Archives of Mechanics • Archiwum Mechaniki Stosowanej • 31, 4, pp. 505-518, Warszawa 1979

The effect of shearing prestress on the response of a thick membrane strip
Part II. The dynamic case

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INVESTIGATION of the effect of shearing prestress on the response of a thick membrane (or thin) plate strip is hereby completed by treating the steady state response under dynamic loading. The singular shear layer discovered for the static case reappears here at exactly the same location. Singular perturbation analysis yields the solution in the core and edge layers although the algebraic computations involved are cumbersome.

Badanie efektu wstępnego naprężenia ścinającego na rozwiązanie pasma traktowanego jako gruba membrana lub cienka płyta, przedstawione w cz. I, uzupełnione jest rozważaniami dotyczącymi obciążenia dynamicznego. Osobliwe warstwy ścinania wykryte w problemie statycznym pojawiają się znowu w tych samych miejscach. Stosując technikę osobliwych perturbacji uzyskuje się rozwiązanie w rdzeniu oraz warstwie brzegowej, chociaż obliczenia algebraiczne stają się bardzo złożone.

Исследование эффекта предварительного напряжения сдвига на решение полосы, трактованной как толстая мембрана или тонкая плита, представленное в I части, дополнено рассуждениями касающимися динамической нагрузки. Особые слои сдвига обнаруженные в статической задаче появляются вновь в этих же самых местах. Применяя технику особых пертурбаций получается решение в сердечнике и в граничном слое, хотя алгебраические расчеты становятся очень сложными.

Introduction

THIS PAPER is a continuation of an earlier work (henceforth called Part I, [1]), under the same title. While attention was restricted to the static problem in [1], here we deal with the dynamic response of a thick membrane strip with small bending rigidity. Because the bending rigidity is taken to be small, its influence could effectively be neglected globally except close to the boundaries and also possibly in the neighbourhood of the applied external load.

In fact we do not intend to render an account of the history of the problem here; neither shall we repeat the outline of the equations governing the deflection of such thick membranes. These questions were adequately treated in Part I where the solution technique to be used was also amply described. We therefore merely assume that the reader is familiar with [1] as far as these issues go, and thereby focus attention on solving the differential equation

$$\varepsilon^2\nabla^4\chi - L\chi = p(x, y, \tau), \tag{A}$$

where

$$L \equiv \beta_x^2 \frac{\partial^2}{\partial x^2} + 2\beta_{xy}^2 \frac{\partial^2}{\partial x \partial y} + \beta_y^2 \frac{\partial^2}{\partial y^2}$$

under the assumption that L is elliptic and in the domain $[-\infty < x < \infty; 0 \leqslant y \leqslant 1]$. We further assume that $\varepsilon > 0$ is a small number thereby allowing us to employ the method of matched asymptotic expansions to solve the above differential equation.

In an attemp to make reference to [1] easier, we commence this paper with Sect. 3. Sections 1 and 2 are contained in Part I and here Eq. (1.11), for example, is to be interpreted as Eq. (1.11) of Sect. 1 (in Part I). Equations starting with (3.1) are therefore to be found in this article.

3. The dynamic response — forced vibrations

In Part I we dealt with the static response of thick membranes. Of equal interest is the response to dynamic loadings. In particular, our interest will be restricted to the steady state response, which turns out in general to be computationally cumbersome. For this reason we shall defer looking at the transient loading phenomena.

The governing differential equation is again Eq. (1.1) (or Eq. (A)), but q must now include the inertia forces $-m\partial^2 w/\partial t^2$ where m denotes the mass of the membrane per unit area. The resulting differential equation can be cast in dimensionless form by introducing the transformations (1.6). Denoting dimensionless time by τ, we set $\tau = \omega t$, where ω is frequency which is chosen such that

$$\frac{N_y}{mb^2\omega^2} = 1.$$

With the abbreviations (1.7) in which we may without loss of generality replace N_0 by $N_y \neq 0$, the governing differential equation can be shown to have the form

$$\varepsilon^2 \nabla^4 \chi + \frac{\partial^2 \chi}{\partial \tau^2} - L\chi = p(x, y, \tau), \tag{3.1a}$$

where x, y are dimensionless coordinates and L is defined by

$$L \equiv \beta_x^2 \frac{\partial^2}{\partial x^2} + 2\beta_{xy}^2 \frac{\partial^2}{\partial x \partial y} + \frac{\partial^2}{\partial y^2}. \tag{3.1b}$$

For steady state forcing functions

$$p(x, y, \tau) = \sum_{j=1}^{\infty} p_j(x, y) \exp(i\lambda_j \tau), \tag{3.2}$$

χ possesses solutions of the form

$$\chi(x, y, \tau) = \sum_{j=1}^{\infty} \phi_j(x, y) \exp(i\lambda_j \tau) \tag{3.3}$$

where each ϕ_j (we henceforth delete the subscript j) satisfies the differential equation

$$\varepsilon^2\nabla^4\phi - L\phi - \lambda^2\phi = p(x, y). \tag{3.4}$$

A solution, uniformly valid in the core region I and the boundary layers II and III but possibly not in the shear layers, is constructed by writing

$$\phi(x, y;\varepsilon) = \psi(x, y;\varepsilon) + \overset{1}{\Xi}(x, y; \varepsilon)e^{-y/\varepsilon} + \overset{2}{\Xi}(x, y; \varepsilon)e^{-(1-y)/\varepsilon}. \tag{3.5}$$

Substituting this representation into Eq. (3.4) results in the following system of differential equations for ψ, $\overset{1}{\Xi}$ and $\overset{2}{\Xi}$:

$$\beta_x^2\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + 2\beta_{xy}^2\frac{\partial^2\psi}{\partial x\partial y} + \lambda^2\psi = \varepsilon^2\nabla^4\psi - p(x, y), \tag{3.6}$$

$$2\left[\frac{\partial\overset{1}{\Xi}}{\partial y} - \beta_{xy}^2\frac{\partial\overset{1}{\Xi}}{\partial x}\right] = \varepsilon^3\nabla^4\overset{1}{\Xi} - 4\varepsilon^2\left[\frac{\partial^3\overset{1}{\Xi}}{\partial y^3} + \frac{\partial^3\overset{1}{\Xi}}{\partial x^2\partial y}\right] + \varepsilon\left[5\frac{\partial^2\overset{1}{\Xi}}{\partial y^2} + (2-\beta_x^2)\frac{\partial^2\overset{1}{\Xi}}{\partial x^2} - 2\beta_{xy}^2\frac{\partial^2\overset{1}{\Xi}}{\partial x\partial y} - \lambda^2\overset{1}{\Xi}\right], \tag{3.7}$$

$$2\left[\frac{\partial\overset{2}{\Xi}}{\partial y} - \beta_{xy}^2\frac{\partial\overset{2}{\Xi}}{\partial x}\right] = -\varepsilon^3\nabla^4\overset{2}{\Xi} - 4\varepsilon^2\left[\frac{\partial^3\overset{2}{\Xi}}{\partial y^3} + \frac{\partial^3\overset{2}{\Xi}}{\partial x^2\partial y}\right] - \varepsilon\left[5\frac{\partial^2\overset{2}{\Xi}}{\partial y^2} + (2-\beta_x^2)\frac{\partial^2\overset{2}{\Xi}}{\partial x^2} - 2\beta_{xy}^2\frac{\partial^2\overset{2}{\Xi}}{\partial x\partial y} - \lambda^2\Xi\right]. \tag{3.8}$$

