Archives of Mechanics • Archiwum Mechaniki Stosowanej • 31, 4, pp. 519-532, Warszawa 1979

Bulk constitutive relations for cracked materials

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WITHIN the frame of continuum infinitesimal theory of a medium with stable random cracks which may open or close (unilateral internal constraints), given in Paper [1], effective form of constitutive equations (1.1) is derived. The geometry of cracks is characterized by their total area and the probability density distribution after direction which suffices for calculation in the case of no interaction of cracks. In calculation one separates macro-strains (called "additional") brought about by response forces at closed cracks. Detailed calculation is carried out for isotropic crack system.

W ramach kontynualnej, infinitezymalnej teorii ośrodka z ustalonym losowym układem rys, które mogą się otwierać lub zamykać (jednostronne więzy wewnętrzne), przedstawionej w pracy [1], wyprowadza się konkretną postać równań konstytutywnych (1.1). Geometria zarysowania jest scharakteryzowana przez ogólną powierzchnię rys i ich rozkład gęstości prawdopodobieństwa podług kierunku, co wystarczy do przeprowadzenia konkretnych obliczeń w przypadku rys bez interakcji. Przy obliczeniu wydziela się makro-odkształcenia (zwane „dodatkowymi") wywołane siłami reakcji w rysach zamkniętych. Szczegółowe obliczenia przeprowadzono dla izotropowego układu rys.

В рамках континуальной, инфинитезимальной теории среды, представленной в работе [1], с установленной случайной системой рисок, которые могут открываться или закрываться (односторонние внутренние связи), выводится конкретный вид определяющих уравнений (1.1). Геометрия системы рисок охарактеризована общей поверхностью рисок и их распределением плотности вероятности по направлению, что достаточно для проведения конкретных расчетов в случае рисок без взаимодействия. При расчете выделяются макродеформации (называемые „дополнительными"), вызванные силами реакции в замкнутых рисках. Подробные расчеты проведены для изотропной системы рисок.

1. Preliminaries, crack geometry

IN PAPER [1](1) we analysed the bulk properties of an elastic material with numerous stable micro-cracks, randomly distributed, where the cracks may open or close depending on loads; consequently, the material exhibits unilateral internal constraints. We have shown that, within the infinitesimal theory, the stress-strain constitutive relations are homogeneous but no more additive. The material has been called pseudo-linear elastic and the relations take the form

$$\boldsymbol{\sigma} = \mathbf{C}(\mathbf{e})\boldsymbol{\epsilon}, \quad \boldsymbol{\epsilon} = \mathbf{S}(\mathbf{s})\boldsymbol{\sigma}, \tag{1.1}$$

where $\boldsymbol{\epsilon}, \boldsymbol{\sigma}$ are macro-strain and stress tensors and $\mathbf{C}, \mathbf{S}$ the bulk elastic stiffness and compliance tensors. The latter are no more constant, instead they are functions of direction in the strain (stress) 9-space,

$$\mathbf{e} = \frac{\boldsymbol{\epsilon}}{\sqrt{\boldsymbol{\epsilon}\cdot\boldsymbol{\epsilon}}}, \quad \mathbf{s} = \frac{\boldsymbol{\sigma}}{\sqrt{\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}}}. \tag{1.2}$$

(1) The reader is recommended to get acquainted beforehand with the said Paper since it explains basic ideas presupposed, in the sequel, to be known.

We have examined the basic properties of the functions $\mathbf{C(e)}$, $\mathbf{S(s)}$, in particular the restrictions imposed by invariance to certain groups of orthogonal transformations (isotropy, orthotropy). This was done with no recourse to intrinsic geometric properties of the crack system. Consequently, so far we cannot answer, for example, the question what crack geometry endows the relations (1.1) with respective symmetry invariance. In general, the problem which will be studied in the present paper consists in specializing the form of constitutive relations (1.1) and expressing those by parameters describing crack geometry.

To begin with, we must select a mode of description of the mentioned geometry for statistically homogeneous crack distributions (assumed throughout in the sequel). Observe that multipoint correlation functions, as used in the theory of composite media, are not suitable, since the cracks unlike say the inclusions, have no finite volume, i.e. are geometric objects of a zero measure. Bearing in mind that a full description is inexhaustible (it requires, for example, an infinite sequence of more and more dimensional correlation functions) we should confine the scope of description according to the following items: (a) available probabilistic information, (b) precision of mechanical assumptions (simplification about non-interacting cracks, say), (c) problems to be solved e.g. finding bulk (mean) strain only, not its probability distribution curve. From among many possible statistics offered by integral stochastic geometry, we select the following simple one taking account of the two main features: (i) the total amout of cracks, (ii) the distribution of cracks after direction (space orientation).

Consider a crack element assimilated to a surface element with the unit normal vector $\mathbf{n}$ and take into account all crack elements with the orientation $\mathbf{n}$ in the unit volume of the material. The total area of these elements amounts to $P\pi(\mathbf{n})d\omega$ where $d\omega$ is the elementary solid angle defining a measure on the space of n's (corresponding to a surface element on the unit sphere); P is the total area of all cracks (of whatever orientation) in the unit volume; $\pi(\mathbf{n})$ defines the probability density of cracks with orientation $\mathbf{n}$, satisfying the normalization condition

$$\int_{\Omega} \pi(\mathbf{n})\,d\omega = 1. \tag{1.3}$$

The domain of integration Ω corresponds to the unit hemisphere with the area 2π since the cracks are not oriented, i.e. the opposite edges of cracks and senses of $\mathbf{n}$ are equivalent (the crack area is singly counted irrespective of whether for the open or the closed crack). No particular assumptions about the shape of the function $\pi(\mathbf{n})$ are made so far, except for the condition (1.3).

If small cracks (with no long range structures) become more and more scattered, for $P \to 0$, we arrive at the model of non-interacting cracks. Then the fields produced in the vicinity of single cracks (and quickly disappearing at larger distances) do not interfere and may be simply added. The quantities connected with cracks will be seen to depend proportionally on the crack concentration, hence it will be useful to employ quantities per unit concentration ($P = 1$); we call them in the sequel resolved quantities.

