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Rayleigh-Taylor instability of a Maxwell's fluid

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THE CHARACTER of equilibrium of an incompressible Maxwell's fluid of variable density has been investigated. Two density configurations have been considered: (i) two superposed fluids of different uniform densities, (ii) a fluid layer having exponentially varying density. In the latter case, an approximate solution has been obtained using the variational principle which characterizes the solution. For unstable arrangement it is shown that the stress relaxation time leads to an increase in the rate at which the arrangement departs from equilibrium. It is further demonstrated that it is possible to have periodic motion in a Maxwell's fluid for some disturbances for which it is not possible to excite waves in a Newtonian fluid.

Rozważa się równowagę nieściśliwej cieczy Maxwella o zmiennej gęstości. Rozpatrywane są dwie szczególne konfiguracje gęstości: (i) superpozycja dwu cieczy o różnych, jednorodnych gęstościach, (ii) warstwa cieczy o wykładniczo zmiennej gęstości. Dla tego ostatniego przypadku otrzymano rozwiązanie przybliżone, wykorzystując odpowiednią zasadę wariacyjną. Dla układu niestatecznego wykazano, że czas relaksacji prowadzi do wzrostu szybkości oddalania się od stanu równowagi. Pokazano też, że w cieczy Maxwella możliwy jest ruch okresowy dla pewnych typów zaburzeń, dla których nie można wywołać fal w cieczy Newtona.

Рассматривается равновесие несжимаемой максвелловской жидкости с переменной плотностью. Рассмотрены две частные конфигурации плотности: 1) суперпозиция двух жидкостей с разными, однородными плотностями, 2) слой жидкости с экспоненциально переменной плотностью. Для этого последнего случая получено приближенное решение, используя соответствующий вариационный принцип. Для неустойчивой системы показано, что время релаксации приводит к росту скорости удаления от состояния равновесия. Показано тоже, что в максвелловской жидкости возможно периодическое движение для некоторых типов возмущений, для которых нельзя вызвать волн в ньютоновской жидкости.

1. Introduction

THE CHARACTER of equilibrium of an incompressible, inviscid fluid of variable density, stratified in the vertical direction was investigated by RAYLEIGH (1883) who derived a result of general validity, namely, that the stratification is stable or unstable as its density decreases everywhere or increases anywhere in the upward direction. Further, he obtained explicit solutions for two density configurations: (i) one fluid of uniform density topped by another fluid of different uniform density, (ii) a fluid having exponentially varying density confined between two horizontal planes.

Taking viscosity of the fluid into account, HARRISON (1908) considered the stability of two superposed fluids and he obtained the dispersion relation in which the growth rate of the disturbance was expressed as a power series in the coefficient of kinematic viscosity. CHANDRASEKHAR (1955) further carried Harrison's work to give a complete treatment of the problem. In addition, he demonstrated that the solution was characterized by a variational principle.

Making use of the variational principle, HIDE (1955) obtained the approximate solutions for the two aforementioned density configurations. His results for two superposed fluids were in fair agreement with the corresponding exact results derived by Chandrasekhar. REID (1962), however, pointed out that in computing the approximate solution. Hide had left out an important term and the closeness he obtained of two solutions was fortuitous. This cast doubts on the usefulness of the variational principle. Nevertheless, SELIG (1964) derived the variational principle due to Chandrasekhar in a manner which was free of any ambiguity.

Although considerable attention has been paid recently to the instabilities of a non-Newtonian fluid, it appears that not enough attention has been paid to the Rayleigh-Taylor instability of these fluids. It is the aim of the present paper to deal with this problem. We have investigated the character of equilibrium of a Maxwell's fluid. Both density configurations, first studied by Rayleigh, have been considered.

2. Formulation of the problem

Consider a Maxwell's fluid of density ϱ, depending on the vertical coordinate z stratified in the vertical direction. For a Maxwell's fluid the constitutive equation is

$$\left(1+\lambda\frac{d}{dt}\right)\tau_{ij}=2\eta e_{ij}, \tag{2.1}$$

where λ is the stress relaxation time, η is the viscosity of the medium, τ_{ij} is the deviatoric stress tensor and e_{ij} is the rate of the strain tensor given by

$$e_{ij}=\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right). \tag{2.2}$$

In Eq. (2.2) u_i denotes the velocity at a point.