We construct solutions for ψ and $\overset{\alpha}{\Xi}$ in terms of the asymptotic series

$$\begin{aligned}\psi(x, y; \varepsilon) &= \sum_{\nu=0}^{\infty}\varepsilon^\nu\psi_\nu(x, y),\\ \overset{\alpha}{\Xi}(x, y; \varepsilon) &= \sum_{\nu=0}^{\infty}\varepsilon^\nu\overset{\alpha}{\Xi}_\nu(x, y).\end{aligned} \tag{3.9}$$

Incorporating these expansions into Eqs. (3.6)–(3.8), the following recurrence relations emerge:

$$\beta_x^2\frac{\partial^2\psi_\nu}{\partial x^2} + \frac{\partial^2\psi_\nu}{\partial y^2} + 2\beta_{xy}^2\frac{\partial^2\psi_\nu}{\partial x\partial y} + \lambda^2\psi_\nu = \begin{cases} p(x, y), & \nu = 0,\\ 0, & \nu = 1,\\ \nabla^4\psi_{\nu-2} & \nu \geqslant 2\end{cases} \tag{3.10}$$

and

$$\nabla^4\overset{\alpha}{\Xi}_{\nu-3} = (-1)^{\alpha+1}\left\{4\left[\frac{\partial^3\overset{\alpha}{\Xi}_{\nu-2}}{\partial y^3} + \frac{\partial^3\overset{\alpha}{\Xi}_{\nu-2}}{\partial x^2\partial y}\right] + 2\frac{\partial\overset{\alpha}{\Xi}_\nu}{\partial y} - 2\beta_{xy}^2\frac{\partial\overset{\alpha}{\Xi}_\nu}{\partial x}\right\} - \left\{5\frac{\partial^2\overset{\alpha}{\Xi}_{\nu-1}}{\partial y^2} + (2-\beta_x^2)\frac{\partial^2\overset{\alpha}{\Xi}_{\nu-1}}{\partial x^2} - 2\beta_{xy}^2\frac{\partial^2\overset{\alpha}{\Xi}_{\nu-1}}{\partial x\partial y} - \lambda^2\overset{\alpha}{\Xi}_{\nu-1}\right\}, \quad \nu = 0, 1, 2, \ldots. \tag{3.11}$$

These differential equations must be complemented by appropriate boundary conditions. We only consider clamped edges here for which we may write

$$\phi(x,0) = \phi(x,1) = \frac{\partial\phi}{\partial y}(x,0) = \frac{\partial\phi}{\partial y}(x,1) = 0.$$

If Eq. (3.5) and the expansions (3.9) are used, we then find

$$\begin{aligned} &\psi_\nu(x,0)+\overset{1}{\Xi}_\nu(x,0) = 0,\\ &\psi_\nu(x,1)+\overset{2}{\Xi}_\nu(x,1) = 0,\\ &\frac{\partial\psi_{\nu-1}}{\partial y}(x,0)+\frac{\partial\overset{1}{\Xi}_{\nu-1}}{\partial y}(x,0)-\overset{1}{\Xi}_\nu(x,0) = 0,\\ &\frac{\partial\psi_{\nu-1}}{\partial y}(x,1)+\frac{\partial\overset{2}{\Xi}_{\nu-1}}{\partial y}(x,1)+\overset{2}{\Xi}_\nu(x,1) = 0. \end{aligned} \tag{3.12}$$

Equations (3.10) through (3.12) describe the chain of boundary value problems for the determination of ϕ which is one of the coefficient functions of the harmonic expansion of χ in Eq. (3.3).

3.1. Solution for $\overset{\alpha}{\Xi}_0\,(x,y)$: $(\alpha = 1,2)$

The lowest order boundary layer solutions are obtained from Eqs. (3.11) and $(3.12)_{3,4}$ if ν is set equal to zero, whereby functions with a negative index vanish. Thus we have

$$\frac{\partial\overset{\alpha}{\Xi}_0}{\partial y}-\beta_{xy}^2\frac{\partial\overset{\alpha}{\Xi}_0}{\partial x} = 0, \tag{3.13}$$

$\overset{\alpha}{\Xi}_0 = 0$, on the boundary $y = 0,\ 1$, respectively.

All C^1-functions of the form $\overset{\alpha}{f}(x+\beta_{xy}^2 y)$ satisfy the differential equation $(3.13)_1$, and from Eq. $(3.13)_2$ it then follows that

$$\overset{1}{f}(x) = 0 \quad \text{and} \quad \overset{2}{f}(x+\beta_{xy}^2) = 0, \quad \forall x\varepsilon(-\infty,\infty).$$

Hence

$$\overset{\alpha}{\Xi}_0(x,y) \equiv 0. \tag{3.14}$$

3.2. Solution for ψ_0

The boundary value problem for ψ_0 can be read off from Eqs. $(3.10)_1$ and $(3.12)_{1,2}$ as

$$\begin{aligned} &\beta_x^2\frac{\partial^2\psi_0}{\partial x^2}+\frac{\partial^2\psi_0}{\partial y^2}+2\beta_{xy}^2\frac{\partial^2\psi_0}{\partial x\partial y}+\lambda^2\psi_0 = -p(x,y),\\ &\psi_0(x,0) = \psi_0(x,1) = 0 \end{aligned} \tag{3.15}$$

and is best solved using Fourier transforms whereby Eq. (3.15) then becomes

$$\frac{\partial^2\overline{\psi}_0}{\partial y^2}+2\beta_{xy}^2 i\xi\frac{\partial\overline{\psi}_0}{\partial y}+(\lambda^2-\xi^2\beta_x^2)\overline{\psi}_0 = -p(\xi, y) = -\sum_{\nu=1}^{\infty} p_\nu(\xi)\sin\nu\pi y$$

with

$$\overline{\psi}_0(\xi, 0) = \overline{\psi}_0(\xi, 1) = 0, \tag{3.16}$$

where we also have introduced the Fourier-sine series

$$\overline{p}(\xi, y) = \sum_{\nu=1}^{\infty} p_\nu(\xi)\sin(\nu\pi y). \tag{3.17}$$

It is a routine matter to construct the solution to Eq. (3.16) which is obtained

$$\overline{\psi}_0 = \sum_{\nu=1}^{\infty}\left\{A_\nu\left[\exp(-i\xi\beta_{xy}^2 y)\left\{\frac{\operatorname{Cosh}(\tilde{\delta}(\xi))-(-1)^\nu\exp(i\xi\beta_{xy}^2)}{\operatorname{Sinh}(\tilde{\delta}(\xi))}\operatorname{Sinh}(\tilde{\delta}(\xi)y)\right.\right.\right.$$
$$\left.\left.\left.-\operatorname{Cosh}(\tilde{\delta}(\xi)y)\right\}\frac{2i\xi\nu\pi\beta_{xy}^2\nu}{\pi^2\nu^2+\xi^2\beta_x^2-\lambda^2}+\sin(\nu\pi y)-\frac{2i\xi\nu\pi\beta_{xy}^2}{\pi^2\nu^2+\xi^2\beta_x^2-\lambda^2}\cos(\nu\pi y)\right]\right\} \tag{3.18a}$$

with

$$A_\nu = \frac{-p_\nu(\xi)(\lambda^2-\xi^2\beta_x^2-\nu^2\pi^2)}{(\xi^2\beta_x^2+2\nu\pi\xi\beta_{xy}^2+\nu^2\pi^2-\lambda^2)(\xi^2\beta_x^2-2\nu\pi\xi\beta_{xy}^2+\nu^2\pi^2-\lambda^2)} \tag{3.18b}$$

and

$$\tilde{\delta}(\xi) = \sqrt{\xi^2(\beta_x^2-\beta_{xy}^4)-\lambda^2}. \tag{3.18c}$$