If all cracks have a fixed orientation $\mathbf{n}$ ($\pi(\mathbf{n})$ becomes a delta function), we shall speak of an oriented system of cracks. Constitutive relations and, in particular, elastic constants for the material with the oriented resolved crack system will be the starting (elementary) ones in the subsequent argument. Knowing this, next to the description of crack geometry by means of concentration P and the distribution $\pi(\mathbf{n})$, we shall be able to find the tensor functions $\mathbf{C}(\mathbf{e}), \mathbf{S}(\mathbf{s})$ in the relations (1.1). Thus the outlined description will prove to be sufficient for non-interacting cracks.

2. Field quantities

Consider a unit volume element containing a great many elementary non-interacting cracks and a reference element of the same material and the same shape without cracks. Let the reference element be loaded by boundary tractions producing a homogeneous field of $\boldsymbol{\sigma}$ (second boundary value problem). The cracked element loaded in the same manner at the external boundary will show the same macro-stress $\boldsymbol{\sigma}$ while the macro-strain amounts to $\boldsymbol{\epsilon} = \boldsymbol{\epsilon}_0+\boldsymbol{\epsilon}_c$ where $\boldsymbol{\epsilon}_0$ corresponds to the plain (non-cracked) material and $\boldsymbol{\epsilon}_c$ is the contribution (macro-perturbation) yielded by cracks. On the other hand let both elements be loaded by the homogeneous macro-strain $\boldsymbol{\epsilon}$ produced by suitable displacement at the boundary (first boundary value problem). Macro-stresses in the plain and the cracked element will be, consecutively, $\boldsymbol{\sigma} = \boldsymbol{\sigma}_0$ and $\boldsymbol{\sigma} = \boldsymbol{\sigma}_0+\boldsymbol{\sigma}_c$ where $\boldsymbol{\sigma}_c$ is the perturbation produced by cracks. For non-interacting cracks, i.e. for $P \to 0$, the quantities $\boldsymbol{\epsilon}_c, \boldsymbol{\sigma}_c$ may be looked upon as infinitesimal, however, we spread the validity of results for finite (not too large) P, in paricular we form resolved quantities for $P = 1$.

The quantities $\boldsymbol{\epsilon}_c, \boldsymbol{\sigma}_c$ are supposed, in turn, to be composed of two contributing parts called in the sequel basic ("b") and additional ("a"), i.e.

$$\boldsymbol{\sigma}_c = \boldsymbol{\sigma}_b+\boldsymbol{\sigma}_a, \quad \boldsymbol{\epsilon}_c = \boldsymbol{\epsilon}_b+\boldsymbol{\epsilon}_a. \tag{2.1}$$

Basic strains (stresses) are by definition the strains (stresses) which would appear if the cracks could not close (imagine these are narrow yet not infinitesimal slits). Additional strains (stresses) are produced by normal forces at opposite boundary surfaces at closed ideal cracks, i.e. cracks with no friction at rest and at motion. The forces in question may be looked upon as response forces to unilateral internal constraints (cf. [1]). The slits loaded in the said manner are equivalent to closed cracks.

For non-interacting cracks the bulk macro-quantities follow from superposition of the partial macro-quantities relative to partial crack systems. Recall that the partial crack system is composed of all elementary cracks with the space orientation $\mathbf{n}$, i.e. it forms an oriented crack system with the area $P\pi(\mathbf{n})d\omega$. Denote by $\boldsymbol{\epsilon}_c^{(n)}$, $\boldsymbol{\sigma}_c^{(n)}$ (and similarly $\boldsymbol{\epsilon}_b^{(n)}, \boldsymbol{\sigma}_b^{(n)}, \boldsymbol{\epsilon}_a^{(n)}, \boldsymbol{\sigma}_a^{(n)}$) respective quantities for the oriented resolved crack system $\mathbf{n}$. Then the superposition yields for bulk macro-quantities

$$\begin{aligned} \boldsymbol{\epsilon}_c &= P \int_{\Omega} \pi(\mathbf{n})\boldsymbol{\epsilon}_c^{(n)}(\mathbf{n})\,d\omega, \\ \boldsymbol{\sigma}_c &= P \int_{\Omega} \pi(\mathbf{n})\,\boldsymbol{\sigma}_c^{(n)}(\mathbf{n})\,d\omega \end{aligned} \tag{2.2}$$

and analogically $\boldsymbol{\epsilon}_b, \ldots$, where the domain of integration, Ω, will be discussed in the sequel.

According to Eq. (2.1) we can decompose the functional relations (1.1) as follows:

$$(2.3)\qquad \begin{aligned} \boldsymbol{\sigma} &= \boldsymbol{\sigma}_0+\boldsymbol{\sigma}_c = \boldsymbol{\sigma}_0+\boldsymbol{\sigma}_b+\boldsymbol{\sigma}_a = [\mathbf{C}_0+\mathbf{C}_b+\mathbf{C}_a(\mathbf{e})]\boldsymbol{\epsilon} = [\hat{\mathbf{C}}+\mathbf{C}_a(\mathbf{e})]\boldsymbol{\epsilon},\\ \boldsymbol{\epsilon} &= \boldsymbol{\epsilon}_0+\boldsymbol{\epsilon}_c = \boldsymbol{\epsilon}_0+\boldsymbol{\epsilon}_b+\boldsymbol{\epsilon}_a = [\mathbf{S}_0+\mathbf{S}_b+\mathbf{S}_a(\mathbf{e})]\boldsymbol{\sigma} = [\hat{\mathbf{S}}+\mathbf{S}_a(\mathbf{s})]\boldsymbol{\sigma},\\ &\hat{\mathbf{C}} = \mathbf{C}_0+\mathbf{C}_b, \quad \hat{\mathbf{S}} = \mathbf{S}_0+\mathbf{S}_b. \end{aligned}$$

The $\mathbf{C}_0, \mathbf{S}_0$ elastic tensors refer to the plain (non-cracked) material whereas $\hat{\mathbf{C}}, \hat{\mathbf{S}}$ to the material with flaws (slits). Both types of the above tensors are material constants since they do not depend on the state of stress. In particular, $\hat{\mathbf{C}}, \hat{\mathbf{S}}$ may be calculated by methods of the theory of multiphase media for a material with "inclusions" in the form of flaws. For the isotropic, both material and crack system, we obtain for $\mathbf{C}_0$ as well as for $\hat{\mathbf{C}}$ two independent elastic constants albeit the numerical values of those are different.