The basic equations of motion are:

The equation of conservation of momentum

$$\varrho\frac{du_i}{dt}=-\frac{\partial p}{\partial x_i}+\frac{\partial}{\partial x_j}\tau_{ij}-g\varrho e_i. \tag{2.3}$$

The equation of incompressibility

$$\frac{\partial \varrho}{\partial t}+u_j\frac{\partial \varrho}{\partial x_j}=0. \tag{2.4}$$

The equation of continuity

$$\frac{\partial u_j}{\partial x_j}=0, \tag{2.5}$$

where p denotes the scalar pressure, g is the acceleration due to gravity and e_i $(=0, 0, 1)$ is a unit vector in the vertically upward direction.

The equilibrium state is characterized by $u_i=0$.

To investigate the character of equilibrium, we give the system a small disturbance, which produces a velocity field u_i $(= u, v, w)$. Let the corresponding perturbations in ϱ, η and p be $\delta\varrho$, $\delta\eta$ and δp, respectively.

The linearized equations of perturbation are

$$\varrho \frac{\partial u_i}{\partial t} = -\frac{\partial}{\partial x_i}\delta p + \frac{\partial}{\partial x_j}\tau_{ij} - g\delta\varrho e_i, \tag{2.6}$$

$$\frac{\partial}{\partial t}\delta\varrho = -w\frac{d\varrho}{dz}, \tag{2.7}$$

and

$$\frac{\partial u_j}{\partial x_j} = 0. \tag{2.8}$$

Analysing the disturbance in normal modes, we seek the solutions of Eqs. (2.6)–(2.8), in which perturbed quantities have the form

$$(\text{some function of } z)\times\exp(ik_x x + ik_y y + nt), \tag{2.9}$$

where k_x and k_y are the horizontal components of the wave vector k_i, and n denotes the rate at which the system departs from equilibrium.

The z-component of the curl of Eq. (2.6) can now be written (on making use of Eqs. (2.1), (2.2), (2.7) and (2.8)) as

$$n[k^2\varrho w - D(\varrho D w)] - \frac{gk^2}{n}(D\varrho)w + \mu(D^2-k^2)^2 w + 2D\mu(D^2-k^2)Dw + D^2\mu(D^2+k^2)w = 0, \tag{2.10}$$

where D stands for d/dz, and

$$\mu = \frac{\eta}{1+\lambda n}; \tag{2.11}$$

μ can be termed as the modified coefficient of viscosity which takes into account the effects of the non-Newtonian parameter λ.

3. Boundary conditions

On a rigid boundary a fluid can not slip. Further, following RAYLEIGH (1883), if we disregard the phenomenon of surface waves, we can take the vertical component of velocity zero at a free surface, thus

$$w = 0 \quad \text{on a boundary rigid or free.} \tag{3.1}$$

Further, on a rigid boundary, in view of the equation of continuity

$$Dw = 0 \quad \text{on a rigid boundary.} \tag{3.2}$$

On a free boundary the tangential stresses τ_{xz} and τ_{yz} must vanish. Now

$$\tau_{xz} = \mu(Du + ik_x w) \quad \text{and} \quad \tau_{yz} = \mu(Dv + ik_y w)$$

therefore,

$$-ik_x\tau_{xz}-ik_y\tau_{yz} = \mu(D^2+k^2)w. \tag{3.3}$$

Since w has been assumed to vanish on a free boundary, it follows that

$$D^2w = 0 \quad \text{on a free boundary.} \tag{3.4}$$

Should there be discontinuities, as in the present problem, we must require the continuity of velocity, tangential stresses and pressure at an interface. This amounts to

$$w, \quad Dw, \quad \mu(D^2+k^2)w \tag{3.5}$$

are continuous across a surface of density discontinuity and the last boundary condition, namely, the continuity of pressure across the interface can be made to satisfy, if we integrate Eq. (2.10) across the interface. This gives

$$n\Delta_s(\varrho Dw)+\frac{gk^2}{n}\Delta_s(\varrho)w_s-\Delta_s[\mu(D^2-2k^2)Dw] = 0, \tag{3.6}$$

where Δ_s denotes the jump a quantity experiences in crossing the surface of discontinuity $z = z_s$, and w_s is the common normal component of velocity there.

It may be remarked here that the eigenvalue problem defined by Eqs. (2.10)–(3.6) is exactly similar to the one considered by Chandasekhar, if μ is interpreted according to Eq. (2.11). We shall, therefore, deal with the next section very briefly.