In the following, we shall also need $(\partial\overline{\psi}_0/\partial y)(\xi, 0)$ and $(\partial\overline{\psi}_0/\partial y)(\xi, 1)$. Thus we record them here for further use:

$$\frac{\partial\overline{\psi}_0}{\partial y}(\xi, 0) = \sum_{\nu=1}^{\infty} A_\nu\left\{\left[i\xi\beta_{xy}^2+\tilde{\delta}(\xi)\frac{\operatorname{Cosh}(\tilde{\delta}(\xi))-(-1)^\nu\exp(i\xi\beta_{xy}^2)}{\operatorname{Sinh}(\tilde{\delta}(\xi))}\right]\times\frac{2i\nu\pi\beta_{xy}^2\xi}{\pi^2\nu^2+\xi^2\beta_x^2-\lambda^2}+\nu\pi\right\},$$

$$\frac{\partial\overline{\psi}_0}{\partial y}(\xi, 1) = \sum_{\nu=1}^{\infty} A_\nu\left\{\left[\exp(-i\xi\beta_{xy}^2)\frac{\tilde{\delta}(\xi)(1-2\operatorname{Sinh}^2\tilde{\delta}(\xi))}{\operatorname{Sinh}\tilde{\delta}(\xi)}\right.\right.$$
$$\left.\left.+(-1)^\nu\left(i\xi\beta_{xy}^2-\tilde{\delta}\operatorname{Coth}(\tilde{\delta}(\xi))\right)\right]\frac{2i\nu\pi\xi\beta_{xy}^2}{\nu^2\pi^2+\beta_x^2\xi^2-\lambda^2}+(-1)^\nu\nu\pi\exp(-i\xi\beta_{xy}^2)\right\}. \tag{3.19}$$

3.3. Solutions for $\overset{\alpha}{\Xi}_1$: $(\alpha = 1, 2)$

The boundary value problems for $\overset{\alpha}{\Xi}_1$ are obtained from Eqs. (3.11) and $(3.12)_{3,4}$ by setting $\nu = 1$. They read

$$\frac{\partial\overset{\alpha}{\Xi}_1}{\partial y}-\beta_{xy}^2\frac{\partial\overset{\alpha}{\Xi}_1}{\partial x} = 0, \quad \alpha = 1, 2,$$
$$\overset{1}{\Xi}_1(x, 0) = \frac{\partial\psi_0}{\partial y}(x, 0), \tag{3.20}$$
$$\overset{2}{\Xi}_1(x, 1) = \frac{\partial\psi_0}{\partial y}(x, 1).$$

Solutions are of the form

$$\overset{1}{\Xi}_1(x, y) = T(x+\beta_{xy}^2 y),$$
(3.21)
$$\overset{2}{\Xi}_1(x, y) = F(x+\beta_{xy}^2 y).$$

The two yet unknown functions $T(\cdot)$ and $F(\cdot)$ are determined by taking Fourier-transforms of the boundary conditions $(3.20)_{2,3}$. Indeed, with the aid of Eqs. (3.19) it follows

$$\overline{\overset{1}{\Xi}}_1(\xi, 1) = \overline{T}(\xi) = \frac{\partial\overline{\psi}_0}{\partial y}(\xi, 0)$$
$$= \sum_{\nu=1}^{\infty} A_\nu\left\{\left[i\xi\beta_{xy}^2 + \tilde{\delta}(\xi)\,\frac{\operatorname{Cosh}\big(\tilde{\delta}(\xi)\big) - (-1)^\nu\exp\big(i\xi\beta_{xy}^2\big)}{\operatorname{Sinh}\big(\tilde{\delta}(\xi)\big)}\right]\frac{2i\nu\pi\beta_{xy}^2\xi}{\nu^2\pi^2+\xi^2\beta_x^2-\lambda^2} + \nu\pi\right\},$$

(3.22) $$\overline{\overset{2}{\Xi}}_1(\xi, 1)\exp(-i\xi\beta_{xy}^2) = F(x+\beta_{xy}^2)\exp(-i\xi\beta_{xy}^2) = \overline{F}(\xi) = \frac{\partial\overline{\psi}_0}{\partial y}(\xi, 1)\exp(-i\xi\beta_{xy}^2)$$
$$= \sum_{\nu=1}^{\infty}\exp(-i\xi\beta_{xy}^2)A_\nu\left\{\left[\exp(-i\xi\beta_{xy}^2)\,\frac{\tilde{\delta}(\xi)\big(1-2\operatorname{Sinh}\big(\tilde{\delta}(\xi)\big)\big)}{\operatorname{Sinh}\big(\tilde{\delta}(\xi)\big)}\right.\right.$$
$$\left.\left. + (-1)^\nu\Big(i\xi\beta_{xy}^2 - \tilde{\delta}(\xi)\coth\big(\tilde{\delta}(\xi)\big)\Big)\right]\frac{2i\nu\pi\xi\beta_{xy}^2}{\nu^2\pi^2+\beta_x^2\xi^2-\lambda^2} + (-1)^\nu\nu\pi\exp(-i\xi\beta_{xy}^2)\right\},$$

where A_ν is given by Eq. (3.18b) and $\tilde{\delta}(\xi)$ is defined in Eq. (3.18c). $F(x)$ and $T(x)$ are determined by Fourier inversion. Once these functions are known, the general solution is obtained by replacing the argument x by $(x+\beta_{xy}^2 y)$.

3.4. Solution for ψ_1

The governing differential equation for ψ_1 reads

(3.23) $$\frac{\partial^2\overline{\psi}_1}{\partial y^2} + 2i\xi\beta_{xy}^2\frac{\partial\overline{\psi}_1}{\partial y} + (\lambda^2-\xi^2\beta_x^2)\overline{\psi}_1 = 0$$

and must satisfy the boundary conditions

$$\overline{\psi}_1(\xi, 0) = -\overline{\overset{1}{\Xi}}_1(\xi, 0) = -\overline{T}(\xi),$$
(3.24)
$$\overline{\psi}_1(\xi, 0) = -\overline{\overset{2}{\Xi}}_1(\xi, 1) = -\exp(i\xi\beta_{xy}^2)\overline{F}(\xi).$$

The solution is easily constructed as

(3.25) $$\overline{\psi}_1(\xi, y) = -\frac{\exp(-i\xi\beta_{xy}^2)}{\operatorname{Sinh}\big(\tilde{\delta}(\xi)\big)}\left\{\overline{T}(\xi)\operatorname{Sinh}\big(\tilde{\delta}(\xi)(1-y)\big) + \overline{F}(\xi)\operatorname{Sinh}\big(\tilde{\delta}(\xi)y\big)\right\}.$$

Further approximations should in principle be constructed by continuing the approach outlined above. We shall not go any further but we must mention here that the above solutions are uniformly valid in the entire strip only if the external loading function is sufficiently smooth. In Part I we found this to be true for the static case and we shall find corroboration for this in the next section where we attempt to construct explicit solutions for a particular loading distribution. Before we turn to that, we might mention that the above results are generalizations of calculations performed by us in 1974. In that paper β_{xy} was set to zero from the outset. That all our results agree for $\beta_{xy} = 0$ with those of that article merely serves as partial check of our computations.