Instead, the tensors $\mathbf{C}_a$, $\mathbf{S}_a$ for additional quantities depend, apart from geometry, on the state of stress which makes the cracks open or closed. The discussed tensor functions, and only these, cause nonlinearity due to local fields produced by unilateral constraint response forces as if the properties of inhomogeneous material changed according to load; therefore the theory exceeds the scope of the classical theory of multiphase bodies. In this respect, the decomposition in oriented crack systems provides a method which requires a possibly most restricted information needed for taking account of the constraints. According to these explanations, in the remaining part of the present section we analyse roughly the tensors $\mathbf{C}_b, \mathbf{S}_b$ (incidental here), while the fundamental tensors $\mathbf{C}_a$, $\mathbf{S}_a$ will be calculated in more detail in Sect. 3.

The constant material tensors $\hat{\mathbf{C}}, \hat{\mathbf{S}}$ are in principle, supposed to be known. All we can do within our restricted theory is to derive $\mathbf{C}_b, \mathbf{S}_b$ from the more fundamental quantities $\mathbf{C}_b^{(n)}, \mathbf{S}_b^{(n)}$ for the oriented resolved crack systems $\mathbf{n}$. For general anisotropy, $\mathbf{C}_b^{(n)}$ (and similarly for $\mathbf{S}_b^{(n)}$) is a function of $\mathbf{n}$, $\mathbf{C}_b^{(n)} = \mathbf{C}_b^{(n)}(\mathbf{n})$, depending on the history of crack formation and on material anisotropy, i.e. on $\mathbf{C}_0$ and is supposed to be preassigned. In the issue $\mathbf{C}_0$ and $\mathbf{C}_b$ may exibit different symmetry properties or, even for the same type of anisotropy, symmetry elements may spatially not coincide.

Consider in more detail the isotropic material (i.e. $\mathbf{C}_0$ is isotropic) with the non-interacting stochastically homogeneous crack system described by the function $\pi(\mathbf{n})$. Assume that for any partial crack system $\mathbf{n}$ there are no preferred directions orthogonal to $\mathbf{n}$ (transversal two-dimensional isotropy). This may occur, for instance, in a composite isotropic material if crack formation depends only on the normal (to crack plane) traction. Consequently, each oriented crack system is invariant under mirror reflection in any of the parallel planes $\mathbf{n}$, under any rotation about an axis $\mathbf{n}$ and reflection in any plane passing through this axis. This is the case of monotropy yielded by invariance under rotation about a hexagonal axis $\mathbf{n}$. With respect to elastic properties this is equivalent to any rotation about $\mathbf{n}$ and the above mentioned reflections, the elastic components being inversion-invariant. In Cartesian coordinates with $\mathbf{n}$ generating the x_3-axis the monotropic elastic matrix, denoted for a while by $\mathbf{C}^{(n)}$, contains 5 independent elastic constants and takes the general form

$$(2.4)\qquad [C^{(n)}_{ijkl}] = \begin{bmatrix} C^{(n)}_{1111} & C^{(n)}_{1122} & C^{(n)}_{1133} & 0 & 0 & 0 \\ & C^{(n)}_{1111} & C^{(n)}_{1133} & 0 & 0 & 0 \\ & & C^{()}_{3333} & 0 & 0 & 0 \\ & & & C^{(n)}_{1313} & 0 & 0 \\ & & & & C^{(n)}_{1313} & 0 \\ & & & & & \frac{1}{2}(C^{(n)}_{1111}-C^{(n)}_{1122}) \end{bmatrix}$$

and similarly for S. The matrix form (2.4) itself as applied to $\mathbf{C}_b$, holds for any $\mathbf{n}$, however, the theory admits of the terms dependent on $\mathbf{n}$, i.e. in general $C^{(n)}_{b\,ijkl} = C^{(n)}_{b\,ijkl}(\mathbf{n})$, since partial tensors depend on geometry of individual cracks.

Usign Eqs. (2.2) and (2.3) we define

$$(2.5)\qquad \mathbf{C}_b = P\int_\Omega \pi(\mathbf{n})\mathbf{C}^{(n)}_b(\mathbf{n})\,d\omega.$$

In a fixed Cartesian coordinate system, as used for $\boldsymbol{\epsilon}, \boldsymbol{\sigma}$ in Eq. (2.3), we have

$$(2.6)\qquad C'^{(n)}_{b\,ijkl} = n_{ip}n_{jq}n_{kr}n_{ls}C^{(n)}_{b\,pqrs},$$

where $C^{(n)}_{b\,pqrs}$ is provided by Eq. (2.4); n_{ip} are cosines of angles between the fixed x_i-axis and the auxiliary one (x_p) and yield the desired orthogonal transformation. In many cases, for "natural" cracks (i.e. not produced in an artificial manner nor by special periodic body forces) the tensors $\mathbf{C}^{(n)}_b$ may be assumed as independent of n, i.e. as material constants. Using Eqs. (2.5) and (2.6) we obtain

$$(2.7)\qquad C_{b\,ijkl} = P\int_\Omega \pi(\mathbf{n})n_{ip}\,n_{jq}\,n_{kr}n_{ls}\,d\omega\,C^{(n)}_{b\,pqrs},$$

where we integrate over the unit hemisphere according to the explanation following Eq. (1.2). Analogous formulae hold for S_b. Thus the problem reduces to averaging elastic tensors. For example, taking account of Eq. (2.4) we calculate (for constant C_b)

$$C_{1111} = a_1 C^{(n)}_{b1111}+a_2 C^{(n)}_{b1122}+a_3 C^{(n)}_{b1133}+a_4 C^{(n)}_{b3333}+a_5 C^{(n)}_{b1313},$$

where

$$a_1 = P\int_\Omega \pi(\mathbf{n})\,(n^4_{11}+n^4_{12}+n^2_{11}n^2_{12})\,d\omega,$$

$$a_2 = P\int_\Omega \pi(\mathbf{n})n^2_{11}n^2_{12}\,d\omega$$

and so on; one must bear in mind that summation takes place over all the terms $C^{(n)}_{bpqrs}$ irrespective of symmetry properties, e.g. for a_1, $C^{(n)}_{b1111}$ appears in terms with the indices 1111, 2222, 1212, 2121.