4. A variational principle

Multiplying Eq. (2.10) by w and integrating across the vertical extent of the fluid (denoted by L), we obtain the following variational formulation after a series of integrations by parts:

$$n\int_L \varrho[(Dw)^2+k^2w^2]dz-\frac{gk^2}{n}\int_L D\varrho w^2dz+\int_L \mu\{[(D^2+k^2)w]^2+4k^2(Dw)^2\}dz = 0 \tag{4.1}$$

the integrated parts vanishing because of the appropriate boundary conditions.

Consider a small change δw in w compatible with the boundary conditions. The corresponding increment δn in n can be found from Eq. (4.1). We have to the first order of smallness.

$$\begin{aligned} -\frac{1}{2}\delta n\Bigg\{\int_L \varrho[(Dw)^2+k^2w^2]dz+\frac{gk^2}{n^2}\int_L D\varrho w^2dz \\ +\int_L \frac{\partial\mu}{\partial n}\big([(D^2+k^2)w]^2+4k^2(Dw)^2\big)dz\Bigg\} = \int_L \Bigg\{n[k^2\varrho w-D(\varrho Dw)] \\ -\frac{gk^2}{n}(D\varrho)w+\mu(D^2-k^2)^2w+2D\mu(D^2-k^2)Dw+D^2\mu(D^2+k^2)w\Bigg\}\delta w\,dz. \end{aligned} \tag{4.2}$$

We observe that a necessary and sufficient condition for δn to be zero to the first order of smallness for a small variation in w compatible with the boundary conditions is that w satisfies the characteristic value problem. Thus the present problem is characterized by a variational principle.

5. The case of two superposed fluids of constant densities separated by a horizontal boundary — an exact solution

In this section we consider the stability of a Maxwell's fluid of density ϱ_1 occupying the region $z < 0$, topped by another Maxwell's fluid of density ϱ_2 occupying the region $z > 0$. The two fluids are separated by a horizontal interface $z = 0$. Let the coefficients of viscosity and the relaxation times for the lower and upper fluids be η_1, λ_1 and η_2, λ_2, respectively.

Following CHANDRASEKHAR (1955) and keeping in mind the new meaning of μ as defined by Eq. (2.11), the following dispersion relation is derived (cf. Chandrasekhar, Eq. (44)):

$$(5.1)\qquad -\left[\frac{gk}{n^2}(\alpha_1-\alpha_2)+1\right](\alpha_2 q_1+\alpha_1 q_2-k)-4k\alpha_1\alpha_2+\frac{4k^2}{n}(\alpha_1\nu_1-\alpha_2\nu_2)[\alpha_2 q_1-\alpha_1 q_2+k(\alpha_1-\alpha_2)]+\frac{4k^3}{n^2}(\alpha_1\nu_1-\alpha_2\nu_2)^2(q_1-k)(q_2-k)=0,$$

where

$$(5.2)\qquad \alpha_1=\frac{\varrho_1}{\varrho_1+\varrho_2},\qquad \alpha_2=\frac{\varrho_2}{\varrho_1+\varrho_2},$$
$$\nu_1=\frac{\eta_1}{\varrho_1(1+\lambda_1 n)},\qquad \nu_2=\frac{\eta_2}{\varrho_2(1+\lambda_2 n)},$$
$$q_1^2=k^2+\frac{\eta}{\nu_1},\qquad q_2^2=k^2+\frac{\eta}{\nu_2}$$

and for boundedness of the solution we require

$$(5.3)\qquad \mathrm{Re}(q_1)>0 \quad\text{and}\quad \mathrm{Re}(q_2)>0.$$

We define

$$(5.4)\qquad \nu=\frac{\eta_1+\eta_2}{\varrho_1+\varrho_2}$$

and measure n, k and q in terms of $(g^2/\nu)^{1/3}$ sec^{-1}, $(g/\nu^2)^{1/3}$ cm^{-1}, respectively. In dimensionless form Eq. (5.1) becomes

$$(5.5)\qquad -\left[\frac{(\alpha_1-\alpha_2)}{n^2}+1\right](\alpha_2 q_1+\alpha_1 q_2-k)-4k\alpha_1\alpha_2+\frac{4k^2}{n}\left[\frac{\delta_1}{1+T_1 n}-\frac{\delta_2}{1+T_2 n}\right][\alpha_2(q_1-k)-\alpha_1(q_1-k)]+\frac{4k^3}{n^2}\left[\frac{\delta_1}{1+T_1 n}-\frac{\delta_2}{1+T_2 n}\right]^2(q_1-k)(q_2-k)=0,$$