3.5. Steady state response to a sinusoidally distributed line load oscillating in time

As loading function we choose

$$p(x, y, \tau) = p_\nu \sin(\nu\pi y)\delta(x-x_0)e^{i\lambda\tau} \tag{3.26}$$

so that (see Eq. (3.17))

$$p_\nu(\xi) = p_\nu \exp(-ix_0). \tag{3.27}$$

Since p is not differentiable at $x = x_0$, we expect the representation (3.5) not to be uniformly convergent in the entire strip.

Substituting Eq. (3.27) into Eq. (3.18b) and the resulting equation into Eq. (3.18a) yields an expression for ψ_0 that can be written as follows:

$$\psi_0 = \sum_{j=1}^{4} I_j,$$

where I_1 through I_4 are the following Fourier inversion integrals:

$$\begin{aligned}
I_1 &= \frac{(-1)^\nu}{2\pi}\int_{-\infty}^{\infty} - \frac{2i\nu\pi p_\nu \beta_{xy}^2 \xi \exp[i\xi(x-x_0+(1-y)\beta_{xy}^2)]}{(\xi^2\beta_x^2+2\nu\pi\beta_{xy}^2\xi+\nu^2\pi^2-\lambda^2)(\xi^2\beta_x^2-2\nu\pi\beta_{xy}^2\xi+\nu^2\pi^2-\lambda^2)} \\
&\qquad\times \frac{\mathrm{Sinh}(\tilde{\delta}(\xi)y)}{\mathrm{Sinh}(\tilde{\delta}(\xi))}\,d\xi, \\
I_2 &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{-2i\nu\pi p_\nu \beta_{xy}^2 \xi \exp[i\xi(x-x_0-\beta_{xy}^2 y)]}{(\xi^2\beta_x^2+2\nu\pi\beta_{xy}^2\xi+\nu^2\pi^2-\lambda^2)(\xi^2\beta_x^2-2\nu\pi\beta_{xy}^2\xi+\nu^2\pi^2-\lambda^2)} \\
&\qquad\times \frac{\mathrm{Sinh}(\tilde{\delta}(\xi)(1-y))}{\mathrm{Sinh}(\tilde{\delta}(\xi))}\,d\xi, \\
I_3 &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{-p_\nu(\lambda^2-\xi^2\beta_x^2-\nu^2\pi^2)\sin(\nu\pi y)\exp(i\xi(x-x_0))}{(\xi^2\beta_x^2+2\nu\pi\beta_{xy}^2\xi+\nu^2\pi^2-\lambda^2)(\xi^2\beta_x^2-2\nu\pi\beta_{xy}^2\xi+\nu^2\pi^2-\lambda^2)}\,d\xi, \\
I_4 &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{-2i\nu\pi\beta_{xy}^2 p_\nu \xi\cos(\nu\pi y)\exp(i(x-x_0))}{(\xi^2\beta_x^2+2\nu\pi\beta_{xy}^2\xi+\nu^2\pi^2-\lambda^2)(\xi^2\beta_x^2-2\nu\pi\beta_{xy}^2\xi+\nu^2\pi^2-\lambda^2)}\,d\xi.
\end{aligned} \tag{3.28}$$

Inspection shows that all integrals but I_3 vanish when $\beta_{xy} = 0$. The latter becomes

$$\begin{aligned}
I_{3|\beta_{xy}=0} &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{p_\nu \sin(\nu\pi y)}{(\lambda^2-\xi^2\beta_x^2-\nu^2\pi^2)}\exp(i\xi(x-x_0))\,d\xi \\
&= -p_\nu \frac{\exp\left(-\frac{1}{\beta_x}\sqrt{\nu^2\pi^2-\lambda^2}\,|x-x_0|\right)}{2\beta_x\sqrt{\nu^2\pi^2-\lambda^2}}, \qquad \lambda^2 < \nu^2\pi^2.
\end{aligned} \tag{3.29}$$

In the construction of this solution we have also used the Sommerfeld radiation condition. Of course, the result (3.29) must also be recovered when I_3 is evaluated for $\beta_{xy} \neq 0$ and β_{xy} is set to zero *a posteriori*.

The evaluation of the inversion integrals (3.28) is best performed using contour integration in the complex ξ-plane. It is readily seen that the integrals do not have branch cuts, but simple poles only. Thus, integration amounts to collecting all residues within the contour considered. The evaluation is a tedious though straightforward matter. We therefore only sketch this evaluation. To this end, let

$$(3.30)\quad \begin{aligned} (\xi^2\beta_x^2+2\nu\pi\beta_{xy}^2\xi+\nu^2\pi^2-\lambda^2) &= \beta_x^2[\xi+(\alpha+i\gamma)][\xi+(\alpha-i\gamma)],\\ (\xi^2\beta_x^2-2\nu\pi\beta_{xy}^2\xi+\nu^2\pi^2-\lambda^2) &= \beta_x^2[\xi-(\alpha-i\gamma)][\xi-(\alpha+i\gamma)], \end{aligned}$$

where

$$(3.31)\quad \alpha = \frac{\nu\pi\beta_{xy}^2}{\beta_x^2} \quad \text{and} \quad \gamma = \frac{\sqrt{\beta_x^2(\nu^2\pi^2-\lambda^2)-\nu^2\pi^2\beta_{xy}^4}}{\beta_x^2}.$$

In order to guarantee exponential decay of the integrals I_1-I_4 away from the line of loading, γ^2 must be larger than zero, which implies

$$(3.32)\quad \lambda^2 < \nu^2\pi^2\left(1-\frac{\beta_{xy}^4}{\beta_x^2}\right),$$

a condition that was also imposed on Eq. (3.29). This further implies that α and γ are real. In the complex ξ-plane, $\xi_p = \pm\alpha\pm i\gamma$ are the locations of the simple poles of I_1-I_4. However, I_1 and I_2 possess additional poles. They are located where $\operatorname{Sinh}(\tilde{\delta}(\xi_p)) = 0$, or $\tilde{\delta}(\xi_p) = i\nu\pi, n = 1, 2, \ldots\ldots$; therefore, in view of Eq. (3.18c)

$$(3.33)\quad \xi_p^2 = \frac{\lambda^2-n^2\pi^2}{\beta_x^2-\beta_{xy}^4},$$

where $\beta_x^2 > \beta_{xy}^4$, since L is elliptic. Thus ξ_p is either real or purely imaginary; explicitly

$$(3.34)\quad \xi_p = \frac{|\lambda^2-n^2\pi^2|}{\beta_x^2-\beta_{xy}^4}, \quad \begin{cases} 1, & n = 1, 2, \ldots, \nu-1,\\ i, & n = \nu, \nu+1, \ldots, \infty \end{cases}$$

and we shall henceforth write

$$(3.35)\quad \tilde{\omega} \equiv \frac{|\lambda^2-n^2\pi^2|}{\beta_x^2-\beta_{xy}^4}.$$

