mathbf{u}}_0$ is assigned. Denoting by $\boldsymbol{\sigma}$ the element stresses at time $t \geqslant 0$, the equilibrium equations are

$$\mathbf{C}^T\boldsymbol{\sigma} + \mathbf{V}\dot{\mathbf{u}} + \mathbf{M}\ddot{\mathbf{u}} = \mathbf{f}, \quad \forall t \geqslant 0, \tag{2.3}$$

where $\mathbf{V}\dot{\mathbf{u}}$ is the vector of the viscous forces and $\mathbf{M}\ddot{\mathbf{u}}$ the vector of the inertia forces. As is known [17], the viscous matrix $\mathbf{V}$ and the mass matrix $\mathbf{M}$ are both symmetric and positive definite.

The plastic behaviour of the structural elements is characterized by the following m simultaneous inequalities:

$$\mathbf{N}^T\boldsymbol{\sigma} - \mathbf{y} \leqslant 0, \tag{2.4}$$

which define the domain of the stress space within which the stress vector $\boldsymbol{\sigma}$ can range. This domain is a polyhedron of m faces having external unit normals collected in the (constant) matrix $\mathbf{N}$ and distances from the origin collected in the vector $\mathbf{y}$. Therefore, introducing the plastic potential vector $\boldsymbol{\varphi}$ as in the following

$$\boldsymbol{\varphi} = \mathbf{N}^T\sigma - \mathbf{y}, \tag{2.5}$$

we can express (plastic) strain rates by the usual flow rule, that is

$$\dot{\boldsymbol{\epsilon}} = \frac{\partial \boldsymbol{\varphi}}{\partial \boldsymbol{\sigma}} \dot{\boldsymbol{\lambda}} = \mathbf{N}\dot{\boldsymbol{\lambda}}, \tag{2.6}$$

to which the following side conditions must be added [18]:

$$\boldsymbol{\varphi} \leqslant \mathbf{0}, \quad \dot{\boldsymbol{\lambda}} \geqslant \mathbf{0}, \quad \boldsymbol{\varphi}^T\dot{\boldsymbol{\lambda}} = 0, \quad \dot{\boldsymbol{\varphi}}^T\dot{\boldsymbol{\lambda}} = 0. \tag{2.7}$$

Let us observe that Eq. (2.6) expresses the strain rate vector $\dot{\boldsymbol{\varepsilon}}$ as a non-negative linear combination of the unit normals of the yield polyhedron, while Eqs. $(2.7)_{1-4}$ are equivalent to the usual concepts of the plasticity theory (convexity, normality). Since the plastic potentials, $\boldsymbol{\varphi}$, and the *plastic activation coefficients*, $\dot{\boldsymbol{\lambda}}$, are sign constrained, Eqs. $(2.7)_{3,4}$ apply componentwise and say that plastic activation of a face of the yield polyhedron

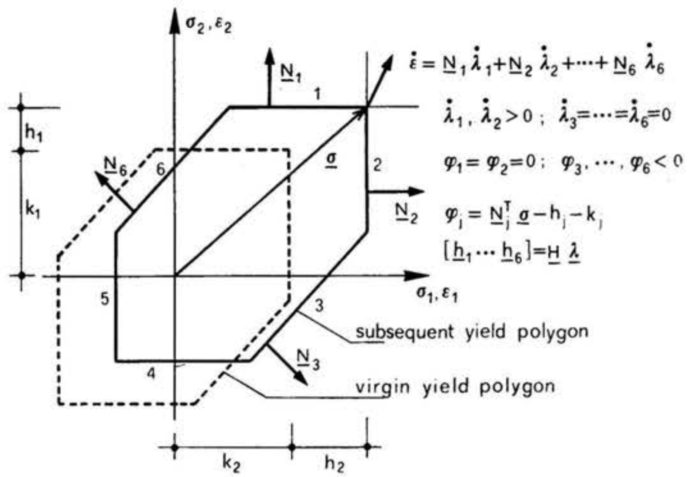

FIG. 1. Typical yield domain in two dimensions and mathematical description of the flow-rule.

occurs only if the corresponding potential and its time derivative are both zero (see Fig. 1).

The vector $\mathbf{y}$ depends on strain (work-hardening) and on strain rate (strain rate sensitivity). Assuming a piecewise linear dependence, we write

$$\mathbf{y} = \mathbf{H}\lambda + \mathbf{R}\dot{\boldsymbol{\lambda}} + \mathbf{k}, \tag{2.8}$$

where $\mathbf{H}$ and $\mathbf{R}$ are symmetric and positive semidefinite (*psd*) matrices and $\mathbf{k}$ is the vector of the *plastic resistances*. The work-hardening matrix $\mathbf{H}$ can be defined in such a way that the most important hardening laws can be allowed for (isotropic, Koiter's, kinematic work-hardening, for instance). In the following the matrix $\mathbf{H}$ is supposed to be *psd.*

In Eq. (2.8) the vector $\mathbf{k}$ is positive ($\mathbf{k} > 0$) and describes the distances of the yield faces from the origin in the virgin state ($t < 0$).

The vector

$$\boldsymbol{\lambda} = \int_0^T \dot{\boldsymbol{\lambda}} dt \tag{2.9}$$

is a non-decreasing and non-negative function of t and can be assumed as a measure of plastic strain. The boundness of $\boldsymbol{\lambda}$ can therefore be assumed as equivalent to the boundedness of plastic strain and then to the adaptability of the structure to the loadings.

3. Further relations and definitions

The following observations and definitions will be useful.

3.1. Statically admissible yield surface (SAYS)

Let us suppose that we can find a time-independendent vector, say $\hat{\boldsymbol{\lambda}}$, such that the following conditions are satisfied:

$$\text{(3.1)} \qquad \begin{gathered} \mathbf{C}\boldsymbol{\sigma} = \mathbf{f}(\boldsymbol{\tau}), \quad \hat{\boldsymbol{\lambda}} \geqslant \mathbf{0}, \\ \mathbf{N}^T\hat{\boldsymbol{\sigma}} - \mathbf{H}\hat{\boldsymbol{\lambda}} - \mathbf{k} \leqslant \mathbf{0}, \quad \forall \boldsymbol{\tau} \in \Pi, \end{gathered}$$

where $\hat{\boldsymbol{\sigma}}$ changes with the load. The vector $\hat{\boldsymbol{\lambda}}$ defines a *subsequent yield surface* such that, for every loading condition, at least one stress vector $\hat{\boldsymbol{\sigma}}$ can be found which is in *statical equilibrium* with the load and *does not exceed the yield surface*. Such a yield surface is called "statically admissible", while it is called "safe statically admissible" when Eq. $(3.1)_3$ applies in the stronger form

$$\text{(3.2)} \qquad \mathbf{N}^T\hat{\boldsymbol{\sigma}} - \mathbf{H}\hat{\boldsymbol{\lambda}} - \mathbf{k} < \mathbf{0}, \quad \forall \boldsymbol{\tau} \in \Pi.$$