In the general case partial crack systems have no transversal symmetry and retain only invariance to mirror reflection in the n plane. Instead of Eq. (2.4) we obtain 13 constants

$$(2.8)\qquad [C^{(n)}_{ijkl}] = \begin{bmatrix} C^{(n)}_{1111} & C^{(n)}_{1122} & C^{(n)}_{1133} & 0 & 0 & C^{(n)}_{1112} \\ & C^{(n)}_{2222} & C^{(n)}_{2233} & 0 & 0 & C^{(n)}_{2212} \\ & & C^{(n)}_{3333} & 0 & 0 & C^{(n)}_{3312} \\ & & & C^{(n)}_{2323} & C^{(n)}_{2313} & 0 \\ & & & & C^{(n)}_{1313} & 0 \\ & & & & & C^{(n)}_{1212} \end{bmatrix}$$

and the components are functions of n; the formula (2.7) is written more explicitly as

$$C_{bijkl} = P \int_{\Omega} n_{ip} n_{jq} n_{kr} n_{ls} [\pi(\mathbf{n}) C^{(n)}_{b\,pqrs}(\mathbf{n})] d\omega \tag{2.9}$$

with tensor components in the brackets according to Eq. (2.8). In general, we integrate in spherical coordinates and select eventual symmetry axes of the functions of n in the brackets in Eq. (2.9) as coordinate axes (for details of calculation cf. Section 3).

3. Additional quantities

The concept of a resolved oriented crack system has made it possible to reveal certain restrictions on the form of the tensors $\mathbf{C}_b, \mathbf{S}_b$ (cf. Eqs. (2.7) and (2.9)), however we would rather find the tensors $\hat{\mathbf{C}}, \hat{\mathbf{S}}$ in Eq. (2.3) in a direct experimental test. Unfortunately, the latter cannot be separated from the quantities $\mathbf{C}_a, \mathbf{S}_a$ except, possibly, for some special modes of loading to be suggested by the theory. Thus the additional terms (which cause nonlinearity) are crucial for constitutive equations.

The basic assumption for the relation (1.1) is that the cracks be ideal, i.e. conservative, with no energy dissipation by friction. It follows that only normal-to-crack forces may appear at crack edges and these are unidirectional (compressive or zero). Consider an oriented crack system $\mathbf{n}$ and cut out a representative unit cube with a facet $\mathbf{n}$. Let the crack edge be loaded in a just explained manner. Then, under averaging, all boundary force vectors cancel except the normal on the facets $\mathbf{n}$. One must realize that, being interested in additional quantities, we take into account the virtual (separate) action of the said response forces while other agencies (say shear stresses) have already been included in material and basic quantities (cf. Eq. (2.3)).

Thus the kinematics is altogether provided by the additional strain tensor depending proportionally on the stress vector $\sigma_n \mathbf{n}$ where

$$\sigma_n = \mathbf{n} \cdot \boldsymbol{\sigma} \mathbf{n} \tag{3.1}$$

is the intensity of normal forces due to the preassigned bulk stress tensor $\boldsymbol{\sigma}$. Let us define the strain $\boldsymbol{\eta}^{(n)}$ as the additional strain tensor for the oriented resolved crack system $\mathbf{n}$ when $\sigma_n = 1$. The said quantity may be looked upon as a material constant and the tensor function $\boldsymbol{\eta}^{(n)}(\mathbf{n})$ (supposed to be known) yields the desired information about the properties of the cracked material in view of additional quantities. Express $\boldsymbol{\eta}^{(n)}$ in the form

$$\boldsymbol{\eta}^{(n)} = \sum_J \eta_J \mathbf{m}_J \otimes \mathbf{m}_J, \quad J = 1, 2, 3, \tag{3.2}$$

where η_J are principal strains and $\mathbf{m}_J$ principal directions of $\boldsymbol{\eta}^{(n)}$; in general $\eta_J = \eta_J(\mathbf{n}), \mathbf{m}_J = \mathbf{m}_J(\mathbf{n})$.

In particular, for an isotropic (plain) material the tensor (3.2) will coincide with crack orientation, consequently say $\mathbf{m}_1 = \mathbf{n}$. If, moreover, the cracks are transversally (statistically) isotropic, the two remaining principal directions will be equivalent, i.e. $\eta_2 = \eta_3$ and we obtain

$$\boldsymbol{\eta}^{(n)} = \eta_1 \mathbf{m}_1 \otimes \mathbf{m}_1 + \eta_2(\mathbf{m}_2 \otimes \mathbf{m}_2 + \mathbf{m}_3 \otimes \mathbf{m}_3) = \eta_1 \mathbf{m}_1 \otimes \mathbf{m}_1 + \eta_2(I - \mathbf{m}_1 \otimes \mathbf{m}_1)$$
$$= \eta_1 \mathbf{n} \otimes \mathbf{n} + \eta_2(\mathbf{I} - \mathbf{n} \otimes \mathbf{n}) = a_n I + b_n \mathbf{n} \otimes \mathbf{n},$$
$$a_n = \eta_2, \quad b = \eta_1 - \eta_2 \tag{3.3}$$

and $\mathbf{I}$ is the unit tensor of δ_{ij}. If, more to it, the above quantities do not depend on $\mathbf{n}$, we have for all resolved oriented systems

$$\boldsymbol{\eta}^{(n)} = a\mathbf{I} + b\mathbf{n} \otimes \mathbf{n}. \tag{3.4}$$

Analogous formulae may of course be derived for additional stresses.

Using Eq. (2.2) and bearing in mind that at no interaction simple superposition holds, we come at the bulk quantities

$$\boldsymbol{\epsilon}_a = P \int_\Omega \pi(\mathbf{n}) \boldsymbol{\eta}^{(n)}(\mathbf{n})\, (\mathbf{n} \cdot \boldsymbol{\sigma}\mathbf{n})\, d\omega. \tag{3.5}$$

The region of integration Ω now depends on the stress $\boldsymbol{\sigma}$ (which is crucial for further analysis) since the integrand strains equal

$$\boldsymbol{\eta}^{(n)}(\mathbf{n})(\mathbf{n} \cdot \boldsymbol{\sigma}\mathbf{n}) \quad \text{for} \quad \mathbf{n} \cdot \boldsymbol{\sigma}\mathbf{n} < 0 \quad \text{(cracks closed)}, \tag{3.6}$$

$$0 \quad \text{for} \quad \mathbf{n} \cdot \boldsymbol{\sigma}\mathbf{n} > 0 \quad \text{(cracks open)}. \tag{3.6}$$

Thus we integrate over all $\mathbf{n}$'s satisfying $\mathbf{n} \cdot \boldsymbol{\sigma}\mathbf{n} < 0$ ($\boldsymbol{\sigma}$ given) i.e. $\Omega = \Omega(\boldsymbol{\sigma})$.