It is not hard to show that the residues at the real poles correspond to solutions periodic in x. Such solutions are unrealistic because they would lead to an infinitely large deformation energy. Hence, poles on the real axis must be suppressed which requires that in Eq. (3.34) $n > \nu$. Figure 1 summarizes the above results and shows the integration contour for the integrals I_1-I_4. Applying the residue theorem now yields for I_3 and I_4:

$$(3.36)\quad \begin{aligned} I_4 &= \frac{\nu\pi p_\nu\beta_{xy}^2}{2\alpha\gamma\beta_x^4}\cos(\nu\pi y)\sin\big(\alpha(x-x_0)\big)\exp(-\gamma|x-x_0|),\\ I_3 &= -\frac{p_\nu\sin(\nu\pi y)\exp(-\gamma|x-x_0|)}{2\alpha\gamma(\alpha^2+\gamma^2)\beta_x^4}\{\big(\alpha(\lambda^2-\nu^2\pi^2)-2\alpha\gamma^2\beta_x^2-\alpha\beta_x^2(\alpha^2-\gamma^2)\big)\\ &\quad\times\cos\big(\alpha(x-x_0)\big)-\big(\gamma(\lambda^2-\nu^2\pi^2)+2\alpha^2\gamma\beta_x^2-\gamma\beta_x^2(\alpha^2-\gamma^2)\big)\sin(\alpha|x-x_0|)\big), \end{aligned}$$

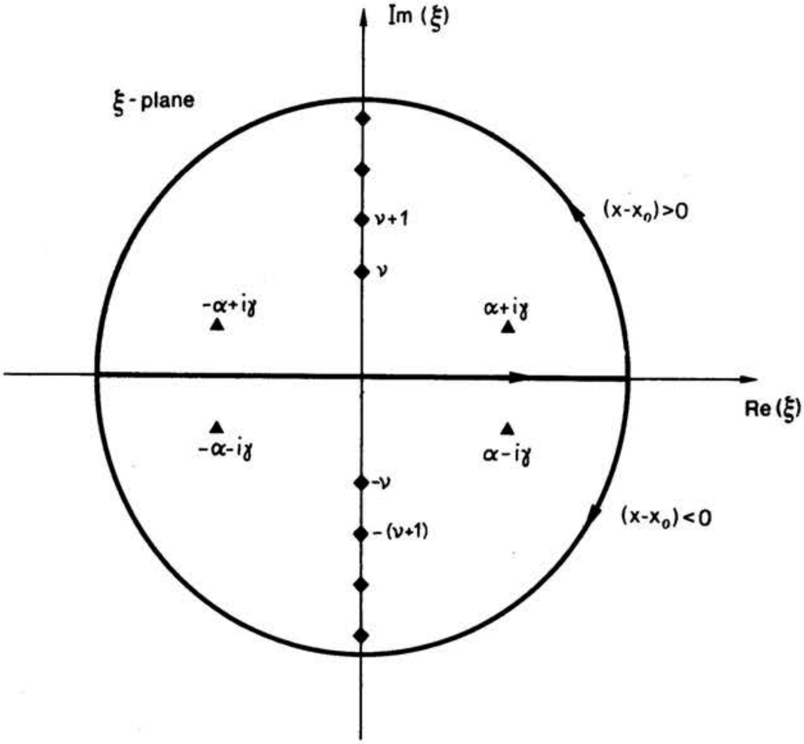

FIG. 1. Poles ($\pm\alpha\pm i\gamma$) (▲) and Sinh $\tilde{\delta}(\xi)=0$ (♦) for the evaluation of the integrals I_1-I_9 and corresponding integration contour.

whereas

$$(3.37)\qquad I_1 = \operatorname{sgn}(x-x_0)\Big\{2\pi\sum_{n=\nu}^{\infty}\frac{(-1)^{n+\nu+1}}{G}p_\nu n\pi\nu\beta_{xy}^2(\beta_x^2-\beta_{xy}^4)\sin(n\pi y)$$
$$\times\exp\{(\operatorname{sgn}(x_0-x)\tilde{\omega}(x-x_0)-\beta_{xy}^2(y-1))\}+\frac{(-1)^\nu}{H}\beta_{xy}^2p_\nu\pi\nu\{\operatorname{sgn}(x-x_0)\tilde{C}$$
$$\times\sin[\alpha((x-x_0)-\beta_{xy}^2(y-1))]+\tilde{D}\cos[\alpha((x-x_0)-\beta_{xy}^2(y-1))]\}$$
$$\times\exp[\operatorname{sgn}(x_0-x)\gamma((x-x_0)-\beta(y-1))]\Big\},$$

$$(3.38)\qquad I_2 = \operatorname{sgn}(x-x_0)\Big\{2\pi\sum_{n=\nu}^{\infty}\frac{(-1)^{n+1}}{G}p_\nu\beta_{xy}^2n\pi\nu(\beta_x^2-\beta_{xy}^4)\sin(n\pi(1-y))$$
$$\times\exp\{\operatorname{sgn}(x_0-x)\tilde{\omega}((x-x_0)-\beta_{xy}^2y)\}+\frac{p_\nu\nu\pi\beta_{xy}^2}{H}\{\operatorname{sgn}(x-x_0)\tilde{A}\sin[\alpha((x-x_0)-\beta_{xy}^2y)]$$
$$+\tilde{B}\cos[\alpha((x-x_0)-\beta_{xy}^2y)]\}\exp[\operatorname{sgn}(x-x_0)\gamma((x-x_0)\beta_{xy}^2y)]\Big\}.$$

Here

$$\tilde{A} = \text{Sinh}(\mu)\,\text{Sinh}\big((1-y)\mu\big)\cos(\sigma)\cos\big((1-y)\sigma\big) + \text{Cosh}(\mu)\,\text{Cosh}\big((1-y)\mu\big)\sin(\sigma)\sin\big((1-y)\sigma\big),$$

$$\tilde{B} = \text{Sinh}(\mu)\,\text{Cosh}\big((1-y)\mu\big)\cos(\sigma)\sin\big((1-y)\sigma\big) - \text{Cosh}(\mu)\,\text{Sinh}\big((1-y)\mu\big)\sin(\sigma)\cos\big((1-y)\sigma\big),$$

(3.39) $$\tilde{C} = \text{Sinh}(\mu)\,\text{Sinh}(\mu y)\cos(\sigma)\cos(\sigma y) + \text{Cosh}(\mu)\,\text{Cosh}(\mu y)\sin(\sigma)\sin(\sigma y),$$

$$\tilde{D} = \text{Sinh}(\mu)\text{Cosh}(\mu y)\cos(\sigma)\sin(\sigma y) - \text{Cosh}(\mu)\,\text{Sinh}(\mu y)\sin(\sigma)\cos(\sigma y),$$

$$G = \{[\beta_x^2\pi^2(\nu^2-n^2)-(\nu^2\pi^2-\lambda^2)\beta_{xy}^4]^2+4\nu^2\pi^2\beta_{xy}^4(n^2\pi^2-\lambda^2)(\beta_x^2-\beta_{xy}^4)\},$$

$$H = 2\alpha\gamma(\sin^2\sigma+\sinh^2\mu)\beta_x^4,$$

with

$$\mu = +\sqrt{r}\cos(\theta_1/2), \qquad \sigma = +\sqrt{r}\sin(\theta_1/2),$$

(3.40) $$r = \sqrt{[(\alpha^2-\gamma^2)(\beta_x^2-\beta_{xy}^4)-\lambda^2]^2+4\alpha^2\gamma^2(\beta_x^2-\beta_{xy}^4)^2},$$

$$\theta_1 = \tan^{-1}\left\{\frac{2\alpha\gamma(\beta_x^2-\beta_{xy}^4)}{(\alpha^2-\gamma^2)(\beta_x^2-\beta_{xy}^4)-\lambda^2}\right\}.$$