where

$$(5.6)\qquad \delta_1=\frac{\eta_1}{\eta_1+\eta_2},\qquad \delta_2=\frac{\eta_2}{\eta_1+\eta_2},$$
$$T_1=\lambda_1\left(\frac{g^2}{\nu}\right)^{1/3},\qquad T_2=\lambda_2\left(\frac{g^2}{\nu}\right)^{1/3}$$

and q_1 and q_2 are now given by

(5.7) $$q_1^2 = k^2 + \frac{n\alpha_1}{\delta_1}(1+T_1 n), \quad q_2^2 = k^2 + \frac{n\alpha_2}{\delta_2}(1+T_2 n).$$

For a full discussion of Eq. (5.5) it must be squared repeatedly; which will result into an equation of the 26th degree in n. This equation will contain some trivial roots and some extraneous roots which will violate the requirement (5.3). The roots left after rejecting these spurious roots will be termed as admissible roots.

If the lighter fluid lies beneath the heavier fluid, i.e. if $\alpha_1 < \alpha_2$, it will be found that Eq. (5.5) has only one admissible root with a positive real part, thereby implying instability. Indeed this root is positive and its asymptotic behaviour for small and large values of k is

(5.8) $$n^2 \to k(\alpha_2 - \alpha_1), \qquad k \to 0,$$

and

$$n \to (\alpha_2 - \alpha_1)/2k, \qquad k \to \infty.$$

It is apparent that n does not depend upon the non-Newtonian parameters either for $k \to 0$ or for $k \to \infty$. Further, we note that n tends to zero for extreme values of k, hence a mode of maximum instability must exist which is expected to assert itself in the initial course of the motion. In order to study the effect of the non-Newtonian parameters

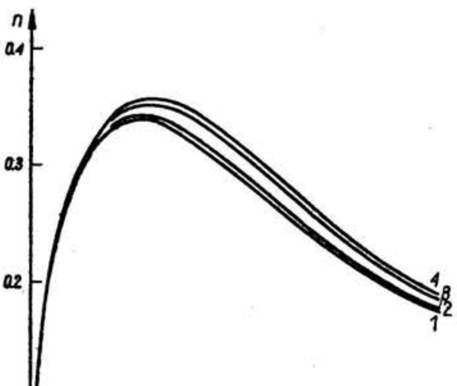

FIG. 1. Illustrating the influence of non-parameters in the unstable case when the heavier fluid lies atop the higher fluid. The growth rate n is plotted against k for $\alpha_1 = 0.2$, $\delta_1 = 0.2$, and various values of T_1 and T_2. For curves *1*: $T_1 = 0.5$, $T_2 = 0.5$; *2*: $T_1 = 1.0$, $T_2 = 0.5$; *3*: $T_1 = 0.5$, $T_2 = 1.0$; *4*: $T_1 = 1.0$, $T_2 = 1.0$.

T_1 and T_2 on the mode of maximum instability, the positive root of Eq. (5.5) was computed on CDC 6000 at the University of Calgary, Canada, for different values of T_1 and T_2. The results are depicted in Fig. 1 in which n has been plotted against k. It can be seen that an increase in the value of either T_1 or T_2 leads to an increase in the value of n, the difference being most pronounced at the maxima of the curve, i.e. for the mode of maximum instability. Hence we conclude that a non-Newtonian unstable arrangement departs at a faster rate from equilibrium as compared to a similar Newtonian arrangement.

When the lighter fluid lies atop the heavier fluid, Eq. (5.5) does not have any admissible root with a positive real part, which is to be expected, and we have the stability of the system. The discussion of every admissible root for all parameters is somewhat cumbersome, therefore, we have restricted ourselves to the investigation of motion of a single Maxwell's fluid occupying the lower z-plane.