Consequently, we obtain $\boldsymbol{\epsilon}_a = \boldsymbol{\epsilon}_a(\boldsymbol{\sigma})$, according to Eq. (2.3); a similar argument holds for $\boldsymbol{\sigma}_a = \boldsymbol{\sigma}_a(\boldsymbol{\epsilon})$. The sign rule is such as to make tension stresses and extension strains positive, as usual in continuum mechanics.

Let us perform a more detailed calculation of $\boldsymbol{\epsilon}_a$ for overall isotropy, i.e. for Eq. (3.4) and $\pi(\mathbf{n}) = 1/2\pi = \text{const}$ (do not confuse the two π's!). Substituting this in Eq. (3.5) we obtain

$$\boldsymbol{\epsilon}_a = \frac{P}{2\pi}\Big[a \int_\Omega \mathbf{I}\,(\mathbf{n} \cdot \boldsymbol{\sigma}\mathbf{n})\, d\omega + b \int_\Omega \mathbf{n} \otimes \mathbf{n}(\mathbf{n} \cdot \boldsymbol{\sigma}\mathbf{n})\, d\omega\Big],$$

$$\varepsilon_{a\,ij} = \frac{P}{2\pi}\Big[a\delta_{ij} \int_\Omega n_k n_l\, d\omega + b \int_\Omega n_i n_j n_k n_l\, d\omega\Big]\sigma_{kl},$$
$$\Omega = \{\mathbf{n} \colon n_k n_l \sigma_{kl} < 0\}. \tag{3.7}$$

Express, conveniently, all quantities in the Cartesian coordinates determined by the principal directions $\boldsymbol{\nu}_K$ of the stress tensor

$$\boldsymbol{\sigma} = \sum_K \sigma_K \boldsymbol{\nu}_K \otimes \boldsymbol{\nu}_K, \quad K = 1, 2, 3, \tag{3.8}$$

where σ_K are principal stresses. Then,

$$\mathbf{n} \cdot \boldsymbol{\sigma}\mathbf{n} = \sum_K \sigma_K (\mathbf{n} \cdot \boldsymbol{\nu}_K)^2 = \sum_K \sigma_K n_K^2,$$

where $n_K = \mathbf{n} \cdot \mathbf{v}_K$ are directional cosines of $\mathbf{n}$ in the basis $\mathbf{v}_K$ and hence, from Eq. $(3.7)_1$, in the said basis

$$\varepsilon_{aIJ} = \frac{P}{2\pi}\left[a\delta_{IJ}\sum_K \sigma_K \int_\Omega n_K^2 d\omega + b\sum_K \sigma_K \int_\Omega n_I n_J n_K^2 d\omega.\right.$$

Now, for overall isotropy, the principal directions of $\boldsymbol{\epsilon}_a$ and $\boldsymbol{\sigma}$ are expected to coincide and, in fact, the terms with $I \neq J$ are readily seen to disappear. Thus only diagonal terms are left which we assimilate to principal strains; hence, using one-index notation we obtain

$$\varepsilon_{aJ} = \frac{P}{2\pi}\sum_K \left(a\int_\Omega n_K^2 d\omega + b\int_\Omega n_J^2 n_K^2 d\omega\right)\sigma_K. \tag{3.9}$$

By Eq. (3.6) we integrate over all $\mathbf{n}$'s satisfying

$$\sum_K s_K n_K^2 < 0, \quad \sum_K n_K^2 = 1, \quad n_{K_0} > 0, \tag{3.10}$$

where σ_K has been replaced by normalized dimensionless $s_K, \sum\limits_K s_K^2 = 1$, in accordance with Eq (1.1), the left hand part of the inequality $(3.10)_1$ being a homogeneous linear form of σ_K; therefore $\boldsymbol{\epsilon}_a = \boldsymbol{\epsilon}_a(\mathbf{s})$. The equality $(3.10)_2$ in conjunction with the inequality $(3.10)_3$, where K_0 is one (arbitrary) of the indices 1, 2, 3, shows that the region of integration lies on a hemisphere (cf. explanations to Eq. (1.2)) which has been chosen in coincidence with the basis $\mathbf{v}_K$.

Thus the constitutive relation for additional strains in the basis $\mathbf{v}_K$ takes the form (cf. Eq. (2.3))

$$\varepsilon_{aJ} = \sum_K S_{aJK}\sigma_K = \frac{P}{2\pi}\sum_K [a\alpha_K(\mathbf{s}) + b\beta_{JK}(\mathbf{s})]\sigma_K, \tag{3.11}$$
$$\alpha_K = \int_\Omega n_K^2 d\omega, \quad \beta_{JK} = \int_\Omega n_J^2 n_K^2 d\omega,$$

the terms α_K, β_{JK} being functions of $\sigma_1:\sigma_2:\sigma_3 = s_1:s_2:s_3$ and a, b — certain material constants; the latter depend, apart from elastic properties, on the individual crack geometry. In invariant form

$$\mathbf{S}_a = \sum_{J,K} S_{aJK}\mathbf{v}_J \otimes \mathbf{v}_J \otimes \mathbf{v}_K \otimes \mathbf{v}_K. \tag{3.12}$$

These formulae are analogous to Eqs. (3.2) and (3.3) of [1] where the reader is referred to for detailed explanation. In particular, in an arbitrary orthogonal coordinate system (cf. [1], Eq. (3.5))

$$S_{a\,ijkl} = \sum_{J,K} S_{aJK}\nu_{Ji}\nu_{Jj}\nu_{Kk}\nu_{Kl}; \tag{3.13}$$

$\nu_{Ji}, \ldots,$ are respective cosines of angles between the coordinate axes and principal axes of the tensor $\boldsymbol{\sigma}$. Thus, while the S_{aJK}'s reflect the dependence on the ratio $\sigma_1:\sigma_2:\sigma_3$ of principal stresses, the remaining factors take account of spatial orientation of the stress tensor. The latter is being followed by the strain ϵ_a according to the "motion" of the closed crack system. The integrals involved. i.e. the functions $\alpha_K(S), \beta_{JK}(S)$ in Eq. (3.11) for different particular cases, are calculated in the Appendix.