The reader may check that $\beta_{xy} = 0$ implies $I_1 = I_2 = I_4 = 0$, and that in this limit I_3 goes over into the expression (3.29) as it must. Furthermore, $\psi_0 = \sum_{j=1}^{4} I_j$ is not uniformly valid everywhere in the strip $-\infty < x < \infty$. Indeed, for $x-x_0 > 0$, I_2 becomes singular, provided $(x-x_0) < \beta_{xy}^2 y$. Similarly, for $(x-x_0) < 0$, I_1 is singular if $(x-x_0) < \beta_{xy}^2(y-1)$. This is the shear layer region already mentioned in Part I.

Next we determine ψ_1. For this $\bar{F}$ and $\bar{T}$ must be known as evidenced from Eq. (3.25). They can be evaluated from Eq. (3.22) if the relations (3.18b) and (3.27) are invoked. We then find that ψ_1 can be expressed as the sum of five Fourier inversion integrals viz:

(3.41) $$\psi_1(x,y) = \sum_{j=5}^{9} I_j,$$

where

(3.42) $$I_5 = p_\nu \nu \beta_{xy}^4 \int_{-\infty}^{\infty} \frac{\xi^2[\text{Sinh}(\tilde{\delta}(\xi)(1-y))-(-1)^\nu e^{i\xi\beta_{xy}^2}\text{Sinh}(\tilde{\delta}(\xi)y)]e^{i\xi[(x-x_0)-\beta_{xy}^2 y]}}{\text{Sinh}(\tilde{\delta}(\xi))\text{J}_1(\xi)\cdot\text{J}_2(\xi)}\,d\xi,$$

(3.43) $$I_6 = -p_\nu\nu\beta_{xy}^2 \int_{-\infty}^{\infty} \frac{i\xi\tilde{\delta}(\xi)\,\text{Sinh}(\tilde{\delta}(\xi)y)e^{i\xi[(x-x_0)-\beta_{xy}^2 y]}}{\text{J}_1(\xi)\cdot\text{J}_2(\xi)}\,d\xi,$$

(3.44) $I_7 =$

$$\frac{p_\nu\nu}{2}\int_{-\infty}^{\infty} \frac{(\lambda^2-\xi^2\beta_x^2-\pi^2\nu^2)[\text{Sinh}(\tilde{\delta}(\xi)(1-y))-(-1)^\nu e^{i\xi\beta_{xy}^2}\text{Sinh}(\tilde{\delta}(\xi)y)]e^{i\xi[(x-x_0)-\beta_{xy}^2 y]}}{\text{Sinh}(\tilde{\delta}(\xi))\,\text{J}_1(\xi)\cdot\text{J}_2(\xi)}\,d\xi,$$

while

$$(3.45)\quad I_8 = -p_\nu \nu \beta_{xy}^2 \int\limits_{-\infty}^{\infty} \frac{i\xi\tilde{\delta}(\xi)[\operatorname{Cosh}(\tilde{\delta}(\xi))-(-1)^\nu e^{i\xi\beta_{xy}^2}]\operatorname{Cosh}(\tilde{\delta}(\xi)y)e^{i\xi[(x-x_0)-\beta_{xy}^2 y]}}{\operatorname{Sinh}(\tilde{\delta}(\xi))\mathrm{J}_1(\xi)\cdot\mathrm{J}_2(\xi)}\,d\xi$$

and

$$(3.46)\quad I_9 =$$

$$p_\nu \nu \beta_{xy}^2 \int\limits_{-\infty}^{\infty} \frac{i\xi\tilde{\delta}(\xi)[\operatorname{Cosh}(\tilde{\delta}(\xi))-(-1)^\nu e^{i\xi\beta_{xy}^2}]\operatorname{Cosh}(\tilde{\delta}(\xi))\operatorname{Sinh}(\tilde{\delta}(\xi)y)e^{i\xi[(x-x_0)-\beta_{xy}^2 y]}}{\operatorname{Sinh}^2(\tilde{\delta}(\xi))\mathrm{J}_1(\xi)\cdot\mathrm{J}_2(\xi)}\,d\xi.$$

Here

$$\mathrm{J}_1(\xi) = (\xi^2\beta_x^2+\xi\pi\nu 2\beta_{xy}^2+\pi^2\nu^2-\lambda^2)$$

and

$$\mathrm{J}_2(\xi) = (\xi^2\beta_x^2-\xi\pi\nu 2\beta_{xy}^2+\pi^2\nu^2-\lambda^2).$$

Direct inspection reveals that when $\beta_{xy} = 0$, all the above integrals vanish except for I_7 which now becomes

$$(3.47)\quad I_{7|\beta_{xy}=0} = \frac{p_\nu \nu}{2}\int\limits_{-\infty}^{\infty} \frac{[\operatorname{Sinh}(\tilde{\delta}(\xi)(1-y))-(-1)^\nu \operatorname{Sinh}(\tilde{\delta}(\xi))y)]e^{i\xi(x-x_0)}}{\operatorname{Sinh}(\tilde{\delta}(\xi))(\xi^2\beta_x^2+\pi^2\nu^2-\lambda^2)}\,d\xi$$

the solution to which was constructed in an earlier paper [2].

If we now focus attention on inverting the integrals involved in ψ_1, we find that the poles remain at the locations identified while treating the integrals I_1-I_4. This implies that all integrals have simple poles at $\xi_p = \pm\tilde{\alpha}\pm i\gamma$ (See Fig. 1). Except in the case of I_6 additional poles lie on the imaginary axis wherever $\operatorname{Sinh}\tilde{\delta}(\xi)$ vanishes. For I_5, I_7 and I_8 these are simple poles while for I_9 we have a double pole. All these integrals can be evaluated by invoking the residue theorem. Unfortunately, the calculation while straightforward is fairly laborious and we can only list the results here. In particular, we find