6. Gravity waves in a Maxwell's fluid

Taking the limit of Eq. (5.1) as $\varrho_2 \to 0$, at the same time assuming that the coefficient of kinematic viscosity of the upper fluid remains finite and non-zero, we obtain the following dispersion relation in the dimensionless form for gravity waves:

$$\begin{aligned} T^3n^7+3T^2n^6+Tn^5(3+2kT^2+8k^2T)+n^4(1+6kT^2+16k^2T) \\ +kn^3(6T+8k+kT^3+8k^2T^2+24k^3T)+k^2n^2(2+3kT^2+16k^2T+24k^3) \\ +k^2n(3T+8k+8k^3T+16k^4)+k^2(1+8k^3) = 0, \end{aligned} \tag{6.1}$$

where

$$T = \lambda\left(\frac{g^2}{\nu}\right)^{1/3}. \tag{6.2}$$

The assumption that the coefficient of kinematic viscosity of the upper fluid remains finite and non-zero when we take the limit is justified on the ground that Eq. (6.1) passes to the corresponding relation obtained by Chandrasekhar for the case $T = 0$.

When $T = 0$, i.e. for a Newtonian fluid, Chandrasekhar has demonstrated that Eq. (6.1) has only two admissible roots. He has further shown that there exists a critical value of k (say k_*) such that for $k < k_*$, these two modes correspond to periodic motion and for $k > k_*$, they correspond to aperiodic motion. In the latter case Chandrasekhar named the two modes as "viscous mode" and "creeping mode", the "viscous" mode decays rapidly and the "creeping mode" decays slowly.

Now we make allowance for non-vanishing values of T. In this case it was found that Eq. (6.1) has three admissible roots. The asymptotic behaviour of these three roots is given by

$$\begin{aligned} n_1 &= -\frac{1}{T}+0.9126k^2, \\ n_2, n_3 &= -(2k^2-2^{\frac{1}{2}}k^{\frac{11}{4}}) \pm i(k^{\frac{1}{2}}+2k^{\frac{5}{2}}T-2^{\frac{1}{2}}k^{\frac{11}{4}}), \quad k \to 0 \end{aligned} \tag{6.3}$$

and

$$\begin{aligned} n_1 &= -\frac{1}{2k}, \\ n_{2,3} &= -\frac{1}{2T} \pm i \times 0.9553\frac{k}{\sqrt{T}}, \quad k \to \infty. \end{aligned} \tag{6.4}$$

From the above it follows that periodic motion takes place for both short wave-lengths and large wave-lengths. In constradistinction, for short wave-lengths there is an aperiodic motion for a Newtonian fluid. Hence we conclude that one effect of non-Newtonian parameters is to excite waves for disturbances corresponding to short wave-lengths. It may be further noted that for $k \to \infty$, we no longer have a "viscous" mode. Instead, we have what we can term a "Maxwell's" mode which is responsible for oscillatory motion for $k \to \infty$. This mode, however, gives rise to the aperiodic motion for $k \to 0$, the oscillatory motion being caused by the "creeping" mode.

Since we have oscillatory motion for both $k \to 0$ and $k \to \infty$, it would be of interest to know the nature of the motion for intermediate values of k. Keeping this in mind, n was computed from Eq. (6.1) for increasing values of T; the results are presentsd in Fig. 2. It appears that there exists a critical value of T (say T^*) such that for $T < T^*$

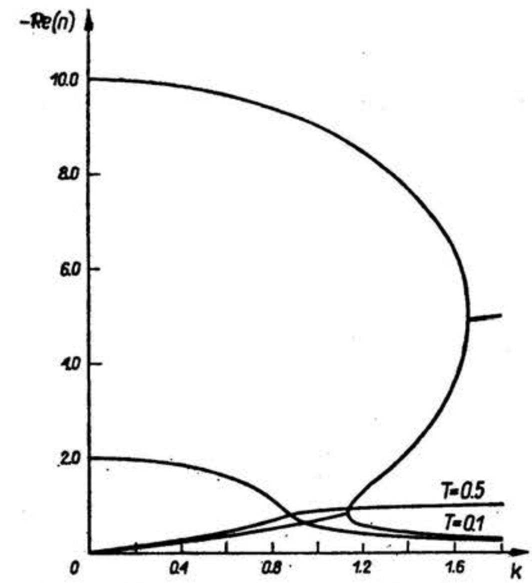

FIG. 2. Illustrating the variation of $-\mathrm{Re}(n)$ the rate of damping of gravity waves with k, the wave-number for a Maxwell's fluid for different values of t, the stress relaxation time.

we do not have oscillatory motion in the intermediate range of values of k; however for $T > T^*$, waves always arise, therefore we can generalize our earlier observation and state that for the value of relaxation time exceeding some critical value, it is always possible to excite waves on a Maxwell's fluid for any disturbance.