4. Final remarks

The outlined theory yields the first approximation solution (for non-interacting cracks); it enables one to calculate effectively constitutive functions in Eq. (1.1). For simple crack geometries the latter are determined with accuracy to a few material constants (e.g. a, b in Eq. (3.11) for overall isotropy). The theory tells us how to determine these constants experimentally.

For instance, according to the Appendix we know that additional quantities disappear at overall tension, therefore we are able to determine the tensors $\hat{\mathbf{C}}, \hat{\mathbf{S}}$; then we find the constants a, b at overall compression tests (cf. (Eq. A4)). Finally, by formulae in the Appendix we determine theoretically the constitutive functions for all other modes of loading. Thus, by means of a few formulae and parameters the theory predicts the behaviour of material for infinity of load paths which, by experimental methods, would require a great many tests for different $\mathbf{e}$'s in Eq. (1.1), and would yield only numerical approximation. Note that in any case the constitutive tensors in Eq. (1.1) co-rotate with the stress tensor ellipsoid, cf. (Eq. (3.12)). However, while at overall tension (resp. compression), i.e. in the first (resp. the opposite) octant of the principal stress space of s_K (cf. Appendix, (I) and (II)), they do not depend on the s_K's and may be represented by a spherical surface, in the remaining octants they depend on stress, e.g. through the argument (A.5) (cf. also [1], Fig. 2).

In the present paper we have not analysed the problem of crack increase and propagation which would involve strength properties of the material apart from the elastic ones. It is obvious that in a step by step construction of the function $\pi(\mathbf{n})$, according to the load sequence, subsequent increments would depend on the momentary $\pi(\mathbf{n})$ and state of stress. Of course, in general the function $\pi(\mathbf{n})$ would not suffice and we would need more information about the relative spatial position of oriented crack elements, i.e. say multipoint correlation functions. However, the total amount of cracks and distribution after direction are the most fundamental characteristics, thus the theory is seen to be also the basic tool for approximate solution of that extended problem.

Appendix

Calculation of integrals in the formula (3.11)

It may be shown, under assumptions more general than in the present paper (cf. [1], Eq. (3.6) and relevant explanations), that constitutive functions, in particular $S_{aJK}(\mathbf{s})$, can be reduced to two only, say S_{a11}, S_{a12}. In view of the quoted equalities each of these functions takes in general 6 different forms for consecutive subdomains, according to the signs of arguments, symbolically,

$$
\text{(A.1)}\quad
\begin{aligned}
&S_{a11}(s_1, s_2, s_3): \quad S_{a11}(+,+,+), S_{a11}(-,-,-), S_{a11}(+,-,-),\\
&\qquad\qquad S_{a11}(-,+,+), S_{a11}(+,+,-), S_{a11}(-,+,-);\\
&S_{a12}(s_1, s_2, s_3): \quad S_{a12}(+,+,+), S_{a12}(-,-,-), S_{a12}(+,-,-),\\
&\qquad\qquad S_{a12}(-,+,+), S_{a12}(+,+,-), S_{a12}(-,+,-).
\end{aligned}
$$

In virtue of Eq. (3.11) calculation reduces to the three integrals

$$\int\limits_{\Omega} n_1^2\, d\omega, \qquad \int\limits_{\Omega} n_1^4\, d\omega, \qquad \int\limits_{\Omega} n_1^2 n_2^2\, d\omega,$$

the first two of which enter into S_{a11} whereas the first and the third one into S_{a12}. For each of these integrals the above sign combinations of s_1, s_2, s_3 are to be considered. Explicit formulae can be derived for uniform lateral stress, i.e. for say arbitrary s_1 and $s_2 = s_3$. In particular, the following cases will be discussed:

(I) Overall tension (not necessarily homogeneous), $s_K \geqslant 0$, $K = 1, 2, 3$. The inequality $(3.10)_1$ is not satisfied for any n_K, consequently Ω is an empty set and $\boldsymbol{\epsilon}_a = 0$, all cracks being open (clearly, this holds only under our general assumption of non-interacting cracks).

(II) Overall compression, $s_K < 0$, $K = 1, 2, 3$. Now Eq. (3.10) is accomplished for all $\mathbf{n}$'s and Ω is the whole hemisphere. Select spherical coordinates (Fig. 1) with $d\omega = \sin\varphi\, d\varphi\, d\vartheta$ and take, without loss of generality.

$$\text{(A.2)}\qquad n_K = n_1 = \cos\varphi, \quad n_2 = \sin\varphi\cos\vartheta, \quad n_3 = \sin\varphi\sin\vartheta.$$

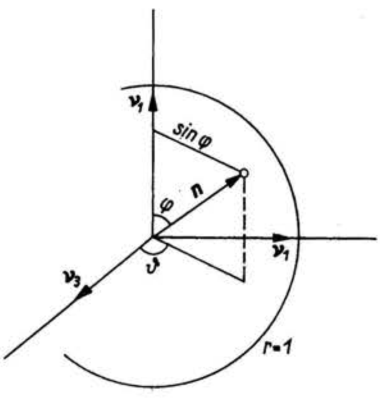

FIG. 1.

Hence

$$\alpha_K(\mathbf{s}) = \int\limits_{\Omega} n_1^2\, d\omega = \int\limits_{\vartheta}\int\limits_{\varphi} \cos^2\varphi \sin\varphi\, d\varphi\, d\vartheta = -\int\limits_0^{2\pi} d\vartheta \int\limits_0^{\pi/2} \cos^2\varphi\, d(\cos\varphi)$$

$$= -2\pi \int\limits_1^0 \zeta^2 d\zeta = \frac{2}{3}\pi$$

(with substitution of the new variable $\zeta = \cos\varphi$), i.e. α_K does not depend on $\mathbf{s}$.