3.2. Perturbances

Given a safe SAYS, we can always transform it in a simple SAYS by introducing the "perturbance" vector, $\mathbf{k}^*$, and writing

$$\text{(3.3)} \qquad \begin{gathered} \mathbf{C}^T\hat{\boldsymbol{\sigma}} = \mathbf{f}(\boldsymbol{\tau}), \quad \hat{\boldsymbol{\lambda}} \geqslant \mathbf{0}, \\ \mathbf{N}^T\hat{\boldsymbol{\sigma}} - \mathbf{H}\hat{\boldsymbol{\lambda}} - \mathbf{k} + \mathbf{k}^* \leqslant \mathbf{0}, \quad \forall \boldsymbol{\tau} \in \Pi. \end{gathered}$$

The perturbance vector $\mathbf{k}^*$ is arbitrary. In particular, the vector $\mathbf{k}^*$ can be given the following forms:

$$\text{(3.4)} \quad \text{i)} \qquad \mathbf{k}^* = \alpha\mathbf{k}, \quad 0 < \alpha < 1,$$

with α suitably selected within the open interval (0,1). This is equivalent to a uniform shrinkage of the given yield surface.

$$\text{(3.5)} \quad \text{ii)} \qquad \mathbf{k}^* = \bar{\mathbf{k}}\omega, \quad \omega > 0,$$

where $\bar{\mathbf{k}} \geqslant \mathbf{0}$ is arbitrary. This is equivalent to a non-uniform shrinkage of the given yield surface.

$$\text{(3.6)} \quad \text{iii)} \qquad \mathbf{k}^* = \mathbf{N}^T\bar{\boldsymbol{\sigma}}\omega, \quad \omega > 0,$$

that is $\bar{\mathbf{k}} = \mathbf{N}^T\bar{\boldsymbol{\sigma}}$ in Eq. (3.5), $\bar{\boldsymbol{\sigma}}$ being arbitrary. In this case the given yield surface undergoes a rigid translation $-\bar{\boldsymbol{\sigma}}\omega$.

$$\text{(3.7)} \quad \text{iv)} \qquad \mathbf{k}^* = \mathbf{N}^T\mathbf{A}^T\bar{\mathbf{f}}\omega, \quad \omega > 0,$$

that is $\bar{\boldsymbol{\sigma}} = \mathbf{A}^T\bar{\mathbf{f}}$ in Eq. (3.6), $\bar{\mathbf{f}}$ being an arbitrary load equilibrated by the stresses $\bar{\boldsymbol{\sigma}}$. Again, the yield surface undergoes a translation $-\mathbf{A}^T\bar{\mathbf{f}}\omega$.

We shall call "admissible perturbances" those perturbances which satisfy Eqs. (3.3). Moreover, for the sake of clarity, we shall call "perturbed" SAYS a yield surface defined by Eqs. (3.3).

4. A general inequality

Let us suppose that a perturbed SAYS exists. Then considering the vector $\boldsymbol{\tau}$ as a time function, let the inequality $(3.3)_3$ be multiplied by the non-negative vector of plastic strain intensity coefficients, $\dot{\boldsymbol{\lambda}}$, relative to the real mechanical problem. We obtain

$$\hat{\boldsymbol{\sigma}}^T\mathbf{N}\dot{\boldsymbol{\lambda}} - \hat{\boldsymbol{\lambda}}\mathbf{H}\dot{\boldsymbol{\lambda}} - \mathbf{k}^T\dot{\boldsymbol{\lambda}} + \mathbf{k}^{*T}\dot{\boldsymbol{\lambda}} \leqslant 0, \quad \forall t \geqslant 0. \tag{4.1}$$

Then, let the equality $(2.7)_3$, which in explicit form reads

$$\boldsymbol{\sigma}^T\mathbf{N}\dot{\boldsymbol{\lambda}} - \boldsymbol{\lambda}^T\mathbf{H}\dot{\boldsymbol{\lambda}} - \dot{\boldsymbol{\lambda}}^T\mathbf{R}\dot{\boldsymbol{\lambda}} - \mathbf{k}^T\dot{\boldsymbol{\lambda}} = 0, \quad \forall t \geqslant 0, \tag{4.2}$$

be subtracted from the inequality (4.1). The result is, remembering Eqs. (2.8) and (2.6),

$$-(\boldsymbol{\sigma} - \hat{\boldsymbol{\sigma}})^T\dot{\boldsymbol{\epsilon}} + (\boldsymbol{\lambda} - \hat{\boldsymbol{\lambda}})^T\mathbf{H}\dot{\boldsymbol{\lambda}} + \dot{\boldsymbol{\lambda}}^T\mathbf{R}\dot{\boldsymbol{\lambda}} + \mathbf{k}^{*T}\dot{\boldsymbol{\lambda}} \leqslant 0, \tag{4.3}$$

whose first term can be put in the form

$$-(\boldsymbol{\sigma} - \hat{\boldsymbol{\sigma}})^T\dot{\boldsymbol{\epsilon}} = \dot{\mathbf{u}}^T\mathbf{V}\dot{\mathbf{u}} + \dot{\mathbf{u}}^T\mathbf{M}\ddot{\mathbf{u}} \tag{4.4}$$

in view of the virtual work principle. Then, substituting from Eq. (4.4) gives for Eq. (4.3)

$$\dot{L} \leqslant -\mathbf{k}^{*T}\dot{\boldsymbol{\lambda}}, \tag{4.5}$$

where L is the *psd* quadratic form

$$L = \frac{1}{2}(\boldsymbol{\lambda} - \hat{\boldsymbol{\lambda}})^T\mathbf{H}(\boldsymbol{\lambda} - \hat{\boldsymbol{\lambda}}) + \frac{1}{2}\dot{\mathbf{u}}^T\mathbf{M}\dot{\mathbf{u}}. \tag{4.6}$$

The inequality (4.5) is a general inequality, in the sense that it is valid for any time instant $t > 0$ and for arbitrary perturbances, $\mathbf{k}^*$, within the class of admissible ones.