$$\begin{aligned}(3.48)\quad I_5 = -\operatorname{sgn}(x-x_0)\Big\{&\sum_{n=\nu}^{\infty}\Big\langle\frac{(-1)^n p_\nu n\pi\beta_{xy}^4(\beta_x^2-\beta_{xy}^4)}{G}16\sqrt{(n^2\pi^2-\lambda^2)(\beta_x^2-\beta_{xy}^4)}\\ &\times\{\sin(n\pi(1-y))-(-1)^\nu\exp[-\operatorname{sgn}(x-x_0)\tilde{\omega}\beta_{xy}^2]\}\exp[-\operatorname{sgn}(x-x_0)\\ &\times\tilde{\omega}((x-x_0)-\beta_{xy}^2 y))]\Big\rangle+\frac{\operatorname{sgn}(x-x_0)}{H_1}\beta_{xy}^4 p_\nu\nu\pi\exp[-\operatorname{sgn}(x-x_0)(\gamma[(x-x_0)-\beta_{xy}^2 y])]\\ &\times\Big\{\operatorname{sgn}(x-x_0)\cdot(\tilde{\alpha}\tilde{A}-\gamma\tilde{B})\cos(\tilde{\alpha}((x-x_0)-\beta_{xy}^2 y))-(\gamma\tilde{A}+\tilde{\alpha}\tilde{B})\sin(\tilde{\alpha}((x-x_0)-\beta_{xy}^2 y))\\ &-(-1)^\nu\exp(-\operatorname{sgn}(x-x_0)\gamma\beta_{xy}^2)[\operatorname{sgn}(x-x_0)(\tilde{\alpha}\tilde{C}-\gamma\tilde{D})\cos(\tilde{\alpha}((x-x_0)\\ &-\beta_{xy}^2(y-1)))-(\gamma\tilde{C}+\tilde{\alpha}\tilde{D})\sin(\tilde{\alpha}((x-x_0)-\beta_{xy}^2(y-1)))]\Big\}\Big\},\end{aligned}$$

$$\begin{aligned}(3.49)\quad I_6 = \frac{p_\nu\nu\pi\beta_{xy}^2}{2\tilde{\alpha}\gamma\beta_x^4}&\exp[-\operatorname{sgn}(x-x_0)\gamma[(x-x_0)-\beta_{xy}^2 y]]\{\sin(\tilde{\alpha}((x-x_0)\\ &-\beta_{xy}^2 y))[\mu\operatorname{Sinh}\mu y\cos\sigma y-\sigma\operatorname{Cosh}\mu y\sin\sigma y]+\operatorname{sgn}(x-x_0)\cos(\tilde{\alpha}((x-x_0)\\ &-\beta_{xy}^2 y))[\mu\operatorname{Cosh}\mu y\sin\sigma y+\sigma\operatorname{Sinh}\mu y\sin\sigma y]\},\end{aligned}$$

$$(3.50)\quad I_7 = -8\sum_{n=\nu}^{\infty}(-1)^n p_\nu \nu n\pi^2 \frac{(\beta_x^2-\beta_{xy}^4)^{3/2}}{\sqrt{n^2\pi^2-\lambda^2 G_1}}\sin n\pi y[(-1)^n+(-1)^\nu$$

$$\times\exp\big(-\operatorname{sgn}(x-x_0)\beta_{xy}^2\big)][\beta_x^2\pi^2(n^2-\nu^2)+\beta_{xy}^4(\nu^2\pi^2-\lambda^2)]\exp\big[-\operatorname{sgn}(x-x_0)\tilde{\omega}$$

$$\times[(x-x_0)-\beta_{xy}^2 y]\big]+\operatorname{sgn}(x-x_0)\frac{p_\nu \nu\pi e^{\gamma[(x-x_0)-\beta_{xy}^2 y]}}{4\tilde{\alpha}\gamma\beta_{xy}^4(\gamma^2-\tilde{\alpha}^2)(\operatorname{Sinh}^2\mu+\sin^2\sigma)}$$

$$\times\{\cos(\tilde{\alpha}[(x-x_0)-\beta_{xy}^2 y])[(\gamma\Lambda+\alpha\Delta)\bar{\bar{B}}-\operatorname{sgn}(x-x_0)(\alpha\Lambda-\gamma\Delta)\bar{\bar{A}}]$$
$$+\operatorname{sgn}(x-x_0)\sin(\tilde{\alpha}[(x-x_0)-\beta_{xy}^2 y])[(\gamma\Lambda+\alpha\Delta)\bar{\bar{A}}+\operatorname{sgn}(x-x_0)(\alpha\Lambda-\gamma\Delta)\bar{\bar{B}}]\}$$

with

$$\Lambda = \lambda^2-\pi^2\nu^2-\beta_x^2(\tilde{\alpha}^2-\gamma^2),$$
$$\Delta = 2\tilde{\alpha}\beta_x^2\gamma,$$
$$\bar{\bar{A}} = \tilde{A}-(-1)^\nu[\tilde{C}\cos(\tilde{\alpha}\beta_{xy}^2)-\operatorname{sgn}(x-x_0)\sin(\tilde{\alpha}\beta_{xy}^2)\exp(-\operatorname{sgn}(x-x_0)\gamma\beta_{xy}^2],$$
$$\bar{\bar{B}} = \tilde{B}-(-1)^\nu[\operatorname{sgn}(x-x_0)\sin(\tilde{\alpha}\beta_{xy}^2)+\cos(\tilde{\alpha}\beta_{xy}^2)\exp(-\operatorname{sgn}(x-x_0)\gamma\beta_{xy}^2)].$$
$$G_1 = [\beta_x^2\pi^2(\nu^2-n^2)-\nu^2\pi^2\beta_{xy}^4-\lambda^2\beta_x^4]^2+4\pi^2\nu^2\beta_{xy}^4(n^2\pi^2-\lambda^2)(\beta_x^2-\beta_{xy}^4),$$
$$H_1 = 2\tilde{\alpha}\gamma(\sin^2\sigma+\operatorname{Sinh}^2\mu)\beta_x^4;$$

$$(3.51)\quad I_8 = \sum_{n=\nu}^{\infty}\frac{(-1)^{n+1}}{G}4p_\nu \nu n^2\pi^3\beta_{xy}^2[(-1)^n-(-1)^\nu\exp\big(-\operatorname{sgn}(x-x_0)\tilde{\omega}\beta_{xy}^2\big)]$$

$$\times\cos n\pi y\exp\big[-\operatorname{sgn}(x-x_0)\tilde{\omega}[(x-x_0)-\beta_{xy}^2 y]\big]$$

$$-\frac{p_\nu \nu\pi\beta_{xy}^2\exp\big[-\operatorname{sgn}(x-x_0)\gamma[(x-x_0)-\beta_{xy}^2 y]\big]}{2\tilde{\alpha}\gamma\beta_{xy}^4(\operatorname{Sinh}^2\mu+\sin^2\sigma)}\big\{\operatorname{sgn}(x-x_0)(\tilde{Q}\tilde{G}-\tilde{H}\tilde{P})$$

$$\times\cos\big(\tilde{\alpha}[(x-x_0)-\beta_{xy}^2 y]\big)-(\tilde{P}\tilde{G}+\tilde{H}\tilde{Q})\sin\big(\tilde{\alpha}[(x-x_0)-\beta_{xy}^2 y]\big)-(-1)^n$$

$$\times\exp[-\operatorname{sgn}(x-x_0)\gamma\beta_{xy}^2]\big[\operatorname{sgn}(x-x_0)(\mu\tilde{Q}-\sigma\tilde{P})\cos\big(\tilde{\alpha}[(x-x_0)-\beta_{xy}^2(y-1)]\big)$$

$$-(\mu\tilde{P}+\sigma\tilde{Q})\sin\big(\tilde{\alpha}[(x-x_0)-\beta_{xy}^2(y-1)]\big)\big]\big\},$$

