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BRIEF NOTES

Corollaries of Ericksen's theorems on the deformations possible in every isotropic hyperelastic body

G. P. PARRY (BATH)

ERICKSEN's theorems on the universal deformations of isotropic hyperelastic materials are extended. It is shown that a deformation need be self-equilibrated only in a well-defined class of materials in order that it be universal.

1. Introduction

IN [1] AND [2], ERICKSEN has analysed the universal deformations of compressible and incompressible isotropic hyperelastic materials. Ericksen's analyses are here used to demonstrate that one need suppose a deformation is self-equilibrated only in a certain limited class of isotropic materials in order that it be self-equilibrated in every isotropic material. However, many of the well-known classes of ideal hyperelastic materials do not constitute such a class, so that non-universal deformations may be possible in such materials. The work applies to both compressible and incompressible materials, but the two cases have to be treated separately.

2. Compressible materials

Let A denote the matrix of deformation gradients, and let A' denote the transpose of A. Let

$$B = AA', \tag{2.1}$$

and let I_j, $(j = 1, 2, 3)$ denote the principal invariants of B^{-1}. Assume that a strain energy function, w say, exists, then, by objectivity and isotropy,

$$w = w(I_j). \tag{2.2}$$

Following ERICKSEN [1], the equations of equilibrium in the absence of body force may be written as

$$\frac{\partial w}{\partial I_k}\left(\sqrt{I_3}\,\frac{\partial I_k}{\partial (B^{-1})^r_s}(B^{-1})^l_s\right)^{,r} + \frac{\partial^2 w}{\partial I_k \partial I_l}\left(\sqrt{I_3}\, I_l^{,r}\,\frac{\partial I_k}{\partial (B^{-1})^r_s}(B^{-1})^l_s\right) = 0. \tag{2.3}$$

If the coefficient of each derivative of w vanishes, then these equations are satisfied for every strain energy function w, and vice versa. Let p denote the 9-tuple consisting of the independent components of $\left(\frac{\partial w}{\partial I_j}, \frac{\partial^2 w}{\partial I_k \partial I_l}\right)$, and let $\mathbf{p}^*$ denote the 12-tuple consisting of all the components of the same derivatives. Likewise, let $\mathbf{q}^i$ and $\mathbf{q}^{*i}$ denote the 9-tuple (12-tuple) consisting of the independent components (all the components) of the coefficients of the derivatives of w in Eq. (2.3). Thus

$$q^i = (a^i_j, b^i_{jk}),$$

where

$$a^i_j = \left(\sqrt{I_3}\ \frac{\partial I_j}{\partial (B^{-1})^r_s}(B^{-1})^i_s\right)^{,r},$$

and

$$b^i_{jk} = \left(I_j{}^{,r}\frac{\partial I_k}{\partial (B^{-1})^r_s} + I_k{}^{,k}\frac{\partial I_j}{\partial (B^{-1})^r_s}\right)(B^{-1})^i_s,$$

supposing, without loss of generality, that

$$b^i_{jk} = b^i_{kj}.$$

According to ERICKSEN [1], if $\mathbf{q}^i = 0$, then the deformation is homogeneous.

The equations of equilibrium have the form

$$\mathbf{p}^{*\prime}\mathbf{q}^{*i} = 0. \tag{2.4}$$

A necessary and sufficient condition that Eq. (2.4) implies $\mathbf{q}^{*i} = 0$ is that one may choose nine linearly independent 9-tuples $\mathbf{p}$. This amounts to supposing that the deformation is possible in a *class* of materials wide enough to allow this choice.

Suppose that the strain energy function is given in terms of different invariants, J_k (say), and that the transformation $I_k \to J_k$ is non-singular. Then it is easy to show that the transformation between the 12-tuples $\left(\frac{\partial w}{\partial I_j}, \frac{\partial w}{\partial I_j \partial I_k}\right)$ and $\left(\frac{\partial w}{\partial J_k}, \frac{\partial^2 w}{\partial J_k \partial J_l}\right)$ is non-singular (having determinant $\left(\left(\frac{\partial J_j}{\partial J_k}\right)^7\right)$, and therefore so is the transformation between the independent components of the same derivatives non-singular (since the obvious symmetry is preserved).

Therefore, any deformation possible in a class of isotropic, hyperelastic materials allowing nine independent choices of the 9-tuple $\left(\frac{\partial w}{\partial J_k}, \frac{\partial^2 w}{\partial J_k \partial J_l}\right)$ is universal.