5. The adaptation criterion

The following theorems can be shown to be true, always in the hypothesis that the work-hardening matrix $\mathbf{H}$ is *psd.*

THEOREM I. *A necessary and sufficient condition for dynamic adaptation to occur is that a statically admissible yield surface exists.*

Proof. The necessity is immediate. In fact, when adaptation occurs, the structure reaches some perfectly rigid state described by the vectors $\mathbf{u}_a$, $\boldsymbol{\lambda}_a$ so that we can take $\hat{\mathbf{u}} = \mathbf{u}_a$ and $\hat{\boldsymbol{\lambda}} = \boldsymbol{\lambda}_a$ and Eqs. (3.1) are certainly satisfied.

It is less simple to prove the sufficiency. It can be rigorously demonstrated only if the yield surface is a safe one, that is if the inequality $(3.1)_3$ is replaced by the inequality (3.2). However, the theorem could be made plausible by using a perturbation technique.

Thus, the hypothesis is that a safe SAYS exists, so that introducing the perturbances $\mathbf{k}^* = \alpha\mathbf{k}$, α being suitably chosen in the open interval (0,1), a perturbed SAYS can be generated and the inequality (4.5) holds. From this we deduce that the *psd* quadratic form L, being $\dot{L} \leqslant 0$ by Eq. (4.5), proves to be a non-increasing function of t. As a result, as $t \to \infty$, the velocity $\dot{\mathbf{u}}$ tends to vanish and the plastic coefficient vector $\boldsymbol{\lambda}$ tends to some time independent vector, $\boldsymbol{\lambda}_a$, which generally is different from $\hat{\boldsymbol{\lambda}}$. Moreover, the vector $\boldsymbol{\lambda}_a$ is finite, as we can deduce from Eq. (4.5) by integrating with respect to time. We have in fact

$$\mathbf{k}^T\boldsymbol{\lambda}(\infty) = \frac{1}{\alpha}[L(0)-L(\infty)] \leqslant \frac{1}{\alpha}L(0) \tag{5.1}$$

where, since $\boldsymbol{\lambda}(0) = \mathbf{0}$, it is

$$L(0) = \frac{1}{2}\hat{\boldsymbol{\lambda}}^T\mathbf{H}\hat{\boldsymbol{\lambda}} + \frac{1}{2}\dot{\mathbf{u}}_0^T\mathbf{M}\dot{\mathbf{u}}_0. \tag{5.2}$$

Since $\mathbf{k}$ is a positive vector and the right hand member of Eq. $(5.1)_2$ is finite, we conclude that $\boldsymbol{\lambda}(\infty)$ is finite too, and this is equivalent to saying that the structure adapts to the loadings. As a result, the energy dissipated, D, is also limited. It is in fact

$$\begin{aligned} D &= \int_0^\infty \boldsymbol{\sigma}^T\dot{\boldsymbol{\epsilon}}\,dt = \int_0^\infty [\varphi + \mathbf{H}\boldsymbol{\lambda} + \mathbf{R}\dot{\boldsymbol{\lambda}} + \mathbf{k}]^T\dot{\boldsymbol{\lambda}}\,dt \\ &= \frac{1}{2}\boldsymbol{\lambda}^T(\infty)\mathbf{H}\boldsymbol{\lambda}(\infty) + \mathbf{k}^T\boldsymbol{\lambda}(\infty) + \int_0^\infty \dot{\boldsymbol{\lambda}}^T\mathbf{R}\dot{\boldsymbol{\lambda}}\,dt, \end{aligned} \tag{5.3}$$

where the latter integral is limited because it satisfies the inequality

$$\int_0^\infty \dot{\boldsymbol{\lambda}}^T\mathbf{R}\dot{\boldsymbol{\lambda}}\,dt \leqslant L(0) \tag{5.4}$$

as we can deduce from Eq. (4.3). So the theorem already given in [4] for continuous bodies has been found.

Let us observe that:

a) If we put $\mathbf{M} = \mathbf{V} = \mathbf{0}$ and $\mathbf{R} = \mathbf{0}$, all the dynamic terms disappear from the problem and the (dynamic) Theorem I transforms into the statical theorem previously given in [3];

b) The conditions required for the validity of Theorem I are expressed in purely statical terms, as appears from Eqs. $(3.1)_{1-3}$, (3.2) and $(3.3)_{1-3}$.

These two remarks permit us to formulate the following second theorem:

THEOREM II. *Dynamic adaptation occurs if, and only if, statical adaptation occurs.*

Proof. If there is dynamic adaptation, then a statically admissible yield surface exists which assures also the statical adaptation, and *vice versa.* (Q.E.D.)

On the basis of Theorem II, there is no difference between the safety factor for a static load and the safety factor for a dynamic load, provided both loads range within the same domain. The evaluation of the safety factor can be made by solving a linear programming problem [3, 5, 6].

A kinematical theorem for inadaptation, that is a theorem analogous to that of KOITER [20] for elastic-perfecly plastic shakedown, could be given in the present context. However, always on the basis of Theorem II, it would prove to be like that given in [3].

On the basis of Theorem I and of the definitions $(3.1)_{1,2}$ and (3.2), the following theorem can also be proved.

THEOREM III. *In the case of kinematic work-hardening and zero initial velocities, if the structure has a safety factor for adaptation $s^* > 1$, a permanent load can always be found such that, applying this load all the time, the structure would adapt to any load $s < s^*$ remaining always perfectly rigid.*

Proof. In the case of kinematic work-hardening, we know [3, 6] that the addition of a permanent load does not produce any change in the value of the safety factor for adaptation and that $\mathbf{H}\hat{\boldsymbol{\lambda}} = \mathbf{N}^T\boldsymbol{\sigma}'$, the stress vector $\boldsymbol{\sigma}'$ being the displacement of the yield surface. By hypothesis, a safe SAYS exists for any load $s\mathbf{f}(\boldsymbol{\tau})$, provided $s < s^*$. Then, setting $\tilde{\boldsymbol{\sigma}} = \hat{\boldsymbol{\sigma}} - \boldsymbol{\sigma}'$, Eqs. $(3.1)_1$ and (3.2) become

$$\mathbf{C}^T\tilde{\boldsymbol{\sigma}} = s\mathbf{f}(\boldsymbol{\tau}) + \mathbf{f}', \qquad \mathbf{N}^T\boldsymbol{\sigma} - \mathbf{k} < \mathbf{0}, \quad \forall \boldsymbol{\tau} \in \Pi, \tag{5.5}$$

where the load

$$\mathbf{f}' = -\mathbf{C}^T\boldsymbol{\sigma}' \tag{5.6}$$

is the permanent load to add. Eqations (5.5) say that the "virgin" yield surface itself is a safe SAYS with respect to the loadings $s\mathbf{f}(\boldsymbol{\tau}) + \mathbf{f}'$. As a result, for any $s < s^*$, we can take $L(0) = 0$ in Eq. $(5.1)_2$ and this implies that $\boldsymbol{\lambda}(\infty) = \mathbf{0}$. The theorem so proved could have some value in engineering practice.