with

$$\tilde{G} = \mu\operatorname{Cosh}\mu\cos\sigma+\sigma\operatorname{Sinh}\mu\sin\sigma,$$
$$\tilde{H} = \sigma\operatorname{Cosh}\mu\cos\sigma-\mu\operatorname{Sinh}\mu\sin\sigma,$$
$$\tilde{P} = \operatorname{Cosh}\mu y\operatorname{Sinh}\mu\cos\sigma\cos\sigma y-\operatorname{Cosh}\mu\operatorname{Sinh}\mu y\sin\sigma\sin\sigma y,$$
$$\tilde{Q} = \operatorname{Cosh}\mu y\operatorname{Cosh}\mu\sin\sigma\cos\sigma y+\operatorname{Sinh}\mu y\operatorname{Sinh}\mu\cos\sigma\sin\sigma y.$$

$$(3.52)\quad I_9 = -\frac{p_\nu \nu\pi\beta_{xy}^2 \exp[-\mathrm{sgn}(x-x_0)\gamma((x-x_0)-\beta_{xy}^2 y)]}{\tilde{\alpha}\gamma\beta_x^4(\sinh^2\mu+\sin^2\sigma)^2}\Big\{-\mathrm{sgn}(x-x_0)\mathscr{A}$$
$$\times\cos\left(\tilde{\alpha}(x-x_0)-\beta_{xy}^2 y\right)+\mathscr{B}\sin\left(\tilde{\alpha}(x-x_0)-\beta_{xy}^2 y\right)-(-1)^\nu e^{-\mathrm{sgn}(x-x_0)\gamma\beta_{xy}^2}\Big[-\mathrm{sgn}(x-x_0)$$
$$\times\mathscr{C}\cos\left(\tilde{\alpha}(x-x_0)-\beta_{xy}^2(y-1)\right)+\mathscr{D}\sin\left(\tilde{\alpha}(x-x_0)-\beta_{xy}^2(y-1)\right)\Big]\Big\}$$
$$-\mathrm{sgn}(x-x_0)\sum_{n=\nu}^{\infty}\Bigg\{\Bigg\{(-1)^n\frac{\pi^4 n^3}{2\tilde{\omega}^2 G}\Bigg\{\Bigg[[(-1)^n-(-1)^\nu\exp\left(-\mathrm{sgn}(x-x_0)\beta_{xy}^2\right)]$$
$$\times\Bigg[\sin(n\pi y)\Bigg(1-\frac{\beta_x^2-\beta_{xy}^2}{n^2\pi^2}+\mathrm{sgn}(x-x_0)\left(y\tilde{\omega}\beta_{xy}^2-\tilde{\omega}(x-x_0)\right)\Bigg)$$
$$+\mathrm{sgn}(x-x_0)\,y\cos(n\pi y)\frac{\beta_x^2-\beta_{xy}^4}{4n\pi}\Bigg]+\mathrm{sgn}(x-x_0)(-1)^\nu\beta_{xy}^2\sin(n\pi y)e^{-\mathrm{sgn}(x-x_0)\tilde{\omega}\beta_{xy}^2}\Bigg]\Bigg\}$$
$$+\frac{4n\pi^2\tilde{\omega}(\beta_x^2-\beta_{xy}^4)[(-1)^n-(-1)^\nu e^{-\mathrm{sgn}(x-x_0)\beta_{xy}^2\tilde{\omega}}]\sin(n\pi y)}{G}\left[\frac{n^2\pi^2\left[1-(n^2\pi^2-\lambda^2)\frac{2}{n^2\pi^2}\right]}{8\tilde{\omega}^3(\beta_x^2-\beta_{xy}^4)^3}\right.$$
$$\left.-\frac{n^2\pi^2[2\tilde{\omega}\beta_x^2(-\tilde{\omega}^2\beta_x^2+\pi^2\nu^2-\lambda^2)-4\pi^2\nu^2\tilde{\omega}^2\beta_{xy}^4]}{\tilde{\omega}^2 G}\right]\Bigg\}\Bigg\}\exp[-\mathrm{sgn}(x-x_0)\tilde{\omega}[(x-x_0)-\beta_{xy}^2 y]]$$

with

$$\tilde{E} = \mathrm{Cosh}\,\mu\,\mathrm{Sinh}\,\mu(\cos^2\sigma-\sin^2\sigma),\qquad \tilde{F} = \cos\sigma\sin\sigma(\mathrm{Sinh}^2\mu+\mathrm{Cosh}^2\mu),$$
$$\mathscr{A} = \tilde{G}(\tilde{C}\tilde{F}-\tilde{D}\tilde{E})-\tilde{H}(\tilde{C}\tilde{E}+\tilde{D}\tilde{F}),\qquad \mathscr{B} = \tilde{G}(\tilde{C}\tilde{E}+\tilde{D}\tilde{F})+\tilde{H}(\tilde{C}\tilde{F}-\tilde{D}\tilde{E}),$$
$$\mathscr{C} = \mu(\tilde{C}\tilde{F}-\tilde{D}\tilde{E})-\sigma(\tilde{C}\tilde{E}+\tilde{D}\tilde{F}),\qquad \mathscr{D} = \mu(\tilde{C}\tilde{E}+\tilde{D}\tilde{F})+\sigma(\tilde{C}\tilde{F}-\tilde{D}\tilde{E}).$$

Inspection of the above expressions reveals that the derived series are not uniformly convergent for all x. In fact, when $\alpha(y-1)\leqslant(x-x_0)\leqslant\alpha y$, these series become divergent. In this singular region (referred to in Part I as the shear layer) we search for a perturbation solution by writing

$$(3.54)\qquad \phi = \begin{cases}\displaystyle\sum_{\nu=0}^{\infty}\varepsilon^\nu\varphi_\nu^{\mathrm{IVA}}(X,y); & \text{for}\quad X\equiv\dfrac{1}{\varepsilon}(x-x_0)>0,\\ \displaystyle\sum_{\nu=0}^{\infty}\varepsilon^\nu\varphi_\nu^{\mathrm{IVA}}(\tilde{X},y); & \text{for}\quad \tilde{X}\equiv\dfrac{1}{\varepsilon}(x_0-x)>0.\end{cases}$$

Following the earlier work [1], we derive the following expressions:

$$(3.55)\qquad \begin{aligned}\varphi_0^{\mathrm{IVA}} &= \mathscr{F}_0+g_0X,\\ \varphi_1^{\mathrm{IVA}} &= \mathscr{F}_1+g_1X-p_\nu\frac{\sin(\nu\pi y)}{2\beta_x^3}e^{-\beta_x X}-\frac{\beta_{xy}^2}{\beta_x^2}\frac{\partial g_0}{\partial y}X^2,\\ \varphi_0^{\mathrm{IVB}} &= \mathscr{F}_0+g_0\tilde{X},\\ \varphi_1^{\mathrm{IVB}} &= \mathscr{F}_1-\left(g_1+p_\nu\frac{\sin(\nu\pi y)}{\beta_2^x}\right)-p_\nu\frac{\sin(\nu\pi y)}{2\beta_x^2}e^{-\beta_x\tilde{X}}-\frac{\beta_{xy}^2}{\beta_x^2}\frac{\partial g_0}{\partial y}\tilde{X}^2\end{aligned}$$

as our results for the first two approximations to the solution in regions A and B of the shear layer (see Fig. 1). Explicit expressions for $g_0(=0)$ as well as the other functions can be derived by carrying out a two-term matching of the shear layer solution (3.55) with the core. The details of such a matching has been illustrated in [1] and in view of the space requirements there is little to be gained in flogging the algebra here.

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Received February 18, 1978.