7. A continuously stratified fluid of finite depth — an approximate solution

Consider a layer of a Maxwell's fluid confined between the planes $z = 0$ and $z = d$, for which ϱ and η have the following dependence on z in the undisturbed state

$$\varrho = \varrho_0 \exp \beta z \tag{7.1}$$

and

(7.2) $$\eta = \nu\varrho_0 \exp\beta z,$$

where β, ν and ϱ_0 are appropriate constants.

We assume the following trial function for $w(z)$:

(7.3) $$w(z) = W\sin lz.$$

Vanishing of w at the upper boundary requires that

(7.4) $$l = \frac{\pi s}{d},$$

where s is an integer.

We shall further make the assumption

(7.5) $$|\beta d| \ll 1$$

which implies that the density variation in the fluid is a good deal lower than the average density of the fluid.

Substituting for ϱ, μ and $w(z)$ in Eq. (4.1) and evaluating the integrals, we obtain the following eigen-value relation between n and k:

(7.6) $$n^2(k^2+l^2)+\frac{\nu n}{1+\lambda n}(k^2+l^2)^2-g\beta k^2 = 0.$$

Equation (7.6) reduces to the corresponding dispersion relation derived by HIDE (1955) for a Newtonian fluid ($\lambda = 0$).

It will be found convenient to deal with Eq. (7.6) in a non-dimensional form. Measuring n and k in terms of $(\pi^2 s^2 \nu/d^2)\,\mathrm{sec}^{-1}$ and $(\pi s/d)\,\mathrm{cm}^{-1}$, Eq. (7.6) takes the following dimensionless form:

(7.7) $$\tau n^3+n^2+n\left(k^2+1-\frac{G\tau k^2}{1+k^2}\right)-\frac{Gk^2}{1+k^2} = 0,$$

where

(7.8) $$G = \frac{g\beta d^4}{\pi^4 s^4 \nu^2},$$

and

(7.9) $$\tau = \frac{\lambda\nu\pi^2 s^2}{d^2}.$$

Here G has the form of the Grashoff number and τ can be regarded as the non-dimensional relaxation time.

7.1. Unstable stratification

If G is positive, i.e. if the density increases in the upward direction, Eq. (7.7) admits one positive real root thereby implying instability; the other two roots being either real and negative or complex conjugate with negative real parts. The behaviour of the positive root for small and large k is

(7.10)
$$n \to Gk^2 \quad \text{for} \quad k \to 0,$$
$$n \to G/k^2 \quad \text{for} \quad k \to \infty.$$

So it can be seen, as in the case of two superposed fluids, that the disturbances corresponding to large or short wave lengths remain unaffected by the non-Newtonian parameter τ. To see the effects of τ on the mode of maximum instability in Fig. 3, n has been

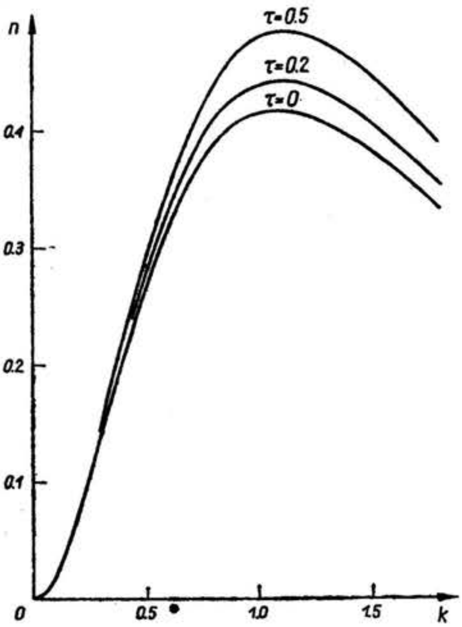

FIG. 3. Illustrating the influence of the non-Newtonian parameter in the unstable case of continuously stratified fluid. The growth rate n is plotted against the wave-number k for $G = 2.0$ and several values of τ.