6. Bounds on plastic deformations

The inequality (4.5) can now be used to deduce bounds on some plastic deformation parameters. Integrating over the time interval $(0, t_1)$ and recalling that $\boldsymbol{\lambda}(0) = \mathbf{0}$ and $\mathbf{u}(0) = \mathbf{0}$, yields

$$\mathbf{k}^{*T}\boldsymbol{\lambda}(t_1) \leqslant L(0) - L(t_1) \leqslant L(0), \tag{6.1}$$

where the non-negative term $L(t_1)$ in the right-hand member of Eq. $(6.1)_1$ has been disregarded. Three kinds of deformation parameters can be bounded using Eq. $(6.1)_2$ and remembering the positions (3.4)–(3.7).

6.1. Bound on plastic strain intensities

Taking $\mathbf{k}^* = \bar{\mathbf{k}}\omega$, then the inequality $(6.1)_2$ becomes

$$\bar{\mathbf{k}}^T\boldsymbol{\lambda}(t_1) \leqslant \omega^{-1}\left(\frac{1}{2}\hat{\boldsymbol{\lambda}}^T\mathbf{H}\hat{\boldsymbol{\lambda}} + K_0\right), \tag{6.2}$$

where K_0 is the given initial kinetic energy, i.e.

$$K_0 = \frac{1}{2}\dot{\mathbf{u}}_0^T\mathbf{M}\dot{\mathbf{u}}_0. \tag{6.3}$$

Since $\bar{\mathbf{k}}$ can be any non-negative vector, the inequality (6.2) represents a bound on any linear non-negative combination of the plastic strain intensity components at any time $t_1 > 0$.

6.2. Bound on plastic strains

If we take $\mathbf{k}^* = \mathbf{N}^T\bar{\boldsymbol{\sigma}}\omega$, then we have the inequality

$$\bar{\boldsymbol{\sigma}}^T\boldsymbol{\epsilon}(t_1) \leqslant \omega^{-1}\left(\frac{1}{2}\hat{\boldsymbol{\lambda}}^T\mathbf{H}\hat{\boldsymbol{\lambda}} + K_0\right), \tag{6.4}$$

which is a bound on strain at time $t_1 > 0$.

6.3. Bound on displacements

Taking $\mathbf{k}^* = \mathbf{N}^T\mathbf{A}^T\bar{\mathbf{f}}\omega$, then the inequality $(6.1)_2$ yields

$$\bar{\mathbf{f}}^T\mathbf{u}(t_1) \leqslant \omega^{-1}\left(\frac{1}{2}\hat{\boldsymbol{\lambda}}^T\mathbf{H}\hat{\boldsymbol{\lambda}} + K_0\right), \tag{6.5}$$

which is a bound on the displacement vector $\mathbf{u}(t_1)$. This recalls the "dummy load" method by PONTER [20].

Since the bounding quantity in Eqs. (6.2), (6.4) and (6.5) contains parameters which have some degree of freedom, we may try to render the bounds most stringent. These parameters (the vectors $\hat{\boldsymbol{\sigma}}$ and $\hat{\boldsymbol{\lambda}}$, as well as the disturbance multiplier ω) must satisfy the conditions $(3.3)_{1-3}$.

Therefore, let us consider the following minimization problem:

$$(6.6)_1 \qquad \text{Minimize } \Phi = \frac{1}{\omega}\left(\frac{1}{2}\boldsymbol{\mu}^T\mathbf{H}\boldsymbol{\mu} + K_0\right) \quad \text{subject to} \quad \boldsymbol{\mu} \geqslant \mathbf{0}, \quad \omega > 0$$

and

$$(6.6)_{2,3} \qquad \begin{gathered} \mathbf{C}^T\varrho = \mathbf{f}(\boldsymbol{\tau}), \\ \mathbf{N}^T\boldsymbol{\rho} - \mathbf{H}\boldsymbol{\mu} - \mathbf{k} + \bar{\mathbf{k}}\omega \leqslant \mathbf{0}, \quad \forall \boldsymbol{\tau} \in \Pi. \end{gathered}$$

This problem is a convex one, with linear equality and inequality constraints. Its optimality conditions (see Appendix) are the following:

$$\mathbf{C}^T\boldsymbol{\rho} = \mathbf{f}(\boldsymbol{\tau}), \quad \forall \boldsymbol{\tau} \in \Pi \quad \text{(equilibrium);} \tag{6.7}$$

$$\begin{gathered} \boldsymbol{\psi} = \mathbf{N}^T\boldsymbol{\rho} - \mathbf{H}\boldsymbol{\mu} - \mathbf{k} + \bar{\mathbf{k}}\omega, \\ \boldsymbol{\psi} \leqslant \mathbf{0}, \quad \mathbf{l} \geqslant \mathbf{0}, \quad \boldsymbol{\psi}^T\mathbf{l} = 0, \quad \forall \boldsymbol{\tau} \in \Pi \quad \text{(conformity);} \end{gathered} \tag{6.8}$$

$$\mathbf{N}\mathbf{l} - \mathbf{C}\mathbf{a} = \mathbf{0}, \quad \forall \boldsymbol{\tau} \in \Pi \quad \text{(compatibility);} \tag{6.9}$$

$$\boldsymbol{\mu} = \int_{\Pi} \mathbf{l}\, d\Pi, \quad \mathbf{v} = \int_{\Pi} \mathbf{a}\, d\Pi \quad \text{(resultant mechanism);} \tag{6.10}$$

$$\omega\bar{\mathbf{k}}^T\boldsymbol{\mu} = \frac{1}{2}\boldsymbol{\mu}^T\mathbf{H}\boldsymbol{\mu} + K_0, \tag{6.11}$$

where the vectors $\mathbf{l}$ and $\mathbf{a}$ are Lagrangian variables having the meanings of plastic intensity coefficients and displacements, respectively.