The integrals β_{JK} reduce to the following ones:

$$\int_{\Omega} n_1^4 d\omega = \int_{\vartheta}\int_{\varphi} \cos^4\varphi \sin\varphi d\varphi d\vartheta = -2\pi \int_0^{\pi/2} \cos^4\varphi d(\cos\varphi) = \frac{2}{5}\pi,$$

$$\int_{\Omega} n_2^2 n_1^2 d\omega = \int_{\vartheta}\int_{\varphi} (\sin\varphi\cos\vartheta)^2 \cos^2\varphi \sin\varphi d\varphi d\vartheta = \int_0^{2\pi} \cos^2\vartheta d\vartheta \int_0^{\pi/2} \sin^3\varphi\cos^2\varphi d\varphi$$

$$= \left[\frac{1}{2}\sin\vartheta\cos\vartheta + \frac{1}{2}\vartheta\right]_0^{2\pi} \left[\frac{1}{16}\left(\frac{1}{5}\cos 5\varphi - \frac{1}{3}\cos 3\varphi - 2\cos\varphi\right)\right]_0^{\pi/2} = \frac{2}{15}\pi.$$

Both results can be combined into the following one:

$$\beta_{JK} = \frac{2}{15}\pi(1+2\delta_{JK}) \tag{A.3}$$

and consequently, according to Eq. (3.11),

$$\varepsilon_{aJ}/P = \left(a + \frac{b}{5}\right)\sigma + \frac{2}{15} b\delta_{JK}\sigma_K, \qquad \sigma = \frac{1}{3}\operatorname{tr}\boldsymbol{\sigma}. \tag{A.4}$$

This holds for overall compression: thus in the $(-,-,-)$ octant of the coordinates (s_1, s_2, s_3) we have obtained an apparently linear-elastic relation with coefficients on respective spheres.

(III) "Mixed" conditions for (s_1, s_2, s_3) of the type $(+,-,-)$ and $(-,+,+)$ where we assume $s_2 = s_3$ (uniform lateral load). According to Eq. $(3.10)_1$ we have

$$\text{case } (+,-,-): \quad n_1^2 < \frac{|s_2|}{s_1}(n_2^2+n_3^2), \quad s_2 = s_3 < 0,$$

$$\text{case } (-,+,+): \quad n_1^2 > \frac{s_2}{|s_1|}(n_2^2+n_3^2), \quad s_2 = s_3 > 0.$$

Substituting Eq. (A.2) we obtain

$$(+,-,-): \quad \frac{s_1}{|s_2|} < \operatorname{tg}^2\varphi \quad \text{or} \quad \varphi > \varphi_0 = \operatorname{arctg}\sqrt{\frac{s_1}{|s_2|}},$$

$$(-,+,+): \quad \frac{|s_1|}{s_2} > \operatorname{tg}^2\varphi \quad \text{or} \quad \varphi < \varphi_0 = \operatorname{arctg}\sqrt{\frac{|s_1|}{s_2}}.$$

Thus integration is performed over the part of the unit sphere cut out by the spherical cone $\varphi = \varphi_0$ and the following results are obtained:

case $(-,+,+)$

$$\int_{\Omega} n_1^2 d\omega = 2\pi \int_0^{\varphi_0} \cos^2\varphi \sin\varphi d\varphi = -\frac{2\pi}{3}\cos^3\varphi\Big|_0^{\varphi_0} = \frac{2}{3}\pi(1-\cos^3\varphi_0),$$

$$\int_{\Omega} n_1^4 d\omega = 2\pi \int_0^{\varphi_0} \cos^4\varphi \sin\varphi d\varphi = -2\pi \int_1^{\cos\varphi_0} \zeta^4 d\zeta = \frac{2}{5}\pi(1-\cos^5\varphi_0),$$

$$\int_{\Omega} n_1^2 n_2^2 d\omega = \int_0^{2\pi} \cos^2\vartheta d\vartheta \int_0^{\varphi_0} \sin^2\varphi \cos^2\varphi \sin\varphi d\varphi = \pi \int_0^{\varphi_0} (\cos^4\varphi - \cos^2\varphi) d(\cos\varphi)$$

$$= \pi \int_1^{\cos\varphi_0} (\zeta^4 - \zeta^2) d\zeta = \frac{\pi}{15}(2 - 5\cos^3\varphi_0 + 3\cos^5\varphi_0);$$

case $(+, -, -)$

$$\int_{\Omega} n_1^2 d\omega = -\frac{2\pi}{3}\cos^3\varphi\Big|_{\varphi_0}^{\pi/2} = \frac{2}{3}\pi\cos^3\varphi_0,$$

$$\int_{\Omega} n_1^4 d\omega = 2\pi \int_{\varphi_0}^{\pi/2} \cos^4\varphi \sin\varphi d\varphi = -2\pi \int_{\cos\varphi_0}^{0} \zeta^4 d\zeta = \frac{2}{5}\pi\cos^5\varphi_0,$$

$$\int_{\Omega} n_1^2 n_2^2 d\omega = \pi \int_{\cos\varphi_0}^{0} (\zeta^4 - \zeta^2) d\zeta = \frac{\pi}{15}(5\cos^3\varphi_0 - 3\cos^5\varphi_0).$$

(IV) Mixed conditions of type $(+, +, -)$ and $(-, +, -)$ with $s_1 = s_2$ and $s_1 = s_3$, respectively, according to the two last terms in Eq. (A.1). However, in that case a direct calculation of the relevant integrals, for $s_2 = s_3$, turns out to be more convenient. Thus we calculate

$$\int_{\Omega} n_2^2 d\omega, \quad \int_{\Omega} n_2^4 d\omega, \quad \int_{\Omega} n_2^2 n_3^2 d\omega$$

in coordinates and under conditions of (III) (in view of $s_2 = s_3$ the respective integrals for n_3 have the same value).