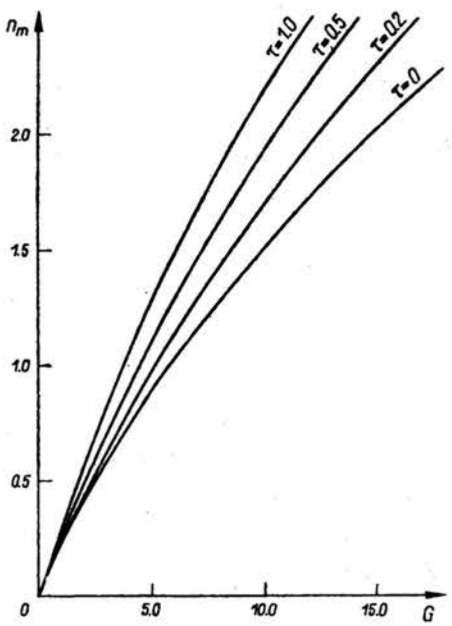

FIG. 4. Illustrating the behavior of n_m. The maximum growth rate against G, the non-dimensional measure of buoyancy forces for different values of τ, the dimensionless stress relaxation time.

plotted against k for $G = 2.0$ and various values of τ. We observe that an increase in τ leads to an increase in the value of n and n_m, the maximum growth rate. Thus, what we stated in the case of two superposed fluids is still valid, namely, that a non-Newtonian stratification departs from the steady state at a faster rate as compared to a Newtonian stratification.

To obtain n_m, we differentiate Eq. (7.7) with respect to k and set $dn/dk = 0$. The equation governing n_m is

(7.11)
$$(1+\tau n_m)(n_m^2-G)^2-4n_mG = 0.$$

The behaviour of n_m against G for different values of τ has been exhibited in Fig. 4.

7.2. Stable stratification

If G is negative, i.e. if the density decreases in the upward direction, Eq. (7.8) admits either three negative real roots or one negative real root and a pair of complex conjugate roots with negative real parts. In any case, the stability of the system is assured. To study the manner in which the equilibrium is restored we proceed as follows.

The cubic

$$n^3+bn^2+cn+d=0$$

with real coefficients has three real roots or one real and a pair of complex roots according to whether

$$X \equiv \frac{4}{27}\left(\frac{b^2}{3}-c\right)^3-\left(d-\frac{bc}{3}+\frac{2b^3}{27}\right)^2 \tag{7.12}$$

is positive or negative.

On substituting for the coefficients from Eq. (7.7), we find that

$$X=-\frac{4}{27\tau^4}\left\{\frac{\tau^4 k G_1^3}{(1+k^2)^3}+\frac{\tau^2 k^4 G_1^2}{(1+k^2)^2}[2+3\tau(1+k^2)]+\frac{k^2 G_1}{1+k^2}[1-5\tau(1+k^2)+6\tau^2(1+k^2)^2]+(1+k^2)^2\left[\tau(1+k^2)-\frac{1}{4}\right]\right\}, \tag{7.13}$$

where

$$G_1=-G. \tag{7.14}$$

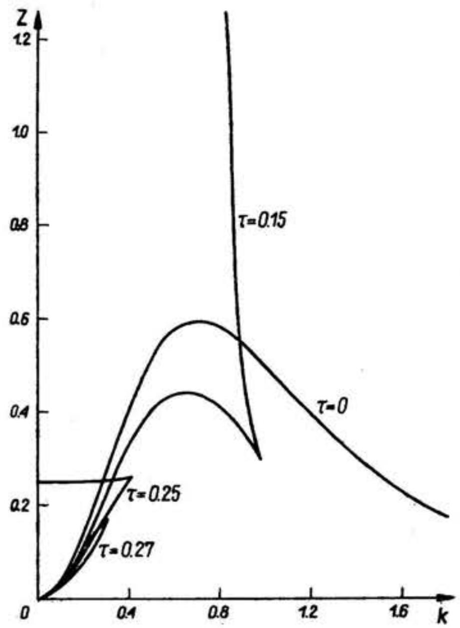

FIG. 5. Illustrating the plot of z against k, the wave number for several values of τ, the dimensionless stress relaxation time.

X can be regarded as a cubic polynomial in G_1. Let any of its zeros be $1/z$. In Fig. 5 z is plotted against k for various values of τ. The curve divides the first quadrant in the $k-z$ plane into two regions one of which is bounded if τ is greater than 1/4. If $\tau < 1/4$, periodic motion is possible in the region lying below the curve, and when $\tau > 1/4$, periodic motion takes place outside the closed region. It can be further seen that if $\tau = \tau_c$ ($= 0.29616$), X is negative for all values of k, hence we can conclude that for the values of relaxation time exceeding some critical value, waves can always be excited in a Maxwell's fluid.

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