The solution (if any) to the problem (6.6) must satisfy the above optimality conditions. It defines a perturbed SAYS determined by the vectors $\boldsymbol{\mu}$, $\mathbf{v}$, and $\bar{\mathbf{k}}\omega$. In fact, for every loading condition, a stress vector $\boldsymbol{\rho}$ exists which is in statical equilibrium with the given load and does not exceed the perturbed yield surface (see Eqs. (6.7) and $(6.8)_{1,2}$). Therefore, Eqs. (3.3) are satisfied if we take $\hat{\boldsymbol{\lambda}} = \boldsymbol{\mu}$, $\hat{\boldsymbol{\sigma}} = \boldsymbol{\rho}$, $\mathbf{k}^* = \bar{\mathbf{k}}\omega$.

In addition to this, we find that the set of Eqs. (6.7) to (6.10) for a given ω looks like a "finite" or "holonomic" description of the original adaptation problem suitably perturbed. Under every load — independently of the real time history and considering the load as applied upon a structure which has already been adapted — (congruent) plastic deformation is produced. The sum of these deformations, considering all the loading conditions (*resultant mechanism*, Eqs. (6.10)), constitutes the deformation associated with the adapted structure. Equation (6.11) is a consequence of the fact that the perturbance multiplier ω has been considered as an additive variable in the framework of the minimization problem. From Eq. (6.11) the optimal perturbance multiplier is deduced.

Comparing Eq. (6.11) with Eqs. (6.2), (6.4) and (6.5) enables us to write the bound inequalities in a more expressive form, i.e.

$$\begin{aligned} \bar{\mathbf{k}}^T\boldsymbol{\lambda}(t_1) &\leqslant \bar{\mathbf{k}}^T\boldsymbol{\mu}, \\ \bar{\boldsymbol{\sigma}}^T\boldsymbol{\epsilon}(t_1) &\leqslant \bar{\boldsymbol{\sigma}}^T\mathbf{q}, \\ \bar{\mathbf{f}}^T\mathbf{u}(t_1) &\leqslant \bar{\mathbf{f}}^T\mathbf{v}, \end{aligned} \tag{6.12}$$

where $\mathbf{q}$ is the strain vector associated with the resultant mechanism, i.e.

$$\mathbf{q} = \mathbf{N}\boldsymbol{\mu} = \mathbf{C}\mathbf{v}. \tag{6.13}$$

Equations (6.12) show that the "finite" solution of the "perturbed" adaptation problem can be used to bound directly the corresponding real quantities at any time $t_1 > 0$. More precisely, if we take all components of $\bar{\mathbf{k}}$ to be zero, except the j-th one which is equal to one, the inequality $(6.12)_1$ transforms into

$$\lambda_j(t_1) \leqslant \mu_j. \tag{6.14}$$

Furthermore, if only a single component of $\bar{\boldsymbol{\sigma}}$, say $\bar{\sigma}_h$, is different from zero, that is $\sigma_h = \pm 1$, then, instead of Eq. $(6.12)_2$, we can write

$$q_h^- \leqslant \varepsilon_h(t_1) \leqslant q_h^+, \tag{6.15}$$

where q_h^- comes from the finite solution for $\bar{\sigma}_h = -1$ while q_h^+ comes from that for $\bar{\sigma}_h = +1$. Finally, if all the components of $\bar{\mathbf{f}}$ are zero except the j-th one, and $\bar{f}_j = \pm 1$, then

$$v_j^- \leqslant u_j(t_1) \leqslant v_j^+, \tag{6.16}$$

where v_j^+ and v_j^- are finite solutions relative to $\bar{f}_j = +1$ and $\bar{f}_j = -1$, respectively.

7. Variational formulation of the adaptation problem

Since the set of Eqs. (6.7) to (6.10) can be viewed as the governing equations of an adaptation problem in finite plasticity, the minimization problem (6.6) can be considered as the corresponding variational formulation. In other words, the problem (6.6) for a given ω is a minimum principle similar to that of Haar-Kármán for elastic-plastic structures [21, 22]. The objective function to minimize is the sum of the initial kinetic energy and of the energy associated with the adaptation strains in connection with their ability to produce hardening. Equilibrium and plasticity (or conformity) conditions are the constraints to impose for every loading condition. If the multiplier ω is included within the variables of the problem, the solution (if any) constitutes the most stringent bound at the selected element of the structure.

The following two statements can be phrased:

THEOREME IV. *The structure adapts to the loadings if and only if the domain of admissible solutions to the unperturbed (i.e.,* $\bar{\mathbf{k}} = \mathbf{0}$, $\omega = 1$) *minimization problem* (6.6) *has at least one interior point.*

THEOREME V. *The admissible solutious to the minimum principle* (6.6), *which characterizes the holonomic adaptation problem suitably perturbed, furnish a class of bounds on deformation parameters at a given point of the structure, while the most stringent bound is given by the true solution to the same minimum principle.*

Following what has been said previously the two above statements are self-evident.

8. Example

Let us consider (Fig. 2) a simple bar whose yield stress resultant is denoted by k. An axial load $f = k\tau$, $(0 \leqslant \tau \leqslant s)$, is applied in traction at the free node where a mass M is

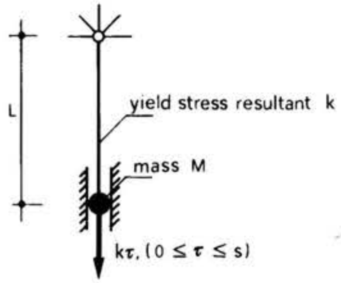

FIG. 2. Rigid-plastic bar subjected to a variable load.

located. The initial velocity of the mass is zero. We assume a linear kinematic hardening law and denote the hardening coefficient by $c > 0$.

If $s \leqslant 1$, the system remains rigid for any load (it is already adapted) while when $s > 1$ it deforms. However, as long as $s < 2$, the system is able to adapt to the load, which means that the bar becomes rigid some time after the beginning. Finally, for $s \geqslant 2$ there is no adaptation.

Referring to the case $1 \leqslant s < 2$, we want to find bounds on some deformation parameters and compare them with the real values produced by a stepwise load history (Fig. 3). A load jump occurs, by hypothesis, when the system is already at rest under the action of the previous load. This is expected to produce the maximum deformation effects.

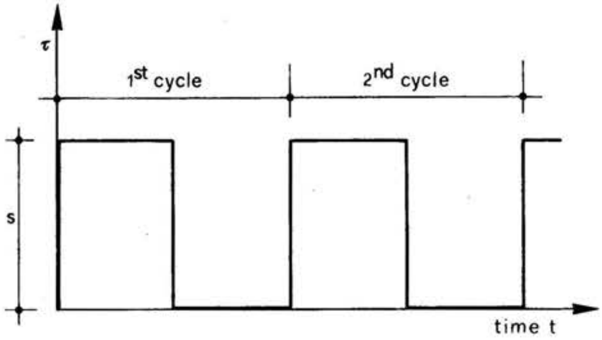

FIG. 3. Stepwise load history.