case $(-, +, +)$

$$\int_{\Omega} n_2^2 d\omega = \int_{\vartheta}\int_{\varphi} (\sin\varphi\cos\vartheta)^2 \sin\varphi d\varphi d\vartheta = \int_0^{2\pi} \cos^2\vartheta d\vartheta \int_0^{\varphi_0} \sin^3\varphi d\varphi$$

$$= \pi\left[\frac{1}{3}\cos^3\varphi - \cos\varphi\right]_0^{\varphi_0} = \frac{\pi}{3}(2 - 3\cos\varphi_0 + \cos^3\varphi_0),$$

$$\int_{\Omega} n_2^4 d\omega = \int_{\vartheta}\int_{\varphi} (\sin\varphi\cos\vartheta)^4 \sin\varphi d\varphi d\vartheta = \int_0^{2\pi} \cos^4\vartheta d\vartheta \int_0^{\varphi_0} \sin^4\varphi \sin\varphi d\varphi$$

$$= \int_0^{2\pi} \cos^2\vartheta(1 - \sin^2\vartheta) d\vartheta \int_0^{\varphi_0} (1 - \cos^2\varphi)^2 \sin\varphi d\varphi = \left[-\frac{1}{8}\left(\frac{1}{4}\sin 4\vartheta - \vartheta\right)\right.$$

$$\left. - \frac{1}{2}\sin\vartheta\cos\vartheta - \frac{1}{2}\vartheta\right]_0^{2\pi} \int_1^{\cos\varphi_0} (1 - 2\zeta^2 + \zeta^4) d\zeta = -\frac{3}{4}\pi\left[\zeta - \frac{2}{3}\zeta^3 + \frac{1}{5}\zeta^5\right]_1^{\cos\varphi_0}$$

$$= \frac{\pi}{20}(8 - 15\cos\varphi_0 + 10\cos^3\varphi_0 - 3\cos^5\varphi_0),$$

$$\int_{\Omega} n_2^2 n_3^2 d\omega = \int_{\vartheta}\int_{\varphi} (\sin\varphi\cos\vartheta)^2(\sin\varphi\sin\vartheta)^2\sin\varphi\, d\varphi\, d\vartheta$$

$$= \int_0^{2\pi} \cos^2\vartheta\sin^2\vartheta\, d\vartheta \int_0^{\varphi_0} (1-\cos^2\varphi)^2\sin\varphi\, d\varphi = \frac{1}{8}\left[\frac{1}{4}\sin 4\vartheta - \vartheta\right]_0^{2\pi} \int_1^{\cos\varphi_0} (1-2\zeta^2+\zeta^4)\, d\zeta$$

$$= \frac{\pi}{60}(8-15\cos\varphi_0+10\cos^3\varphi_0-3\cos^5\varphi_0);$$

case $(+, -, -)$

$$\int_{\Omega} n_2^2 d\omega = \int_0^{2\pi} \cos^2\vartheta\, d\vartheta \int_{\varphi_0}^{\pi/2} \sin^3\varphi\, d\varphi = \pi\left[\frac{1}{3}\cos^3\varphi - \cos\varphi\right]_{\varphi_0}^{\pi/2} = \frac{\pi}{3}(3\cos\varphi_0 - \cos^3\varphi_0),$$

$$\int_{\Omega} n_2^4 d\omega = \int_0^{2\pi} \cos^4\vartheta\, d\vartheta \int_{\varphi_0}^{\pi/2} \sin^4\varphi\sin\varphi\, d\varphi = -\frac{3}{4}\pi\left[\zeta - \frac{2}{3}\zeta^3 + \frac{1}{5}\zeta^5\right]_{\cos\varphi_0}^{0}$$

$$= \frac{\pi}{20}(15\cos\varphi_0 - 10\cos^3\varphi_0 + 3\cos^5\varphi_0),$$

$$\int_{\Omega} n_2^2 n_3^2 d\omega = \int_0^{2\pi} \cos^2\vartheta\sin^2\vartheta\, d\vartheta \int_{\varphi_0}^{\pi/2} \sin^4\varphi\sin\varphi\, d\varphi = -\frac{\pi}{4}\left[\zeta - \frac{2}{3}\zeta^3 + \frac{1}{5}\zeta^5\right]_{\cos\varphi_0}^{0}$$

$$= \frac{\pi}{60}(15\cos\varphi_0 - 10\cos^3\varphi_0 + 3\cos^5\varphi_0).$$

All integrals under (III) and (IV) are seen to be certain polynomials of the argument

$$\cos\left(\operatorname{arctg}\sqrt{\left|\frac{s_1}{s_2}\right|}\right) \tag{A.5}$$

and, according to Eq. (3.11), so are the components of S_{aJK}.

(V) The general case of unequal s_1, s_2, s_3. The analysis leads to elliptic-type integrals which can be calculated numerically, so we restrict ourselves to one example. Suppose $s_1 < 0$, $s_2 > s_3 > 0$; the condition $(3.10)_1$ reads

$$s_1(\cos\varphi)^2 + s_2(\sin\varphi\cos\vartheta)^2 + s_3(\sin\varphi\sin\vartheta)^2 < 0,$$

i.e.

$$s_2\cos^2\vartheta + s_3\sin^2\vartheta < |s_1|\operatorname{cotg}^2\varphi,$$

or

$$\varphi < \operatorname{arccotg}\sqrt{\frac{s_2}{|s_1|}\cos^2\vartheta + \frac{s_3}{|s_1|}\sin^2\vartheta} = \operatorname{arcotg}\left(\beta\sqrt{1-\varkappa^2\sin^2\vartheta}\right),$$

$$\beta = \sqrt{\frac{s_2}{|s_1|}}, \quad \varkappa^2 = \frac{s_2 - s_1}{s_2}.$$

This time the domain of integration is cut out by an elliptic cone since, in Cartesian coordinates (cf. Fig. 1).

$$s_2 x^2 + s_3 y^2 < |s_1|\varrho^2\operatorname{cotg}^2\varphi = |s_1| z^2,$$

where ϱ is the generating radius of the ellipse (the same follows also immediately from Eq. $(3.10)_1$ upon replacing the n_K's by x, y, z, respectively).

Let us calculate, for example,

$$\int_{\Omega} n_1^2\,d\omega = \int_{\vartheta}\int_{\varphi}\cos^2\varphi\sin\varphi\,d\varphi\,d\vartheta = -\int_{\vartheta}\left[\int_{\varphi}\cos^2\varphi\,d(\cos\varphi)\right]d\vartheta$$

$$= -\int_0^{2\pi}\left[\int_1^{\cos\varphi(\vartheta)}\zeta^2 d\zeta\right]d\vartheta = \frac{2}{3}\pi - \frac{1}{3}\int_0^{2\pi}\cos^3\varphi(\vartheta)\,d\vartheta$$

$$= \frac{2}{3}\pi - \frac{1}{3}\int_0^{2\pi}\cos^3\left[\operatorname{arcotg}\left(\beta\sqrt{1-\varkappa^2\sin^2\vartheta}\right)\right]d\vartheta$$

(further numerical calculation).

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Received April 6, 1978.