Detailed inspection of Ericksen's analysis reveals that of the nine equations $\mathbf{q}^i = 0$ only three are employed in proving that the deformation is homogeneous. These three equations are

$$a^i_1 = 0, \quad a^i_3 = 0, \quad b^i_{13} = 0. \tag{2.5}$$

Suppose that the strain energy does not depend on I_2, then clearly one may conclude that any deformation possible in such a class of isotropic, hyperelastic materials, allowing five independent choices of the 5-tuple $\left(\frac{\partial w}{\partial I_j}, \frac{\partial^2 w}{\partial I_j \partial I_k}\right)$, $(i, k = 1, 3)$, is universal.

In illustration, note that the "two-term" isotropic strain energy functions of OGDEN [3], which are of the form

$$w = \mu J_1 + \nu J_2 + f(J_3), \tag{2.6}$$

where μ, ν are constants and f is some known function, may *allow* non-universal deformations.

3. Incompressible materials

Here there are two independent invariants, I_j, $(j = 1, 2)$. Following ERICKSEN [2], the condition of equilibrium is that

$$A_i = \left(\frac{\partial w}{\partial I_2} B_i^j\right)_{,j} - \left(\frac{\partial w}{\partial I_1} (B^{-1})_i^j\right)_{,j}, \tag{3.1}$$

be the gradient of a scalar function, for which it is necessary and sufficient that

$$A_{i,j} = A_{j,i}. \tag{3.2}$$

Let $r(r^*)$ be the 9-tuple (14-tuple) consisting of the independent components (all the components) of

$$\left(\frac{\partial w}{\partial I_j}, \frac{\partial^2 w}{\partial I_j \partial I_k}, \frac{\partial^3 w}{\partial I_j \partial I_k \partial I_l}\right).$$

Then Eq. (3.2) may be written in the form

$$\mathbf{r}^{*\prime}\mathbf{s}_{ij}^* = \mathbf{r}^{*\prime}\mathbf{s}_{ji}^*, \tag{3.3}$$

where each component of $\mathbf{s}_{ij}^*$ is a known function of B and its space derivatives. According to Ericksen [2], the nine conditions

$$\mathbf{s}_{ij}^* = \mathbf{s}_{ji}^*, \tag{3.4}$$

are necessary and sufficient for the deformation to be universal. But one may deduce Eq. (3.4) provided only that the deformation is supposed to be self-equilibrated in a class of material which allows nine linearly independent choices of $\mathbf{r}$. Therefore, any deformation possible in such a class of materials must be universal.

Suppose that the strain energy is given as a function of different invariants J_k, $k = 1, 2$, and suppose that the transformation $I_j \to I_k$ is non-singular. Then the transformation between the 14-tuples

$$\left(\frac{\partial w}{\partial I_k}, \frac{\partial^2 w}{\partial I_k \partial I_l}, \frac{\partial^3 w}{\partial I_k \partial I_l \partial I_m}\right) \quad \text{and} \quad \left(\frac{\partial w}{\partial J_k}, \frac{\partial^2 w}{\partial J_k \partial J_l}, \frac{\partial^3 w}{\partial J_k \partial J_l \partial J_m}\right)$$

is also non-singular, with determinant $\left(\frac{\partial J_j}{\partial J_k}\right)^{17}$, and, therefore, so is the transformation between the corresponding 9-tuples.

Hence, any deformation possible in a class of isotropic, hyperelastic incompressible materials allowing nine independent choices of the 9-tuple $\left(\frac{\partial w}{\partial J_k}, \frac{\partial^2 w}{\partial J_k \partial J_l}, \frac{\partial^3 w}{\partial J_k \partial I_l \partial J_m}\right)$ is universal.

Examples of classes of materials which do *not* allow this choice are the neo-Hookean, Mooney-Rivlin and "two-term" solids of Ogden (as in (2.6) with $f = 0$). That non-universal deformations are possible in Mooney-Rivlin solids is evidenced by the construction of "composite" controlled deformations by HILL and PETROSKI [4].

In passing, notice that if a different isotropic material, with strain energy $\overline{w}$, is defined by

$$\overline{w}(A) = w(A^{-1\prime}),$$

or, equivalently, by

$$\overline{w}(I_1, I_2) = w(I_2, I_1),$$

then, from Eq. (3.1)

$$A_i = -\overline{A}_i,$$

in the obvious notation. It follows that if A is self-equilibrated in the material with strain energy w (say "in w"), then $A^{-1\prime}$ is self-equilibrated "in $\overline{w}$". Noting a result of SHIELD [5] which claims that if A is self-equilibrated "in w", then A^{-1} is self-equilibrated "in $\overline{w}$", it follows that if A is self-equilibrated "in w", so is A^1.

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SCHOOL OF MATHEMATICS
UNIVERSITY OF BATH, CLAVERTON DOWN, BATH, GREAT BRITAIN.

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