For each load cycle there is a deformation cycle (Fig. 4) in which the bar first yields in traction under the load sk, then yields in compression under zero load. The mass oscillates around the final rest position which is reached at, say, the n-th cycle. During the first half of the r-th cycle ($r = 1, 2, \ldots, n$) the mass moves from the rest position u_{r-1}^-

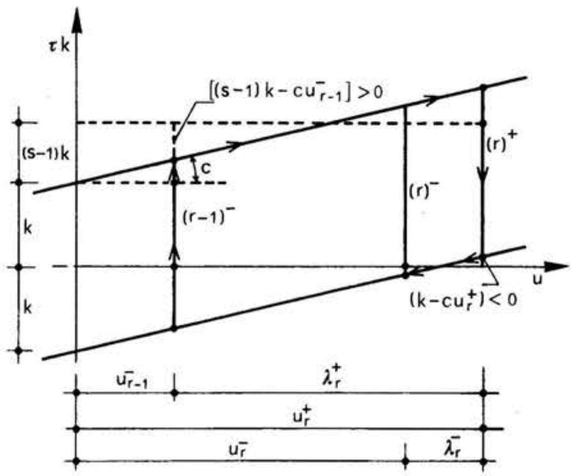

FIG. 4. Typical deformation cycle.

(produced during the compression phase of the previous cycle, $u_0^- = 0$) to a further position u_r^+, causing the bar to suffer a plastic strain intensity in traction, λ_r^+. During this phase it is

$$sk - M\ddot{u} - cu - k = 0, \tag{8.1}$$

where u indicates the current position of the mass. Multiplying Eq. (8.1) by $\dot{u}$ and then integrating with respect to t along the entire step furnishes an algebraic equation of second degree whose solution, $u = u_r^+$, is

$$u_r^+ = \frac{2k}{c}(s-1) - u_{r-1}^- = \frac{2k}{c}\left(s-1-\frac{c}{k}u_{r-1}^-\right) + u_{r-1}^-. \tag{8.2}$$

Thus

(8.3) $$\lambda_r^+ = u_r^+ - u_{r-1}^- = \frac{2k}{c}(s-1) - 2u_{r-1}^-.$$

If

(8.4) $$k - cu_r^+ = k(3-2s) + cu_{r-1}^- > 0,$$

the system is already adapted; otherwise, further deformation is produced during the second half of the r-th cycle. In this case, in fact, the mass moves from the position u_r^+ to a nearer position u_r^- causing the bar to suffer a plastic deformation intensity in compression, λ_r^-. During this phase it must be

(8.5) $$M\ddot{u} + cu - k = 0,$$

and again multiplying by $\dot{u}$ and then integrating with respect to t along the complete phase gives us a second degree equation whose solution, $u = u_r^-$, is

(8.6) $$u_r^- = \frac{2k}{c} - u_r^+ = \frac{2k}{c}(2-s)r$$

and the associated plastic strain intensity proves to be

(8.7) $$\begin{aligned} \lambda_r^- &= u_r^+ - u_r^- = \frac{2k}{c}(2s-3) - 2u_{r-1}^- \\ &= \frac{2k}{c}(2rs+1-4r). \end{aligned}$$

No further deformation is produced during the subsequent loading cycle if

(8.8) $$sk \leqslant k + cu_r^-,$$

otherwise, a new deformation cycle is to be considered.

It is easily shown that $u_r^+ > u_{r+1}^+$ and that $u_{r-1}^- < u_r^-$. Thus the maximum displacement of the mass is given by

(8.9) $$u_{\max} = u_1^+ = \frac{2k}{c}(s-1).$$

The maximum plastic strain intensity in traction is

(8.10) $$\lambda^+ = \lambda_1^+ + \lambda_2^+ + \ldots + \lambda_n^+ = \frac{2k}{c}n(ns+1-2n),$$

where the multiplier s must satisfy the condition (8.8) for $r = n$, i.e.

(8.11) $$s \leqslant \frac{4n+1}{2n+1}, \quad (n = 1, 2, 3, \ldots),$$

while the analogous quantity in compression is

(8.12) $$\lambda^- = \lambda_1^- + \lambda_2^- + \ldots + \lambda_{n-1}^- = \frac{2k}{c}(n-1)(ns+1-2n),$$

where s must satisfy the inequality (8.4) for $r = n$, i.e.

(8.13) $$s \leqslant \frac{4n-1}{2n}, \quad (n = 1, 2, 3, \ldots).$$

In Figs. 5 and 6 the dimensionless parameters $c\lambda^+/2k$ and $c\lambda^-/2k$ are plotted as functions of s.

Now we want to apply the present method in order to determine upper bounds on λ^+ and λ^-.

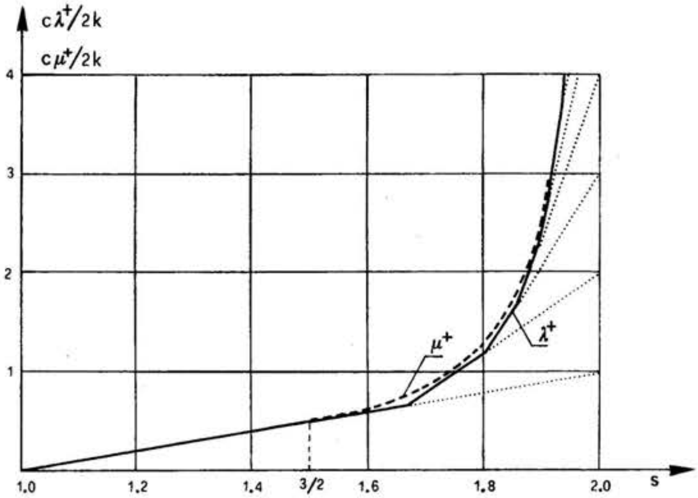

FIG. 5. Maximum plastic strain intensity in traction (λ^+) and upper bound (μ^+).

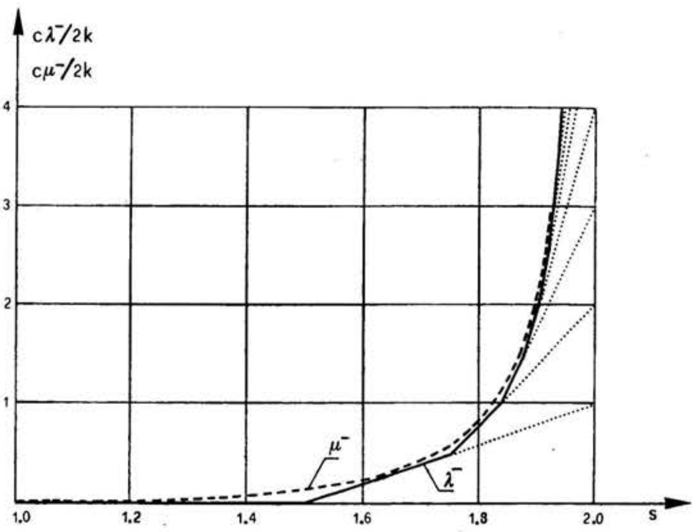

FIG. 6. Maximum plastic strain intensity in compression (λ^-) and upper bound (μ^-).

The bound on λ^+ is obtained by diminishing the positive yield stress of the bar by an amount ωk, $\omega > 0$ being unknown. We have to solve a problem of finite plasticity with quasi-static loads. The yield conditions are

$$(8.14)\qquad \begin{aligned} &\varphi^+ = \tau k - c(\mu^+ - \mu^-) - k + \omega k \leqslant 0, \quad \mu^+ \geqslant 0, \quad \varphi^+\mu^+ = 0,\\ &\varphi^- = -\tau k + c(\mu^+ - \mu^-) - k \leqslant 0, \quad \mu^- \geqslant 0, \quad \varphi^-\mu^- = 0. \end{aligned}$$

It must obviously be $\psi^+ = 0$ for $\tau = s$ and $\psi^- \leqslant 0$ for $\tau = 0$. Thus

$$\omega = \frac{c}{k}(\mu^+ - \mu^-) - (s-1), \tag{8.15}$$

$$\mu^+ - \mu^- \leqslant \frac{k}{c}. \tag{8.16}$$

The optimality condition (6.11) gives

$$\frac{1}{2} c(\mu^+ - \mu^-)^2 = \omega k \mu^+ \tag{8.17}$$

and substituting from Eq. (8.15)

$$\frac{1}{2}\frac{c}{k}(\mu^+ - \mu^-)^2 = \left[\frac{c}{k}(\mu^+ - \mu^-) - (s-1)\right]\mu^+. \tag{8.18}$$

If $\mu^- = 0$, the latter equation furnishes

$$\mu^+ = \frac{2k}{c}(s-1), \tag{8.19}$$

which is valid for $s \leqslant \frac{3}{2}$, as we can deduce by verifyng the inequality (8.16). If $\mu^- > 0$, then $\psi^- = 0$ and the inequality (8.16) must be verified as an equality, i.e.

$$\mu^+ - \mu^- = \frac{k}{c}. \tag{8.20}$$

Substituting this result in Eqs. (8.18) gives

$$\mu^+ = \frac{2k}{c}\frac{1}{4(2-s)}. \tag{8.21}$$

The curve $\mu^+ = \mu^+(s)$ is plotted in Fig. 5 and we see that μ^+ proves to be a good upper bound for the parameter λ^+ $\left(\text{they coincide for } s \leqslant \frac{3}{2}\right)$.

An analogous procedure is to be adopted for obtaining the upper bound on λ^-, but this time we diminish the negative yield stress of the bar. Thus the yield conditions now read

$$\begin{aligned} \psi^+ &= \tau k - c(\mu^+ - \mu^-) - k \leqslant 0, \quad \mu^+ \geqslant 0, \quad \psi^+\mu^+ = 0, \\ \psi^- &= -\tau k + c(\mu^+ - \mu^-) - k + \omega k \leqslant 0, \quad \mu^- \geqslant 0, \quad \psi^-\mu^- = 0. \end{aligned} \tag{8.22}$$

Again $\psi^+ = 0$ under the load $\tau = s$, and $\psi^- \leqslant 0$ under the load $\tau = 0$. Thus we deduce

$$\mu^+ - \mu^- = \frac{k}{c}(s-1), \tag{8.23}$$

$$\omega \leqslant 2 - s. \tag{8.24}$$

The optimality condition (6.11) is now written as

$$\frac{1}{2}c(\mu^+ - \mu^-)^2 = \omega k \mu^-, \tag{8.25}$$

which shows that μ^- must be different from zero, and hence $\psi^- = 0$ and Eq. (8.24) holds as an equality. Thus Eq. (8.25) becomes

$$\frac{1}{2}\frac{k}{c}(s-1)^2 = (2-s)\mu^- \tag{8.26}$$

from which

$$\mu^- = \frac{2k}{c}\frac{(s-1)^2}{4(2-s)}. \tag{8.27}$$

In Fig. 6 this parameter is plotted as a function of s. We see again that μ^- proves to be a good upper bound for the parameter λ^-.

Finally, we observe that if the bar were permanently prestressed by the force $-sk/2$ (compression), it would remain rigid under any loading history, provided $s < 2$.

9. Conclusion

In the present paper the dynamic work-hardening adaptation problem has been studied in the case of infinitesimal displacements, taking into account viscous forces and strain rate sensitivity. A perturbation method, recently given by the author [15, 16], is used for the treatment of adaptation criteria together with bounds on plastic strains and displacements. The concept of "statically admissible yield surface" (SAYS) proves to be crucial for the given adaptation criterion. It is shown that dynamic adaptation implies static adaptation and *vice versa*, as previously found for continuous bodies by the author [4]. The extreme simplicity of this result perhaps warns of the caution with which the rigid-plastic model is to be handled in dynamics, however, it indicates also the existance in the structure of a strength reservoir and, in the author's opinion, it may be valuable to take it into account.

Bounds on plastic strain, on plastic strain intensity and on displacement at any point of the adapted structure are formulated. These bounds can be made the most stringent by solving a holonomic adaptation problem suitably perturbed, and the holonomic solution can be used as the bounding quantity at the selected point.

The effectiveness of the proposed bound techniques has not yet been sufficiently assessed. However, the simple numerical application worked out permits us to expect the present method to furnish bounds which seem to be good, also under a load approaching the maximum value for adaptation.

A variational formulation of the holonomic adaptation problem, similar to Haar-Kármán's principle, is shown to play a role in characterizing the adaptability of the structure, as well as the bound optimality. In the author's opinion, however, this is a point to be clarified better in the future.

Appendix

The optimality conditions (6.7)–(6.11) are deduced in the present Appendix. To this purpose, let us introduce for every loading condition the vectors

$$\mathbf{w}^T = \left[\frac{1}{2} z_1^2 \;\; \frac{1}{2} z_2^2 \;\; \ldots\right], \quad \mathbf{z}^T = [z_1 \;\; z_2 \;\; \ldots] \tag{A.1}$$

both having as many components as there are plastic potentials. Considering $\mathbf{w}$ as a vector of slack variables, the inequality $(6.6)_3$ transforms into an equality, i.e.

$$\mathbf{N}^T\boldsymbol{\rho} - \mathbf{H}\boldsymbol{\mu} - \mathbf{k} + \bar{\mathbf{k}}\omega + \mathbf{w} = \mathbf{0}, \quad \forall \boldsymbol{\tau} \in \Pi. \tag{A.2}$$

Then, let us consider the following Lagrangian functional:

$$\Phi_L = \frac{1}{\omega}\left[\frac{1}{2}\boldsymbol{\mu}^T\mathbf{H}\boldsymbol{\mu} + K_0\right] + \int_\Pi \bar{\mathbf{a}}^T[\bar{\mathbf{f}}(\boldsymbol{\tau}) - \mathbf{C}^T\boldsymbol{\rho}]d\Pi + \int \bar{\mathbf{l}}^T[\mathbf{N}^T\boldsymbol{\rho} - \mathbf{H}\boldsymbol{\mu} + \mathbf{k} + \bar{\mathbf{k}}\omega +]d\Pi, \tag{A.3}$$

where $\bar{\mathbf{a}}$ and $\bar{\mathbf{l}}$ are Lagrangian vector variables. The first variation is

$$\delta\Phi_L = \delta\boldsymbol{\mu}^T\left[\frac{1}{\omega}\mathbf{H}\boldsymbol{\mu} - \mathbf{H}\int_\Pi \bar{\mathbf{l}}d\Pi\right] + \int_\Pi \delta\boldsymbol{\rho}^T[-\mathbf{C}\bar{\mathbf{a}} + \mathbf{N}\bar{\mathbf{l}}]d\Pi + \int_\Pi \bar{\mathbf{l}}^T\mathbf{z}\delta\mathbf{z}d\Pi + \delta\omega\left[-\frac{1}{\omega^2}\left(\frac{1}{2}\boldsymbol{\mu}^T\mathbf{H}\boldsymbol{\mu} + K_0\right)\right.$$
$$\left. + \bar{\mathbf{k}}^T\int_\Pi \bar{\mathbf{l}}d\Pi\right] + \int_\Pi \delta\bar{\mathbf{a}}^T[\bar{\mathbf{f}}(\boldsymbol{\tau}) - \mathbf{C}^T\boldsymbol{\rho}]d\Pi + \int_\Pi \delta\bar{\mathbf{l}}^T[\mathbf{N}^T\boldsymbol{\rho} - \mathbf{H}\boldsymbol{\mu} - \mathbf{k} + \bar{\mathbf{k}}\omega + \mathbf{w}]d\Pi. \tag{A.4}$$

Then, letting

$$\mathbf{a} = \omega\bar{\mathbf{a}}, \quad \mathbf{l} = \omega\bar{\mathbf{l}}, \tag{A.5}$$

the following equations are deduced from Eq. (A.4):

$$\mathbf{C}^T\boldsymbol{\rho} = \mathbf{f}(\boldsymbol{\tau}), \quad \forall\boldsymbol{\tau} \in \Pi, \tag{A.6}$$

$$\mathbf{N}^T\boldsymbol{\rho} - \mathbf{H}\boldsymbol{\mu} - \mathbf{k} + \mathbf{k}\omega + \mathbf{w} = 0, \quad \mathbf{l}^T\mathbf{z} = 0, \quad \forall\tau \in \Pi, \tag{A.7}$$

$$\mathbf{N}\mathbf{l} - \mathbf{C}\mathbf{a} = \mathbf{0}, \quad \forall\boldsymbol{\tau} \in \Pi, \tag{A.8}$$

$$\mathbf{H}\boldsymbol{\mu} - \mathbf{H}\mathbf{l}_R \geqslant \mathbf{0}, \quad \boldsymbol{\mu} \geqslant 0, \quad \boldsymbol{\mu}^T\mathbf{H}\boldsymbol{\mu} - \boldsymbol{\mu}^T\mathbf{H}\mathbf{l}_R = 0, \tag{A.9}$$

$$\omega\bar{\mathbf{k}}^T\mathbf{l}_R = \frac{1}{2}\boldsymbol{\mu}^T\mathbf{H}\boldsymbol{\mu} + K_0, \tag{A.10}$$

$$\mathbf{l}_R = \int_\Pi \mathbf{l}\, d\Pi, \quad \mathbf{a}_R = \int_\Pi \mathbf{a}\, d\Pi. \tag{A.11}$$

It is easy to show that the inequality $(A.9)_1$ applies always as an equality. In fact, multiplying it by $\mathbf{l}_R \geqslant \mathbf{0}$ we have

$$\mathbf{l}_R^T\mathbf{H}\mathbf{l}_R - \mathbf{l}_R^T\mathbf{H}\boldsymbol{\mu} \leqslant 0. \tag{4.12}$$

Then, summing the latter with the equality $(A.8)_3$ yields

$$(\boldsymbol{\mu}-\mathbf{l}_R)^T\mathbf{H}(\boldsymbol{\mu}-\mathbf{l}_R) \leqslant 0 \tag{A.13}$$

and hence, **H** being *psd*,

$$(\boldsymbol{\mu}-\mathbf{l}_R)^T\mathbf{H}(\boldsymbol{\mu}-\mathbf{l}_R) = 0 \tag{A.14}$$

which is equivalent to

$$\mathbf{H}(\boldsymbol{\mu}-\mathbf{l}_R) = \mathbf{0}. \tag{A.15}$$

If **H** is *pd*, then it must be $\boldsymbol{\mu} = \mathbf{l}_R$; while if **H** is strictly *psd*, it is in general $\boldsymbol{\mu} \neq \mathbf{l}_R$. However, in connection with the problem we are studying, we can consider them as equivalent because $\mathbf{H}\boldsymbol{\mu}$ and $\mathbf{H}\mathbf{l}_R$ are interchangeable. Thus, by eliminating the vector **z** from Eqs. (A.6) to (A.11), the optimality conditions take the form given in Eq. (6.7) to Eq. (6.11).

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